Some Remarks on the Affinely Connected Areal Space

著者	IGARASHI Takanori
journal or publication title	Memoirs of the Muroran Institute of Technology
volume	5
number	2
page range	925-928
year	1966-08-25
URL	http://hdl.handle.net/10258/3286

Some Remarks on the Affinely Connected Areal Space

Takanori Igarashi※

Abstract

An areal space $A_n^{(m)}$ is defined as an n-dimensional space in which an m-dimensional areal metric is endowed *a priori* in the form

$$S=\int_{(m)}^{\cdots}\int F(x,p)\,du^1\cdots du^m \quad \text{with} \quad p\equiv(p_\alpha^i)\equiv\left(\frac{\partial x^i}{\partial u^\alpha}\right)^{※※}$$

over a region of the subspece V_m given parametrically by $x^i=x^i(u^\alpha)$.

If a metric tensor g_{ij} can be constructed algebraically from the fundamental function $F(x,p)$ and its first and second derivatives with respect to p's, then the space is called to be of the submetric class. In the present paper, we adopt the normalized metric tensor as the metric tensor in a space of submetric class.

The aim of the present paper is to characterize the covariant derivative with respect to x in a special case. In §1, we find that the ecmetric tensor $L^{*\alpha\beta}_{ij}$ is covariant constant when the space $A_n^{(m)}$ is affinely connected. In §2, we require the necessary and sufficient condition in order that the metric bitensor $g_{ij,kl}\equiv\frac{1}{2}\frac{\partial^2 F}{\partial p^{ij}\partial p^{kl}}$ in $A_n^{(2)}$ is covariant constant.

§ 1. The connection theory of the areal space has been discussed by many scholars.[1-10] We use the connection which has been defined by A. Kawaguchi and K. Tandai[4] by means of the normalized metric tensor g_{ij}.

g_{ij} is defined as follows:

$$(1.1)\qquad g_{ij}=\left(\frac{1}{m}L_{ij}^{\alpha\beta}+p_i^\alpha p_j^\beta\right)g_{\alpha\beta}, \qquad g_{\alpha\beta}=g_{ij}p_\alpha^i p_\beta^j \quad \text{and} \quad |g_{\alpha\beta}|=F^2$$

where $p_i^\alpha\equiv F^{-1}\dfrac{\partial F}{\partial p_\alpha^i}$ satisfies the relation

$$(1.2)\qquad p_i^\alpha=g_{ij}g^{\alpha\beta}p_\beta^j, \qquad g^{\alpha\beta}g_{\beta\gamma}=\delta_\gamma^\alpha$$

and $L_{ij}^{\alpha\beta}\equiv p_{i;j}^{\alpha\ \beta}+p_i^\beta p_j^\alpha$ is the Legendre's form of F.

The covariant differential of the contravatiant vector $X^i(x,p)$ which is homogeneous of degree 0 in p's is in the form

$$(1.3)\qquad DX^i=X^i{}_{|h}dx^h+X^i|_l^\alpha Dp_\alpha^l,$$

putting

$$(1.4)\qquad X^i{}_{|h}=X^i{}_{;h}-X^i{}_{;l}^{\ \alpha}\Gamma^{*l}{}_{\alpha h}+X^j\Gamma^{*i}{}_{jh}, \qquad X^i|_l^\alpha=X^i{}_{;l}^{\ \alpha}+C_{jl}^{i\alpha}X^j,$$

where the notations $;k$ and $\overset{\alpha}{;l}$ mean e.g. $X^i{}_{;k}=\partial X^i/\partial x^k$, $X^i{}_{;l}^{\ \alpha}=\partial X^i/\partial p_\alpha^l$.

※ 五十嵐敬典

※※ Latin indices run over $1,2,\cdots,n$; Greek indices over $1,2,\cdots,m$ (in §2, especially over 1, 2).

We need two postulates: (1) $Dg_{ij}=0$, (2) $\Gamma^*_{ijk}=\Gamma^*_{jik}$, then the connection parameters Γ^{*i}_{jk} and $C^{i}_{j,l}{}^{\alpha}$ satisfy the following conditions

(1.5) $$g_{ij;k}-g_{ij;l}{}^{\alpha}\Gamma^{*l}{}_{\alpha k}=g_{hj}\Gamma^{*h}{}_{ik}+g_{ih}\Gamma^{*h}{}_{jk}, \qquad g_{ij;l}{}^{\alpha}=g_{hj}C^{h}_{i,l}{}^{\alpha}+g_{ih}C^{h}_{j,l}{}^{\alpha}.$$

Let us name $X^i{}_{|h}$ the *covariant derivative of X^i with respect to x* according to M. Gama[8]. By means of (1.4), we have

$$(X^i{}_{|h})_{;j}^{\beta}=X^i{}_{;h;j}^{\ \ \beta}-X^i{}_{;l;j}^{\ \alpha\beta}\Gamma^{*l}{}_{\alpha h}-X^i{}_{;l}^{\alpha}\Gamma^{*l}{}_{kh;j}^{\ \ \beta}p^k_\alpha-X^i{}_{;l}^{\beta}\Gamma^{*l}{}_{jh}+X^k{}_{;j}^{\beta}\Gamma^{*i}{}_{kh}+X^k\Gamma^{*i}{}_{kh;j}^{\ \ \beta},$$
$$(X^i{}_{;j}^{\beta})_{|h}=X^i{}_{;j;h}^{\beta}-X^i{}_{;j;l}^{\beta\alpha}\Gamma^{*l}{}_{\alpha h}+X^k{}_{;j}^{\beta}\Gamma^{*i}{}_{kh}-X^i{}_{;k}^{\beta}\Gamma^{*k}{}_{jh}.$$

Hence, the commutation of operators $|h$ and $\overset{\beta}{;j}$ for X^i is that

(1.6) $$(X^i{}_{|h})_{;j}^{\beta}-(X^i{}_{;j}^{\beta})_{|h}=-X^i{}_{;l}^{\alpha}\Gamma^{*l}{}_{kh;j}^{\ \ \beta}p^k_\alpha+X^k\Gamma^{*i}{}_{kh;j}^{\ \ \beta}$$
$$=(\delta^i_l X^k-X^i{}_{;l}^{\alpha}p^k_\alpha)\Gamma^{*l}{}_{kh;j}^{\ \ \beta}.$$

Applying (1.6) to the metric tensor g_{ij}, we have

$$(g_{ik|h})_{;j}^{\beta}-(g_{ik;j}^{\ \ \beta})_{|h}=(-\delta^s_i g_{lk}-\delta^s_k g_{il}-g_{ik;l}^{\ \ \alpha}p^s_\alpha)\Gamma^{*l}{}_{sh;j}^{\ \ \beta}.$$

This relation give us

(1.7) $$(g_{ik;j}^{\ \ \beta})_{|h}=(\delta^s_i g_{lk}+\delta^s_k g_{il}+g_{ik;l}^{\ \ \alpha}p^s_\alpha)\Gamma^{*l}{}_{sh;j}^{\ \ \beta},$$

because of $g_{ik|h}=0$ from (1.4) and the postulate (1).

The space in which the connection parameter Γ^{*i}_{jk} is independent of p's is called an *affinely connected* one. In such a space, $\Gamma^{*i}{}_{jk;l}^{\ \ \alpha}=0$ holds good. Thus, from (1.7), we have

Theorem 1.1. When the areal space is affinely connected, then the partial derivatives of the metric tensor g_{ij} with respect to p's are covariant constant.

In his paper[11], A. Kawaguchi has represented the *ecmetric tensor* $L^{*\alpha\beta}_{ij}\equiv L^{\alpha\beta}_{ij}-g^{\alpha\beta}g_{\gamma\delta}L^{\gamma\delta}_{ij}$ in the form

(1.8) $$L^{*\alpha\beta}_{ij}=g^{\alpha\gamma}g_{sk;j}^{\ \ \beta}\gamma^s_i p^k_\gamma.$$

In view of $g^{\alpha\gamma}{}_{|h}=0$, $p^k_{\gamma|h}=0$, $\gamma^s_{i|h}=0$, it follows from (1.9) that

(1.9) $$L^{*\alpha\beta}_{ij|h}=g^{\alpha\gamma}(\delta^t_s g_{lk}+\delta^t_k g_{sl}+g_{sk;l}^{\ \ \alpha}p^t_\alpha)\Gamma^{*l}{}_{th;j}^{\ \ \beta}\gamma^s_i p^k_\gamma$$
$$=(g^{\alpha\gamma}g_{lk}\gamma^t_i p^k_\gamma+\hat{L}^{\alpha\gamma}_{il}p^t_\gamma+L^{*\alpha\gamma}_{il}p^t_\gamma)\Gamma^{*l}{}_{th;j}^{\ \ \beta},$$

where we use the notation $\hat{L}^{\alpha\beta}_{ij}\equiv g^{\alpha\beta}g_{ik}\gamma^k_i$ according to E. T. Davies[12].

Substituting the relation $L^{\alpha\beta}_{ij}=\hat{L}^{\alpha\beta}_{ij}+L^{*\alpha\beta}_{ij}$ into (1.9), we get

$$L^{*\alpha\beta}_{ij|h}=(p^\alpha_l\gamma^t_i+L^{\alpha\gamma}_{il}p^t_\gamma)\Gamma^{*l}{}_{th;j}^{\ \ \beta}=(p^\alpha_l\delta^t_i+p^\gamma_{l;i}{}^{\alpha}p^t_\gamma)\Gamma^{*l}{}_{th;j}^{\ \ \beta}$$
$$=(p^\gamma_l p^t_\gamma)_{;i}^{\alpha}\Gamma^{*l}{}_{th;j}^{\ \ \beta},$$

finally

(1.10) $$L^{*\alpha\beta}_{ij|h}=\beta^{t\ \alpha}_{l;i}\Gamma^{*l}{}_{th;j}^{\ \ \beta}.$$

From this fact, we can conclude

Theorem 1.2. *When the areal space is affinely connected, then the ecmetric* *tensor* $L^{*\alpha\beta}_{ij}$ *is covariant constant.*

§ 2. Let us consider an areal space $A_n^{(2)}$ based on the area of 2-dimensional surface $x^i = x^i(u^1, u^2)$.

The *metric bitensor*[3] $g_{ij,kl}$ is expressed in the form

$$(2.1) \qquad g_{ij,kl} = 4F^2 (L^{[1[1}_{[i[k} L^{2]2]}_{j]l]} + 4L^{[1[1}_{[i[k} p^{2]}_{j]} p^{2]}_{l]} + p^{[1}_{[i} p^{[1}_{[k} p^{2)}_{j)} p^{2]}_{l]}) \,.$$

In view of (1.2), on differentiating $g_{\delta\gamma} p^\delta_i = g_{ik} p^k_\gamma$ with respect to p^j_β, and contracting $g^{\gamma\alpha}$, we have

$$(2.2) \qquad \begin{aligned} L^{\alpha\beta}_{ij} &= (g_{ij} - g_{\delta\gamma} p^\delta_i p^\gamma_j) g^{\alpha\beta} + \gamma^h_i g_{hk;}{}^{\beta}_{j} p^k_\gamma g^{\gamma\alpha} \\ &= (g_{ij} - g_{\delta\gamma} p^\delta_i p^\gamma_j) g^{\alpha\beta} + L^{*\alpha\beta}_{ij} \end{aligned}$$

making use of (1.8). Substituting (2.2) into (2.1), it follows that

$$(2.3) \qquad g_{ij,kl} = g_{ik} g_{jl} - g_{il} g_{;k} + 4F^2 L^{*[1[1}_{[i[k} (L^{*2]2]}_{j]l]} + 4p^{2]}_{j]} p^{2]}_{l]}) \,.$$

The covariant derivative of $g_{ij,kl}$ with respect to x is given from (2.3) by

$$(2.4) \qquad \begin{aligned} g_{ij,kl|h} &= 4F^2 L^{*[1[1}_{[i[k]|h|} (L^{*2]2]}_{j]l]} + 4p^{2]}_{j]} p^{2]}_{l]}) + 4F^2 L^{*[1[1}_{[i[k} L^{*2]2]}_{j]l]|h} \\ &= 8F^2 L^{*[1[1}_{[i[k||h|} (L^{*2]2]}_{j]l]} + 2p^{2]}_{j]} p^{2]}_{l]}) \,, \end{aligned}$$

making use of $g_{ij|h} = 0$ and $p^\alpha_{i|h} = 0$.

Thus, we have

Theorem 2.1. *In the areal space* $A_n^{(2)}$, *it is necessary and sufficient that the ecmetric tensor satisfies* $L^{*\alpha\beta}_{ij|h} = 0$ *in order that the covariant derivatives of the metric bitennsor* $g_{ij,kl}$ *with respect to* x*'s vanish.*

Corollary. *In* $A_n^{(2)}$, *the covariant derivatives of* $g_{ij,kl}$ *with respect to* x*'s vanish when and only when* $\beta^t_{l;}{}^{\alpha}_{i} \Gamma^{*l}{}_{th;}{}^{\beta}_{j} = 0$ *holds good.*

Because we can get from (2.4) that

$$(2.5) \qquad g_{ij,kl|h} = 8F^2 \beta^t_{s;}{}^{[1}_{[i} \Gamma^{*|s|}{}_{|th|;}{}^{[1}_{[k} (L^{*2]2]}_{j]l]} + 2p^{2]}_{j]} p^{2]}_{l]})$$

by means of (1.10).

From (2.5), we can conclude

Theorem 2.2. *When the areal space* $A_n^{(2)}$ *is affinely connected, then the metric bitensor* $g_{ij,kl}$ *is covariant constant.*

Acknowledgement. The present author wishes to express his best thanks to Prof. A. Kawaguchi and Mr. M. Gama who kindly gave him guidance and advice about the study of this problem.

Seminar of Mathematics,
Muroran Institute of Technology,
Muroran Japan.

(Received Apr. 27, 1966)

References

1) A. Kawaguchi und S. Hokari: Die Grundlegen der Geometrie der fünfdimensionalen metrischen Räume auf Grund des Begriffs des zweidimensionalen Flächeninhalt, *Proc. of Imp. Acad.*, vol. 16 (1940), 313–319.

2) A. Kawaguchi: Connection parameters of areal spaces, *Tensor* (*O. S.*), vol. 9 (1949), 38–40, (in Japanese).

3) A. Kawaguchi: On areal space I, Metric tensors in n-dimensional space based on the notion of two-dimensional area, *Tensor* (*N. S.*), vol. 1 (1950), 14–45.

4) A. Kawaguchi and K. Tandai: On areal space V, Normalized metric tensor and connection parameters in a space of the submetric class, *Tensor* (*N. S.*), vol. 2 (1952), 47–58.

5) K. Tandai: On general connections in an areal space I, General connection on a fibre of the tangent m-frame bundle, *Tensor* (*N. S.*), vol. 13 (1963), 277–291.

6) Su Buchin: Certain affinely connected spaces with areal metrics, *Acta Math. Sinica*, vol. 7 (1957), 285–294.

7) E. T. Davies: Areal spaces, *Annali di Matematica*, (IV), vol. 55 (1961), 22–30.

8) M. Gama: On areal spaces of the submetric class, *Tensor* (*N. S.*), vol. 16 (1965), 262–268.

9) R. Debever: Sur une classe de formes quadratiques extérieurs et la géométrie fondés sur la notion d'aire, *Comptes Rendus*, 224 (1947), 1269–1271.

10) H. Iwamoto: On the geometry in a space based on the notion of area, *Tensor* (*O. S.*), vol. 9 (1949), 7–17, (in Japanese).

11) A. Kawaguchi: A remark to the theory of areal spaces, *Nieuw Archief voor Wiskunde*, (3), vol. 2 (1954), 115–117.

12) E. T. Davies: Vedute generali sugli spazi variazionali, *Geometria del Calcolo dell variazioni*, vol. 1 (1961), C.I.M.E., Roma-Instituto Matematico dell'Università.