https://doi.org/10.1016/j.geomphys.2022.104728

Journal of Geometry and Physics 185 (2023) 104728

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/geomphys

A filtration associated to an abelian inner ideal of a Lie algebra ☆

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ARTICLE INFO

Article history:
Received 11 February 2021
Received in revised form 28 October 2022
Accepted 7 December 2022
Available online 14 December 2022

MSC:
17B60
17B65
17B05

Keywords:
Lie algebra
Inner ideal
Subquotient

ABSTRACT

Let B be an abelian inner ideal and let $\mathrm{Ker}_L B$ be the kernel of B. In this paper we show that when there exists $n \in \mathbb{N}$ with $[B, \mathrm{Ker}_L B]^n \subset B$, the inner ideal B induces a bounded filtration in L where B is the first nonzero submodule of the filtration and where the wings of the Lie algebra associated to the filtration coincide with the subquotient determined by B. This filtration extends the principal filtration induced by ad-nilpotent elements of index less than or equal to three defined in [22].

1. Introduction

Inner ideals of Lie algebras are the Lie analogues of one-sided ideals of the associative setting. They are submodules B of a Lie algebra L (over a ring of scalars) such that $[B, [B, L]] \subset B$. An abelian inner ideal is an inner ideal which is also an abelian subalgebra, i.e., such that $[B, B] = 0$. They were first introduced in 1977 by G. Benkart [6], where she stated that "it is hoped that inner ideals will play a role analogous to Jordan inner ideals in the development of an Artinian theory for Lie algebras". The inner ideals of a Lie algebra are closely related to the ad-nilpotent elements, and certain restrictions on the ad-nilpotent elements yield an elementary criterion for distinguishing the nonclassical from the classical simple Lie algebras over algebraically closed fields of characteristic $p > 5$. Inner ideals of classical Lie algebras were classified by G. Benkart and A. Fernández López [5] and [7], using the fact that these algebras can be obtained as the derived Lie subalgebras of (involution) simple Artinian associative rings.

As G. Benkart had predicted in her seminal paper, an Artinian theory for Lie algebras was obtained in [18] and [19] by making use of inner ideals; inner ideals of simple finite dimensional Lie algebras were classified in [14], and the classification of inner ideals of finitary simple Lie algebras was obtained in [17]. We highlight the paper by A. Baranov and J. Rowley [1]

☆ All authors were partially supported by the Junta de Andalucía FQM264, and by the Universidad de Málaga B4: Ayudas para Proyectos Puente UMA "Sistemas de Jordan, álgebras de Lie y estructuras relacionadas".

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https://doi.org/10.1016/j.geomphys.2022.104728
0393-0440/

in which locally finite simple Lie algebras over an algebraically closed field of characteristic zero are characterized in terms of the existence of nonzero proper inner ideals.

In the last years, as announced in a communication presented by A. M. Cohen at the conference *Buildings and Symmetry* celebrated at the University of Western Australia in 2017, inner ideals of Lie algebras have achieved a relevant role in the study of certain geometries related to algebraic groups, recovering a research line started in 1973 by J. Faulkner, [15]. In this sense, we highlight the works of H. Cuyper, J. Meulewaeter and T. De Medts, see [11], [12], [13]. As A. M. Cohen mentions in his work [11], "the explicit classification of inner ideals for simple Lie algebras related to algebraic groups given in [16] confirms that the notion of inner ideal is well suited for the study of algebraic groups".

Further evidence of the usefulness of inner ideals comes from [20], where it was shown that an abelian inner ideal B of finite length in a nondegenerate Lie algebra L over a ring of scalars with $\frac{1}{2}, \frac{1}{3}$ gives rise to a finite $\mathbb{Z}$-grading in L with B being a wing of this grading. In general, a grading on a Lie algebra is a very strong condition, as it is the hypothesis of finite length. However, in this article we will show that any abelian inner ideal B of a Lie algebra L with some extra condition yields a bounded filtration on L with B being the first nonzero submodule of this filtration.

Principal filtrations associated to a single element were first introduced by O. Loos in 1990 in the context of Jordan systems, see [26], in a way analogous to the associative notions. In 2005, D. Passman studied maximal bounded $\mathbb{Z}$-filtrations of Artinian semisimple rings [27] and, together with Y. Barnea, described filtrations in semisimple Lie algebras in a series of papers published between 2006 and 2010, see [2], [3] and [28]. Later on, in 2012 the first two authors of this paper showed in [22] that any ad-nilpotent element a of index less than or equal to 3 in a Lie algebra L over a ring of scalars Φ with $\frac{1}{2}, \frac{1}{3} \in \Phi$ gives rise to a 5-filtration in L, such that when considering the 5-graded Lie algebra induced by this filtration, the Jordan pair determined by the wings of that Lie algebra coincides with the Jordan algebra of the Lie algebra.

These facts were used in [23] to give conditions under which the Jordan algebras determined by ad-nilpotent elements of index less than or equal to 3 are special as Jordan objects. In the same paper, they also provided conditions under which the (Jordan) subquotients associated to abelian inner ideals are special, but the specializing homomorphisms were built *ad-hoc* because filtrations associated to inner ideals were not known.

In this article we will show that any abelian inner ideal B of a Lie algebra L for which there exists $n \in \mathbb{N}$ with $[B, \mathrm{Ker}_L B]^n \subset B$ yields a bounded filtration on L. With this tool in hand, we will easily obtain the specializing homomorphisms of [23] under which the subquotients attached abelian inner ideals are special.

We highlight the condition one has to require to any abelian inner ideal B to give rise to such filtration: there must exist $n \in \mathbb{N}$ such that $[B, \mathrm{Ker}_L B]^n \subset B$. When B is the abelian inner ideal generated by a single element, this condition is trivially fullfilled. It also holds for abelian inner ideals of Lie algebras of the form R^- or $\mathrm{Skew}(R, *)$ for a semiprime associative algebra R with or without involution $*$, and for abelian inner ideals of finite length of nondegenerate Lie algebras (in particular, for nondegenerate finite-dimensional Lie algebras). Nevertheless, we do not know if there exist nondegenerate (even simple) Lie algebras with abelian inner ideals B that do not satisfy this condition.

2. Preliminaries

Throughout this paper we will be dealing with Lie algebras L, associative algebras R and linear Jordan pairs V over a ring of scalars Φ containing $\frac{1}{2}$ and $\frac{1}{3}$. As usual, $[x, y]$ will denote the Lie bracket of two elements x, y of L, and the product of elements of R will be written by juxtaposition. Any associative algebra R gives rise to a Lie algebra R^- with Lie bracket $[x, y] := xy - yx$, for all $x, y \in R$. If R has an involution $*$ we will consider the Lie subalgebra of R^-, $\mathrm{Skew}(R, *) = \{x \in R \mid x^* = -x\}$. The set of symmetric elements of R is denoted by $H(R, *)$, and since $\frac{1}{2} \in \Phi$, $R = H(R, *) \oplus \mathrm{Skew}(R, *)$. Jordan triple products of a Jordan pair $V = (V^+, V^-)$ will be written by $\{x, y, z\}$ for any $x, z \in V^\sigma$, $y \in V^{-\sigma}$, $\sigma = \pm$. The reader is referred to [24] and [25] for basic results, notation and terminology on Lie algebras and Jordan pairs.

2.1. A Φ-module B of a Lie algebra L is called an abelian inner ideal if $[B, B] = 0$ and $[B, [B, L]] \subset B$. The kernel of an abelian inner ideal is

$$\mathrm{Ker}_L B = \{x \in L \mid [B, [B, x]] = 0\}.$$

Associated to an abelian inner ideal B of L we can consider the subquotient $(B, L/\mathrm{Ker}_L B)$, which is a linear Jordan pair with products

$$\{b_1, \bar{x}, b_2\} = [[b_1, x], b_2] \qquad \{\bar{x}, b_1, \bar{y}\} = \overline{[[x, b_1], y]}$$

for every $b_1, b_2 \in B$ and every $\bar{x}, \bar{y} \in L/\mathrm{Ker}_L B$, see [20, 3.2].

2.2. Let us single out some relations concerning the abelian inner ideal $B \subseteq L$ and $K = \mathrm{Ker}_L B$:

$$[B, L] \cup [K, [B, L]] \cup [[B, K], L] \subseteq K \text{ (by [20, 3.2])}$$

$$[[B, K], [B, K]] \subset [B, K].$$

As a consequence, for all $b_i \in B$, $z, z_i \in K$, $k \geq 1$,

$$\mathrm{ad}_{b_1}\mathrm{ad}_{b_2}\mathrm{ad}_{z}\mathrm{ad}_{b_3} = \mathrm{ad}_{b_1}\mathrm{ad}_{z}\mathrm{ad}_{b_2}\mathrm{ad}_{b_3} = 0,$$

and, by a recursive argument and the Jacobi identity,

$$\mathrm{ad}_{[b_1,z_1]}\cdots\mathrm{ad}_{[b_k,z_k]}[b_{k+1}, z_{k+1}] = \mathrm{ad}_{b_1}\mathrm{ad}_{z_1}\cdots\mathrm{ad}_{b_k}\mathrm{ad}_{z_k}[b_{k+1}, z_{k+1}].$$

Moreover, since $\frac{1}{2} \in \Phi$, for every $b_0, \ldots, b_i, \ldots, b_j, \ldots, b_s \in B$ and $z_0, \ldots, z_s \in K$, if $1 < i < j$ and $b_i = b_j$ we have

$$\mathrm{ad}_{[b_1,z_1]}\cdots\mathrm{ad}_{[b_i,z_i]}\cdots\mathrm{ad}_{[b_i,z_j]}\cdots\mathrm{ad}_{[b_s,z_s]}([b_0, z_0]) = 0 \tag{$*$}$$

because

$$\begin{aligned}&\mathrm{ad}_{[b_1,z_1]}\cdots\mathrm{ad}_{[b_i,z_i]}\cdots\mathrm{ad}_{[b_i,z_j]}\cdots\mathrm{ad}_{[b_s,z_s]}([b_0, z_0]) = \\ &= \frac{1}{2}[b_i, [b_i, \mathrm{ad}_{[b_1,z_1]}\cdots\mathrm{ad}_{z_i}\cdots\mathrm{ad}_{z_j}\cdots\mathrm{ad}_{[b_s,z_s]}([b_0, z_0])]] \in [b_i, [b_i, K] = 0\end{aligned}$$

since $[[b_1, z_1], L] \subset K$.

2.3. Let L be a Lie algebra. A finite $\mathbb{Z}$-grading is a non-trivial $\mathbb{Z}$-grading of L such that the support $\operatorname{supp} L = \{m \in \mathbb{Z} \mid L_m \neq 0\}$ is finite. In this case

$$L = L_{-n} \oplus L_{-(n-1)} \oplus \ldots \oplus L_0 \oplus \ldots \oplus L_{n-1} \oplus L_n$$

for some positive integer n. If $L_{-n} + L_n \neq 0$, we will call such a grading a $(2n+1)$-grading. Note that if L is nondegenerate then both L_{-n} and L_n are non-zero. If $L = L_{-n} \oplus \ldots \oplus L_n$ is a $(2n+1)$-graded Lie algebra, then $V = (L_{-n}, L_n)$ is a Jordan pair with products

$$\{x, y, z\} = [[x, y], z] \qquad \{y, x, t\} = [[y, x], t]$$

for every $x, z \in L_{-n}$ and every $y, t \in L_n$, which is called the associated Jordan pair of L.

2.4. Let L be a Lie algebra over Φ. A $\mathbb{Z}$-filtration $\{\mathcal{F}_i\}_{i\in\mathbb{Z}}$ is a chain of submodules of L

$$\ldots \subset \mathcal{F}_{-2} \subset \mathcal{F}_{-1} \subset \mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots$$

such that $[\mathcal{F}_i, \mathcal{F}_j] \subset \mathcal{F}_{i+j}$ for every $i, j \in \mathbb{Z}$. A $\mathbb{Z}$-filtration $\{\mathcal{F}_i\}_{i\in\mathbb{Z}}$ is bounded if there exist $n, m \in \mathbb{Z}$, with $n < m$, such that $\mathcal{F}_i = 0$ for every $i \leq n$ and $\mathcal{F}_j = L$ for every $j \geq m$. If $\{\mathcal{F}_i\}_{i\in\mathbb{Z}}$ is a $\mathbb{Z}$-filtration of a Lie algebra L over Φ, we can consider the Φ-module

$$\hat{L} = \ldots \oplus \underbrace{\mathcal{F}_{i-1}/\mathcal{F}_{i-2}}_{\hat{L}_{i-1}} \oplus \underbrace{\mathcal{F}_i/\mathcal{F}_{i-1}}_{\hat{L}_i} \oplus \underbrace{\mathcal{F}_{i+1}/\mathcal{F}_i}_{\hat{L}_{i+1}} \oplus \ldots \tag{$\star$}$$

with product $[\bar{x}, \bar{y}] = \overline{[x, y]} \in \mathcal{F}_{i+j}/\mathcal{F}_{i+j-1}$ for every $\bar{x} = x + \mathcal{F}_{i-1} \in \mathcal{F}_i/\mathcal{F}_{i-1}$ and every $\bar{y} = y + \mathcal{F}_{j-1} \in \mathcal{F}_j/\mathcal{F}_{j-1}$. Thereby $\hat{L}$ has structure of $\mathbb{Z}$-graded Lie algebra and it is called the induced graded Lie algebra.

2.5. An associative algebra R is semiprime if for every nonzero ideal I of R, $I^2 \neq 0$, and it is prime if $IJ \neq 0$ for every pair of nonzero ideals I, J of R. It is well known that an associative algebra R is prime if and only if $aRb \neq 0$ for arbitrary nonzero elements $a, b \in R$, and it is semiprime if and only if it is nondegenerate, i.e., $aRa \neq 0$ for every nonzero element $a \in R$.

The extended centroid of R will be denoted by $C(R)$ (see [4, §2.3] for its definition and main properties). When R is semiprime, $C(R)$ is von Neumann regular, and when R is prime, $C(R)$ is a field. The central closure of R is $\hat{R} = C(R) + C(R)R$ and R is centrally closed if it coincides with its central closure. When R has an involution $*$, this involution extends to $C(R)$ and to $\hat{R}$.

If R is a prime associative algebra with involution $*$, the involution is of the first kind when every element in $C(R)$ is symmetric with respect to $*$, and it is of the second kind if there are nonzero skew-symmetric elements in $C(R)$. For every $0 \neq \lambda \in \mathrm{Skew}(C(R), *)$, $0 \neq \lambda^2$ is invertible in $H(C(R), *)$, so $R = H(R, *) \oplus \mathrm{Skew}(R, *) = \lambda^2 H(R, *) \oplus \mathrm{Skew}(R, *) \subseteq \lambda\,\mathrm{Skew}(R, *) \oplus \mathrm{Skew}(R, *) \subset R$, hence $R = \lambda\,\mathrm{Skew}(R, *) \oplus \mathrm{Skew}(R, *)$. If R is a $*$-prime associative algebra that is not prime, there always exist a nonzero $\lambda \in \mathrm{Skew}(C(R), *)$ (see [10, 2.4]) and therefore $R = \lambda\,\mathrm{Skew}(R, *) \oplus \mathrm{Skew}(R, *)$.

3. Filtration associated to an abelian inner ideal

In this section we will construct a filtration associated to an abelian inner ideal. If B is an abelian inner ideal of a Lie algebra L such that $[B, \mathrm{Ker}_L B]^n \subset B$ for some $n \in \mathbb{N}$, we will show that B induces a bounded filtration of L starting on B and whose second last submodule coincides with $\mathrm{Ker}_L B$.

Theorem 3.1. *If L is a Lie algebra and B is an abelian inner ideal of L, then the chain $\cdots \subset \mathcal{F}_{-n} \subset \mathcal{F}_{-n+1} \subset \cdots \subset \mathcal{F}_0 \subset \cdots \subset \mathcal{F}_{n-1} \subset \mathcal{F}_n \subset \cdots$, where $\mathcal{F}_{-m} := \{0\}$ and $\mathcal{F}_m := L$ for all $m \geq n$, and*

$$\begin{aligned}
\mathcal{F}_{-n} &:= B \\
&\vdots \\
\mathcal{F}_{-k} &:= [B, \mathrm{Ker}_L B]^k + B \text{ for } k = 1, \dots, n-1 \\
&\vdots \\
\mathcal{F}_0 &:= \{x \in L \mid [x, B] \subset B\} \\
&\vdots \\
\mathcal{F}_s &:= \mathrm{ad}^{n-s-1}_{[B, \mathrm{Ker}_L B]}(\mathrm{Ker}_L B) + \mathcal{F}_0 \text{ for } s = 1, \dots, n-1 \\
&\vdots \\
\mathcal{F}_n &:= L
\end{aligned}$$

is a bounded filtration of L if and only if there exists $n \in \mathbb{N}$ such that $[B, \mathrm{Ker}_L B]^n \subset B$. The induced graded Lie algebra $\hat{L}$ has associated Jordan pair $(\hat{L}_{-n}, \hat{L}_n)$ equal to the subquotient $(B, L/\mathrm{Ker}_L B)$.

Proof. Let us denote by $K := \mathrm{Ker}_L B$. If the above chain $\{\mathcal{F}_i\}$ is a filtration, then $[\mathcal{F}_{-n}, \mathcal{F}_{n-1}] \subset \mathcal{F}_{-1}$ so $[B, \mathrm{Ker}_L B]^n \subset B$.

For the converse, firstly let us check that each $\mathcal{F}_i \subset \mathcal{F}_{i+1}$ for all $i \in \mathbb{Z}$. Clearly $\mathrm{ad}_{[B,K]}[B, K] \subset [B, [K, [B, K]] \subset [B, K]$ so $\mathcal{F}_{-k} \subset \mathcal{F}_{-k+1}$ for all $k = 2, \dots, n-1$. Moreover, $\mathcal{F}_{-1} = [B, K] + B \subset \mathcal{F}_0$ because $[B, [B, K]] + [B, B] = 0 \in B$. The containment $\mathcal{F}_0 \subset \mathcal{F}_1$ follows by definition of $\mathcal{F}_1$. Furthermore $[[B, K], K] \subset K$ implies $\mathrm{ad}^{n-s-1}_{[B,K]}(K) \subset \mathrm{ad}^{n-s}_{[B,K]}(K)$ for all $s = 1, \dots, n-2$, so $\mathcal{F}_s \subset \mathcal{F}_{s+1}$ for all $s = 1, \dots, n-2$. Finally $\mathcal{F}_{n-1} \subset \mathcal{F}_n = L$.

By using the Jacobi identity, we get that for each $k \in \mathbb{N}$,

$$[\mathrm{ad}^{k-1}_{[B,K]}([B, K]), u] \subset \mathrm{ad}^k_{[B,K]}(u) \text{ for every } u \in L. \tag{a}$$

We also have that

$$[\mathcal{F}_0, K] \subset K. \tag{b}$$

Indeed, given $x \in \mathcal{F}_0$ and $k \in K$, we have that for any $b, b' \in B$,

$$[b, [b', [x, k]]] = [[b, [b', x]], k] + [[b', x], [b, k]] + [[b, x], [b', k]] + [x, [b, [b', k]]] = 0$$

because $[b, x], [b', x] \in B$ and B is abelian. Moreover,

$$\mathrm{ad}^{n-1}_{[B,K]}(K) \subset \mathcal{F}_0 \tag{c}$$

because for any $b \in B$,

$$[\mathrm{ad}^{n-1}_{[B,K]}(K), b] = \mathrm{ad}^{n-1}_{[B,K]}([K, b]) \subset B.$$

Now let us prove that $[\mathcal{F}_i, \mathcal{F}_j] \subset \mathcal{F}_{i+j}$ for all i, j:

(1) $[\mathcal{F}_{-s}, \mathcal{F}_{-r}] \subset \mathcal{F}_{-r-s}$ for any $r, s \in \{1, \dots, n\}$: by (a)

$$\begin{aligned}
[\mathcal{F}_{-s}, \mathcal{F}_{-r}] &= [\mathrm{ad}^{s-1}_{[B,K]}([B, K]) + B, \mathrm{ad}^{r-1}_{[B,K]}([B, K]) + B] = \\
&= [\mathrm{ad}^{s-1}_{[B,K]}([B, K]), \mathrm{ad}^{r-1}_{[B,K]}([B, K])] \subset \mathrm{ad}^{s+r-1}_{[B,K]}([B, K])
\end{aligned}$$

(2) $[\mathcal{F}_r, \mathcal{F}_0] \subset \mathcal{F}_r$ for any $r \in \{-n, \dots, n\}$: by (b) and the Jacobi identity.

(3) $[\mathcal{F}_{-r}, \mathcal{F}_s] \subset \mathcal{F}_{s-r}$ for any $r, s \in \{1, \ldots, n\}$: by (a) and (b) we have that

$$[\mathrm{ad}^{r-1}_{[B,K]}([B,K]) + B,\ \mathrm{ad}^{n-s-1}_{[B,K]}(K) + \mathcal{F}_0] \subset$$
$$\subset \mathrm{ad}^{n-s+r-1}_{[B,K]}(K) + B + \mathrm{ad}^{r-1}_{[B,K]}([B,K]) + \mathrm{ad}^{n-s-1}_{[B,K]}([B,K]).$$

Then

- If $r = s$, by (c)

$$\mathrm{ad}^{n-s+r-1}_{[B,K]}(K) + B + \mathrm{ad}^{r-1}_{[B,K]}([B,K]) + \mathrm{ad}^{n-s-1}_{[B,K]}([B,K]) \subset$$
$$\subset \mathrm{ad}^{n-1}_{[B,K]}(K) + B + [B,K] \subset \mathcal{F}_0.$$

- If $r > s$,

$$\mathrm{ad}^{n-s+r-1}_{[B,K]}(K) + B + \mathrm{ad}^{r-1}_{[B,K]}([B,K]) + \mathrm{ad}^{n-s-1}_{[B,K]}([B,K]) \subset$$
$$\subset \mathrm{ad}^{r-s}_{[B,K]}(\mathrm{ad}^{n-1}_{[B,K]}(K)) + B + \mathrm{ad}^{r-1}_{[B,K]}([B,K]) + \mathrm{ad}^{n-s-1}_{[B,K]}([B,K]) \subset$$
$$\subset \mathrm{ad}^{r-s}_{[B,K]}(\mathcal{F}_0) + B + [B,K] \subset$$
$$\subset \mathrm{ad}^{r-s-1}_{[B,K]}(([B,K],\mathcal{F}_0]) + B + [B,K] \subset \mathcal{F}_{s-r}.$$

- If $r < s$,

$$\mathrm{ad}^{n-s+r-1}_{[B,K]}(K) + B + \mathrm{ad}^{r-1}_{[B,K]}([B,K]) + \mathrm{ad}^{n-s-1}_{[B,K]}([B,K]) \subset \mathcal{F}_{s-r}.$$

(4) $[\mathcal{F}_s, \mathcal{F}_r] \subset \mathcal{F}_{r+s}$ for any $r, s \in \{1, \ldots, n\}$:

- If $r + s \geq n$ there is nothing to prove because $[\mathcal{F}_r, \mathcal{F}_s] \subset \mathcal{F}_{r+s} = L$.
- If $r + s < n$, then $2n - 2 - r - s \geq n - 1$. Our proof goes by induction on r from $n-2$ to 1: if $r = n-2$, we just need to study $s = 1$ (the rest of the cases follow trivially)

$$[\mathcal{F}_{n-2}, \mathcal{F}_1] = [\mathrm{ad}_{[B,K]}(K) + \mathcal{F}_0, \mathrm{ad}^{n-2}_{[B,K]}(K) + \mathcal{F}_0] \subset$$
$$\subset [\mathrm{ad}_{[B,K]}(K), \mathrm{ad}^{n-2}_{[B,K]}(K)] + \mathrm{ad}_{[B,K]}(K) + \mathrm{ad}^{n-2}_{[B,K]}(K) + \mathcal{F}_0 \subset$$
$$\subset \mathrm{ad}_{[B,K]}([K, \mathrm{ad}^{n-2}_{[B,K]}(K)]) + K \subset$$
$$\subset [[B,K],L] + K \subset K = \mathcal{F}_{n-1}.$$

Let us suppose that $[\mathcal{F}_r, \mathcal{F}_s] \subset \mathcal{F}_{r+s}$ for every s and let us prove it for $r-1$:

$$[\mathcal{F}_{r-1}, \mathcal{F}_s] = [\mathrm{ad}^{n-r}_{[B,K]}(K) + \mathcal{F}_0, \mathrm{ad}^{n-s-1}_{[B,K]}(K) + \mathcal{F}_0] \subset$$
$$\subset [\mathrm{ad}^{n-r}_{[B,K]}(K), \mathrm{ad}^{n-s-1}_{[B,K]}(K)] + \mathrm{ad}^{n-r}_{[B,K]}(K) + \mathrm{ad}^{n-s-1}_{[B,K]}(K) + \mathcal{F}_0 \subset$$
$$\subset \mathrm{ad}_{[B,K]}([\mathrm{ad}^{n-r-1}_{[B,K]}(K), \mathrm{ad}^{n-s-1}_{[B,K]}(K)]) +$$
$$+ [\mathrm{ad}^{n-r-1}_{[B,K]}(K), \mathrm{ad}^{n-s}_{[B,K]}(K)] + \mathcal{F}_{r-1} + \mathcal{F}_s \subset$$
$$\subset \mathrm{ad}_{[B,K]}(\mathcal{F}_{r+s}) + \mathcal{F}_{r-1+s} + \mathcal{F}_{r-1} + \mathcal{F}_s \subset \mathcal{F}_{r+s}.$$

Finally, since $\mathcal{F}_0 \subset K$ we have that $\mathcal{F}_{n-1} = K$ and the associated Jordan pair $(\hat{L}_{-n}, \hat{L}_n)$ coincides with the subquotient $(B, L/K)$. $\square$

Remark 3.2. In [22, Theorem 1.2] the principal filtration associated to an ad-nilpotent element of index ≤ 3 of a Lie algebra was introduced: Let L be a Lie algebra over a ring of scalars Φ with $\frac{1}{2}, \frac{1}{3} \in \Phi$ and let $x \in L$ be a Jordan element (an element with $\mathrm{ad}_x^3 L = 0$). Let $\mathrm{Ker}\, x = \{a \in L \mid \mathrm{ad}_x^2 a = 0\}$ and define

$$\mathcal{G}_i = 0,\ i \leq -3, \qquad \mathcal{G}_{-2} = [x,[x,L]] + \Phi x \qquad \mathcal{G}_{-1} = [x, \mathrm{Ker}\, x] + \Phi x$$
$$\mathcal{G}_0 = \{a \in L \mid [a,x] \in [x,[x,L]]\} \qquad \mathcal{G}_1 = \mathrm{Ker}\, x \qquad \mathcal{G}_j = L,\ j \geq 2.$$

Then $\{\mathcal{G}_i\}_i$ is a bounded filtration of L, called the principal filtration of L defined by x.

Every Jordan element $x \in L$ generates the abelian inner ideal $(x) = \Phi x + [x,[x,L]]$. It is shown in [20, Lemma 3.7] that when L is nondegenerate,

$$\operatorname{Ker} x = \operatorname{Ker}_L(x),$$

where $\operatorname{Ker}_L(x)$ denotes the kernel of the abelian inner ideal (x). From now on let us suppose that L is nondegenerate and let us denote $K = \operatorname{Ker} x = \operatorname{Ker}_L(x)$.

The condition $[[B, K], [B, K]] \subset B$ is fulfilled for the abelian inner ideal $B = (x)$ because $[[B, K], [B, K]] = [[\mathcal{G}_{-2}, \mathcal{G}_1], [\mathcal{G}_{-2}, \mathcal{G}_1]] \subset [\mathcal{G}_{-1}, \mathcal{G}_{-1}] \subset \mathcal{G}_{-2} = B$. Therefore we can consider the filtration $\{\mathcal{F}_i\}_i$ associated to the abelian inner ideal (x) given in Theorem 3.1.

Let us compare this filtration $\{\mathcal{F}_i\}_i$ with the principal filtration $\{\mathcal{G}_i\}_i$ associated to x:

(i) $\mathcal{F}_{-2} = (x) = \mathcal{G}_{-2}$.

(ii) $\mathcal{F}_{-1} = [(x), K] + (x) \subset [x, K] + (x) = \mathcal{G}_{-1}$ because for every $a \in L$ and every $z \in K$

$$[[x, [x, a]], z] = [x, [[x, a], z]] - [x, [a, [x, z]]] \in [x, K].$$

Conversely, $\mathcal{G}_{-1} = [x, K] + \Phi x \subset [(x), K] + (x) = \mathcal{F}_{-1}$.

(iii) $[\mathcal{G}_0, (x)] \subset (x)$ because $\{\mathcal{G}_i\}_i$ is a filtration, so $\mathcal{G}_0 \subset \mathcal{F}_0$.

(iv) By construction, $\mathcal{F}_i = \mathcal{G}_i$ for every $i \geq 1$.

We have obtained

$$\begin{array}{ccccccccc}
\mathcal{F}_{-2} & \subset & \mathcal{F}_{-1} & \subset & \mathcal{F}_0 & \subset & \mathcal{F}_1 = K & \subset & \mathcal{F}_2 = L \\
\shortparallel & & \shortparallel & & \cup & & \shortparallel & & \shortparallel \\
\mathcal{G}_{-2} & \subset & \mathcal{G}_{-1} & \subset & \mathcal{G}_0 & \subset & \mathcal{G}_1 = K & \subset & \mathcal{G}_2 = L
\end{array}$$

and both filtrations coincide when $x \in [x, [x, L]]$.

The reason for the containment of $\mathcal{G}_0$ into $\mathcal{F}_0$ is that the filtration $\{\mathcal{F}_i\}_i$ defined in Theorem 3.1 constructs the largest possible 0-submodule, while the principal filtration $\{\mathcal{G}_i\}_i$ attached to x has a non-maximal $\mathcal{G}_0$ submodule.

Remark 3.3. In [23, Proposition 3.4] it was shown the following result:

If L is a strongly prime Lie algebra, B is an abelian inner ideal such that $[B, \operatorname{Ker}_L B]$ is nilpotent and $\operatorname{Ker}_L B$ is not a subalgebra of L, then the subquotient $(B, L/\operatorname{Ker}_L B)$ is a special strongly prime Jordan pair.

The key point in the proof of this result was an explicit construction of the specializing homomorphisms [23, Proposition 3.1]. In view of the filtration associated to B, these homomorphisms can be directly obtained from a more general result about graded Lie algebras.

Since B satisfies the condition $[B, \operatorname{Ker}_L B]^n = 0 \subset B$ for some n, one can consider the filtration associated to B as in Theorem 3.1. The induced graded Lie algebra $\hat{L}$ given in $(\star)$ of 2.4 has associated Jordan pair $(\hat{L}_{-n}, \hat{L}_n)$ equal to the subquotient $(B, L/\operatorname{Ker}_L B)$. In [16, Theorem 11.34], a family of homomorphisms was built for any graded Lie algebra $\hat{L}$. In particular, the pair (Ψ_{n-1}, Ψ_{1-n})

$$\Psi_{n-1} : \hat{L}_{-n} \to \operatorname{Hom}(\hat{L}_{n-1}, \hat{L}_{-1}) \qquad \Psi_{1-n} : \hat{L}_n \to \operatorname{Hom}(\hat{L}_{-1}, \hat{L}_{n-1})$$

defined by $\Psi_{\sigma(n-1)}(x)(y) = \operatorname{ad}_x y$ for any $x \in \hat{L}_{-\sigma n}$ and any $y \in \hat{L}_{n-1}$ if $\sigma = +$ or $y \in \hat{L}_{-1}$ if $\sigma = -$ is a homomorphism of Jordan pairs between $V = (\hat{L}_{-n}, \hat{L}_n)$ and the special Jordan pair $(\operatorname{Hom}(\hat{L}_{n-1}, \hat{L}_{-1}), \operatorname{Hom}(\hat{L}_{-1}, \hat{L}_{n-1}))^{(+)}$.

Then the condition of $\operatorname{Ker}_L B$ not being a subalgebra of L implies the injectivity of the specializing homomorphisms (Ψ_{n-1}, Ψ_{1-n}) (see [23, Proposition 3.3]), so the subquotient $(B, L/\operatorname{Ker}_L B)$ is special as a Jordan pair.

4. The condition $[B, \operatorname{Ker}_L B]^n \subset B$

4.1. In [23, Proposition 3.5(d)] it was shown that for any abelian inner ideal B of a centrally closed prime associative algebra R, $[B, \operatorname{Ker}_{R^-} B]$ is nilpotent of index k with $k \leq 3$. This result easily extends to semiprime associative algebras. Indeed, given an abelian inner ideal of a semiprime associative algebra R, since R is a subdirect product of prime associative algebras R_i, B decomposes into a subdirect product of abelian inner ideals B_i of R_i. For each i, let us consider the central closure $\hat{R}_i$, and let us extend B_i to an abelian inner ideal $\hat{B}_i = C(R_i)B_i$ of $\hat{R}_i$. Then $[\hat{B}_i, \operatorname{Ker}_{\hat{R}_i} \hat{B}_i]$ is nilpotent of index ≤ 3 for each i, and therefore $[B, \operatorname{Ker}_{R^-} B]$ is nilpotent of index ≤ 3.

In particular, every abelian inner ideal B of a semiprime associative algebra R satisfies $[B, \operatorname{Ker}_{R^-} B]^k = 0 \subset B$ for some $k \leq 3$, so by Theorem 3.1 it gives rise to the bounded filtration in R^-:

$$\mathcal{F}_{-k} \subset \cdots \subset \mathcal{F}_0 \subset \cdots \subset \mathcal{F}_k$$

where $\mathcal{F}_{-k} = B$ and $\mathcal{F}_k = R^-$.

Proposition 4.2. *Let R be a $*$-prime centrally closed associative algebra with involution $*$ and suppose that $\frac{1}{2}, \frac{1}{3}, \frac{1}{5} \in \Phi$. Let $L = \operatorname{Skew}(R, *)$ and let B be an abelian inner ideal of L. Then $[B, \operatorname{Ker}_L B]$ is a nilpotent subalgebra of L of index n, where we have the following possibilities for n:*

(a) *If R is $*$-prime not prime or if it is prime and the involution is of the second kind, $n \le 3$.*
(b) *If R is prime and the involution is of the first kind, then $n \le 4$. In particular, if $[L, L] = 0$, $n = 1$; otherwise $b^3 = 0$ for every $b \in B$ and either*
  (b1) *there exists $b \in B$ such that $b^3 = 0$, $b^2 \neq 0$. In this case $n = 2$ and L admits a 3-grading $L = L_{-1} \oplus L_0 \oplus L_1$ with $B = L_{-1}$ and $\mathrm{Ker}_L B = L_{-1} \oplus L_0$, or*
  (b2) *$B^2 = 0$ and $n \le 4$. If, moreover, $B(\mathrm{Ker}_L B)B = 0$, then $n \le 3$.*

Proof. (a) If R is $*$-prime not prime or if it is prime and the involution is of the second kind, there exists a nonzero skew-symmetric element λ in the extended centroid $C(R)$ of R and $R = \mathrm{Skew}(R, *) \oplus \lambda\, \mathrm{Skew}(R, *)$. Then $\hat{B} = B + \lambda B$ is an inner ideal of R^- and $\mathrm{Ker}_{R^-} \hat{B} = \mathrm{Ker}_L B + \lambda\, \mathrm{Ker}_L B$. By [23, Proposition 3.5(d)] we have that $[\hat{B}, \mathrm{Ker}_{R^-} \hat{B}]$ is nilpotent of index less than or equal to three, so in particular $[B, \mathrm{Ker}_L B]$ is nilpotent of index less or equal to 3.
(b) If the involution is of the first kind and $[L, L] = 0$, then clearly $n = 1$. Otherwise $[L, L] \neq 0$, and by [8, Proposition 6.2] we have that $b^3 = 0$ for every $b \in B$.

(b1) If there exists $b \in B$ with $b^2 \neq 0$, then B is a Clifford inner ideal of L. Let us see that L is 3-graded with $L_{-1} = B$: by [9, Definition 3.1], b is a Clifford element of R, b^2 is von Neumann regular in R by [9, Proposition 3.1(4)] and there exists $d \in H(R, *)$ such that $d^2 = 0$, $b^2 d b^2 = b^2$ and $d = d b^2 d$. The element $e = d b^2$ is a $*$-orthogonal idempotent, i.e., $e^2 = e$, $e^* e = e e^* = 0$ (see [9, pag. 289]) and by [9, Proposition 3.5] induces a 3-grading in L with

$$
\begin{aligned}
&L_{-1} = \kappa((1-e)Le),\\
&L_0 = \kappa(eRe) + (1-e-e^*)L(1-e-e^*),\\
&L_1 = \kappa(eL(1-e)),
\end{aligned}
$$

where $\kappa(x) := x - x^*$ for every $x \in R$. By [9, Proposition 3.6(4)] $b \in \kappa((1-e)Le) = L_{-1}$. On the other hand, the inner ideal $[b, [b, L]]$ contains b because $[b, [b, -(bd + db)] = -b^2 db - bdb^2 + 2b^2 db + 2bdb^2 = b^2 db + bdb^2 = b$ by [9, Proposition 3.6(3)]. So by [8, Proposition 6.2], $[b, [b, L]]$ is a Clifford inner ideal and it is a maximal abelian inner ideal of L by [8, Proposition 5.1(b)]. Therefore, $[b, [b, L]] = B = L_{-1}$.

(b2) Otherwise, $b^2 = 0$ for every $b \in B$. In this case, for every $b_1, b_2 \in B$, $0 = (b_1 + b_2)^2 = b_1^2 + b_2^2 + 2b_1 b_2 = 2b_1 b_2$ because $[B, B] = 0$, so

$$B^2 = 0. \tag{1}$$

Moreover, for every $z \in \mathrm{Ker}_L B$,

$$0 = [b_1, [b_2, z]] = -b_1 z b_2 - b_2 z b_1, \quad \text{i.e.,} \quad b_1 z b_2 = -b_2 z b_1. \tag{2}$$

In particular,

$$bzb = 0 \tag{3}$$

for every $b \in B$ and every $z \in \mathrm{Ker}_L B$.

For every $b_1, b_2, b_3 \in B$ and every $z_1, z_2 \in \mathrm{Ker}_L B$, we have that

$$b_1 z_1 b_2 z_2 b_3 = 0. \tag{4}$$

Indeed, we are going to show that $b_1 z_1 b_2 z_2 b_3$ satisfies $(b_1 z_1 b_2 z_2 b_3) R (b_1 z_1 b_2 z_2 b_3) = 0$, and since R is semiprime, the element $b_1 z_1 b_2 z_2 b_3$ must be zero: for any $x \in R$ we have

$$
\begin{aligned}
(b_1 z_1 b_2 z_2 b_3) x (b_1 z_1 b_2 z_2 b_3) &= b_3 z_1 b_1 z_2 b_2 x b_1 z_1 b_2 z_2 b_3\\
&= b_3 z_1 b_1 x^* b_2 z_2 b_1 z_1 b_2 z_2 b_3 + b_3 z_1 b_1 (z_2 b_2 x - x^* b_2 z_2) b_1 z_1 b_2 z_2 b_3\\
&=^{(\star)} 0 - \frac{1}{2} b_3 z_1 [b_1, [b_1, z_2 b_2 x - x^* b_2 z_2]] z_1 b_2 z_2 b_3\\
&= -\frac{1}{2} b_3 z_1 [b_1, [b_1, z_2 b_2 x - (z_2 b_2 x)^*]] z_1 b_2 z_2 b_3 =^{(\star\star)} 0,
\end{aligned}
$$

where $(\star)$ follows because $b_2 z_2 b_1 z_1 b_2 = 0$ by (2) and (3), and $(\star\star)$ is true because $[b_1, [b_1, z_2 b_2 x - (z_2 b_2 x)^*]] \in B$, so we can apply (2) and (3).

We claim that $[B, \mathrm{Ker}_L B]^4 = 0$. Since L is nondegenerate ([5, Theorem 2.10] and [21, Proposition 4.6]) it suffices to show that for every $b_1, b_2, b_3, b_4 \in B$ and $z_1, z_2, z_3, z_4 \in \mathrm{Ker}_L B$ we have that

$$a := [[b_1, z_1], [[b_2, z_2], [[b_3, z_3], [b_4, z_4]]]]$$

is an absolute zero divisor of L: for any $u \in L$ we have that $[a, [a, u]]$ is a sum of monomials of the form

$$[B_i, Z_i][B_j, Z_j][B_k, Z_k][B_l, Z_l][B_m, Z_m][B_n, Z_n][B_o, Z_o][B_p, Z_p](u), \tag{5}$$

$i, j, k, l, m, n, o \in \{1, 2, 3, 4\}$, where by capital letters we denote adjoint maps, i.e., $A = \mathrm{ad}_a$ for every $a \in R$. Since $[B_p, Z_p](u) = z' \in \mathrm{Ker}_L B$, we reduce each of the monomials in (5) to monomials of the form

$$[B_i, Z_i][B_j, Z_j][B_k, Z_k][B_l, Z_l][B_m, Z_m][B_n, Z_n][B_o, Z_o](z'). \tag{6}$$

In view of (1)–(4),

- $[b_{i_1}, z_{j_1}][b_{i_2}, z_{j_2}][b_{i_3}, z_{j_3}] =$
  $= (b_{i_1}z_{j_1} - z_{j_1}b_{i_1})(b_{i_2}z_{j_2} - z_{j_2}b_{i_2})(b_{i_3}z_{j_3} - z_{j_3}b_{i_3})$
  $= (b_{i_1}z_{j_1}b_{i_2}z_{j_2} - b_{i_1}z_{j_1}z_{j_2}b_{i_2} + z_{j_1}b_{i_1}z_{j_2}b_{i_2})(b_{i_3}z_{j_3} - z_{j_3}b_{i_3})$
  $= -b_{i_1}z_{j_1}b_{i_2}z_{j_2}z_{j_3}b_{i_3}$, so
- $[b_{i_1}, z_{j_1}][b_{i_2}, z_{j_2}][b_{i_3}, z_{j_3}][b_{i_4}, z_{j_4}] =$
  $= (b_{i_1}z_{j_1} - z_{j_1}b_{i_1})(b_{i_2}z_{j_2} - z_{j_2}b_{i_2})(b_{i_3}z_{j_3} - z_{j_3}b_{i_3})(b_{i_4}z_{j_4} - z_{j_4}b_{i_4})$
  $= b_{i_1}z_{j_1}b_{i_2}z_{j_2}z_{j_3}b_{i_3}z_{j_4}b_{i_4}$, and
- $[b_{i_1}, z_{j_1}][b_{i_2}, z_{j_2}][b_{i_3}, z_{j_3}][b_{i_4}, z_{j_4}] = 0.$

Therefore the possible non-zero variants of the associative monomials in (6) have the form

$$\pm (b_{i_1}z_{j_1}b_{i_2}z_{j_2}z_{j_3}b_{i_3}z_{j_4}b_{i_4})z'(b_{i_5}z_{j_5}b_{i_6}z_{j_6}z_{j_7}b_{i_7}),$$
$$\pm (b_{i_1}z_{j_1}b_{i_2}z_{j_2}z_{j_3}b_{i_3})z'(b_{i_4}z_{j_4}b_{i_5}z_{j_5}z_{j_6}b_{i_6}z_{j_7}b_{i_7})$$

and are equal to 0 due to (4).

We have shown that all the possible monomials in the associative expansion of $[a, [a, u]]$ are zero for every $u \in L$, so $a = 0$. Thus $[B, \mathrm{Ker}_L B]$ is nilpotent of index less than or equal to 4.

Suppose that $B(\mathrm{Ker}_L B)B = 0$: if we consider the abelian inner ideal $\hat{B} := B + BRB$ of R^-, then $\mathrm{Ker}_L B = \mathrm{Ker}_{R^-} \hat{B} \cap L$. By [23, Proposition 3.5(d)], $[\hat{B}, \mathrm{Ker}_{R^-} \hat{B}]$ is nilpotent of index less than or equal to 3, and therefore $[B, \mathrm{Ker}_L B]$ is also nilpotent of index less than or equal to 3. □

Since every semiprime associative algebra R with involution $*$ is a subdirect product of $*$-prime associative algebras R_i, given an abelian inner ideal B of $L = \mathrm{Skew}(R, *)$ we can consider the projections B_i of B onto $L_i = \mathrm{Skew}(R_i, *)$. Let $\hat{R}_i$ be the central closure of each R_i and let $\hat{B}_i = H(C(R_i), *)B_i$ be the abelian inner ideal generated by B_i in $\hat{L}_i = \mathrm{Skew}(\hat{R}_i, *)$. By 4.2, $[\hat{B}_i, \mathrm{Ker}_{\hat{L}_i} \hat{B}_i]$ is nilpotent of index ≤ 4, so $[B_i, \mathrm{Ker}_{L_i} B_i]$ is also nilpotent of index ≤ 4 for each i. Thus $[B, \mathrm{Ker}_L B]$ is nilpotent of index ≤ 4 and we have shown the following result.

Corollary 4.3. *Let R be a semiprime associative algebra over Φ with involution $*$ and suppose that $\frac{1}{2}, \frac{1}{3}, \frac{1}{5} \in \Phi$. Let $L = \mathrm{Skew}(R, *)$ and let B be an abelian inner ideal of L. Then $[B, \mathrm{Ker}_L B]$ is a nilpotent subalgebra of L of index $k \le 4$. Therefore, $[B, \mathrm{Ker}_L B]^k = 0 \subset B$ and B gives rise to a bounded filtration*

$$\mathcal{F}_{-k} \subset \cdots \subset \mathcal{F}_0 \subset \cdots \subset \mathcal{F}_k,$$

where $\mathcal{F}_{-k} = B$ and $\mathcal{F}_k = \mathrm{Skew}(R, *)$.

Remark 4.4. Let L be nondegenerate. Then for every nonzero abelian inner ideal B of finite length of L there exists a finite $\mathbb{Z}$-grading $L = L_{-n} \oplus \cdots \oplus L_0 \oplus \cdots \oplus L_n$ such that $B = L_n$ (this is always the case when L is nondegenerate finite dimensional), see [20, Corollary 6.2]. With respect to this grading, $\mathrm{Ker}\, B = L_{-(n-1)} \oplus \cdots \oplus L_0 \oplus \cdots \oplus L_n$, so $[B, \mathrm{Ker}\, B] \subset L_1 \oplus \cdots \oplus L_n$ implies that $[B, \mathrm{Ker}_L B]^n \subset B$.

Remark 4.5. Every finitely generated subalgebra of $[B, \mathrm{Ker}_L B]$ is contained into the subalgebra generated by some finite set of the form

$$\{[b_1, k_1], \ldots [b_s, k_s]\}.$$

Here every product

$$\mathrm{ad}_{[b_{i_1}, k_{i_1}]} \cdots \mathrm{ad}_{[b_{i_t}, k_{i_t}]}([b_{i_0}, k_{i_0}])$$

for $b_{i_0}, \ldots, b_{i_t} \in \{b_1, \ldots, b_s\}$ and $k_{i_0}, \ldots, k_{i_t} \in \{k_1, \ldots, k_s\}$ with $t \ge s + 2$ must be zero by 2.2($*$) because there occurs at least a repetition among $b_{i_2}, \ldots, b_{i_t}$.

Notice that the algebra $[B, \mathrm{Ker}_L B]$ is not only locally nilpotent, but it is also the sum of the ideals

$$\mathrm{id}_{[B,\mathrm{Ker}_L B]}([b, z])$$

generated by $[b, z]$, $b \in B$, $z \in \mathrm{Ker}_L B$, and all of them are nilpotent of index at most 4 by 2.2$(*)$ again.

In particular, this also implies that $[B, \mathrm{Ker}_L B]$ is nilpotent in case B is finite dimensional.

Remark 4.6. Recall that a Lie algebra L is called special if it can be seen as a subalgebra of some R^- where R is an associative PI-algebra; therefore, if L is semiprime and special, then L can be embedded into a subdirect product of prime associative algebras with central closures of bounded dimension over their extended centroids; hence, for every inner ideal B of L, the image of $[B, \mathrm{Ker}_L B]$ in each of these factors is nilpotent of index at most some n, and the same is true for $[B, \mathrm{Ker}_L B]$.

Remark 4.7. Given a Lie algebra L over a field F and an abelian inner ideal B, since in the Lie subalgebra $[B, \mathrm{Ker}_L B]$ every element is ad-nilpotent, if this index of ad-nilpotence is bounded by some k and the characteristic of F is zero or high enough depending on k, then $[B, \mathrm{Ker}_L B]$ is nilpotent by [29, Theorem].

We have shown general situations where, given an abelian inner ideal B, the condition $[B, \mathrm{Ker}_L B]^n \subset B$ is fulfilled for some n. This justifies the following conjecture.

Conjecture 4.8. *For every nondegenerate (strongly prime or simple) Lie algebra L and every abelian inner ideal B of L there exists $n \in \mathbb{N}$ (possibly $n = 4$) such that $[B, \mathrm{Ker}_L B]^n \subset B$.*

Data availability

No data was used for the research described in the article.

Acknowledgements

We would like to thank the referee for his/her careful reading of the paper and his/her useful comments and remarks.

References

[1] A.A. Baranov, J. Rowley, Inner ideals of simple locally finite Lie algebras, J. Algebra 379 (2013) 11–30.
[2] Y. Barnea, D.S. Passman, Filtrations in semisimple Lie algebras. I, Trans. Am. Math. Soc. 358 (5) (2006) 1983–2010.
[3] Y. Barnea, D.S. Passman, Filtrations in semisimple Lie algebras. II, Trans. Am. Math. Soc. 360 (2) (2008) 801–817.
[4] K.I. Beidar, W.S. Martindale III, A.V. Mikhalev, Rings with Generalized Identities, Monographs and Textbooks in Pure and Applied Mathematics, vol. 196, Marcel Dekker, Inc., New York, 1996.
[5] G. Benkart, The Lie inner ideal structure of associative rings, J. Algebra 43 (2) (1976) 561–584.
[6] G. Benkart, On inner ideals and ad-nilpotent elements of Lie algebras, Trans. Am. Math. Soc. 232 (1977) 61–81.
[7] G. Benkart, A. Fernández López, The Lie inner ideal structure of associative rings revisited, Commun. Algebra 37 (11) (2009) 3833–3850.
[8] J. Brox, A. Fernández López, M. Gómez Lozano, Inner ideals of Lie algebras of skew elements of prime rings with involution, Proc. Am. Math. Soc. 144 (7) (2016) 2741–2751.
[9] J. Brox, A. Fernández López, M. Gómez Lozano, Clifford elements in Lie algebras, J. Lie Theory 27 (1) (2017) 283–294.
[10] J. Brox, E. García, M. Gómez Lozano, R. Muñoz Alcázar, G. Vera de Salas, A description of ad-nilpotent elements in semiprime rings with involution, Bull. Malays. Math. Sci. Soc. 44 (2021) 2577–2602.
[11] A.M. Cohen, Inner ideals in Lie algebras and spherical buildings, Indag. Math. 32 (5) (2021) 1115–1138.
[12] H. Cuypers, J. Meulewaeter, Extremal elements in Lie algebras, buildings and structurable algebras, J. Algebra 580 (2021) 1–42.
[13] T. De Medts, J. Meulewaeter, Inner ideals and structurable algebras: Moufang sets, triangles and hexagons, arXiv:2008.02700v2.
[14] C. Draper, A. Fernández López, E. García, M. Gómez Lozano, The inner ideals of the simple finite dimensional Lie algebras, J. Lie Theory 22 (4) (2012) 907–929.
[15] J.R. Faulkner, On the geometry of inner ideals, J. Algebra 26 (1973) 1–9.
[16] A. Fernández López, Jordan Structures in Lie Algebras, Mathematical Surveys and Monographs, vol. 240, American Mathematical Society, Providence, RI, 2019.
[17] A. Fernández López, E. García, M. Gómez Lozano, Inner ideals of finitary simple Lie algebras, J. Lie Theory 16 (1) (2006) 97–114.
[18] A. Fernández López, E. García, M. Gómez Lozano, An Artinian theory for Lie algebras, J. Algebra 319 (3) (2008) 938–951.
[19] A. Fernández López, E. García, M. Gómez Lozano, Inner ideal structure of nearly Artinian Lie algebras, Proc. Am. Math. Soc. 137 (1) (2009) 1–9.
[20] A. Fernández López, E. García, M. Gómez Lozano, E. Neher, A construction of gradings of Lie algebras, Int. Math. Res. Not. (16) (2007) rnm051.
[21] E. García, M. Gómez Lozano, A characterization of the Kostrikin radical of a Lie algebra, J. Algebra 346 (2011) 266–283.
[22] E. García, M. Gómez Lozano, Principal filtrations of Lie algebras, Commun. Algebra 40 (10) (2012) 3622–3628.
[23] E. García, M. Gómez Lozano, R. Muñoz Alcázar, On the speciality of Jordan algebras and subquotients of Lie algebras, J. Algebra 563 (2020) 426–441.
[24] N. Jacobson, Lie Algebras, Dover Publications, Inc., New York, 1979. Republication of the 1962 original.
[25] O. Loos, Jordan Pairs, Lecture Notes in Mathematics, vol. 460, Springer-Verlag, Berlin-New York, 1975.
[26] O. Loos, Filtered Jordan systems, Commun. Algebra 18 (6) (1990) 1899–1924.
[27] D.S. Passman, Filtrations in semisimple rings, Trans. Am. Math. Soc. 357 (12) (2005) 5051–5066.
[28] D.S. Passman, Filtrations in semisimple Lie algebras, III, in: Advances in Ring Theory, in: Trends Math., Birkhäuser/Springer, Basel AG, Basel, 2010, pp. 257–268.
[29] E.I. Zelmanov, Engel Lie algebras, Dokl. Akad. Nauk SSSR 292 (2) (1987) 265–268.