Nonlinear Analysis: Modelling and Control, Online First, 1–12
https://doi.org/10.15388/namc.2024.29.33534

Vilnius University Press

Fixed points of generalized cyclic contractions without continuity and application to fractal generation

Subhadip Roy[1], **Parbati Saha**, **Sumon Ghosh**[2,3], **Binayak S. Choudhury**

Department of Mathematics,
Indian Institute of Engineering Science and Technology, Shibpur,
Howrah-711103, India

subhadip_123@yahoo.com; parbati_saha@yahoo.co.in;
ghoshsumon33@gmail.com; binayak12@yahoo.co.in

Received: December 23, 2022 / **Revised:** September 6, 2023 / **Published online:** November 7, 2023

Abstract. In this paper, we define a generalized cyclic contraction and prove a unique fixed point theorem for these contractions. An illustrative example is given, which shows that these contraction mappings may admit the discontinuities and also that an existing result in the literature is effectively generalized by the theorem. We apply the fixed point result for generation of fractal sets through construction of an iterated function system and the corresponding Hutchinsion–Barnsley operator. The above construction is illustrated by an example. The study here is in the context of metric spaces.

Keywords: Hausdorff metric, fixed point, iterated function system, Hutchinson–Barnsley operator, fractal.

1 Introduction and mathematical preliminaries

Contractions of various kinds appear in a large way in metric fixed point theory. In fact, metric fixed point theory is widely held to have originated in the work of Banach [5] published in 1922, where the notion of contraction and the famous contraction mapping principle was introduced. In the following hundred years, various classes of mappings satisfying different types of contractive inequalities have been studied in the context of fixed point theory. The handbook [17] provides a comprehensive description of this development till the year 2000. Some of the more recent references are noted in [4, 9, 15, 23, 28, 29]. Contraction mappings are well known for their applications [7, 24]. One such domain of application is in Hutchinson–Barnsley's theory, where families of

[1]The author was supported by Indian Institute of Engineering Science and Technology, Shibpur as JRF.
[2]The author was supported by UGC, India as SRF.
[3]Corresponding author.

contractions are utilized for the generation of fractals [20], which are sets characterized by self-similarity appearing in different domains of mathematics, physics and computer science [10, 27]. It is a method originally proposed by Hutchinson [12] and further elaborated by Barnsley [6] in which a finite family of Banach contractions was used for the generation of fractal sets in arbitrary metric spaces. In later research the theory has been extended, modified and applied in the framework of different mathematical spaces for obtaining fractals. Some instances of these works are in references [3, 8, 22, 25, 26].

In this paper, we primarily introduce a cyclic contraction by generalizing the well-known θ-contraction. Further, it is illustrated that the class of cyclic contractions introduced here may contain discontinuous functions. We establish a unique fixed point theorem for these cyclic contractions, which is an actual generalization of certain existing results as demonstrated through an example.

In the second part of the present research, we construct an iterated function system (IFS) by utilizing a finite family of the cyclic contractions mentioned above. Then, by an application of the fixed point theorem proved here to the Hutchinson–Barnsley operator constructed out of the IFS, we establish the existence of fractal sets. Further, the iterations leading to the fractal sets are also obtained. The above method of generating fractals is demonstrated.

Throughout the paper, we use the following notations: $\mathbb{N}_n$ will denote the set of first n natural numbers, $CB(X)$ will denote the set of all nonempty, closed and bounded subsets, while $K(X)$ will denote the set of compact subsets of the metric space (X, d), respectively.

A self-mapping $T : X \to X$ on a metric space (X, d) is said to be a *contraction mapping* if there exists a constant $k \in [0, 1)$ such that, for every $x, y \in X$,

$$d(Tx, Ty) \leqslant kd(x, y). \tag{1}$$

The Banach contraction principle [5, 17] guarantees the existence of a unique fixed point for contraction mappings in a complete metric space.

A remarkable fixed point result was established by Jleli et al. [15] for the generalized contractions given through the condition that

$$d(Tx, Ty) \neq 0 \quad \Longrightarrow \quad \theta\bigl(d(Tx, Ty)\bigr) \leqslant \bigl[\theta\bigl(d(x, y)\bigr)\bigr]^k, \quad k \in (0, 1), \tag{2}$$

where $\theta : (0, \infty) \to (1, \infty)$ is a function with certain properties. The contraction defined in (2) is called θ-contraction because of its dependence on the function θ.

The class of function $\theta : (0, \infty) \to (1, \infty)$, satisfying

(Θ1) θ is nondecreasing;
(Θ2) for any sequence $\{x_n\} \subset (0, \infty)$, $\lim_{n\to\infty} \theta(x_n) = 1$ if and only if $\lim_{n\to\infty} x_n = 0$;
(Θ3) there exists $r \in (0, 1)$ and $l \in (0, \infty]$ such that $\lim_{t\to 0^+} (\theta(t) - 1)/t^r = l$,

is denoted by Θ [15]. It was proved in [15] that in a complete metric space the contraction given in (2) has a unique fixed point under the assumption that $\theta \in \Theta$. There are many generalizations and extensions of the above mentioned result like those in [11, 18, 21].

It was noted by Ahmad et al. [1] that condition ($\Theta 3$) neither implies nor is implied by the condition of continuity on θ. Further, they obtained the same fixed point result as in [15] with condition ($\Theta 3$) being replaced by the continuity of θ. Imdad et al. [13] further dropped condition ($\Theta 1$) that is the nondecreasing property of θ and showed that the result still remains valid. This result has several extensions in works like [2, 14, 19]. We generalize a result of [13] in the next section.

Below are certain notions associated with Hutchinson–Barnsley's theory for fractal generation, which is used in the subsequent discussion.

Definition 1. (See [6].) Let (X, d) be a metric space. The mapping $h : K(X) \times K(X) \to \mathbb{R}$, defined as

$$h\bigl(A, B\bigr) = \max\bigl\{D(A, B), D(B, A)\bigr\},$$

where

$$D(P, Q) = \sup_{p \in P} \inf_{q \in Q} d(p, q),$$

is a metric on $K(X)$ and called the Hausdorff metric induced by d.

Theorem 1. (*See* [6].) *If* (X, d) *is complete, then* $(K(X), h)$ *is also complete.*

Lemma 1. (*See* [6].) *If* $\{A_i\colon 1 \leqslant i \leqslant N\}$ *and* $\{B_i\colon 1 \leqslant i \leqslant N\}$ *are two finite collections of subsets of* $K(X)$ *for some* $i \in \mathbb{N}_N$*, then*

$$h\Biggl(\bigcup_{1 \leqslant i \leqslant N} A_i, \bigcup_{1 \leqslant i \leqslant N} B_i\Biggr) \leqslant \max_{1 \leqslant i \leqslant N} h(A_i, B_i).$$

Definition 2. (See [6].) An iterated function system (IFS) consists of a complete metric space (X, d) and a finite set of contraction mappings $T_i : X \to X$. It is denoted by $\{X; T_i, i \in \mathbb{N}_N\}$.

Definition 3. (See [6].) Let X be a metric space, and let $\{T_i, i \in \mathbb{N}_N\}$ be a finite set of mappings on X. The Hutchinson–Barnsley operator $F : K(X) \to K(X)$ is defined as $F(A) = \bigcup_{i \in \mathbb{N}_N} \hat{T}_i(A)$, where $\hat{T}_i(A) = \{T_i(a)\colon\ a \in A\}$.

Definition 4. (See [6].) Let $\{X; T_i, i \in \mathbb{N}_N\}$ be an IFS. A set $A \in K(X)$ is called an attractor or a deterministic fractal of the IFS if $F(A) = A$ and for all $B \in K(X)$, $\lim_{n\to\infty} F^n(B) = A$, where F is the corresponding Hutchinson–Barnsley operator.

In Hutchinson–Barnsley's theory, IFS is the basic instrument for generation of fractal sets. It has many versions in which different types of contractions have been utilized. Some instances of these works are noted in [8, 22, 25].

Let Θ' denotes the class of functions $\theta : (0, \infty) \to (1, \infty)$ satisfying the following conditions:

- ($\Theta 1'$) for any sequence $\{x_n\} \subset (0, \infty)$, $\lim_{n\to\infty} \theta(x_n) = 1$ if and only if $\lim_{n\to\infty} x_n = 0^+$;
- ($\Theta 2'$) θ is continuous.

Definition 5. Let (X,d) be a metric space, m be a positive integer, and let $A_1, A_2, \dots, A_m$ by nonempty subsets of X. Suppose $T : \bigcup_{i=1}^m A_i \to \bigcup_{i=1}^m A_i$ be an operator. Then T is said to a generalized cyclic θ-contraction if

(i) $T(A_i) \subseteq T(A_{i+1})$ for all $i \in \mathbb{N}_m$, where $A_{m+1} = A_1$;
(ii) there exists $\theta \in \Theta'$ and $k \in (0,1)$ such that

$$d(Tx,Ty) \neq 0 \quad \Longrightarrow \quad \theta\bigl(d(Tx,Ty)\bigr) \leqslant \bigl[\theta\bigl(d(x,y)\bigr)\bigr]^k \tag{3}$$

for any $x \in A_i,\ y \in A_{i+1}$, where $A_{m+1} = A_1$.

2 Fixed point result

Theorem 2. *Let (X,d) be a complete metric space, m be a positive integer, and let $A_1, A_2, \dots, A_m$ by nonempty closed subsets of X. Suppose $T : \bigcup_{i=1}^m A_i \to \bigcup_{i=1}^m A_i$ be a generalized cyclic θ-contraction with respect to some $\theta \in \Theta'$. Then $\bigcap_{i=1}^m A_i$ is nonempty, and T has a unique fixed point $x^* \in \bigcap_{i=1}^m A_i$. Further, the sequence $\{x_n\}$, where $x_{n+1} = Tx_n$ converges to x^* for any initial choice of $x_0 \in \bigcup_{i=1}^m A_i$.*

Proof. Let $x_0 \in \bigcup_{i=1}^m A_i$ and $x_{n+1} = Tx_n$, $n \geqslant 0$. If for some n, $d(x_{n+1}, x_n) = 0$, then $Tx_n = x_n$, which shows that x_n is a fixed point of T. So, we assume that $d(x_{n+1}, x_n) \neq 0$ for all n. For any $n \geqslant 0$, there exists $i(l) \in \{1, 2, \dots, m\}$ such that $x_n \in A_{i(l)}$, $x_{n+1} \in A_{i(l)+1}$. Since T is a generalized cyclic θ-contraction,

$$\begin{aligned} 1 < \theta\bigl(d(x_{n+1}, x_n)\bigr) &= \theta\bigl(d(Tx_n, Tx_{n-1})\bigr) \\ &\leqslant \bigl[\theta\bigl(d(x_n, x_{n-1})\bigr)\bigr]^k = \bigl[\theta\bigl(d(Tx_{n-1}, Tx_{n-2})\bigr)\bigr]^k \\ &\leqslant \bigl[\theta\bigl(d(x_{n-1}, x_{n-2})\bigr)\bigr]^{k^2} \leqslant \cdots \leqslant \bigl[\theta\bigl(d(x_1, x_0)\bigr)\bigr]^{k^n}. \end{aligned}$$

Taking limit as $n \to \infty$ in the above inequality, since $0 < k < 1$, we get

$$\lim_{n\to\infty} \theta\bigl(d(x_{n+1}, x_n)\bigr) = 1.$$

Then, by the property $(\Theta 1')$ of the function θ,

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{4}$$

Next, we show that $\{x_n\}$ is a Cauchy sequence. If not, then there exists $\epsilon > 0$ for which we can find two subsequences $\{x_{m(k)}\}$, $\{x_{n(k)}\}$ with $n(k) > m(k) \geqslant k$ such that

$$d(x_{m(k)}, x_{n(k)}) \geqslant \epsilon. \tag{5}$$

Let $n(k)$ be the smallest integer with $n(k) > m(k)$ satisfying (5). Then

$$d(x_{m(k)}, x_{n(k)-1}) < \epsilon.$$

Then we have

$$\begin{aligned}\epsilon \leqslant d(x_{m(k)}, x_{n(k)}) &\leqslant d(x_{m(k)}, x_{n(k)-1}) + d(x_{n(k)}, x_{n(k)-1}) \\ &< \epsilon + d(x_{n(k)-1}, x_{n(k)}).\end{aligned} \tag{6}$$

Taking limit as $n \to \infty$ in (6), we get

$$\lim_{n\to\infty} d(x_{m(k)}, x_{n(k)}) = \epsilon. \tag{7}$$

Also,

$$\begin{aligned} d(x_{m(k)}, x_{n(k)}) &\leqslant d(x_{n(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{m(k)-1}) \\ &\quad + d(x_{m(k)-1}, x_{m(k)}); \\ d(x_{m(k)-1}, x_{n(k)-1}) &\leqslant d(x_{n(k)}, x_{n(k)-1}) + d(x_{n(k)}, x_{m(k)}) \\ &\quad + d(x_{m(k)-1}, x_{m(k)}).\end{aligned} \tag{8}$$

Taking limit as $n \to \infty$ in (8) and using (4), (7), we get

$$\lim_{n\to\infty} d(x_{m(k)-1}, x_{n(k)-1}) = \epsilon. \tag{9}$$

Putting $x = x_{m(k)-1}$ and $y = x_{n(k)-1}$ in (3),

$$\theta\bigl(d(x_{m(k)}, x_{n(k)})\bigr) \leqslant \bigl(\theta(d(x_{m(k)-1}, x_{n(k)-1})\bigr)^k. \tag{10}$$

Taking limit as $n \to \infty$ in (10) and using (Θ2), (7), (9), we get

$$\theta(\epsilon) \leqslant \bigl[\theta(\epsilon)\bigr]^k,$$

which is a contraction as $0 < k < 1$. This shows that $\{x_n\}$ is a Cauchy sequence and hence convergent in the complete metric space (X, d). Suppose $x_n \to x^*$ as $n \to \infty$. Also, from the cyclic representation of $\{A_i;\ i = 1, 2, \dots, m\}$ it is possible to construct a subsequence of $\{x_n\}$ from each A_i, which converges to x^*. Therefore, $x^* \in \bigcap_{i=1}^m A_i$.

Let $Y = \bigcap_{i=1}^m A_i$ and $f : Y \to Y$ be the restriction of T on the complete subspace Y. Then, since $0 < k < 1$ and $\theta(d(x_n, x^*)) > 1$,

$$\begin{aligned}\theta\bigl(d\bigl(x_{n+1}, f(x^*)\bigr)\bigr) = \theta\bigl(d\bigl(x_{n+1}, T(x^*)\bigr)\bigr) &\leqslant \bigl[\theta\bigl(d(x_n, x^*)\bigr)\bigr]^k \\ &< \theta\bigl(d(x_n, x^*)\bigr).\end{aligned}$$

Since $d(x_n, x^*) \to 0$ as $n \to \infty$, by (Θ1) we have $\theta(d(x_n, x^*)) \to 1$ as $n \to \infty$. From the above relation we get $\theta(d(x_{n+1}, fx^*)) \to 1$ as $n \to \infty$. Therefore, by (Θ1), $d(x_n, Tx^*) \to 0$ as $n \to \infty$. Then it follows that $T(x^*) = x^*$.

Let $x^* \neq z^* \in X$ be such that $T(z^*) = z^*$. Then

$$\theta\bigl(d(x^*, z^*)\bigr) = \theta\bigl(d\bigl(T(x^*), T(z^*)\bigr)\bigr) \leqslant \bigl[\theta\bigl(d(x^*, z^*)\bigr)\bigr]^k,$$

which is a contradiction. Hence T has a unique fixed point $x^* \in \bigcap_{i=1}^m A_i$. □

Example 1. Consider the complete metric space (X, d), where $X = \mathbb{R}$, $d(x, y) = |x-y|$. Let $A_1 = [0, 1]$ and $A_2 = [0, 2]$. Define a mapping $T : A_1 \cup A_2 \to A_1 \cup A_2$ by

$$T(x) = \begin{cases} \frac{x}{2} & \text{if } x \in [0, 1], \\ \frac{1}{2} & \text{if } x \in \left[1, \frac{3}{2}\right], \\ \frac{1}{4} & \text{if } x \in \left(\frac{3}{2}, 2\right]. \end{cases}$$

Then T is a generalized cyclic θ-contraction with the choice of $\theta(t) = \mathrm{e}^t$ and $k = 1/2$. Therefore, by Theorem 2, T has a unique fixed point $x^* = 0$.

Remark 1. If we take $A_i = X$ for all i in Theorem 2, then Eq. (2) is satisfied for all $x, y \in X$. Therefore, generalized cyclic θ-contraction is a generalization of a theorem in [13].

It can be noted in Example 1 that T is not a contraction in the sense of Imdad et al. [13] as for $x = 3/2$ and $y = 7/4$, we have $d(Tx, Ty) = d(x, y)$. So, there does not exist any $k \in (0, 1)$ such that Eq. (2) is satisfied. Hence Theorem 2 is a nontrivial generalization of the above mentioned result of [13], and therefore, is also a proper generalization of the main theorem in [1] in turn.

Also, θ-contraction is continuous, whereas generalized cyclic θ-contraction mappings can be discontinuous as can be concluded from the observation that in Example 1 the mapping T is discontinuous at $x = 3/2$.

A similar type of result is established in the work [16], where the function θ is assumed continuous in addition to assumptions ($\Theta1$)–($\Theta3$). Theorem 2 is derived here with θ satisfying only ($\Theta1$) and ($\Theta2$), which is without ($\Theta1$) and ($\Theta3$) utilized in the above mentioned work.

3 Fractal generation

Let $\{A_i\}_{i=1}^m$ be a collection of nonempty subsets of a metric space (X, d), and let $T : \bigcup_{i=1}^m A_i \to \bigcup_{i=1}^m A_i$ be a continuous generalized cyclic θ-contraction. For any $C \in K(\bigcup_{i=1}^m A_i)$, $T(C) \in K(\bigcup_{i=1}^m A_i)$. Using the fact that $\bigcup_{i=1}^m K(A_i) \subseteq K(\bigcup_{i=1}^m (A_i))$, we define the operator generated by the continuous mapping T as $\widehat{T} : \bigcup_{i=1}^m K(A_i) \to \bigcup_{i=1}^m K(A_i)$ is defined by $\widehat{T}(C) = T(C) = \{T(x)\colon x \in C\}$ for all $C \in \bigcup_{i=1}^m K(A_i)$.

The above construction is possible since T is cyclic.

Lemma 2. *If A is closed subset of the complete metric space (X, d), then $K(A)$ is a closed subset of the complete metric space $(K(X), h)$.*

In the following, we obtain a theorem, which is an application of Theorem 2 under the additional condition that θ is nondecreasing.

Lemma 3. *If $\{A_i\}_{i=1}^m$ is a collection of nonempty subsets of a metric space (X, d) and $T : \bigcup_{i=1}^m A_i \to \bigcup_{i=1}^m A_i$ is a continuous generalized cyclic θ-contraction, then the map $\widehat{T} : \bigcup_{i=1}^m K(A_i) \to \bigcup_{i=1}^m K(A_i)$ is also a generalized cyclic θ-contraction in the metric space $(K(X), h)$, where $\theta \in \Theta'$ is nondecreasing.*

Proof. Let $C \in K(A_i)$ for some $i \in \mathbb{N}_m$. Now, $T(C)$ is a compact set since T is a continuous function. Also, by cyclic representation of T, $T(C) \subseteq A_{i+1}$. Therefore, $T(C) \in K(A_{i+1})$.

This implies that $\widehat{T}(C) \in K(A_{i+1})$. Therefore, $\widehat{T}(K(A_i)) \subseteq K(A_{i+1})$ for each $i \in \mathbb{N}_m$.

Let $A \in K(A_i)$ and $B \in K(A_{i+1})$ for some $i \in \mathbb{N}_m$. We have to show

$$\theta\bigl(h\bigl(\widehat{T}(A), \widehat{T}(B)\bigr)\bigr) \leqslant \bigl[\theta(h(A, B)\bigr]^k.$$

Let $x_0 \in A$. Since B is compact, there exists $y_0 \in B$ such that $d(x_0, y_0) = \inf_{y \in B} d(x_0, y)$. Since θ is nondecreasing,

$$\begin{aligned}\theta\bigl(d\bigl(Tx_0, T(B)\bigr)\bigr) &= \theta\Bigl(\inf_{y \in B} d(Tx_0, Ty),\Bigr) \leqslant \theta\bigl(d(Tx_0, Ty_0)\bigr)\\ &\leqslant \bigl[\theta\bigl(d(x_0, y_0)\bigr)\bigr]^k = \Bigl[\theta\Bigl(\inf_{y \in B} d(x_0, y)\Bigr)\Bigr]^k\\ &\leqslant \Bigl[\theta\Bigl(\sup_{x \in A} \inf_{y \in B} d(x, y)\Bigr)\Bigr]^k = \bigl[\theta\bigl(D(A, B)\bigr)\bigr]^k.\end{aligned}$$

Since $x_0 \in A$ is arbitrary and θ is nondecreasing,

$$\sup_{x \in A} \theta\bigl(d\bigl(Tx, T(B)\bigr)\bigr) \leqslant \bigl[\theta\bigl(D(A, B)\bigr)\bigr]^k \leqslant \bigl[\theta\bigl(h(A, B)\bigr)\bigr]^k.$$

Also,

$$\begin{aligned}\theta\bigl(D\bigl(\widehat{T}(A), \widehat{T}(B)\bigr)\bigr) &= \theta\bigl(D\bigl(T(A), T(B)\bigr)\bigr) = \theta\Bigl(\sup_{x \in A} d\bigl(Tx, T(B)\bigr)\Bigr)\\ &= \sup_{x \in A} \theta\bigl(d\bigl(Tx, T(B)\bigr)\bigr) \leqslant \bigl[\theta\bigl(h(A, B)\bigr)\bigr]^k.\end{aligned}$$

Similarly, we can show that $\theta(D(\widehat{T}(B), \widehat{T}(A))) \leqslant [\theta(h(A, B))]^k$. Then

$$\begin{aligned}\theta\bigl(h\bigl(\widehat{T}(A), \widehat{T}(B)\bigr)\bigr) &= \theta\bigl(\max\bigl\{D\bigl(\widehat{T}(A), \widehat{T}(B)\bigr), D\bigl(\widehat{T}(B), \widehat{T}(A)\bigr)\bigr\}\bigr)\\ &\leqslant \bigl[\theta\bigl(h(A, B)\bigr)\bigr]^k.\end{aligned}$$

This completes the proof. □

Theorem 3. *If $\{A_i\}_{i=1}^m$ is a collection of nonempty subsets of a metric space (X, d) and $T_n : \bigcup_{i=1}^m A_i \to \bigcup_{i=1}^m A_i$ for each $n \in \mathbb{N}_N$ are continuous generalized cyclic θ-contractions with the values $k_1, k_2, \ldots, k_N$, respectively, then the Hutchinson operator $F : \bigcup_{i=1}^m K(A_i) \to \bigcup_{i=1}^m K(A_i)$ defined by $F(C) = \bigcup_{n=1}^N \widehat{T}_n(C)$ is a generalized cyclic θ-contraction map in $(K(X), h)$ with the value of $k = \max\{k_n;\ n \in \mathbb{N}_N\}$.*

Proof. Let $C \in K(A_i)$ for some $i \in \mathbb{N}_m$. By Lemma 3, for each $n \in \mathbb{N}_N$, $\widehat{T}_n$ is a generalized cyclic θ-contraction on $(K(X), h)$. Therefore, $\widehat{T}_n(C) \in K(A_{i+1})$ for all $n \in \mathbb{N}_N$. So, $F(C) = \bigcup_{n=1}^N \widehat{T}_n(C) \in K(A_{i+1})$. Thus we have $F(K(A_i)) \subseteq K(A_{i+1})$ for $i \in \mathbb{N}_m$.

We have to show that $\theta(h(F(A),F(B))) \leqslant [\theta(h(A,B))]^k$ for $A \in K(A_i)$ and $B \in K(A_{i+1})$, $i \in \mathbb{N}_m$, $A_{m+1} = A_m$. Using Lemma 1,

$$\begin{aligned} h\big(F(A),F(B)\big) &= h\left(\bigcup_{n=1}^{N} \widehat{T}_n(A), \bigcup_{n=1}^{N} \widehat{T}_n(B)\right) \\ &\leqslant \max_{1\leqslant n\leqslant N} h\big(\widehat{T}_n(A),\widehat{T}_n(B)\big). \end{aligned}$$

Since θ is nondecreasing,

$$\begin{aligned} \theta\big(h\big(F(A),F(B)\big)\big) &\leqslant \theta\Big(\max_{1\leqslant n\leqslant N} h\big(\widehat{T}_n(A),\widehat{T}_n(B)\big)\Big) \\ &= \max_{1\leqslant n\leqslant N} \theta\big(h\big(T_n(A),T_n(B)\big)\big) \\ &\leqslant \max_{1\leqslant n\leqslant N} \big[\theta\big(h(A,B)\big)\big]^{k_n}. \end{aligned}$$

Now,

$$\begin{aligned} \log\Big(\max_{1\leqslant n\leqslant N} \big[\theta\big(h(A,B)\big)\big]^{k_n}\Big) &= \max_{1\leqslant n\leqslant N} \big(\log\big[\theta\big(h(A,B)\big)\big]^{k_n}\big) \\ &= \max_{1\leqslant n\leqslant N} k_n \log\big(\theta\big(h(A,B)\big)\big) \\ &= k\log\big(\theta\big(h(A,B)\big)\big) \\ &= \log\big(\big[\theta\big(h(A,B)\big)\big]^k\big). \end{aligned}$$

Since $\log$ is injective,

$$\max_{1\leqslant n\leqslant N} \big[\theta\big(h(A,B)\big)\big]^{k_n} = \big[\theta\big(h(A,B)\big)\big]^k.$$

Therefore,

$$\theta\big(h\big(F(A),F(B)\big)\big) \leqslant \big[\theta\big(h(A,B)\big)\big]^k.$$

Hence the proof. □

Theorem 4. *Let (X,d) be a complete metric space and $\{A_i\}_{i=1}^m$ be a collection of nonempty closed subsets of X. Let $T_n : \bigcup_{i=1}^m A_i \to \bigcup_{i=1}^m A_i$ be continuous generalized cyclic θ-contractions for each $n \in \mathbb{N}_N$. Then the Hutchinson–Barnsley operator $F : \bigcup_{i=1}^m K(A_i) \to \bigcup_{i=1}^m K(A_i)$ has a unique fixed point $A \in K(X)$, and the limit $\lim_{n\to\infty} F^n(B) = A$ for any $B \in \bigcup_{i=1}^m K(A_i)$, which is the fractal generated by the IFS $\{\bigcup_{i=1}^m A_i, T_n, n \in \mathbb{N}_N\}$.*

Proof. Since (X,d) is complete, $(K(X),h)$ is also complete. Again, A_i is closed subset of X for each $i \in \mathbb{N}_m$. Hence $K(A_i)$ is also closed subset of $K(X)$. By Theorem 3, $F : \bigcup_{i=1}^m K(A_i) \to \bigcup_{i=1}^m K(A_i)$ is a generalized cyclic θ-contraction. Then the theorem follows by an application of Theorem 2. □

Example 2. Let $X = \mathbb{R}$ be equipped with the usual metric d. Let $A_1 = [1,3]$ and $A_2 = [2,4]$. Define $T_1 : A_1 \cup A_2 \to A_1 \cup A_2$ by

$$T_1(x) = \begin{cases} \frac{26-x}{8} & \text{for } x \in [1,3], \\ \frac{23}{8} & \text{for } x \in [3, \frac{15}{4}], \\ \frac{53-8x}{8} & \text{for } x \in [\frac{15}{4}, 4]. \end{cases}$$

Now, $T_1(A_1) = [23/8, 25/8] \subseteq A_2$, and $T_1(A_2) = [11/4, 3] \subseteq A_1$.

Let $x \in A_1, y \in A_2$. Consider $k = 2/3$ and $\theta(t) = \mathrm{e}^t \in \Theta'$. Then

$$\theta\bigl(d(T_1x, T_1y)\bigr) \leqslant \bigl(\theta\bigl(d(x,y)\bigr)\bigr)^k \quad \text{for all } x \in A_1,\ y \in A_2.$$

Therefore, T_1 is a generalized cyclic θ-contraction.

Define $T_2 : A_1 \cup A_2 \to A_1 \cup A_2$ by

$$T_2(x) = \begin{cases} \frac{19-x}{8} & \text{for } x \in [1,3], \\ 2 & \text{for } x \in [3, \frac{15}{4}], \\ \frac{23-4x}{4} & \text{for } x \in [\frac{15}{4}, 4]. \end{cases}$$

Then $T_2(A_1) = [2, 9/4] \subseteq A_2$, and $T_2(A_2) = [15/8, 17/8] \subseteq A_1$. It is also easy to check that T_2 is a generalized cyclic θ-contraction. Also, it is noted that both T_1 and T_2 are continuous.

Note that for $x = 15/4$ and $y = 4$, we have $d(T_1x, T_1y) = d(T_2x, T_2y) = d(x,y) = 1/4$. Therefore, both T_1 and T_2 do not satisfy (1) with some $k \in (0,1)$. Hence they are not Banach contractions. Also inequality (3) is not satisfied for all $x, y \in A_1 \cup A_2$. Since both T_1 and T_2 are continuous generalized cyclic θ-contraction, by Theorem 4, the IFS $\{(A_1 \cup A_2); T_1, T_2\}$ admits a fractal set, that is, there exists a set A such that $A = F(A)$, where F is the Hutchinson–Barnsley operator.

Here the set A is similar to a Cantor set for $[2,3]$ with 8 subintervals and retaining first and last subintervals at each stage. We have shown the first four iterations of the same in Fig. 1.

Let $A_0 = [2,3]$. Then

$$C_1 = F(A_0) = T_1(A_0) \cup T_2(A_0) = \left[2, \frac{17}{8}\right] \cup \left[\frac{23}{8}, 3\right],$$
$$C_2 = F^2(A_0) = F(C_1) = \left[2, \frac{129}{64}\right] \cup \left[\frac{135}{64}, \frac{17}{8}\right] \cup \left[\frac{23}{8}, \frac{185}{64}\right] \cup \left[\frac{191}{64}, 3\right]$$

and so on...

So,

$$\lim_{n\to\infty} F^n(A_0) = \lim_{n\to\infty} C_n = \bigcap_{j\in\mathbb{N}} C_j = A.$$

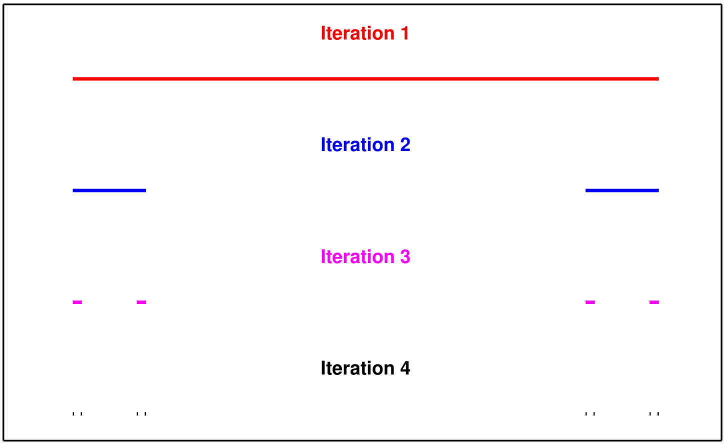

Figure 1. Attractor of the IFS in Example 2.

4 Conclusion

The unique fixed point Theorem 2 proved here is an actual generalization of a result in [13], which in turn is a generalization of some other results including the Banach contraction mapping principle. Thus, in effect, we have been able to generalize the contraction mapping principle through our theorem, which also applies to certain functions with discontinuities. The second part of our paper is a contribution to the Hutchinson–Barnsley's theory. It is our perception that there are large scopes of research towards the goal of fractal generation by the construction of IFS through other types of cyclic contractions as well. Such efforts are supposed to be taken up in our future work.

Acknowledgment. The authors gratefully acknowledge the suggestions made by the referee.

References

1. J. Ahmad, A. Al-Mazrooei, Y.J. Cho, Y.O. Yang, Fixed point results for generalized Θ-contractions, *J. Nonlinear Sci. Appl.*, **10**(5):2350–2358, 2017, https://doi.org/10.22436/jnsa.010.05.07.
2. A. Al-Rawashdeh, J. Ahmad, Common fixed point theorems for JS-contractions, *Bull. Math. Anal. Appl.*, **8**(4):12–22, 2016.
3. I. Altun, H. Sahin, M. Aslantas, A new approach to fractals via best proximity point, *Chaos Solitions Fractals*, **146**:110850, 2021, https://doi.org/10.1016/j.chaos.2021.110850.
4. J. Andres, J. Fiser, L. Gorniewicz, Fixed points and sets of multivalued contractions: An advanced survey with some new results, *Fixed Point Theory*, **22**(1):15–30, 2021, https://doi.org/10.24193/fpt-ro.2021.1.02.

5. S. Banach, Sur les opérations dans les ensembles abstraits et leurs applications aux équations intégrales, *Fundam. Math.*, **3**(1):133–181, 1922, http://eudml.org/doc/213289.
6. M.F. Barnsley, *Fractals Everywhere,* 2nd ed., Academic Press, Boston, 1993, https://doi.org/10.1016/C2013-0-10335-2.
7. K.C. Border, *Fixed Point Theorems with Applications to Economics and Game Theory*, Cambridge Univ. Press, Cambridge, 2010, https://doi.org/10.1017/CBO9780511625756.
8. B.S. Choudhury, P. Chakraborty, Strong fixed points of Φ-couplings and generation of fractals, *Chaos Solitions Fractals*, **163**:112514, 2022, https://doi.org/10.1016/j.chaos.2022.112514.
9. P.N. Dutta, B.S. Choudhury, A generalisation of contraction principle in metric spaces, *Fixed Point Theory Appl.*, **2008**:406368, 2008, https://doi.org/10.1155/2008/406368.
10. I.V. Grossu, O. Savencu, A.I. Miron, C. Besliu, N. Verga, Medical module for hyper-fractal analysis, *Comput. Phys. Commun.*, **273**:108255, 2022, https://doi.org/10.1016/j.cpc.2021.108255.
11. N. Hussain, A. Al-Mazrooei, J. Ahmad, Fixed point results for generalized (α-η)-Θ contractions with applications, *J. Nonlinear Sci. Appl.*, **10**(8):4197–4208, 2017, https://doi.org/10.22436/jnsa.010.08.15.
12. J. Hutchinson, Fractals and self-similarity, *Indiana Univ. Math. J.*, **30**(5):713–747, 1981, https://doi.org/10.1512/iumj.1981.30.30055.
13. M. Imdad, W.M. Alfaqih, I.A. Khan, Weak θ-contractions and some fixed point results with applications to fractal theory, *Adv. Difference Equ.*, **2018**:439, 2018, https://doi.org/10.1186/s13662-018-1900-8.
14. M. Jleli, E. Karapınar, B. Samet, Further generalizations of the Banach contraction principle, *J. Inequal. Appl.*, **2014**:439, 2014, https://doi.org/10.1186/1029-242X-2014-439.
15. M. Jleli, B. Samet, A new generalization of the Banach contraction principle, *J. Inequal. Appl.*, **2014**:38, 2014, https://doi.org/10.1186/1029-242X-2014-38.
16. M. Jleli, B. Samet, A generalized θ-contraction and related fixed point theorems, *J. Nonlinear Sci. Appl.*, **10**(8):4724–4733, 2017, https://doi.org/10.22436/jnsa.010.09.15.
17. W.A. Kirk, B. Sims (Eds.), *Handbook of Metric Fixed Point Theory*, Springer, Dordrecht, 2001, https://doi.org/10.1007/978-94-017-1748-9.
18. Z. Li, S. Jiang, Fixed point theorems of JS-quasi-contractions, *Fixed Point Theory Appl.*, **2016**:40, 2016, https://doi.org/10.1186/s13663-016-0526-3.
19. X.-D. Liu, S.-S. Chang, Y. Xiao, L.C. Zhao, Existence of fixed points for Θ-type contraction and Θ-type Suzuki contraction in complete metric spaces, *Fixed Point Theory Appl.*, **2016**:8, 2016, https://doi.org/10.1186/s13663-016-0496-5.
20. B.B. Mandelbrot, *The Fractal Geometry of Nature*, Freeman, New York, 1982.
21. V. Parvaneh, N. Hussain, A. Mukheimer, H. Aydi, On fixed point results for modified JS-contractions with applications, *Axioms*, **8**(3):84, 2018, https://doi.org/10.3390/axioms8030084.

22. R. Pasupathi, A.K.B. Chand, M.A. Navascués, Cyclic iterated function systems, *J. Fixed Point Theory Appl.*, **22**(3):58, 2020, https://doi.org/10.1007/s11784-020-00790-9.

23. V. Pata, A fixed point theorem for metric spaces, *J. Fixed Point Theory Appl.*, **10**:299–305, 2011, https://doi.org/10.1007/s11784-011-0060-1.

24. V. Pata, *Fixed Point Theorems and Applications*, Unitext, Vol. 116, Springer, Cham, 2019, https://doi.org/10.1007/978-3-030-19670-7.

25. A. Petruşel, G. Petruşel, Coupled fractal dynamics via Meir–Keeler operators, *Chaos Solitons Fractals*, **122**:206–212, 2019, https://doi.org/10.1016/j.chaos.2019.03.011.

26. M. Priya, R. Uthayakumar, Fractal set of generalized countable partial iterated function system with generalized contractions via partial Hausdorff metric, *Topol. Appl.*, **308**:108000, 2022, https://doi.org/10.1016/j.topol.2022.108000.

27. A.M.M. Sharif Ullah, D.M. D'Addona, Y. Seto, S. Yonehara, A. Kubo, Utilizing fractals for modeling and 3D printing of porous structures, *Fractal Fract.*, **5**(2):40, 2021, https://doi.org/10.3390/fractalfract5020040.

28. T. Suzuki, W. Takahashi, Fixed point theorems and characterizations of metric completeness, *Topol. Methods Nonlinear Anal.*, **8**(2):371–382, 1996.

29. D. Wardowski, Fixed points of a new type of contractive mappings in complete metric spaces, *Fixed Point Theory Appl.*, **2012**:94, 2012, https://doi.org/10.1186/1687-1812-2012-94.