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Multipoint boundary value problem for a coupled system of ψ-Hilfer nonlinear implicit fractional differential equation

Muthukrishnan Latha Maheswari[a], **Kolathur Srinivasan Keerthana Shri**[a], **Elsayed M. Elsayed**[b,c,1]

[a]Department of Mathematics,
PSG College of Arts & Science,
Coimbatore, Tamil Nadu, India
lathamahespsg@gmail.com; keerthanashriks@gmail.com

[b]Mathematics Department, Faculty of Science,
King AbdulAziz University,
P.O. Box 80203, Jeddah 21589, Saudi Arabia

[c]Department of Mathematics, Faculty of Science,
Mansoura University,
Mansoura 35516, Egypt
emmelsayed@yahoo.com

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Abstract. This study examines the existence and uniqueness of the solution to the coupled system of the ψ-Hilfer nonlinear implicit fractional multipoint boundary value problem. The uniqueness is shown by the Banach contraction principle, and the existence is shown by Krasnosel'skii's fixed point theorem in a special working space. An example is presented to verify our results. The existence and uniqueness of the solution are analysed graphically.

Keywords: existence, fixed point technique, multipoint boundary value problem, ψ-Hilfer fractional derivative.

1 Introduction

Fractional differential equations (FDEs) are used extensively in the technical and scientific fields of chemistry, physics, control theory, aerodynamics, economics, polymer rheology, signal and image processing, finance, blood flow phenomena, etc. to mathematically describe systems and processes [4, 17, 18]. Also, FDEs are considered more efficient compared to integer-order differential equations for describing the hereditary attributes of many materials and processes. Coupled systems of FDEs have proven to be of huge significance and interest and are employed in the areas of bioengineering, ecology, financial

[1]Corresponding author.

economics, chaotic and fractional dynamics, and many more [6, 14]. Coupled systems of FDEs are nonlocal in nature and occurs in systems including distributed-order dynamical systems [15], the Duffing system [5], quantum evolution of complex systems [20], the Lorenz system [11], the Chua circuit [13], anomalous diffusion [22], synchronisation of coupled fractional-order chaotic systems [8, 29], systems of nonlocal thermoelasticity [7], secure communication and control processing [21], etc. Several theoretical studies and research findings on coupled systems of FDEs are presented in [2, 3]. The research of FDEs with ψ-Hilfer fractional derivative has notable development. The advantage of ψ-Hilfer fractional derivative is the freedom of choice of the classical differential operator [27]. The qualitative analysis of solutions to the initial and boundary value problems is a prominent area of research [12,16,26,28,30]. In [1], Abdo investigated a coupled system of fractional terminal value problem involving generalized Hilfer fractional derivative of the type

$$\begin{aligned} &D_{a^+}^{\theta_1,\eta_1;\psi}y(t) = f_1\big(t,x(t)\big), \quad a<t\leqslant T,\ a>0,\\ &D_{a^+}^{\theta_2,\eta_2;\psi}x(t) = f_2\big(t,y(t)\big), \quad a<t\leqslant T,\ a>0,\\ &y(T)=w_1\in\mathbb{R}, \qquad x(T)=w_2\in\mathbb{R}, \end{aligned}$$

where $0<\theta_i<1$, $0\leqslant\eta_i\leqslant1$, $D_{a^+}^{\theta_i,\eta_i;\psi}$ $(i=1,2)$ is the Hilfer fractional derivative of order θ_i and type η_i with respect to ψ, and $f:(a,T]\times\mathbb{R}\to\mathbb{R}$ is a continuous function.

Sitho et al. in [24] proved the existence and uniqueness of solutions for the following class of boundary value problems of ψ-Hilfer-type fractional differential equations supplemented with nonlocal integro-multipoint boundary conditions of the form

$$\begin{aligned} &\big({}^HD_{a^+}^{\alpha_1,\beta_1;\psi} + k\,{}^HD_{a^+}^{\alpha_1-1,\beta_1;\psi}\big)u(z) = f\big(z,u(z)\big), \quad k\in\mathbb{R},\ z\in[c,d],\\ &u(c)=0, \qquad u(d)=\sum_{i=1}^m\mu_i\int_a^{\eta_i}\psi'(s)u(s)\,\mathrm{d}s+\sum_{j=1}^m\sigma_j u(\xi_j), \end{aligned}$$

where ${}^HD_{a^+}^{\alpha_1,\beta_1;\psi}$ is the ψ-Hilfer fractional derivative of order α_1, $1<\alpha_1<2$, $0\leqslant\beta_1\leqslant1$, $c\geqslant0$, $\mu_i,\lambda_j\in\mathbb{R}$, $\eta_i,\xi_j\in(c,d)$, $f:[c,d]\times\mathbb{R}\to\mathbb{R}$ is a continuous function.

Recently, in [23], the boundary value problem was extended to the coupled system of ψ-Hilfer-type fractional differential equations with integro-multipoint boundary conditions of the form

$$\begin{aligned} &\big({}^HD_{a^+}^{\alpha_1,\beta_1;\psi} + k\,{}^HD_{a^+}^{\alpha_1-1,\beta_1;\psi}\big)u(z) = f\big(z,u(z),v(z)\big), \quad z\in[c,d],\\ &\big({}^HD_{a^+}^{\bar{\alpha}_1,\beta_1;\psi} + k\,{}^HD_{a^+}^{\bar{\alpha}_1-1,\beta_1;\psi}\big)v(z) = g\big(z,u(z),v(z)\big), \quad z\in[c,d],\\ &u(c)=0, \qquad u(d)=\sum_{i=1}^m\mu_i\int_a^{\eta_i}\psi'(s)v(s)\,\mathrm{d}s+\sum_{j=1}^m\sigma_j v(\xi_j),\\ &v(c)=0, \qquad v(d)=\sum_{r=1}^p\nu_r\int_a^{\zeta_r}\psi'(s)u(s)\,\mathrm{d}s+\sum_{s=1}^q\tau_s u(\sigma_s), \end{aligned}$$

where ${}^{H}D_{a^+}^{\alpha_1,\beta_1;\psi}$, ${}^{H}D_{a^+}^{\bar{\alpha_1},\beta_1;\psi}$ are the ψ-Hilfer fractional derivatives of order α_1, and $\bar{\alpha_1}$, $1 < \alpha_1, \bar{\alpha_1} < 2$, $0 \leqslant \beta_1 \leqslant 1$, $c \geqslant 0$, $\mu_i, \lambda_j, \nu_r, \tau_s \in \mathbb{R}^+$, $\eta_i, \xi_j, \zeta_r, \sigma_s \in (c,d)$, $f, g\colon [c,d] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions. The authors employed fixed point technique to establish the existence and uniqueness of solution. The results were presented using the Banach and Krasnosel'skii's fixed point theorems and the Leray–Schauder alternative.

Motivated by the above results, our objective is to investigate the coupled system of ψ-Hilfer nonlinear implicit fractional multipoint boundary value problem of the form

$$\begin{aligned} &{}^{H}D_{a^+}^{\alpha_1,\beta:\psi} x(t) = f\bigl(t, y(t), {}^{H}D_{a^+}^{u,v;\psi} y(t)\bigr), \quad t \in \mathbb{J} = [a,b], \\ &{}^{H}D_{a^+}^{\alpha_2,\beta:\psi} y(t) = g\bigl(t, x(t), {}^{H}D_{a^+}^{u,v;\psi} x(t)\bigr), \quad t \in \mathbb{J} = [a,b], \\ &x(a) = 0, \qquad x(b) = \sum_{i=1}^{m} \varphi_i {}^{H}D_{a^+}^{\delta_i,\beta:\psi} y(\eta_i) + \sum_{j=1}^{n} \sigma_j y(\omega_j), \\ &y(a) = 0, \qquad y(b) = \sum_{r=1}^{p} \lambda_r {}^{H}D_{a^+}^{\theta_r,\beta:\psi} x(\zeta_r) + \sum_{s=1}^{q} \mu_s x(\xi_s), \end{aligned} \tag{1}$$

where ${}^{H}D_{a^+}^{\alpha_1,\beta:\psi}$, ${}^{H}D_{a^+}^{\alpha_2,\beta:\psi}$, ${}^{H}D_{a^+}^{u,v;\psi}$, ${}^{H}D_{a^+}^{\delta_i,\beta:\psi}$, and ${}^{H}D_{a^+}^{\theta_r,\beta:\psi}$ are the ψ-Hilfer fractional derivatives of order α_1, α_2, u, δ_i, and θ_r, respectively, with $1 < \delta_i, \theta_r < u < \alpha_1, \alpha_2 < 2$ and type $0 \leqslant \beta, v \leqslant 1$, $\varphi_i, \sigma_j, \lambda_r, \mu_s, \in \mathbb{R}^+$, $\eta_i, \omega_j, \zeta_r, \xi_s \in \mathbb{J}$, $f, g : \mathbb{J} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

This paper has the following structure. Fundamental concepts and theorems essential for the investigation are given in Section 2. In Section 3, we develop a lemma that is significant for determining the main results. In Section 4, the existence and uniqueness of the solution to (1) are established, and an example is provided to validate our results.

2 Preliminaries

Let us denote $C([a,b],\mathbb{R})$ to be the space of all continuous functions from $[a,b]$ to $\mathbb{R}$ and $AC([a,b],\mathbb{R})$ to be the space of all absolutely continuous functions from $[a,b]$ to $\mathbb{R}$.

Definition 1. (See [17].) Let (a,b) $(-\infty \leqslant a < b \leqslant \infty)$ be a finite or infinite interval of the real line $\mathbb{R}$ and $\nu > 0$. Let $\psi(t)$ be an increasing and positive monotone function on $(a,b]$ having a continuous derivative $\psi'(t)$ on (a,b). The ψ-Riemann–Liouville fractional integral $I_{a+}^{\nu;\psi}(\cdot)$ of a function $h \in AC^n([a,b],\mathbb{R})$ with respect to another function ψ on $[a,b]$ is defined by

$$I_{a+}^{\nu;\psi} h(t) = \frac{1}{\Gamma(\nu)} \int_a^t \psi'(s)\bigl(\psi(t) - \psi(s)\bigr)^{\nu-1} h(s)\,\mathrm{d}s, \quad t > a > 0,$$

where $\Gamma(\cdot)$ represents the gamma function.

Definition 2. (See [17].) Let $\psi'(t) \neq 0$ and $\nu > 0$, $n \in \mathbb{N}$. The Riemann–Liouville fractional derivative of order ν of a function $h \in AC^n([a,b],\mathbb{R})$ with respect to another function ψ is defined by

$$D_{a^+}^{\nu,\rho;\psi}h(t) = \left(\frac{1}{\psi'(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right)^n I_{a+}^{n-\nu;\psi}h(t)$$
$$= \frac{1}{\Gamma(n-\nu)}\left(\frac{1}{\psi'(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_a^t \psi'(s)\big(\psi(t)-\psi(s)\big)^{n-\nu-1}h(s)\,\mathrm{d}s,$$

where $n = [\nu]+1$, $[\nu]$ represents the integer part of the real number ν.

Definition 3. (See [27].) Let $n-1 < \nu < n$ with $n \in \mathbb{N}$, $[a,b]$ is the interval such that $-\infty \leqslant a < b \leqslant \infty$, and $h, \psi \in C^n([a,b],\mathbb{R})$ are two functions such that $\psi(t)$ is increasing and $\psi'(t) \neq 0$ for all $t \in [a,b]$. The ψ-Hilfer fractional derivative ${}^{H}D_{a^+}^{\nu,\rho;\psi}(\cdot)$ of a function h of order ν and type $0 \leqslant \rho \leqslant 1$ is defined by

$${}^{H}D_{a^+}^{\nu,\rho;\psi}h(t) = I_{a+}^{\rho(n-\nu);\psi}\left(\frac{1}{\psi'(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right)^n I_{a+}^{(1-\rho)(n-\nu);\psi}h(t),$$

where $n = [\nu]+1$, $[\nu]$ represents the integer part of the real number ν with $\gamma = \nu + \rho(n-\nu)$.

Lemma 1. (*See* [17].) *Let* $\nu, s > 0$. *Then we have the following semigroup property*:

$$I_{a^+}^{\nu;\psi}I_{a^+}^{s;\psi}h(t) = I_{a^+}^{\nu+s;\psi}h(t), \quad t > a.$$

Lemma 2. (*See* [27].) *If* $h \in C^n([a,b],\mathbb{R})$, $n-1 < \nu < n$, $0 \leqslant \rho \leqslant 1$, *and* $\gamma = \nu + \rho(n-\nu)$, *then*

$$I_{a+}^{\nu;\psi}\,{}^{H}D_{a^+}^{\nu,\rho;\psi}h(t) = h(t) - \sum_{k=1}^{n}\frac{(\psi(t)-\psi(a))^{\gamma-k}}{\Gamma(\gamma-k+1)}h_{\psi}^{[n-k]}I_{a^+}^{(1-\rho)(n-\nu);\psi}h(a)$$

for all $t \in \mathbb{J}$, *where*

$$h_{\psi}^{[n]}h(t) = \left(\frac{1}{\psi'(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right)^n h(t).$$

Proposition 1. (*See* [17,27].) *Let* $\nu \geqslant 0$, $l > 0$, *and* $t > a$. *Then the* ψ*-fractional integral and derivative of a power function are given by*

$$I_{a+}^{\nu;\psi}\big(\psi(s)-\psi(a)\big)^{l-1}(t) = \frac{\Gamma(l)}{\Gamma(l+\nu)}\big(\psi(t)-\psi(a)\big)^{l+\nu-1},$$
$$D_{a^+}^{\nu,\rho;\psi}\big(\psi(s)-\psi(a)\big)^{l-1}(t) = \frac{\Gamma(l)}{\Gamma(l-\nu)}\big(\psi(t)-\psi(a)\big)^{l-\nu-1},$$
$${}^{H}D_{a^+}^{\nu,\rho;\psi}\big(\psi(s)-\psi(a)\big)^{l-1}(t) = \frac{\Gamma(l)}{\Gamma(l-\nu)}\big(\psi(t)-\psi(a)\big)^{l-\nu-1},$$

$l > \gamma = \nu + \rho(n-\nu)$.

Lemma 3. (*See* [26].) *Let* $n-1<\nu<n$, $m-1<u<m\leqslant n$, $m,n\in\mathbb{N}$, $0\leqslant v\leqslant 1$, *and* $\nu\geqslant u+v(m-u)$. *If* $h\in C^m(\mathbb{J},\mathbb{R})$, *then*

$$ {}^{H}D_{a^+}^{u,v;\psi} I_{a+}^{\nu;\psi} h(t) = I_{a^+}^{u-\nu;\psi} h(t). $$

Lemma 4 [Banach contraction principle]. (*See* [9].) *Let* A *be the closed nonempty subset of a Banach space* B. *Then any contraction mapping* $\mathcal{T}: A\to A$ *has a unique fixed point.*

Theorem 1 [Krasnosel'skii's fixed point theorem]. (*See* [19].) *Let* D *be a closed, bounded, convex, and nonempty subset of a Banach space* $(B,\|\cdot\|)$. *Suppose that* P *and* Q *are operators from* D *to* D *such that*

(i) $Px+Qy\in D$ *for all* $x,y\in D$,
(ii) P *is continuous and compact,*
(iii) Q *is a contraction mapping.*

Then there exist a $z\in D$ *such that* $z=Pz+Qz$.

3 An auxiliary result

Let us define a special working space,

$$ \mathbb{B}=\left\{x \mid x(t)\in C\big([a,b],\mathbb{R}\big),\ {}^{H}D_{a^+}^{u,v;\psi}x(t)\in C\big([a,b],\mathbb{R}\big)\right\} $$

with the associated norm

$$ \|x\|_{\mathbb{B}}=\max\Big\{\sup_{t\in\mathbb{J}}\big\|x(t)\big\|,\ \sup_{t\in\mathbb{J}}\big\|{}^{H}D_{a^+}^{u,v;\psi}x(t)\big\|\Big\}. $$

It is clear from [25] that $\mathbb{B}$ is a Banach space, and consequently, the product space $\mathbb{B}\times\mathbb{B}$ is a Banach space with the norm

$$ \big\|(x,y)\big\|_{\mathbb{B}}=\|x\|_{\mathbb{B}}+\|y\|_{\mathbb{B}}. $$

For the existence and uniqueness results, it is necessary to develop the following lemma.

Lemma 5. *Let* $a\geqslant 0$, $1<\delta_i,\theta_r<u<\alpha_1,\alpha_2<2$, $0\leqslant\beta, v\leqslant 1$, $\gamma_1=\alpha_1+\beta(2-\alpha_1)$, $\gamma_2=\alpha_2+\beta(2-\alpha_2)$, *and* $\curlywedge\neq 0$. *Then for* $f,g:\mathbb{J}\times\mathbb{B}\times\mathbb{B}\to\mathbb{B}$, *the solution of the coupled system*

$$ \begin{aligned} &{}^{H}D_{a^+}^{\alpha_1,\beta:\psi}x(t)=f\big(t,y(t),{}^{H}D_{a^+}^{u,v;\psi}y(t)\big), \quad t\in\mathbb{J}=[a,b],\\ &{}^{H}D_{a^+}^{\alpha_2,\beta:\psi}y(t)=g\big(t,x(t),{}^{H}D_{a^+}^{u,v;\psi}x(t)\big), \quad t\in\mathbb{J}=[a,b],\\ &x(a)=0, \qquad x(b)=\sum_{i=1}^{m}\varphi_i{}^{H}D_{a^+}^{\delta_i,\beta:\psi}y(\eta_i)+\sum_{j=1}^{n}\sigma_j y(\omega_j),\\ &y(a)=0, \qquad y(b)=\sum_{r=1}^{p}\lambda_r{}^{H}D_{a^+}^{\theta_r,\beta:\psi}x(\zeta_r)+\sum_{s=1}^{q}\mu_s x(\xi_s) \end{aligned} \tag{2} $$

is given by

$$\begin{aligned} x(t) = {} & I_{a^+}^{\alpha_1;\psi} f\big(t, y(t), {}^H D_{a^+}^{u,v;\psi} y(t)\big) \\ & + \frac{(\psi(t) - \psi(a))^{\gamma_1 - 1}}{\Gamma(\gamma_1) \cdot \curlywedge} \Bigg[\varPhi \Bigg(\sum_{i=1}^{m} \varphi_i I_{a^+}^{\alpha_2 - \delta_i;\psi} g\big(\eta_i, x(\eta_i), {}^H D_{a^+}^{u,v;\psi} x(\eta_i)\big) \\ & + \sum_{j=1}^{n} \sigma_j I_{a^+}^{\alpha_2;\psi} g\big(\omega_j, x(\omega_j), {}^H D_{a^+}^{u,v;\psi} x(\omega_j)\big) - I_{a^+}^{\alpha_1;\psi} f\big(b, y(b), {}^H D_{a^+}^{u,v;\psi} y(b)\big) \Bigg) \\ & + D \Bigg(\sum_{r=1}^{p} \lambda_r I_{a^+}^{\alpha_1 - \theta_r;\psi} f\big(\zeta_r, y(\zeta_r), {}^H D_{a^+}^{u,v;\psi} y(\zeta_r)\big) \\ & + \sum_{s=1}^{q} \mu_s I_{a^+}^{\alpha_1;\psi} f\big(\xi_s, y(\xi_s), {}^H D_{a^+}^{u,v;\psi} y(\xi_s)\big) - I_{a^+}^{\alpha_2;\psi} g\big(b, x(b), {}^H D_{a^+}^{u,v;\psi} x(b)\big) \Bigg) \Bigg], \end{aligned} \tag{3}$$

$$\begin{aligned} y(t) = {} & I_{a^+}^{\alpha_2;\psi} g\big(t, x(t), {}^H D_{a^+}^{u,v;\psi} x(t)\big) \\ & + \frac{(\psi(t) - \psi(a))^{\gamma_2 - 1}}{\Gamma(\gamma_2) \cdot \curlywedge} \Bigg[\varOmega \Bigg(\sum_{i=1}^{m} \varphi_i I_{a^+}^{\alpha_2 - \delta_i;\psi} g\big(\eta_i, x(\eta_i), {}^H D_{a^+}^{u,v;\psi} x(\eta_i)\big) \\ & + \sum_{j=1}^{n} \sigma_j I_{a^+}^{\alpha_2;\psi} g\big(\omega_j, x(\omega_j), {}^H D_{a^+}^{u,v;\psi} x(\omega_j)\big) - I_{a^+}^{\alpha_1;\psi} f\big(b, y(b), {}^H D_{a^+}^{u,v;\psi} y(b)\big) \Bigg) \\ & + C \Bigg(\sum_{r=1}^{p} \lambda_r I_{a^+}^{\alpha_1 - \theta_r;\psi} f\big(\zeta_r, y(\zeta_r), {}^H D_{a^+}^{u,v;\psi} y(\zeta_r)\big) \\ & + \sum_{s=1}^{q} \mu_s I_{a^+}^{\alpha_1;\psi} f\big(\xi_s, y(\xi_s), {}^H D_{a^+}^{u,v;\psi} y(\xi_s)\big) - I_{a^+}^{\alpha_2;\psi} g\big(b, x(b), {}^H D_{a^+}^{u,v;\psi} x(b)\big) \Bigg) \Bigg], \end{aligned} \tag{4}$$

where

$$\begin{aligned} C &= \frac{(\psi(b) - \psi(a))^{\gamma_1 - 1}}{\Gamma(\gamma_1)}, \\ D &= \sum_{i=1}^{m} \varphi_i \frac{(\psi(\eta_i) - \psi(a))^{\gamma_2 - \delta_i - 1}}{\Gamma(\gamma_2 - \delta_i)} + \sum_{j=1}^{n} \sigma_j \frac{(\psi(\omega_j) - \psi(a))^{\gamma_2 - 1}}{\Gamma(\gamma_2)}, \\ \varOmega &= \sum_{r=1}^{p} \lambda_r \frac{(\psi(\zeta_r) - \psi(a))^{\gamma_1 - \theta_r - 1}}{\Gamma(\gamma_1 - \theta_r)} + \sum_{s=1}^{q} \mu_s \frac{(\psi(\xi_s) - \psi(a))^{\gamma_1 - 1}}{\Gamma(\gamma_1)}, \\ \varPhi &= \frac{(\psi(b) - \psi(a))^{\gamma_2 - 1}}{\Gamma(\gamma_2)}, \end{aligned} \tag{5}$$

and

$$\curlywedge = C\varPhi - D\varOmega.$$

Proof. Taking operator $I_{a^+}^{\alpha_1;\psi}$, $I_{a^+}^{\alpha_2;\psi}$ on both sides of (2) and using Lemma 2, we have

$$\begin{aligned} x(t) &= c_1 \frac{(\psi(t)-\psi(a))^{\gamma_1-1}}{\Gamma(\gamma_1)} + c_2 \frac{(\psi(t)-\psi(a))^{\gamma_1-2}}{\Gamma(\gamma_1-1)} \\ &\quad + I_{a^+}^{\alpha_1;\psi} f\bigl(t, y(t), {}^H D_{a^+}^{u,v;\psi} y(t)\bigr), \\ y(t) &= d_1 \frac{(\psi(t)-\psi(a))^{\gamma_2-1}}{\Gamma(\gamma_2)} + d_2 \frac{(\psi(t)-\psi(a))^{\gamma_2-2}}{\Gamma(\gamma_1-1)} \\ &\quad + I_{a^+}^{\alpha_2;\psi} g\bigl(t, x(t), {}^H D_{a^+}^{u,v;\psi} x(t)\bigr). \end{aligned}$$

When $x(a)=0$, $y(a)=0$, we get $c_2=0$, $d_2=0$. Then the above equations become

$$x(t) = c_1 \frac{(\psi(t)-\psi(a))^{\gamma_1-1}}{\Gamma(\gamma_1)} + I_{a^+}^{\alpha_1;\psi} f\bigl(t, y(t), {}^H D_{a^+}^{u,v;\psi} y(t)\bigr), \tag{6}$$

$$y(t) = d_1 \frac{(\psi(t)-\psi(a))^{\gamma_2-1}}{\Gamma(\gamma_2)} + I_{a^+}^{\alpha_2;\psi} g\bigl(t, x(t), {}^H D_{a^+}^{u,v;\psi} x(t)\bigr). \tag{7}$$

Applying the boundary condition at the point b, we have

$$\begin{aligned} & c_1 \frac{(\psi(b)-\psi(a))^{\gamma_1-1}}{\Gamma(\gamma_1)} + I_{a^+}^{\alpha_1;\psi} f\bigl(b, y(b), {}^H D_{a^+}^{u,v;\psi} y(b)\bigr) \\ &\quad = d_1 \sum_{i=1}^{m} \varphi_i \frac{(\psi(\eta_i)-\psi(a))^{\gamma_2-\delta_i-1}}{\Gamma(\gamma_2-\delta_i)} + \sum_{i=1}^{m} \varphi_i I_{a^+}^{\alpha_2-\delta_i;\psi} g\bigl(\eta_i, x(\eta_i), {}^H D_{a^+}^{u,v;\psi} x(\eta_i)\bigr) \\ &\qquad + d_1 \sum_{j=1}^{n} \sigma_j \frac{(\psi(\omega_j)-\psi(a))^{\gamma_2-1}}{\Gamma(\gamma_2)} + \sum_{j=1}^{n} \sigma_j I_{a^+}^{\alpha_2;\psi} g\bigl(\omega_j, x(\omega_j), {}^H D_{a^+}^{u,v;\psi} x(\omega_j)\bigr) \end{aligned}$$

and

$$\begin{aligned} & d_1 \frac{(\psi(b)-\psi(a))^{\gamma_2-1}}{\Gamma(\gamma_2)} + I_{a^+}^{\alpha_2;\psi} g\bigl(b, x(b), {}^H D_{a^+}^{u,v;\psi} x(b)\bigr) \\ &\quad = c_1 \sum_{r=1}^{p} \lambda_r \frac{(\psi(\zeta_r)-\psi(a))^{\gamma_1-\theta_r-1}}{\Gamma(\gamma_1-\theta_r)} + \sum_{r=1}^{p} \lambda_r I_{a^+}^{\alpha_1-\theta_r;\psi} f\bigl(\zeta_r, y(\zeta_r), {}^H D_{a^+}^{u,v;\psi} y(\zeta_r)\bigr) \\ &\qquad + c_1 \sum_{s=1}^{q} \mu_s \frac{(\psi(\xi_s)-\psi(a))^{\gamma_1-1}}{\Gamma(\gamma_1)} + \sum_{s=1}^{q} \mu_s I_{a^+}^{\alpha_1;\psi} f\bigl(\xi_s, y(\xi_s), {}^H D_{a^+}^{u,v;\psi} y(\xi_s)\bigr). \end{aligned}$$

The above equations can be written in the form

$$\begin{aligned} C c_1 - D d_1 &= M, \\ -\Omega c_1 + \Phi d_1 &= N, \end{aligned} \tag{8}$$

where C, D, $\varOmega$, and $\varPhi$ are given in (5), and

$$M = \sum_{i=1}^{m} \varphi_i I_{a^+}^{\alpha_2-\delta_i;\psi} g\big(\eta_i, x(\eta_i), {}^H D_{a^+}^{u,v;\psi} x(\eta_i)\big) + \sum_{j=1}^{n} \sigma_j I_{a^+}^{\alpha_2;\psi} g\big(\omega_j, x(\omega_j), {}^H D_{a^+}^{u,v;\psi} x(\omega_j)\big) - I_{a^+}^{\alpha_1;\psi} f\big(b, y(b), {}^H D_{a^+}^{u,v;\psi} y(b)\big),$$

$$N = \sum_{r=1}^{p} \lambda_r I_{a^+}^{\alpha_1-\theta_r;\psi} f\big(\zeta_r, y(\zeta_r), {}^H D_{a^+}^{u,v;\psi} y(\zeta_r)\big) + \sum_{s=1}^{q} \mu_s I_{a^+}^{\alpha_1;\psi} f\big(\xi_s, y(\xi_s), {}^H D_{a^+}^{u,v;\psi} y(\xi_s)\big) - I_{a^+}^{\alpha_2;\psi} g\big(b, x(b), {}^H D_{a^+}^{u,v;\psi} x(b)\big).$$

From equations (8) we get

$$c_1 = \frac{\varPhi M + DN}{人}, \quad d_1 = \frac{CN + \varOmega M}{人}, \quad \text{where } 人 = C\varPhi - D\varOmega.$$

Now substituting the values of c_1 and d_1 in (6) and (7), we obtain solutions (3) and (4).

Conversely, by direct computation we obtain that (3) and (4) satisfy (2). □

4 Existence and uniqueness results

To establish the existence and uniqueness results, we present our coupled system (2) as a fixed point problem.

First, we define the operator $\mathcal{T} : \mathbb{B} \times \mathbb{B} \to \mathbb{B} \times \mathbb{B}$ by

$$\mathcal{T}(x, y)(t) = \big(\mathcal{T}_1(x, y)(t), \mathcal{T}_2(x, y)(t)\big), \tag{9}$$

where

$$\begin{aligned} \mathcal{T}_1(x, y)(t) &= I_{a^+}^{\alpha_1;\psi} f\big(t, y(t), {}^H D_{a^+}^{u,v;\psi} y(t)\big) \\ &\quad + \frac{(\psi(t)-\psi(a))^{\gamma_1-1}}{\Gamma(\gamma_1)\cdot 人} \Bigg[\varPhi\Bigg(\sum_{i=1}^{m} \varphi_i I_{a^+}^{\alpha_2-\delta_i;\psi} g\big(\eta_i, x(\eta_i), {}^H D_{a^+}^{u,v;\psi} x(\eta_i)\big) \\ &\quad + \sum_{j=1}^{n} \sigma_j I_{a^+}^{\alpha_2;\psi} g\big(\omega_j, x(\omega_j), {}^H D_{a^+}^{u,v;\psi} x(\omega_j)\big) - I_{a^+}^{\alpha_1;\psi} f\big(b, y(b), {}^H D_{a^+}^{u,v;\psi} y(b)\big)\Bigg) \\ &\quad + D\Bigg(\sum_{r=1}^{p} \lambda_r I_{a^+}^{\alpha_1-\theta_r;\psi} f\big(\zeta_r, y(\zeta_r), {}^H D_{a^+}^{u,v;\psi} y(\zeta_r)\big) \\ &\quad + \sum_{s=1}^{q} \mu_s I_{a^+}^{\alpha_1;\psi} f\big(\xi_s, y(\xi_s), {}^H D_{a^+}^{u,v;\psi} y(\xi_s)\big) - I_{a^+}^{\alpha_2;\psi} g\big(b, x(b), {}^H D_{a^+}^{u,v;\psi} x(b)\big)\Bigg)\Bigg] \end{aligned} \tag{10}$$

and

$$\begin{aligned}
&\mathcal{T}_2(x,y)(t)\\
&\quad = I_{a^+}^{\alpha_2;\psi} g\big(t, x(t), {}^H D_{a^+}^{u,v;\psi} x(t)\big)\\
&\qquad + \frac{(\psi(t)-\psi(a))^{\gamma_2-1}}{\Gamma(\gamma_2)\cdot 人}\Bigg[\varOmega\Bigg(\sum_{i=1}^{m}\varphi_i I_{a^+}^{\alpha_2-\delta_i;\psi} g\big(\eta_i, x(\eta_i), {}^H D_{a^+}^{u,v;\psi} x(\eta_i)\big)\\
&\qquad + \sum_{j=1}^{n}\sigma_j I_{a^+}^{\alpha_2;\psi} g\big(\omega_j, x(\omega_j), {}^H D_{a^+}^{u,v;\psi} x(\omega_j)\big) - I_{a^+}^{\alpha_1;\psi} f\big(b, y(b), {}^H D_{a^+}^{u,v;\psi} y(b)\big)\Bigg)\\
&\qquad + C\Bigg(\sum_{r=1}^{p}\lambda_r I_{a^+}^{\alpha_1-\theta_r;\psi} f\big(\zeta_r, y(\zeta_r), {}^H D_{a^+}^{u,v;\psi} y(\zeta_r)\big)\\
&\qquad + \sum_{s=1}^{q}\mu_s I_{a^+}^{\alpha_1;\psi} f\big(\xi_s, y(\xi_s), {}^H D_{a^+}^{u,v;\psi} y(\xi_s)\big) - I_{a^+}^{\alpha_2;\psi} g\big(b, x(b), {}^H D_{a^+}^{u,v;\psi} x(b)\big)\Bigg)\Bigg]. \quad (11)
\end{aligned}$$

For readability, we use the notations listed below:

$$\varTheta(l,s) = \frac{(\psi(l)-\psi(a))^s}{\Gamma(s+1)},$$

$$\begin{aligned}
C_1 &= \varTheta(b,\kappa) + \frac{\varTheta(b,\ \gamma_1-1)}{|\ 人\ |}\\
&\quad \times \Bigg[|\varPhi|\varTheta(b,\kappa) + |D|\Bigg(\sum_{r=1}^{p}\lambda_r\varTheta(\zeta_r,\kappa-\theta_r) + \sum_{s=1}^{q}\mu_s\varTheta(\xi_s,\kappa)\Bigg)\Bigg], \quad (12)\\
C_2 &= \frac{\varTheta(b,\ \gamma_1-1)}{|\ 人\ |}\\
&\quad \times \Bigg[|\varPhi|\Bigg(\sum_{i=1}^{m}\varphi_i\varTheta(\eta_i,\bar\kappa-\delta_i) + \sum_{j=1}^{n}\sigma_j\varTheta(\omega_j,\bar\kappa)\Bigg) + |D|\varTheta(b,\bar\kappa)\Bigg], \quad (13)\\
D_1 &= \frac{\varTheta(b,\gamma_2-1)}{|\ 人\ |}\\
&\quad \times \Bigg[|C|\Bigg(\sum_{r=1}^{p}\lambda_r\varTheta(\zeta_r,\kappa-\theta_r) + \sum_{s=1}^{q}\mu_s\varTheta(\xi_s,\kappa)\Bigg) + |\varOmega|\varTheta(b,\kappa)\Bigg], \quad (14)\\
D_2 &= \varTheta(b,\bar\kappa) + \frac{\varTheta(b,\gamma_2-1)}{|\ 人\ |}\\
&\quad \times \Bigg[|C|\varTheta(b,\bar\kappa) + |\varOmega|\Bigg(\sum_{i=1}^{m}\varphi_i\varTheta(\eta_i,\bar\kappa-\delta_i) + \sum_{j=1}^{n}\sigma_j\varTheta(\omega_j,\bar\kappa)\Bigg)\Bigg], \quad (15)
\end{aligned}$$

where $\kappa = \alpha_1$ or $\alpha_1 - u$, $\bar\kappa = \alpha_2$ or $\alpha_2 - u$.

4.1 Uniqueness of solution

Theorem 2. *Assume that* $\curlywedge \neq 0$ *and* $f, g : \mathbb{J} \times \mathbb{B} \times \mathbb{B} \to \mathbb{B}$ *are two functions satisfying the condition*

(H1) *There exists constants* $L_1, L_2 > 0$ *such that, for all* $t \in \mathbb{J}$ *and* $x_i, y_i \in \mathbb{R}$, $i = 1, 2$, *we have*

$$\|f(t, y_1, \bar{y_1}) - f(t, y_2, \bar{y_2})\| \leqslant L_1(\|y_1(t) - y_2(t)\| + \|\bar{y_1}(t) - \bar{y_2}(t)\|),$$
$$\|g(t, x_1, \bar{x_1}) - g(t, x_2, \bar{x_2})\| \leqslant L_2(\|x_1(t) - x_2(t)\| + \|\bar{x_1}(t) - \bar{x_2}(t)\|).$$

Then system (2) *has a unique solution on* $\mathbb{J}$, *provided that*

$$\Upsilon = L_1(C_1 + D_1) + L_2(C_2 + D_2) < 1, \tag{16}$$

where C_1, C_2, D_1, *and* D_2 *are given by* (12), (13), (14), *and* (15), *respectively.*

Proof. Let us consider the operator $\mathcal{T}(x, y)(t)$ defined as in (9), where $\mathcal{T}_1(x, y)(t)$ and $\mathcal{T}_2(x, y)(t)$ are given by (10) and (11), respectively.

Let $\sup_{t\in\mathbb{J}} \|f(t, 0, 0)\| = M_1 < \infty$, $\sup_{t\in\mathbb{J}} \|g(t, 0, 0)\| = M_2 < \infty$, and let us set

$$\mathbb{B}_r = \{(x, y) \in \mathbb{B} \times \mathbb{B}\colon \|(x, y)\|_{\mathbb{B}} \leqslant r\},$$

where

$$r \geqslant \frac{(C_1 + D_1)M_1 + (C_2 + D_2)M_2}{1 - [L_1(C_1 + D_1) + L_2(C_2 + D_2)]}.$$

Clearly, $\mathbb{B}_r$ is bounded, closed, and convex subset of $\mathbb{B}$.

Step 1. We prove that $\mathcal{T}\mathbb{B}_r \subset \mathbb{B}_r$.

For any $(x, y) \in \mathbb{B}_r$, $t \in \mathbb{J}$, using hypothesis (H1), we have

$$\begin{aligned}\|f(t, y, \bar{y})\| &\leqslant \|f(t, y, \bar{y}) - f(t, 0, 0)\| + \|f(t, 0, 0)\| \\ &\leqslant L_1(\|y(t)\| + \|\bar{y}(t)\|) + M_1\end{aligned}$$

and

$$\begin{aligned}\|g(t, x, \bar{x})\| &\leqslant \|g(t, x, \bar{x}) - g(t, 0, 0)\| + \|g(t, 0, 0)\| \\ &\leqslant L_2(\|x(t)\| + \|\bar{x}(t)\|) + M_2.\end{aligned}$$

Then we obtain

$$\begin{aligned}&\|\mathcal{T}_1(x, y)(t)\| \\ &\quad\leqslant I_{a^+}^{\alpha_1;\psi}\|f(t, y(t), {}^H D_{a^+}^{u,v;\psi} y(t))\| \\ &\qquad+ \frac{\Theta(b,\ \gamma_1 - 1)}{|\curlywedge|}\Bigg[|\Phi|\Bigg(\sum_{i=1}^{m} |\varphi_i| I_{a^+}^{\alpha_2-\delta_i;\psi}\|g(\eta_i, x(\eta_i), {}^H D_{a^+}^{u,v;\psi} x(\eta_i))\| \\ &\qquad+ \sum_{j=1}^{n} |\sigma_j| I_{a^+}^{\alpha_2;\psi}\|g(\omega_j, x(\omega_j), {}^H D_{a^+}^{u,v;\psi} x(\omega_j))\| + I_{a^+}^{\alpha_1;\psi}\|f(b, y(b), {}^H D_{a^+}^{u,v;\psi} y(b))\|\Bigg)\end{aligned}$$

$$
\begin{aligned}
&+|D|\Bigg(\sum_{r=1}^{p}|\lambda_r| I_{a^+}^{\alpha_1-\theta_r;\psi}\big\|f\big(\zeta_r, y(\zeta_r), {}^H D_{a^+}^{u,v;\psi} y(\zeta_r)\big)\big\| \\
&+\sum_{s=1}^{q}|\mu_s| I_{a^+}^{\alpha_1;\psi}\big\|f\big(\xi_s, y(\xi_s), {}^H D_{a^+}^{u,v;\psi} y(\xi_s)\big)\big\| + I_{a^+}^{\alpha_2;\psi}\big\|g\big(b, x(b), {}^H D_{a^+}^{u,v;\psi} x(b)\big)\big\|\Bigg)\Bigg] \\
\leqslant\ & \Theta(b,\alpha_1)\big(L_1\|v\|_{\mathbb{B}} + M_1\big) \\
&+\frac{\Theta(b,\ \gamma_1-1)}{|\curlywedge|}\Bigg[|\Phi|\Bigg(\sum_{i=1}^{m}\varphi_i\Theta(\eta_i,\ \alpha_2-\delta_i)\big(L_2\|u\|_{\mathbb{B}} + M_2\big) \\
&+\sum_{j=1}^{n}\sigma_j\Theta(\omega_j,\alpha_2)\big(L_2\|u\|_{\mathbb{B}} + M_2\big) + \Theta(b,\alpha_1)\big(L_1\|v\|_{\mathbb{B}} + M_1\big)\Bigg) \\
&+|D|\Bigg(\sum_{r=1}^{p}\lambda_r\Theta(\zeta_r,\ \alpha_1-\theta_r)\big(L_1\|v\|_{\mathbb{B}} + M_1\big) \\
&+\sum_{s=1}^{q}\mu_s\Theta(\xi_s,\alpha_1)\big(L_1\|v\|_{\mathbb{B}} + M_1\big) + \Theta(b,\alpha_2)\big(L_2\|u\|_{\mathbb{B}} + M_2\big)\Bigg)\Bigg] \\
\leqslant\ & \Bigg(\Theta(b,\alpha_1) + \frac{\Theta(b,\ \gamma_1-1)}{|\curlywedge|}\Bigg[|\Phi|\Theta(b,\alpha_1) \\
&+|D|\Bigg(\sum_{r=1}^{p}\lambda_r\Theta(\zeta_r,\ \alpha_1-\theta_r) + \sum_{s=1}^{q}\mu_s\Theta(\xi_s,\alpha_1)\Bigg)\Bigg]\Bigg)(L_1 r + M_1) \\
&+\frac{\Theta(b,\ \gamma_1-1)}{|\curlywedge|}\Bigg[|\Phi|\Bigg(\sum_{i=1}^{m}\varphi_i\Theta(\eta_i,\ \alpha_2-\delta_i) + \sum_{j=1}^{n}\sigma_j\Theta(\omega_j,\alpha_2)\Bigg) \\
&+|D|\Theta(b,\alpha_2)\Bigg](L_2 r + M_2)
\end{aligned}
$$

and

$$
\begin{aligned}
&\big\|{}^H D_{a^+}^{u,v;\psi}\mathcal{T}_1(x,y)(t)\big\| \\
&\quad\leqslant\Bigg(\Theta(b,\ \alpha_1-u) + \frac{\Theta(b,\ \gamma_1-1)}{|\curlywedge|}\Bigg[|\Phi|\Theta(b,\ \alpha_1-u) \\
&\qquad+|D|\Bigg(\sum_{r=1}^{p}\lambda_r\Theta(\zeta_r,\ \alpha_1-u-\theta_r) + \sum_{s=1}^{q}\mu_s\Theta(\xi_s,\ \alpha_1-u)\Bigg)\Bigg]\Bigg)(L_1 r + M_1) \\
&\qquad+\frac{\Theta(b,\ \gamma_1-1)}{|\curlywedge|}\Bigg[|\Phi|\Bigg(\sum_{i=1}^{m}\varphi_i\Theta(\eta_i,\ \alpha_2-u-\delta_i) + \sum_{j=1}^{n}\sigma_j\Theta(\omega_j,\ \alpha_2-u)\Bigg) \\
&\qquad+|D|\Theta(b,\ \alpha_2-u)\Bigg](L_2 r + M_2).
\end{aligned}
$$

Hence

$$\left\|\mathcal{T}_1(x,y)\right\|_{\mathbb{B}} \leqslant C_1(L_1 r + M_1) + C_2(L_2 r + M_2).$$

Similarly,

$$\left\|\mathcal{T}_2(x,y)\right\|_{\mathbb{B}} \leqslant D_1(L_1 r + M_1) + D_2(L_2 r + M_2).$$

Consequently,

$$\begin{aligned}\left\|\mathcal{T}(x,y)\right\|_{\mathbb{B}} &\leqslant \left[L_1(C_1+D_1) + L_2(C_2+D_2)\right]r \\ &\quad + (C_1+D_1)M_1 + (C_2+D_2)M_2 \\ &\leqslant r\end{aligned}$$

implies $\mathcal{T}\mathbb{B}_r \subset \mathbb{B}_r$.

Step 2. We prove that $\mathcal{T}$ is a contraction.

For any $(x_1,y_1),(x_2,y_2) \in \mathbb{B}\times\mathbb{B}$ and for each $t \in \mathbb{J}$, using (H1), we have

$$\begin{aligned}
&\left\|\mathcal{T}_1(x_1,y_1)(t) - \mathcal{T}_1(x_2,y_2)(t)\right\| \\
&\quad\leqslant I_{a^+}^{\alpha_1;\psi}\left\|f\left(t,y_1(t),{}^{H}D_{a^+}^{u,v;\psi}y_1(t)\right) - f\left(t,y_2(t),{}^{H}D_{a^+}^{u,v;\psi}y_2(t)\right)\right\| \\
&\qquad+\frac{\Theta(b,\,\gamma_1-1)}{|\curlywedge|}\Biggl[|\Phi|\Biggl(\sum_{i=1}^{m}|\varphi_i| I_{a^+}^{\alpha_2-\delta_i;\psi}\bigl\|g\bigl(\eta_i,x_1(\eta_i),{}^{H}D_{a^+}^{u,v;\psi}x_1(\eta_i)\bigr) \\
&\qquad- g\bigl(\eta_i,x_2(\eta_i),{}^{H}D_{a^+}^{u,v;\psi}x_2(\eta_i)\bigr)\bigr\| \\
&\qquad+\sum_{j=1}^{n}|\sigma_j| I_{a^+}^{\alpha_2;\psi}\bigl\|g\bigl(\omega_j,x_1(\omega_j),{}^{H}D_{a^+}^{u,v;\psi}x_1(\omega_j)\bigr) - g\bigl(\omega_j,x_2(\omega_j),{}^{H}D_{a^+}^{u,v;\psi}x_2(\omega_j)\bigr)\bigr\| \\
&\qquad+I_{a^+}^{\alpha_1;\psi}\bigl\|f\bigl(b,y_1(b),{}^{H}D_{a^+}^{u,v;\psi}y_1(b)\bigr) - f\bigl(b,y_2(b),{}^{H}D_{a^+}^{u,v;\psi}y_2(b)\bigr)\bigr\|\Biggr) \\
&\qquad+|D|\Biggl(\sum_{r=1}^{p}|\lambda_r| I_{a^+}^{\alpha_1-\theta_r;\psi}\bigl\|f\bigl(\zeta_r,y_1(\zeta_r),{}^{H}D_{a^+}^{u,v;\psi}y_1(\zeta_r)\bigr) \\
&\qquad- f\bigl(\zeta_r,y_2(\zeta_r),{}^{H}D_{a^+}^{u,v;\psi}y_2(\zeta_r)\bigr)\bigr\| \\
&\qquad+\sum_{s=1}^{q}|\mu_s| I_{a^+}^{\alpha_1;\psi}\bigl\|f\bigl(\xi_s,y_1(\xi_s),{}^{H}D_{a^+}^{u,v;\psi}y_1(\xi_s)\bigr) - f\bigl(\xi_s,y_2(\xi_s),{}^{H}D_{a^+}^{u,v;\psi}y_2(\xi_s)\bigr)\bigr\| \\
&\qquad+I_{a^+}^{\alpha_2;\psi}\bigl\|g\bigl(b,x_1(b),{}^{H}D_{a^+}^{u,v;\psi}x_1(b)\bigr) - g\bigl(b,x_2(b),{}^{H}D_{a^+}^{u,v;\psi}x_2(b)\bigr)\bigr\|\Biggr)\Biggr] \\
&\quad\leqslant \Theta(b,\alpha_1)L_1\|y_1-y_2\|_{\mathbb{B}} + \frac{\Theta(b,\,\gamma_1-1)}{|\curlywedge|}\Biggl[|\Phi|\Biggl(\sum_{i=1}^{m}\varphi_i\Theta(\eta_i,\,\alpha_2-\delta_i)L_2\|x_1-x_2\|_{\mathbb{B}} \\
&\qquad+\sum_{j=1}^{n}\sigma_j\Theta(\omega_j,\alpha_2)L_2\|x_1-x_2\|_{\mathbb{B}} + \Theta(b,\alpha_1)L_1\|y_1-y_2\|_{\mathbb{B}}\Biggr)
\end{aligned}$$

$$+ |D|\Bigg(\sum_{r=1}^{p}\lambda_r\Theta(\zeta_r,\,\alpha_1-\theta_r)L_1\|y_1-y_2\|_{\mathbb{B}}$$
$$+\sum_{s=1}^{q}\mu_s\Theta(\xi_s,\alpha_1)L_1\|y_1-y_2\|_{\mathbb{B}}+\Theta(b,\alpha_2)L_2\|x_1-x_2\|_{\mathbb{B}}\Bigg)\Bigg]$$

and

$$\begin{aligned}
&\big\|{}^HD_{a^+}^{u,v;\psi}\mathcal{T}_1(x_1,y_1)(t)-{}^HD_{a^+}^{u,v;\psi}\mathcal{T}_1(x_2,y_2)(t)\big\| \\
&\quad\leqslant \Theta(b,\,\alpha_1-u)L_1\|y_1-y_2\|_{\mathbb{B}} \\
&\qquad+\frac{\Theta(b,\,\gamma_1{-}1)}{|\,\curlywedge\,|}\Bigg[|\Phi|\Bigg(\sum_{i=1}^{m}\varphi_i\Theta(\eta_i,\,\alpha_2-u-\delta_i)L_2\|x_1-x_2\|_{\mathbb{B}} \\
&\qquad+\sum_{j=1}^{n}\sigma_j\Theta(\omega_j,\,\alpha_2-u)L_2\|x_1-x_2\|_{\mathbb{B}}+\Theta(b,\,\alpha_1-u)L_1\|y_1-y_2\|_{\mathbb{B}}\Bigg) \\
&\qquad+|D|\Bigg(\sum_{r=1}^{p}\lambda_r\Theta(\zeta_r,\,\alpha_1-u-\theta_r)L_1\|y_1-y_2\|_{\mathbb{B}} \\
&\qquad+\sum_{s=1}^{q}\mu_s\Theta(\xi_s,\,\alpha_1-u)L_1\|y_1-y_2\|_{\mathbb{B}}+\Theta(b,\,\alpha_2-u)L_2\|x_1-x_2\|_{\mathbb{B}}\Bigg)\Bigg].
\end{aligned}$$

Hence

$$\begin{aligned}
\big\|\mathcal{T}_1(x,y)-\mathcal{T}_1(x,y)\big\|_{\mathbb{B}}&\leqslant C_1L_1\|y_1-y_2\|_{\mathbb{B}}+C_2L_2\|x_1-x_2\|_{\mathbb{B}} \\
&\leqslant (C_1L_1+C_2L_2)\big(\|x_1-x_2\|_{\mathbb{B}}+\|y_1-y_2\|_{\mathbb{B}}\big).
\end{aligned}$$

Similarly,

$$\big\|\mathcal{T}_2(x,y)-\mathcal{T}_2(x,y)\big\|_{\mathbb{B}}\leqslant (D_1L_1+D_2L_2)\big(\|x_1-x_2\|_{\mathbb{B}}+\|y_1-y_2\|_{\mathbb{B}}\big).$$

Consequently,

$$\begin{aligned}
\big\|\mathcal{T}(x,y)-\mathcal{T}(x,y)\big\|_{\mathbb{B}}&\leqslant \big[(C_1+D_1)L_1+(C_2+D_2)L_2\big] \\
&\quad\times\big(\|x_1-x_2\|_{\mathbb{B}}+\|y_1-y_2\|_{\mathbb{B}}\big).
\end{aligned}$$

Since $(C_1+D_1)L_1+(C_2+D_2)L_2<1$ by (16), the operator $\mathcal{T}$ is a contraction.

Therefore, by Lemma 4 we observe that $\mathcal{T}$ has a unique fixed point. Thus, system (2) has a unique solution on $\mathbb{J}$. □

4.2 Existence of solution

Theorem 3. *Let $\curlywedge\neq 0$ and $f,g:\mathbb{J}\times\mathbb{B}\times\mathbb{B}$ be continuous functions satisfying condition* (H1) *of Theorem* 2*. Moreover, assume that*

$$\Delta = L_1\big(C_1+D_1-\Theta(b,\nu)\big)+L_2\big(C_2+D_2-\Theta(b,\bar\nu)\big)<1 \tag{17}$$

for $\nu=\alpha_1$ or α_1-u, $\bar\nu=\alpha_2$ or α_2-u. Then system (2) *has at least one solution on $\mathbb{J}$.*

Proof. We decompose the operator $\mathcal{T}$ into four operators:

$$\begin{aligned}
&\mathcal{P}(x,y)(t) = I_{a^+}^{\alpha_1;\psi} f\big(t, y(t), {}^H D_{a^+}^{u,v;\psi} y(t)\big),\\
&\mathcal{Q}(x,y)(t)\\
&\quad = \frac{\Theta(b,\,\gamma_1-1)}{\curlywedge}\Bigg[\varPhi\Bigg(\sum_{i=1}^{m}\varphi_i I_{a^+}^{\alpha_2-\delta_i;\psi} g\big(\eta_i, x(\eta_i), {}^H D_{a^+}^{u,v;\psi} x(\eta_i)\big)\\
&\qquad + \sum_{j=1}^{n}\sigma_j I_{a^+}^{\alpha_2;\psi} g\big(\omega_j, x(\omega_j), {}^H D_{a^+}^{u,v;\psi} x(\omega_j)\big) - I_{a^+}^{\alpha_1;\psi} f\big(b, y(b), {}^H D_{a^+}^{u,v;\psi} y(b)\big)\Bigg)\\
&\qquad + D\Bigg(\sum_{r=1}^{p}\lambda_r I_{a^+}^{\alpha_1-\theta_r;\psi} f\big(\zeta_r, y(\zeta_r), {}^H D_{a^+}^{u,v;\psi} y(\zeta_r)\big)\\
&\qquad + \sum_{s=1}^{q}\mu_s I_{a^+}^{\alpha_1;\psi} f\big(\xi_s, y(\xi_s), {}^H D_{a^+}^{u,v;\psi} y(\xi_s)\big) - I_{a^+}^{\alpha_2;\psi} g\big(b, x(b), {}^H D_{a^+}^{u,v;\psi} x(b)\big)\Bigg)\Bigg],\\
&\mathcal{R}(x,y)(t) = I_{a^+}^{\alpha_2;\psi} g\big(t, x(t), {}^H D_{a^+}^{u,v;\psi} x(t)\big),\\
&\mathcal{S}(x,y)(t)\\
&\quad = \frac{\Theta(b,\gamma_2-1)}{\curlywedge}\Bigg[\varOmega\Bigg(\sum_{i=1}^{m}\varphi_i I_{a^+}^{\alpha_2-\delta_i;\psi} g\big(\eta_i, x(\eta_i), {}^H D_{a^+}^{u,v;\psi} x(\eta_i)\big)\\
&\qquad + \sum_{j=1}^{n}\sigma_j I_{a^+}^{\alpha_2;\psi} g\big(\omega_j, x(\omega_j), {}^H D_{a^+}^{u,v;\psi} x(\omega_j)\big) - I_{a^+}^{\alpha_1;\psi} f\big(b, y(b), {}^H D_{a^+}^{u,v;\psi} y(b)\big)\Bigg)\\
&\qquad + C\Bigg(\sum_{r=1}^{p}\lambda_r I_{a^+}^{\alpha_1-\theta_r;\psi} f\big(\zeta_r, y(\zeta_r), {}^H D_{a^+}^{u,v;\psi} y(\zeta_r)\big)\\
&\qquad + \sum_{s=1}^{q}\mu_s I_{a^+}^{\alpha_1;\psi} f\big(\xi_s, y(\xi_s), {}^H D_{a^+}^{u,v;\psi} y(\xi_s)\big) - I_{a^+}^{\alpha_2;\psi} g\big(b, x(b), {}^H D_{a^+}^{u,v;\psi} x(b)\big)\Bigg)\Bigg].
\end{aligned}$$

This shows that $\mathcal{T}_1(x,y)(t) = \mathcal{P}(x,y)(t) + \mathcal{Q}(x,y)(t)$ and $\mathcal{T}_2(x,y)(t) = \mathcal{R}(x,y)(t) + \mathcal{S}(x,y)(t)$.

Consider the closed ball $\mathbb{B}_r$ defined in Theorem 2.

Step 1. We prove that $\mathcal{T}_1(x,y) + \mathcal{T}_2(u,v) \in \mathbb{B}_r$ for all $(x,y), (u,v) \in \mathbb{B}_r$.

Considering the proof of Theorem 2, we observe that

$$\begin{aligned}
\big\|\mathcal{P}(x,y) + \mathcal{Q}(u,v)\big\|_{\mathbb{B}} &\leqslant C_1(L_1 r + M_1) + C_2(L_2 r + M_2),\\
\big\|\mathcal{R}(x,y) + \mathcal{S}(u,v)\big\|_{\mathbb{B}} &\leqslant D_1(L_1 r + M_1) + D_2(L_2 r + M_2).
\end{aligned}$$

This implies $\mathcal{T}_1(x,y) + \mathcal{T}_2(u,v) \in \mathbb{B}_r$.

Step 2. We prove that $(\mathcal{P}, \mathcal{R})$ is continuous and compact on $\mathbb{B}_r$.

Let y_n be a sequence such that $y_n \to y$ as $n \to \infty$.

For any $(x_n, y_n), (x, y) \in \mathbb{B}_r$ and $t \in \mathbb{J}$, we have

$$\begin{aligned}
&\big\|\mathcal{P}(x_n, y_n)(t) - \mathcal{P}(x, y)(t)\big\| \\
&\quad \leqslant I_{a^+}^{\alpha_1;\psi}\big\|f\big(t, y_n(t), {}^H D_{a^+}^{u,v;\psi} y_n(t)\big) - f\big(t, y(t), {}^H D_{a^+}^{u,v;\psi} y(t)\big)\big\| \\
&\quad \leqslant \Theta(b, \alpha_1) L_1\big(\big\|y_n(t) - y(t)\big\| + \big\|{}^H D_{a^+}^{u,v;\psi} y_n(t) - {}^H D_{a^+}^{u,v;\psi} y(t)\big\|\big) \\
&\quad \leqslant \Theta(b, \alpha_1) L_1 \|y_n - y\|_{\mathbb{B}} \to 0 \quad \text{as } n \to \infty,
\end{aligned}$$

$$\begin{aligned}
&\big\|{}^H D_{a^+}^{u,v;\psi} \mathcal{P}(x_n, y_n)(t) - {}^H D_{a^+}^{u,v;\psi} \mathcal{P}(x, y)(t)\big\| \\
&\quad \leqslant I_{a^+}^{\alpha_1;\psi}\big\|f\big(t, y_n(t), {}^H D_{a^+}^{u,v;\psi} y_n(t)\big) - f\big(t, y(t), {}^H D_{a^+}^{u,v;\psi} y(t)\big)\big\| \\
&\quad \leqslant \Theta(b,\ \alpha_1 - u) L_1\big(\big\|y_n(t) - y(t)\big\| + \big\|{}^H D_{a^+}^{u,v;\psi} y_n(t) - {}^H D_{a^+}^{u,v;\psi} y(t)\big\|\big) \\
&\quad \leqslant \Theta(b,\ \alpha_1 - u) L_1 \|y_n - y\|_{\mathbb{B}} \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Hence

$$\big\|\mathcal{P}(x_n, y_n) - \mathcal{P}(x, y)\big\|_{\mathbb{B}} \to 0 \quad \text{as } n \to \infty.$$

Similarly,

$$\big\|\mathcal{R}(x_n, y_n) - \mathcal{R}(x, y)\big\|_{\mathbb{B}} \to 0 \quad \text{as } n \to \infty.$$

Consequently, $(\mathcal{P}, \mathcal{R})$ is continuous.

Next, we have

$$\begin{aligned}
\big\|\mathcal{P}(x, y)(t)\big\| &\leqslant \Theta(b, \alpha_1)(L_1 r + M_1), \\
\big\|\mathcal{R}(x, y)(t)\big\| &\leqslant \Theta(b, \alpha_2)(L_2 r + M_2), \\
\big\|{}^H D_{a^+}^{u,v;\psi} \mathcal{P}(x, y)(t)\big\| &\leqslant \Theta(b,\ \alpha_1 - u)(L_1 r + M_1), \\
\big\|{}^H D_{a^+}^{u,v;\psi} \mathcal{R}(x, y)(t)\big\| &\leqslant \Theta(b,\ \alpha_2 - u)(L_2 r + M_2).
\end{aligned}$$

Hence

$$\begin{aligned}
\big\|\mathcal{P}(x, y)\big\|_{\mathbb{B}} &\leqslant \max\big\{\Theta(b, \alpha_1), \Theta(b,\ \alpha_1 - u)\big\}(L_1 r + M_1) = \mathcal{G}^*, \\
\big\|\mathcal{R}(x, y)\big\|_{\mathbb{B}} &\leqslant \max\big\{\Theta(b, \alpha_2), \Theta(b,\ \alpha_2 - u)\big\}(L_2 r + M_2) = \mathcal{H}^*.
\end{aligned}$$

Therefore, $\|(\mathcal{P}, \mathcal{R})(x, y)\|_{\mathbb{B}} \leqslant \mathcal{G}^* + \mathcal{H}^*$.

This shows that $(\mathcal{P}, \mathcal{R})\mathbb{B}_r$ is uniformly bounded.

Now consider

$$\begin{aligned}
&\big\|\mathcal{P}(x, y)(t_2) - \mathcal{P}(x, y)(t_1)\big\| \\
&\quad = \big\|I_{a^+}^{\alpha_1;\psi} f\big(t, y(t_2), {}^H D_{a^+}^{u,v;\psi} y(t_2)\big) - I_{a^+}^{\alpha_1;\psi} f\big(t, y(t_1), {}^H D_{a^+}^{u,v;\psi} y(t_1)\big)\big\| \\
&\quad = \Bigg\|\frac{1}{\Gamma(\alpha_1)} \int_a^{t_2} \psi'(s)\big(\psi(t_2) - \psi(s)\big)^{\alpha_1 - 1} f\big(t, y(s), {}^H D_{a^+}^{u,v;\psi} y(s)\big)\,\mathrm{d}s \\
&\qquad - \frac{1}{\Gamma(\alpha_1)} \int_a^{t_1} \psi'(s)\big(\psi(t_1) - \psi(s)\big)^{\alpha_1 - 1} f\big(t, y(s), {}^H D_{a^+}^{u,v;\psi} y(s)\big)\,\mathrm{d}s\Bigg\|
\end{aligned}$$

$$
\begin{aligned}
&= \Bigg\| \frac{1}{\Gamma(\alpha_1)} \int_a^{t_1} \psi'(s)\big[\big(\psi(t_2)-\psi(s)\big)^{\alpha_1-1} - \big(\psi(t_1)-\psi(s)\big)^{\alpha_1-1}\big] \\
&\qquad \times f\big(t, y(s), {}^H D_{a^+}^{u,v;\psi} y(s)\big)\,\mathrm{d}s \\
&\quad + \frac{1}{\Gamma(\alpha_1)} \int_{t_1}^{t_2} \psi'(s)\big(\psi(t_2)-\psi(s)\big)^{\alpha_1-1} f\big(t, y(s), {}^H D_{a^+}^{u,v;\psi} y(s)\big)\,\mathrm{d}s \Bigg\| \\
&\leqslant \frac{1}{\Gamma(\alpha_1+1)}\big[2\big(\psi(t_2)-\psi(t_1)\big)^{\alpha_1} + \psi(t_2)-\psi(a)\big)^{\alpha_1} - \psi(t_1)-\psi(a)\big)^{\alpha_1}\big] \\
&\quad \times \big\| f\big(t, y(s), {}^H D_{a^+}^{u,v;\psi} y(s)\big)\big\|.
\end{aligned}
$$

We know that $f(t, y(s), {}^H D_{a^+}^{u,v;\psi} y(s))$ is bounded on $\mathbb{J}$, and hence the r.h.s. $\to 0$ as $t_2 \to t_1$.

Also,

$$
\begin{aligned}
&\big\| {}^H D_{a^+}^{u,v;\psi} \mathcal{P}(x,y)(t_2) - {}^H D_{a^+}^{u,v;\psi} \mathcal{P}(x,y)(t_1)\big\| \\
&\quad \leqslant \frac{1}{\Gamma(\alpha_1-u+1)}\big[2\big(\psi(t_2)-\psi(t_1)\big)^{\alpha_1-u} \\
&\qquad + \big(\psi(t_2)-\psi(a)\big)^{\alpha_1-u} - \psi(t_1)-\psi(a))^{\alpha_1-u}\big] \\
&\qquad \times \big\| f\big(t, y(s), {}^H D_{a^+}^{u,v;\psi} y(s)\big)\big\| \to 0 \quad \text{as } t_2 \to t_1.
\end{aligned}
$$

Hence

$$
\big\|\mathcal{P}(x,y)(t_2) - \mathcal{P}(x,y)(t_1)\big\|_{\mathbb{B}} \to 0 \quad \text{as } t_2 \to t_1.
$$

Similarly,

$$
\big\|\mathcal{R}(x,y)(t_2) - \mathcal{R}(x,y)(t_1)\big\|_{\mathbb{B}} \to 0 \quad \text{as } t_2 \to t_1.
$$

Consequently,

$$
\big\|(\mathcal{P},\mathcal{R})(x,y)(t_2) - (P,R)(x,y)(t_1)\big\|_{\mathbb{B}} \to 0 \quad \text{as } t_2 \to t_1,
$$

i.e., $(\mathcal{P},\mathcal{R})\mathbb{B}_r$ is equicontinuous. By Arzelà–Ascoli theorem [10] the operator $(\mathcal{P},\mathcal{R})$ will be compact on $\mathbb{B}_r$.

Step 3. We prove that $(\mathcal{Q},\mathcal{S})$ is a contraction mapping.

For any $(x_1,y_1),(x_2,y_2) \in \mathbb{B}\times\mathbb{B}$ and for each $t \in \mathbb{J}$, using (H1), we have

$$
\begin{aligned}
&\big\|\mathcal{Q}(x_1,y_1)(t) - \mathcal{Q}(x_2,y_2)(t)\big\| \\
&\quad \leqslant \frac{\Theta(b,\, \gamma_1-1)}{|\curlywedge|}\Bigg[|\Phi|\Bigg(\sum_{i=1}^{m} \varphi_i \Theta(\eta_i,\, \alpha_2-\delta_i) L_2 \|x_1-x_2\|_{\mathbb{B}} \\
&\qquad + \sum_{j=1}^{n} \sigma_j \Theta(\omega_j,\alpha_2) L_2 \|x_1-x_2\|_{\mathbb{B}} + \Theta(b,\alpha_1) L_1 \|y_1-y_2\|_{\mathbb{B}}\Bigg)
\end{aligned}
$$

$$+ |D|\left(\sum_{r=1}^{p} \lambda_r \Theta(\zeta_r,\, \alpha_1-\theta_r) L_1 \|y_1-y_2\|_{\mathbb{B}} + \sum_{s=1}^{q} \mu_s \Theta(\xi_s, \alpha_1) L_1 \|y_1-y_2\|_{\mathbb{B}} + \Theta(b, \alpha_2) L_2 \|x_1-x_2\|_{\mathbb{B}}\right)\Bigg]$$

and

$$\begin{aligned}
&\left\| {}^{H}D^{u,v;\psi}_{a^+} \mathcal{Q}(x_1,y_1)(t) - {}^{H}D^{u,v;\psi}_{a^+} \mathcal{Q}(x_2,y_2)(t)\right\| \\
&\quad \leqslant \frac{\Theta(b,\, \gamma_1-1)}{|\, \curlywedge \,|}\Bigg[|\Phi|\left(\sum_{i=1}^{m} \varphi_i \Theta(\eta_i,\, \alpha_2-u-\delta_i) L_2 \|x_1-x_2\|_{\mathbb{B}}\right. \\
&\qquad \left. + \sum_{j=1}^{n} \sigma_j \Theta(\omega_j,\, \alpha_2-u) L_2 \|x_1-x_2\|_{\mathbb{B}} + \Theta(b,\, \alpha_1-u) L_1 \|y_1-y_2\|_{\mathbb{B}}\right) \\
&\qquad + |D|\left(\sum_{r=1}^{p} \lambda_r \Theta(\zeta_r,\, \alpha_1-u-\theta_r) L_1 \|y_1-y_2\|_{\mathbb{B}}\right. \\
&\qquad \left. + \sum_{s=1}^{q} \mu_s \Theta(\xi_s,\, \alpha_1-u) L_1 \|y_1-y_2\|_{\mathbb{B}} + \Theta(b,\, \alpha_2-u) L_2 \|x_1-x_2\|_{\mathbb{B}}\right)\Bigg].
\end{aligned}$$

Hence, for $l = \alpha_1$ or $\alpha_1 - u$,

$$\begin{aligned}
&\left\|\mathcal{Q}(x,y) - \mathcal{Q}(x,y)\right\|_{\mathbb{B}} \\
&\quad \leqslant \left[C_1 - \Theta(b,l)\right] L_1 \|y_1-y_2\|_{\mathbb{B}} + C_2 L_2 \|x_1-x_2\|_{\mathbb{B}} \\
&\quad \leqslant \left[\left(C_1 - \Theta(b,l)\right) L_1 + C_2 L_2\right]\left(\|x_1-x_2\|_{\mathbb{B}} + \|y_1-y_2\|_{\mathbb{B}}\right).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\left\|\mathcal{S}(x,y) - \mathcal{S}(x,y)\right\|_{\mathbb{B}} \\
&\quad \leqslant \left[D_1 L_1 + \left(D_2 - \Theta(b,\bar{l})\right) L_2\right]\left(\|x_1-x_2\|_{\mathbb{B}} + \|y_1-y_2\|_{\mathbb{B}}\right)
\end{aligned}$$

for $l = \alpha_1$ or $\alpha_1 - u$. Consequently,

$$\begin{aligned}
&\left\|(\mathcal{Q},\mathcal{S})(x,y) - (\mathcal{Q},\mathcal{S})(x,y)\right\|_{\mathbb{B}} \\
&\quad \leqslant \left[\left(C_1 + D_1 - \Theta(b,l)\right) L_1 + \left(C_2 + D_2 - \Theta(b,\bar{l})\right) L_2\right] \\
&\qquad \times \left(\|x_1-x_2\|_{\mathbb{B}} + \|y_1-y_2\|_{\mathbb{B}}\right)
\end{aligned}$$

for $l = \alpha_1$ or $\alpha_1 - u$. Since $L_1(C_1 + D_1 - \Theta(b,l)) + L_2(C_2 + D_2 - \Theta(b,\bar{l})) < 1$ by (17) the operator $(\mathcal{Q},\mathcal{S})$ is a contraction. Therefore, by Theorem 1 we observe that problem (2) has at least one solution on $\mathbb{J}$. □

Example 1. Consider the coupled system of ψ-Hilfer nonlinear implicit fractional multipoint boundary value problem

$$
\begin{aligned}
&{}^{H}D^{13/7,1/4;\mathrm{e}^{t/3}}x(t) = f\big(t, y(t), {}^{H}D^{16/11,1/6;\mathrm{e}^{t/3}}y(t)\big), \quad t \in \mathbb{J} = \left[\frac{1}{15}, \frac{16}{15}\right],\\
&{}^{H}D^{12/7,1/4;\mathrm{e}^{t/3}}y(t) = g\big(t, x(t), {}^{H}D^{16/11,1/6;\mathrm{e}^{t/3}}x(t)\big), \quad t \in \mathbb{J} = \left[\frac{1}{15}, \frac{16}{15}\right],\\
&x\left(\frac{1}{15}\right) = 0, \qquad y\left(\frac{1}{15}\right) = 0,\\
&x\left(\frac{16}{15}\right) = \sum_{i=1}^{2}\left(\frac{2i-1}{2i+1}\right){}^{H}D^{(2i+9)/10,\,1/4;\mathrm{e}^{t/3}}y\left(\frac{i}{3}\right) + \sum_{j=1}^{3}\left(\frac{j}{j^2+1}\right)y\left(\frac{j}{4}\right),\\
&y\left(\frac{16}{15}\right) = \sum_{r=1}^{3}\frac{2r}{(r+1)^2}{}^{H}D^{(r+7)/7,\,1/4;\mathrm{e}^{t/3}}x\left(\frac{r}{3}\right) + \sum_{s=1}^{4}\left(\frac{s}{s+3}\right)^2 x\left(\frac{s}{5}\right).
\end{aligned}
$$

Here $\alpha_1 = 13/7$, $\alpha_2 = 12/7$, $\beta = 1/4$, $u = 16/11$, $v = 1/6$, $a = 1/15$, $b = 16/15$, $m = 2$, $n = 3$, $p = 3$, $q = 4$, $\varphi_i = (2i-1)/2i+1)$, $\eta_i = i/3$, $\delta_i = (2i+9)/10$, $\sigma_j = j/(j^2+1)$, $\omega_j = j/4$, $\lambda_r = 2r/(r+1)^2$, $\zeta_r = r/3$, $\theta_r = (r+7)/7$, $\mu_s = [s/(s+3)]^2$, $\xi_s = s/5$, $\psi(t) = \mathrm{e}^{t/3}$, $\psi'(t) = \mathrm{e}^{t/3}/3$.

From the data we compute that $\gamma_1 = 53/28$, $\gamma_2 = 29/16$, $C \approx 0.4645842367$, $D \approx 1.5117061858$, $\varPhi \approx 0.5293694247$, $\varOmega \approx 1.8103669378$, and $\curlywedge = C\varPhi - D\varOmega \approx -2.4908062083$.

Consider the functions

$$
\begin{aligned}
f\big(t, y, {}^{H}D^{16/11,1/6;\mathrm{e}^{t/3}}y\big) &= \frac{5}{48+30t}\,\frac{3|y|}{1+|y|} + \frac{5t+1}{38} + \frac{9t\cos|{}^{H}D^{16/11,1/6;\mathrm{e}^{t/3}}y|}{32\mathrm{e}^{(15t-1)}},\\
g\big(t, x, {}^{H}D^{16/11,1/6;\mathrm{e}^{t/3}}x\big) &= \frac{3}{12+\log 15t}\tan^{-1}|x| + \frac{2}{15t}\,\frac{|{}^{H}D^{16/11,1/6;\mathrm{e}^{t/3}}x|}{|{}^{H}D^{16/11,1/6;\mathrm{e}^{t/3}}x|+8} + \frac{1}{2}.
\end{aligned}
$$

Uniqueness. For $x_1, \bar{x_1}, y_1, \bar{y_1}, x_2, \bar{x_2}, y_2, \bar{y_2} \in \mathbb{B}$ and $t \in [1/15, 16/15]$, we have

$$
\begin{aligned}
\big\|f(t, y_1, \bar{y_1}) - f(t, y_2, \bar{y_2})\big\| &\leqslant \frac{3}{10}\big[\|y_1 - y_2\| + \|\bar{y_1} - \bar{y_2}\|\big],\\
\big\|g(t, x_1, \bar{x_1}) - g(t, x_2, \bar{x_2})\big\| &\leqslant \frac{1}{4}\big[\|x_1 - x_2\| + \|\bar{x_1} - \bar{x_2}\|\big].
\end{aligned}
$$

From (H1) we have $L_1 = 3/10$, $L_2 = 1/4$. Hence

$$L_1(C_1 + D_1) + L_2(C_2 + D_2) \approx 0.9270164955 < 1.$$

Consequently, the hypothesis of Theorem 2 is satisfied, and system (2) has a unique solution on $\mathbb{J}$. Moreover, Table 1 presents the numerical results of $\varUpsilon$ for a variety of $t \in [1/15, 16/15]$ and different values of α_1 and α_2. The results are represented graphically in Fig. 1.

Table 1. Numerical results of Υ for different values of α_1 and α_2.

t	Υ		
	$\alpha_1 = 13/7$ $\alpha_2 = 12/7$	$\alpha_1 = 16/9$ $\alpha_2 = 15/9$	$\alpha_1 = 19/11$ $\alpha_2 = 17/11$
0.0667	0	0	0
0.1367	0.2256	0.2765	0.3716
0.2067	0.3074	0.3606	0.4512
0.2767	0.3738	0.4272	0.5130
0.3467	0.4329	0.4854	0.5665
0.4167	0.4877	0.5388	0.6149
0.4867	0.5396	0.5888	0.6601
0.5567	0.5894	0.6365	0.7028
0.6267	0.6379	0.6825	0.7438
0.6967	0.6852	0.7272	0.7834
0.7667	0.7318	0.7710	0.8220
0.8367	0.7779	0.8140	0.8597
0.9067	0.8235	0.8564	0.8968
0.9767	0.8689	0.8984	0.9334
1.0467	0.9141	0.9401	0.9695

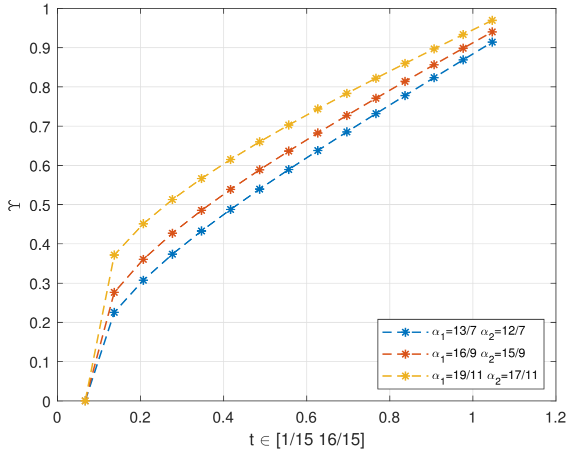

Figure 1. Graphical representation of Υ for different values of α_1 and α_2.

Existence. For $x, \bar{x}, y, \bar{y} \in \mathbb{B}$ and $t \in [1/15, 16/15]$, we have

$$\left\|f(t, y, \bar{y})\right\| \leqslant \frac{23}{30}, \qquad \left\|g(t, x, \bar{x})\right\| \leqslant \frac{\pi}{8} + \frac{3}{4} = \frac{\pi + 6}{8}.$$

From (H1) we have $L_1 = 1/15$ and $L_2 = 1/4$.

Now,

$$L_1\big(C_1 + D_1 - \Theta(b, \alpha_1)\big) + L_2\big(C_2 + D_2 - \Theta(b, \alpha_2)\big) \\ \approx 0.4736128552 < 1.$$

Table 2. Numerical results of $\varDelta$ for different values of α_1 and α_2.

t	$\varDelta$		
	$\alpha_1 = 13/7$ $\alpha_2 = 12/7$	$\alpha_1 = 16/9$ $\alpha_2 = 15/9$	$\alpha_1 = 19/11$ $\alpha_2 = 17/11$
0.0667	0	0	0
0.1367	0.0450	0.0519	0.0646
0.2067	0.0809	0.0902	0.1063
0.2767	0.1146	0.1252	0.1428
0.3467	0.1471	0.1584	0.1766
0.4167	0.1790	0.1906	0.2088
0.4867	0.2105	0.2220	0.2397
0.5567	0.2419	0.2531	0.2699
0.6267	0.2732	0.2838	0.2994
0.6967	0.3046	0.3144	0.3285
0.7667	0.3361	0.3449	0.3573
0.8367	0.3678	0.3753	0.3858
0.9067	0.3997	0.4059	0.4142
0.9767	0.4318	0.4365	0.4425
1.0467	0.4643	0.4673	0.4708

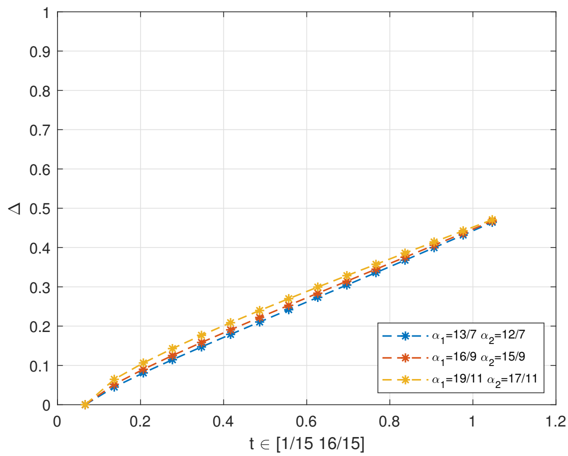

Figure 2. Graphical representation of $\varDelta$ for different values of α_1 and α_2.

Consequently, the hypothesis of Theorem 3 is satisfied, and system (2) has at least one solution on $\mathbb{J}$. Moreover, Table 2 presents the numerical results of $\varDelta$ for a variety of $t \in [1/15, 16/15]$ and different values of α_1 and α_2. The results are represented graphically in Fig. 2.

5 Conclusion

In this study, the existence and uniqueness of the solution to the coupled system of ψ-Hilfer nonlinear implicit fractional multipoint boundary value problem were taken into consideration. In a special working space, the existence and uniqueness of solution to the boundary value problem is investigated. The uniqueness result is investigated using the Banach contraction principle, and the existence result is examined using Krasnosel'skii's fixed point theorem. An example has been developed to demonstrate our results. Furthermore, the uniqueness and existence of the solution for the considered example are analysed graphically.

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