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Abstract— **Granular reasoning proposed by Murai et al. is a mechanism for reasoning using granular computing, and the concept of "focus" has been proposed as a key concept of granular reasoning. On the other hand, the authors have proposed another concept of granularity, called "visibility". In this paper, we try to capture the concepts of visibility and focus as modalities of modal logics by introducing Scott-Montague models that illustrate the visibility and focus by modal operators, respectively.**

I. Introduction

Granular computing based on rough set theory (Pawlak [14], [15]) has been widely studied as a new paradigm of computing (for example, see [7], [17]). In particular, Murai et al. has proposed *granular reasoning* as a mechanism for reasoning using granular computing [8], and developed a framework of granular reasoning, called a zooming reasoning system [9], [10], [11]. The key concept of the zooming reasoning system is focus, which represents sentences we use in some step of reasoning. The focus provides a three-valued truth valuation that assigns the truth value "true" or "false" to atomic sentences that appear in the focus, and assigns the truth value "unknown" to other atomic sentences.

On the other hand, the authors have proposed another concept of granularity, called visibility [5]. Visibility separates all sentences into "visible" sentences, that is, sentences we consider, and "invisible" sentences which are out of consideration. The authors also have constructed four-valued truth valuations based on visibility and focus, which illustrate the concepts of "clearly visible", "obscurely visible" and "invisible" [6].

In this paper, we try to capture the concepts of visibility and focus as modalities. In particular, we produce Scott–Montague models that illustrate some properties of visibility and focus, and represent the concept "A sentence p is visible" and "p is clearly visible" as modal sentences Vp and Cp, respectively.

II. Backgrounds

A. Scott – Montague Models for Modal Logics

Let $\mathcal{P}$ be a set of (at most countably infinite) atomic sentences. We construct a language $\mathcal{L}_{\mathrm{ML}}(\mathcal{P})$ for modal logic from $\mathcal{P}$ using logical operators $\top$ (the truth constant), $\perp$ (the falsity constant), $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (material implication), $\leftrightarrow$ (equivalence) and two modal operators $\square$ (necessity) and $\diamond$ (possibility) by the following rules: (1) $\mathsf{p} \in \mathcal{P} \Rightarrow \mathsf{p} \in \mathcal{L}_{\mathrm{ML}}(\mathcal{P})$ (2) $p \in \mathcal{L}_{\mathrm{ML}}(\mathcal{P}) \Rightarrow \neg p \in \mathcal{L}_{\mathrm{ML}}(\mathcal{P})$ (3) $p, q \in \mathcal{L}_{\mathrm{ML}}(\mathcal{P}) \Rightarrow p \wedge q, p \vee q, p \rightarrow q, p \leftrightarrow q \in \mathcal{L}_{\mathrm{ML}}(\mathcal{P})$ (4) $p \in \mathcal{L}_{\mathrm{ML}}(\mathcal{P}) \Rightarrow \square p, \diamond p \in \mathcal{L}_{\mathrm{ML}}(\mathcal{P})$. A sentence is called *non-modal* if the sentence does not contain any modal operators. We denote $\mathcal{L}(\mathcal{P})$ to mean the set of all non-modal sentences.

Scott-Montague models (or minimal models; see Chellas [1] for details) are a generalization of well-known Kripke models, and provide possible worlds semantics for modal logics. A *Scott-Montague model* $\mathcal{M}$ is a triple

$$\langle W, N, v \rangle,$$

where W is a non-empty set of possible worlds, N is a function from W to 2^{2^W}, and v is a valuation that assigns either the truth value t (true) or f (false) to each atomic sentence $\mathsf{p} \in \mathcal{P}$ at each world $w \in W$.

We denote $\models_w^{\mathcal{M}} p$ to mean that the sentence p is true at the possible world w in the model $\mathcal{M}$. $\models_w^{\mathcal{M}}$ is obtained by extending the valuation v by the usual way. For any sentence $p \in \mathcal{L}_{\mathrm{ML}}(\mathcal{P})$, we define the *truth set* of p in $\mathcal{M}$ as $\|p\|^{\mathcal{M}} = \{w \in W \mid \models_w^{\mathcal{M}} p\}$. The truth condition of modal sentences is given by

$$\models_w^{\mathcal{M}} \square p \stackrel{\mathrm{def}}{\Longleftrightarrow} \|p\|^{\mathcal{M}} \in N(w). \tag{1}$$

Various conditions of N are considered such that

(m)	$X \cap Y \in N(w) \Rightarrow X \in N(w)$ and $Y \in N(w)$,
(c)	$X, Y \in N(w) \Rightarrow X \cap Y \in N(w)$,
(n)	$W \in N(w)$,
(d)	$X \in N(w) \Rightarrow X^c \notin N(w)$,
(t)	$X \in N(w) \Rightarrow w \in X$,
(4)	$X \in N(w) \Rightarrow \{x \in W \mid X \in N(x)\} \in N(w)$,
(5)	$X \notin N(w) \Rightarrow \{x \in W \mid X \notin N(x)\} \in N(w)$.

The smallest classical modal logic E is proved to be both sound and complete with respect to the class of all Scott - Montague models, where E contains the schema $Df\diamond$. $\diamond p \leftrightarrow \neg \square \neg p$ and the rule of inference

$$\textbf{RE: from } p \leftrightarrow q \textbf{ infer } \square p \leftrightarrow \square q$$

with the rules and axiom schemata of propositional logic. Each condition of N corresponds to axiom schema such that

M. $\Box(p \wedge q) \rightarrow (\Box p \wedge \Box q)$,

C. $(\Box p \wedge \Box q) \rightarrow \Box(p \wedge q)$,

N. $\Box\top$,

D. $\Box p \rightarrow \Diamond p$,

T. $\Box p \rightarrow p$,

4. $\Box p \rightarrow \Box\Box p$,

5. $\Diamond p \rightarrow \Box\Diamond p$.

B. Visibility and Focus: Two Concepts of Granular Reasoning

1) Granularized possible worlds based on visibility: Let Γ be a set of non-modal sentences considered in the current step of reasoning. Using Γ, we define the *visibility* relative to Γ. Moreover, we redefine the the concept of the focus, and proposed the *focus* relative to Γ. The definitions of the visibility $Vs(\Gamma)$ and focus $Fc(\Gamma)$ relative to Γ are as follows:

$$Vs(\Gamma) \stackrel{\text{def}}{=} \mathcal{P} \cap \mathrm{Sub}(\Gamma) = \mathcal{P}_\Gamma, \qquad (2)$$

$$Fc(\Gamma) \stackrel{\text{def}}{=} \{\mathsf{p} \in \mathcal{P} \mid \text{either } \Gamma \vdash \mathsf{p} \text{ or } \Gamma \vdash \neg\mathsf{p}\}, \quad (3)$$

where $\mathrm{Sub}(\Gamma)$ is the union of the sets of subsentences of each sentence in Γ. Using a (given) valuation v, we construct the agreement relation $R_{Vs(\Gamma)}$ based on the visibility $Vs(\Gamma)$ as follows:

$$xR_{Vs(\Gamma)}y \stackrel{\text{def}}{\Longleftrightarrow} v(\mathsf{p}, x) = v(\mathsf{p}, y), \ \forall \mathsf{p} \in Vs(\Gamma). \qquad (4)$$

The agreement relation $R_{Vs(\Gamma)}$ induce the set of granularized possible worlds $\tilde{W} \stackrel{\text{def}}{=} W/R_{Vs(\Gamma)}$. We also construct a truth valuation $\tilde{v}_{Vs(\Gamma)}$ for granularized possible worlds $\tilde{x} \stackrel{\text{def}}{=} [x]_{R_{Vs(\Gamma)}} \in \tilde{W}$. The valuation $\tilde{v}_{Vs(\Gamma)}$ becomes the following three-valued one:

$$\tilde{v}_{Vs(\Gamma)} : \mathcal{P} \times \tilde{W} \longrightarrow 2^{\{\mathbf{t},\mathbf{f}\}} \setminus \{\{\mathbf{t},\mathbf{f}\}\}. \qquad (5)$$

The three-valued valuation $\tilde{v}_{Vs(\Gamma)}$ is defined by:

$$\tilde{v}_{Vs(\Gamma)}(\mathsf{p}, \tilde{w}) \stackrel{\text{def}}{=} \begin{cases} \{\mathbf{t}\}, & \text{if } v(\mathsf{p}, x) = \mathbf{t}, \ \forall x \in \tilde{w}, \\ \{\mathbf{f}\}, & \text{if } v(\mathsf{p}, x) = \mathbf{f}, \ \forall x \in \tilde{w}, \\ \emptyset, & \text{otherwise.} \end{cases} \qquad (6)$$
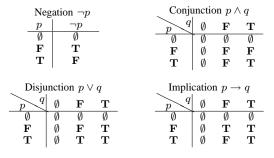
Hereafter, we use the following notations: $\mathbf{T} \stackrel{\text{def}}{=} \{\mathbf{t}\}$ and $\mathbf{F} \stackrel{\text{def}}{=} \{\mathbf{f}\}$, respectively. Using $\tilde{v}_{Vs(\Gamma)}$, we define the visibility of atomic sentences.

Definition 1: An atomic sentence p is *visible* at $\tilde{w}$ if and only if either $\tilde{v}_{Vs(\Gamma)}(\mathsf{p}, \tilde{w}) = \mathbf{T}$ or $\tilde{v}_{Vs(\Gamma)}(\mathsf{p}, \tilde{w}) = \mathbf{F}$. On the other hand, p is *invisible* at $\tilde{w}$ if and only if $\tilde{v}_{Vs(\Gamma)}(\mathsf{p}, \tilde{w}) = \emptyset$.

The three-valued valuation $\tilde{v}_{Vs(\Gamma)}$ is extended to any non-modal sentences by truth assignments of connectives $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction) and $\rightarrow$ (implication) illustrated in Table I. We denote the extended three-valued truth valuation by the same notation $\tilde{v}_{Vs(\Gamma)}$. Similar to the case of atomic sentences, for any non-modal sentence p, we call p is visible at $\tilde{w}$ if and only if either $\tilde{v}_{Vs(\Gamma)}(p, \tilde{w}) = \mathbf{T}$ or $\tilde{v}_{Vs(\Gamma)}(p, \tilde{w}) = \mathbf{F}$. On the other hand, p is invisible at $\tilde{w}$ if and only if $\tilde{v}_{Vs(\Gamma)}(p, \tilde{w}) = \emptyset$. Hence, if both p and q are visible, it is clear that $\neg p$, $p \wedge q$, $p \vee q$ and $p \rightarrow q$ are also visible.

Negation $\neg p$	
p	$\neg p$
$\emptyset$	$\emptyset$
$\mathbf{F}$	$\mathbf{T}$
$\mathbf{T}$	$\mathbf{F}$

Conjunction $p \wedge q$			
p \ q	$\emptyset$	$\mathbf{F}$	$\mathbf{T}$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\mathbf{F}$	$\emptyset$	$\mathbf{F}$	$\mathbf{F}$
$\mathbf{T}$	$\emptyset$	$\mathbf{F}$	$\mathbf{T}$

Disjunction $p \vee q$			
p \ q	$\emptyset$	$\mathbf{F}$	$\mathbf{T}$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\mathbf{F}$	$\emptyset$	$\mathbf{F}$	$\mathbf{T}$
$\mathbf{T}$	$\emptyset$	$\mathbf{T}$	$\mathbf{T}$

Implication $p \rightarrow q$			
p \ q	$\emptyset$	$\mathbf{F}$	$\mathbf{T}$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\mathbf{F}$	$\emptyset$	$\mathbf{T}$	$\mathbf{T}$
$\mathbf{T}$	$\emptyset$	$\mathbf{F}$	$\mathbf{T}$

2) Equivalence classes of granularized possible worlds based on focus: Using the focus $Fc(\Gamma)$ relative to Γ, we construct an agreement relation $R_{Fc(\Gamma)}$ on the *set of granularized possible worlds* $\tilde{W}$. If $Fc(\Gamma) \neq \emptyset$, we define the agreement relation $R_{Fc(\Gamma)}$ as follows:

$$\tilde{x}R_{Fc(\Gamma)}\tilde{y} \stackrel{\text{def}}{\Longleftrightarrow} \tilde{v}_{Vs(\Gamma)}(\mathsf{p}, \tilde{x}) = \tilde{v}_{Vs(\Gamma)}(\mathsf{p}, \tilde{y}), \ \forall \mathsf{p} \in Fc(\Gamma). \ (7)$$

The agreement relation $R_{Fc(\Gamma)}$ on $\tilde{W}$ induce the quotient set of granularized possible worlds $\widehat{W} \stackrel{\text{def}}{=} \tilde{W}/R_{Fc(\Gamma)}$. We treat each equivalence class $\widehat{w} \stackrel{\text{def}}{=} [\tilde{w}]_{R_{Fc(\Gamma)}}$ as a unit of consideration as if each $\widehat{w}$ were a "possible world". On the other hand, if $Fc(\Gamma) = \emptyset$, we can not construct the agreement relation. In this case, we define $\widehat{W} \stackrel{\text{def}}{=} \{\tilde{W}\}$.

We consider a valuation function $\widehat{v}_{Fc(\Gamma)}$ for equivalence classes of granularized possible worlds as the following four-valued one:

$$\widehat{v}_{Fc(\Gamma)} : \mathcal{P} \times \widehat{W} \longrightarrow 2^{\{\mathbf{T},\mathbf{F}\}}. \qquad (8)$$

The valuation $\widehat{v}_{Fc(\Gamma)}$ is defined by:

$$\widehat{v}_{Fc(\Gamma)}(\mathsf{p}, \widehat{w}) \stackrel{\text{def}}{=} \begin{cases} \{\mathbf{T}\}, & \tilde{v}_{Vs(\Gamma)}(\mathsf{p}, \tilde{x}) = \mathbf{T}, \ \forall \tilde{x} \in \widehat{w}, \\ \{\mathbf{F}\}, & \tilde{v}_{Vs(\Gamma)}(\mathsf{p}, \tilde{x}) = \mathbf{F}, \ \forall \tilde{x} \in \widehat{w}, \\ \{\mathbf{T},\mathbf{F}\}, & \exists \tilde{x}, \tilde{y} \in \widehat{w} \text{ s.t. } \tilde{v}_{Vs(\Gamma)}(\mathsf{p}, \tilde{x}) = \mathbf{T}, \\ & \text{and } \tilde{v}_{Vs(\Gamma)}(\mathsf{p}, \tilde{y}) = \mathbf{F}, \\ \emptyset, & \text{otherwise.} \end{cases}$$
$$(9)$$

Definition 2: An atomic sentence p is *clearly visible* (or *in focus*) at $\widehat{w}$ if and only if either $\widehat{v}_{Fc(\Gamma)}(\mathsf{p}, \widehat{w}) = \{\mathbf{T}\}$ or $\widehat{v}_{Fc(\Gamma)}(\mathsf{p}, \widehat{w}) = \{\mathbf{F}\}$. On the other hand, p is *obscurely visible* at $\widehat{w}$ if and only if $\widehat{v}_{Fc(\Gamma)}(\mathsf{p}, \widehat{w}) = \{\mathbf{T},\mathbf{F}\}$. Moreover, p is *invisible* at $\widehat{w}$ if and only if $\widehat{v}_{Fc(\Gamma)}(\mathsf{p}, \widehat{w}) = \emptyset$.

From this definition, it is clear that, for all $\mathsf{p} \in Fc(\Gamma)$, p is clearly visible at all $\widehat{w} \in \widehat{W}$.

Similar to the case of the three-valued valuation $\tilde{v}_{Vs(\Gamma)}$, The four-valued valuation $\widehat{v}_{Fc(\Gamma)}$ is extended to any non-modal sentences by truth assignments illustrated in Table II. We denote the extended four-valued truth valuation by the same notation $\widehat{v}_{Fc(\Gamma)}$. Similar to the three-valued case, for any clearly visible sentences p and q, it is clear that $\neg p$, $p \wedge q$, $p \vee q$ and $p \rightarrow q$ are also clearly visible. Thus, there is at least one equivalence class $\widehat{w} \in \widehat{W}$ such that $\widehat{v}_{Fc(\Gamma)}(p, \widehat{w}) = \{\mathbf{T}\}$ for all $p \in \Gamma$.

Negation $\neg p$

p	$\neg p$
$\emptyset$	$\emptyset$
$\{\mathbf{F}\}$	$\{\mathbf{T}\}$
$\{\mathbf{T}\}$	$\{\mathbf{F}\}$
$\{\mathbf{T},\mathbf{F}\}$	$\{\mathbf{T},\mathbf{F}\}$

Disjunction $p \vee q$

p \ q	$\emptyset$	$\{\mathbf{F}\}$	$\{\mathbf{T}\}$	$\{\mathbf{T},\mathbf{F}\}$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{\mathbf{F}\}$	$\emptyset$	$\{\mathbf{F}\}$	$\{\mathbf{T}\}$	$\{\mathbf{T},\mathbf{F}\}$
$\{\mathbf{T}\}$	$\emptyset$	$\{\mathbf{T}\}$	$\{\mathbf{T}\}$	$\{\mathbf{T}\}$
$\{\mathbf{T},\mathbf{F}\}$	$\emptyset$	$\{\mathbf{T},\mathbf{F}\}$	$\{\mathbf{T}\}$	$\{\mathbf{T},\mathbf{F}\}$

Conjunction $p \wedge q$

p \ q	$\emptyset$	$\{\mathbf{F}\}$	$\{\mathbf{T}\}$	$\{\mathbf{T},\mathbf{F}\}$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{\mathbf{F}\}$	$\emptyset$	$\{\mathbf{F}\}$	$\{\mathbf{F}\}$	$\{\mathbf{F}\}$
$\{\mathbf{T}\}$	$\emptyset$	$\{\mathbf{F}\}$	$\{\mathbf{T}\}$	$\{\mathbf{T},\mathbf{F}\}$
$\{\mathbf{T},\mathbf{F}\}$	$\emptyset$	$\{\mathbf{F}\}$	$\{\mathbf{T},\mathbf{F}\}$	$\{\mathbf{T},\mathbf{F}\}$

Implication $p \rightarrow q$

p \ q	$\emptyset$	$\{\mathbf{F}\}$	$\{\mathbf{T}\}$	$\{\mathbf{T},\mathbf{F}\}$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{\mathbf{F}\}$	$\emptyset$	$\{\mathbf{T}\}$	$\{\mathbf{T}\}$	$\{\mathbf{T}\}$
$\{\mathbf{T}\}$	$\emptyset$	$\{\mathbf{F}\}$	$\{\mathbf{T}\}$	$\{\mathbf{T},\mathbf{F}\}$
$\{\mathbf{T},\mathbf{F}\}$	$\emptyset$	$\{\mathbf{T},\mathbf{F}\}$	$\{\mathbf{T}\}$	$\{\mathbf{T},\mathbf{F}\}$

Note that, however, not all two-valued tautologies are satisfied by $\tilde{v}_{Fc(\Gamma)}$. For example, for any invisible sentence p and obscurely visible sentence q, exclusive middle is not satisfied: $\tilde{v}_{Fc(\Gamma)}(p \vee \neg p, \widehat{w}) = \emptyset$ and $\tilde{v}_{Fc(\Gamma)}(q \vee \neg q, \widehat{w}) = \{\mathbf{T},\mathbf{F}\}$ for all $\widehat{w} \in \widetilde{W}$.

Example 1: Let $\mathcal{P} = \{p, q, r\}$ be a set of atomic sentences, and W be a non-empty set that has the following eight possible worlds:

$$w_1 = \{p,q,r\}, \quad w_2 = \{p,q\}, \quad w_3 = \{p,r\}, \quad w_4 = \{p\},$$
$$w_5 = \{q,r\}, \quad w_6 = \{q\}, \quad w_7 = \{r\}, \quad w_8 = \emptyset.$$

We define the truth value of each atomic sentence $p \in \mathcal{P}$ at each world $w \in W$ by $v(p, w) = \mathbf{t} \Longleftrightarrow p \in w$. By this truth assignment, for example, all atomic sentences are true at w_1. On the other hand, all atomic sentences are false at w_8.

Suppose we have the following set of non-modal sentences considered in the current step of reasoning: $\Gamma = \{q, p \rightarrow q\}$. Hence, we have the visibility $Vs(\Gamma)$ and focus $Fc(\Gamma)$ relative to Γ as follows, respectively: $Vs(\Gamma) = \{p, q\}$, $Fc(\Gamma) = \{q\}$.

Constructing the agreement relation $R_{Vs(\Gamma)}$ by equation (4), we have the following four granularized possible worlds:

$$\tilde{w}_1 = \{w_1, w_2\}, \quad \tilde{w}_3 = \{w_3, w_4\},$$
$$\tilde{w}_5 = \{w_5, w_6\}, \quad \tilde{w}_7 = \{w_7, w_8\}.$$

Each atomic sentence has the following three-valued truth value:

$$\tilde{v}_{Fc(\Gamma)}(p, \tilde{w}_1) = \mathbf{T}, \quad \tilde{v}_{Fc(\Gamma)}(q, \tilde{w}_1) = \mathbf{T}, \quad \tilde{v}_{Fc(\Gamma)}(r, \tilde{w}_1) = \emptyset,$$
$$\tilde{v}_{Fc(\Gamma)}(p, \tilde{w}_3) = \mathbf{T}, \quad \tilde{v}_{Fc(\Gamma)}(q, \tilde{w}_3) = \mathbf{F}, \quad \tilde{v}_{Fc(\Gamma)}(r, \tilde{w}_3) = \emptyset,$$
$$\tilde{v}_{Fc(\Gamma)}(p, \tilde{w}_5) = \mathbf{F}, \quad \tilde{v}_{Fc(\Gamma)}(q, \tilde{w}_5) = \mathbf{T}, \quad \tilde{v}_{Fc(\Gamma)}(r, \tilde{w}_5) = \emptyset,$$
$$\tilde{v}_{Fc(\Gamma)}(p, \tilde{w}_7) = \mathbf{F}, \quad \tilde{v}_{Fc(\Gamma)}(q, \tilde{w}_7) = \mathbf{F}, \quad \tilde{v}_{Fc(\Gamma)}(r, \tilde{w}_7) = \emptyset.$$

These truth values indicate that p and q are visible, while r is invisible.

Next, we construct the agreement relation $R_{Fc(\Gamma)}$ on $\tilde{W}$, and get the following two equivalence classes:

$$\widehat{w_1} = \{\tilde{w}_1, \tilde{w}_5\} = \{\{w_1, w_2\}, \{w_5, w_6\}\},$$
$$\widehat{w_3} = \{\tilde{w}_3, \tilde{w}_7\} = \{\{w_3, w_4\}, \{w_7, w_8\}\}.$$

By (9), each atomic sentence has the following four-valued truth value:

$$\widehat{v}_{Fc(\Gamma)}(p, \widehat{w}_1) = \{\mathbf{T},\mathbf{F}\}, \quad \widehat{v}_{Fc(\Gamma)}(q, \widehat{w}_1) = \{\mathbf{T}\},$$
$$\widehat{v}_{Fc(\Gamma)}(r, \widehat{w}_1) = \emptyset.$$
$$\widehat{v}_{Fc(\Gamma)}(p, \widehat{w}_3) = \{\mathbf{T},\mathbf{F}\}, \quad \widehat{v}_{Fc(\Gamma)}(q, \widehat{w}_3) = \{\mathbf{F}\},$$
$$\widehat{v}_{Fc(\Gamma)}(r, \widehat{w}_3) = \emptyset.$$

This means that q is clearly visible, but p is obscurely visible. Similar to the three-valued case, r is invisible. Four-valued truth values of any non-modal sentences are calculated based on Table II. For example, the truth value of $p \rightarrow q$ is: $\widehat{v}_{Fc(\Gamma)}(p \rightarrow q, \widehat{w}_1) = \{\mathbf{T}\}$ and $\widehat{v}_{Fc(\Gamma)}(p \rightarrow q, \widehat{w}_3) = \{\mathbf{T},\mathbf{F}\}$. Thus, all non-modal sentences in Γ is true, that is, clearly visible, at $\widehat{w}_1$.

III. GRANULAR REASONING AND SCOTT – MONTAGUE MODELS

A. Visibility as Modality

In this subsection, we try to capture the concept of visibility by modality based on Scott – Montague models. Suppose we have a set of granularized possible worlds $\tilde{W}$ based on the visibility $Vs(\Gamma)$ relative to Γ, and a non-modal sentence p is visible at $\tilde{w} \in \tilde{W}$. Instead of the modal operator $\Box$, we use a modal operator V, and we read Vp as "p is visible". We intend to illustrate the visibility by some Scott – Montague model $\mathcal{M}$ as follows: for each possible world $x \in \tilde{w}$,

$$\models_x^{\mathcal{M}} Vp, \text{ if } p \text{ is visible at } \tilde{w}.$$

For illustrating the concept of visibility as modality, we use the following simple function $N_{Vs(\Gamma)}$.

Definition 3: Let $\tilde{W} = \{\tilde{w}_1, \cdots, \tilde{w}_n\}$ be the set of granularized possible worlds based on the visibility $Vs(\Gamma)$. A function $N_{Vs(\Gamma)} : W \rightarrow 2^{2^W}$ is defined by

$$N_{Vs(\Gamma)}(x) \stackrel{\text{def}}{=} \left\{ \bigcup A \mid A \subseteq \tilde{W} \right\}, \forall x \in W, \quad (10)$$

where $\bigcup A$ means the union of all granularized possible worlds in A. If $A = \emptyset$, we define $\bigcup A \stackrel{\text{def}}{=} \emptyset$.

This definition means that, for any $x \in W$, each element $X \in N_{Vs(\Gamma)}(x)$ is constructed by union of some granularized possible worlds. Each $\tilde{w} \in \tilde{W}$ is an equivalence class $[w]_{R_{Vs(\Gamma)}} \subseteq W$, thus the function $N_{Vs(\Gamma)}$ is well-defined. $N_{Vs(\Gamma)}$ satisfies the following conditions.

Lemma 1: The constructed function $N_{Vs(\Gamma)}$ by (10) satisfies the condition (c), (n), (4) and (5). Moreover, $N_{Vs(\Gamma)}$ satisfies the following properties:

$$\text{(v!)} \quad X \in N_{Vs(\Gamma)} \Leftrightarrow X^c \in N_{Vs(\Gamma)}.$$

Next, using $N_{Vs(\Gamma)}$, we construct a Scott - Montague model $\mathcal{M} = \langle W, N_{Vs(\Gamma)}, v \rangle$. Lemma 1 indicates that the model $\mathcal{M}$ validates schemata **C**, **N**, **4** and **5**. Moreover, the condition (v!) corresponds the following schema:

$$\mathbf{V!.} \quad Vp \leftrightarrow V\neg p.$$

The truth condition of modal sentences, (1), captures part of the concept of visibility. The following lemma captures the property that if both p and q are visible, then $\neg p$, $p \wedge q$, $p \vee q$ and $p \to q$ are also visible.

Lemma 2:
1) If $\mathsf{p} \in Vs(\Gamma)$, then $\models_w^{\mathcal{M}} \mathsf{V}\mathsf{p}$ for all $w \in W$.
2) For any $p \in \mathcal{L}(\mathcal{P})$ and all $w \in W$, if $\models_w^{\mathcal{M}} \mathsf{V}p$, then $\models_w^{\mathcal{M}} \mathsf{V}\neg p$.
3) For any $p, q \in \mathcal{L}(\mathcal{P})$ and all $w \in W$, if both $\models_w^{\mathcal{M}} \mathsf{V}p$ and $\models_w^{\mathcal{M}} \mathsf{V}q$, then $\models_w^{\mathcal{M}} \mathsf{V}(p \wedge q)$, $\models_w^{\mathcal{M}} \mathsf{V}(p \vee q)$, and $\models_w^{\mathcal{M}} \mathsf{V}(p \to q)$, respectively.

Combining these lemmas, we have the following result.

Theorem 1: Let Γ be a non-empty set of non-modal sentences, $\tilde{W}$ be the set of granularized possible worlds based on the visibility $Vs(\Gamma)$, and $\mathcal{M} = \langle W, N_{Vs(\Gamma)}, v \rangle$ be a Scott-Montague model that has a function $N_{Vs(\Gamma)}$ by Definition 3. For any non-modal sentence $p \in \mathcal{L}(\mathcal{P})$, if p is visible at $\tilde{w} \in \tilde{W}$, then $\models_x^{\mathcal{M}} \mathsf{V}p$ for all $x \in \tilde{w}$.

However, the converse of Theorem 1 is not satisfied. This is because, in our formulation, any "invisible" tautology p becomes $\models_w^{\mathcal{M}} \mathsf{V}p$. For example, suppose that an atomic sentence r is invisible by $\tilde{v}_{Vs(\Gamma)}$. A tautology $\mathsf{r} \vee \neg \mathsf{r}$ is also invisible by the definition of visibility. However, the truth set $\|\mathsf{r} \vee \neg \mathsf{r}\|^{\mathcal{M}} = W$ is an element of $N_{Vs(\Gamma)}(w)$ for all $w \in W$, therefore $\models_w^{\mathcal{M}} \mathsf{V}(\mathsf{r} \vee \neg \mathsf{r})$. Unfortunately, we can not avoid this difficulty. This is because it causes that the schema **N** is satisfied using $N_{Vs(\Gamma)}$. Therefore we need to restrict our formulation to satisfiable sentences.

Example 2: We use the same setting of Example 1. Using $\tilde{W} = \{\tilde{w}_1, \tilde{w}_3, \tilde{w}_5, \tilde{w}_7\}$, we construct a Scott - Montague model $\mathcal{M} = \langle W, N_{Vs(\Gamma)}, v \rangle$ by Definition 3, where W and v are the same ones defined in Example 1.

We have $Vs(\Gamma) = \{\mathsf{p}, \mathsf{q}\}$, thus atomic sentences p and q are visible but r is invisible. For these atomic sentences, we have the following truth sets, respectively:

$$
\begin{aligned}
\|\mathsf{p}\|^{\mathcal{M}} &= \{w_1, w_2, w_3, w_4\}, \\
\|\mathsf{q}\|^{\mathcal{M}} &= \{w_1, w_2, w_5, w_6\}, \\
\|\mathsf{r}\|^{\mathcal{M}} &= \{w_1, w_3, w_5, w_7\}.
\end{aligned}
$$

Here, it holds that $\|\mathsf{p}\|^{\mathcal{M}} = \{w_1, w_2\} \cup \{w_3, w_4\} = \tilde{w}_1 \cup \tilde{w}_3$ and $\|\mathsf{q}\|^{\mathcal{M}} = \{w_1, w_2\} \cup \{w_5, w_6\} = \tilde{w}_1 \cup \tilde{w}_5$. Therefore, for example, $\models_x^{\mathcal{M}} \mathsf{V}\mathsf{p}$ and $\models_x^{\mathcal{M}} \mathsf{V}\mathsf{q}$ for all $x \in \tilde{w}_1$ by the truth condition (1), respectively. On the other hand, we can not construct $\|\mathsf{r}\|^{\mathcal{M}}$ by union of $\tilde{w}_i$, we have $\not\models_w^{\mathcal{M}} \mathsf{V}\mathsf{r}$ for all $w \in W$.

B. Focus as Modality

Similar to the case of visibility, we try to capture the concept of focus by modality based on Scott - Montague models. The focus $Fc(\Gamma)$ relative to Γ divides all "visible" sentences into "clearly visible" ones and "obscurely visible" ones. For any clearly visible sentence p, using a modal operator C, we denote $\mathsf{C}p$ to mean that "p is clearly visible". We intend to illustrate the focus by some Scott - Montague model $\mathcal{M}$ as follows: Let $\widehat{W}$ be the quotient set of granularized possible worlds based on the focus $Fc(\Gamma)$, and $\widehat{w} \in \widehat{W}$ be an equivalence class of

granularized possible worlds. For each possible world $y \in \tilde{x}$ such that $\tilde{x} \in \widehat{w}$,

$$\models_y^{\mathcal{M}} \mathsf{C}p, \quad \text{if } p \text{ is clearly visible at } \widehat{w}.$$

To construct a function $N_{Fc(\Gamma)}$ that illustrates the concept of focus as modality, we take the following two steps:

1) Constructing a function $N_{Fc(\Gamma)}^{\widehat{w}}$ for each $\widehat{w} \in \widehat{W}$.
2) Combining all $N_{Fc(\Gamma)}^{\widehat{w}}$.

First, we define the function $N_{Fc(\Gamma)}^{\widehat{w}}$.

Definition 4: For each $\widehat{w} \in \widehat{W}$, a function $N_{Fc(\Gamma)}^{\widehat{w}} : \bigcup \widehat{w} \to 2^{2^W}$ is defined by:

$$N_{Fc(\Gamma)}^{\widehat{w}}(x) \stackrel{\text{def}}{=} \left\{ \bigcup A \mid A \subseteq U(\widehat{w}) \right\}, \ \forall x \in \bigcup \widehat{w}, \quad (11)$$

where $U(\widehat{w}) = \left(\tilde{W} \setminus \widehat{w} \right) \cup (\bigcup \widehat{w})$. If $A = \emptyset$, then $\bigcup A \stackrel{\text{def}}{=} \emptyset$.

Next, combining all functions $N_{Fc(\Gamma)}^{\widehat{w}}$, we define the function $N_{Fc(\Gamma)}$.

Definition 5: For all $x \in W$, a function $N_{Fc(\Gamma)} : W \to 2^{2^W}$ is defined by:

$$N_{Fc(\Gamma)}(x) \stackrel{\text{def}}{=} \begin{cases} N_{Fc(\Gamma)}^{\widehat{w}}(x), & \text{if } Fc(\Gamma) \neq \emptyset, \\ & \text{and } x \in \cup\widehat{w} \\ \{W, \emptyset\}, & \text{otherwise.} \end{cases} \quad (12)$$

It is easy to check that the function $N_{Fc(\Gamma)}$ is well-defined by Definition 5. The key of this construction is the set $U(\widehat{w})$, which provides "units" of construction at each possible world $x \in \widehat{w}$. $U(\widehat{w})$ does not contain any granularized possible worlds $\tilde{y}$ in $\widehat{w}$. This is because we need to capture the property that an atomic sentence p is visible if and only if p is true at *all* granularized possible worlds in $\widehat{w}$ or false at *all* granularized possible worlds in $\widehat{w}$. Hence, if some $\tilde{y} \in \widehat{w}$ are contained in $U(\widehat{w})$, some atomic sentence $q \in Vs(\Gamma) \setminus Fc(\Gamma)$ may become "clearly visible". Thus, any $\tilde{y} \in \widehat{w}$ should not be included in $U(\widehat{w})$.

The differences between $N_{Vs(\Gamma)}$ and $N_{Fc(\Gamma)}$ are the following: (1) $N_{Vs(\Gamma)}$ treats all combinations of unions of granularized possible worlds as "unit" of consideration, while $N_{Fc(\Gamma)}$ treats some restricted parts of combinations of unions of granularized possible worlds. This is because we need to distinguish "clearly visible" sentences and "obscurely visible" sentences by using the fucntion $N_{Fc(\Gamma)}$, and the concept "p is clearly visible" (or "in focus") requires that p is either **T** or **F** at *all* granularized possible worlds in $\widehat{w}$. (2) $N_{Fc(\Gamma)}$ needs to treat the case $Fc(\Gamma) = \emptyset$. In the case of $N_{Vs(\Gamma)}$, by the definition of $Vs(\Gamma)$ we need not to consider the case that $Vs(\Gamma) =$. However, in $N_{Fc(\Gamma)}$, we have to consider $Fc(\Gamma) = \emptyset$, that is, the case that "nothing is clear".

$N_{Fc(\Gamma)}$ satisfies the following conditions.

Lemma 3: The constructed function $N_{Fc(\Gamma)}$ by (12) satisfies the condition (c), (n) and (v!).

However, in general, the conditions (4) and (5) are not satisfied.

Next, using $N_{Fc(\Gamma)}$, we construct a Scott - Montague model $\mathcal{M} = \langle W, N_{Fc(\Gamma)}, v \rangle$. Lemma 4 indicates that the model $\mathcal{M}$

validates schemata **C**, **N** and **V**!. Moreover, all properties illustrated in Lemma 2 are also valid for the operator C, that is, if both p and q are clearly visible, then $\neg p$, $p \wedge q$, $p \vee q$ and $p \rightarrow q$ are also clearly visible.

Lemma 4: 1) If $\mathsf{p} \in Vs(\Gamma)$, then $\models^{\mathcal{M}}_w \mathsf{C}\mathsf{p}$ for all $w \in W$.

2) For any $p \in \mathcal{L}(\mathcal{P})$ and all $w \in W$, if $\models^{\mathcal{M}}_w \mathsf{C}p$, then $\models^{\mathcal{M}}_w \mathsf{C}\neg p$.

3) For any $p, q \in \mathcal{L}(\mathcal{P})$ and all $w \in W$, if both $\models^{\mathcal{M}}_w \mathsf{C}p$ and $\models^{\mathcal{M}}_w \mathsf{C}q$, then $\models^{\mathcal{M}}_w \mathsf{C}(p \wedge q)$, $\models^{\mathcal{M}}_w \mathsf{C}(p \vee q)$, and $\models^{\mathcal{M}}_w \mathsf{C}(p \rightarrow q)$, respectively.

Thus, we have the following result.

Theorem 2: Let Γ be a non-empty set of non-modal sentences, $\widehat{W}$ be the set of equivalence classes of granularized possible worlds based on $Fc(\Gamma)$, and $\mathcal{M} = \langle W, N_{Fc(\Gamma)}, v \rangle$ be a Scott-Montague model that has a function $N_{Fc(\Gamma)}$ by Definition 5. For any non-modal sentence p, if p is clearly visible at $\widehat{w} \in \widehat{W}$, then $\models^{\mathcal{M}}_x \mathsf{C}p$ for all $x \in \tilde{y}$ such that $\tilde{y} \in \widehat{w}$.

However, by the same reason of the case of Vp, the converse of Theorem 2 is not satisfied.

Example 3: We use the same setting of Example 1. Using $\widehat{w_1} = \{\tilde{w}_1, \tilde{w}_5\} = \{\{w_1, w_2\}, \{w_5, w_6\}\}$, $\widehat{w_3} = \{\tilde{w}_3, \tilde{w}_7\} = \{\{w_3, w_4\}, \{w_7, w_8\}\}$, we get the sets $U(\widehat{w_1})$ and $U(\widehat{w_3})$ as follows:

$$U(\widehat{w_1}) \stackrel{\text{def}}{=} \{\{w_1, w_2, w_5, w_6\}, \{w_3, w_4\}, \{w_7, w_8\}\},$$

$$U(\widehat{w_3}) \stackrel{\text{def}}{=} \{\{w_3, w_4, w_7, w_8\}, \{w_1, w_2\}, \{w_5, w_6\}\}.$$

Thus, using $U(\widehat{w_1})$ and $U(\widehat{w_3})$, we construct functions $N^{\widehat{w_1}}_{Fc(\Gamma)}$ and $N^{\widehat{w_3}}_{Fc(\Gamma)}$ by Definition 4, and a Scott - Montague model $\mathcal{M} = \langle W, N_{Fc(\Gamma)}, v \rangle$ with the function $N_{Fc(\Gamma)}$ by Definition 5.

We have $Vs(\Gamma) = \{\mathsf{p}, \mathsf{q}\}$ and $Fc(\Gamma) = \{\mathsf{q}\}$, thus q is clearly visible but p is obscurely visible. Here, by the model $\mathcal{M}$, it holds that $\|\mathsf{q}\|^{\mathcal{M}} = \{w_1, w_2, w_5, w_6\} \in N_{Fc(\Gamma)}(x)$ for all $x \in W$. Therefore, for example, $\models^{\mathcal{M}}_{w_2} \mathsf{C}\mathsf{q}$ and $\models^{\mathcal{M}}_{w_2} \mathsf{C}\neg\mathsf{q}$. On the other hand, we can not construct $\|\mathsf{p}\|^{\mathcal{M}}$ at either $\widehat{w_1}$ or $\widehat{w_3}$, therefore we have $\not\models^{\mathcal{M}}_{w_i} \mathsf{C}\mathsf{p}$ and $\not\models^{\mathcal{M}}_{w_i} \mathsf{C}\neg\mathsf{p}$ at any $w_i \in W$.

We can also treat complex non-modal sentences. For example, in Example 1, $\mathsf{p} \rightarrow \mathsf{q}$ is clearly visible at $\widehat{w}$, but obscurely visible at $\widehat{w_3}$. Here, $\|\mathsf{p} \rightarrow \mathsf{q}\|^{\mathcal{M}} = \{w_1, w_2, w_3, w_4, w_5, w_6\} = \{w_1, w_2, w_5, w_6\} \cup \{w_3, w_4\} \in N_{Fc(\Gamma)}(w_i)$ for all $w_i \in \bigcup \widehat{w_1}$. Thus, for example, we have $\models^{\mathcal{M}}_{w_6} \mathsf{C}(\mathsf{p} \rightarrow \mathsf{q})$. On the other hand, we can not construct $\|\mathsf{p} \rightarrow \mathsf{q}\|^{\mathcal{M}}$ at $\widehat{w_3}$, therefore $\not\models^{\mathcal{M}}_{w_i} \mathsf{C}(\mathsf{p} \rightarrow \mathsf{q})$ at any $w_i \in \bigcup \widehat{w_3}$.

IV. CONCLUSION

In this paper, we tried to capture the concepts of visibility and focus as modalities. First, we proposed a modal operator V that means "visible", and constructed the function $N_{Vs(\Gamma)}$ to illustrate some properties of visibility. Moreover, we proposed a Scott–Montague models $\mathcal{M}$ such that if p is visible at $\tilde{w}$, then $\models^{\mathcal{M}}_x \mathsf{V}p$ at all $x \in \tilde{w}$. Next, we proposed another modal operator C that means "clearly visible", and constructed the function $N_{Fc(\Gamma)}$. Moreover, we proposed another Scott–Montague models $\mathcal{M}$ such that if p is clearly visible at $\widehat{w}$, then $\models^{\mathcal{M}}_x \mathsf{C}p$ at all $x \in \bigcup \tilde{w}$.

There are many future issues. First, we need to explore connections between V and C by multi modal Scott – Montague models and axiomatic characterization of visibility and focus. Combination with other modal logics, in particular, logics of knowledge and belief (for example, see [3]), and logics of time (for example, see [16]) are also interest. Moreover, we need to consider relationship among our framework and zooming reasoning systems [9], [10], [11] and belief change (for example, see [2], [4]).

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