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Abstract

We introduce discrete and p-adic continuous versions of the FitzHugh-Nagumo system on the one-dimensional p-adic unit ball. We provide criteria for the existence of Turing patterns. We present extensive simulations of some of these systems. The simulations show that the Turing patterns are traveling waves in the p-adic unit ball.

1 Introduction

Several models involving parabolic equations have been used in neuroscience for the propagation of nerve impulses. Among these models, the one of FitzHugh-Nagumo

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plays a central role. Proposed in the 1950s by FitzHugh, this model accurately explains the propagation of electric impulses along the nerve axon of the giant squid, see [16], [20] and the references therein. Nowadays FitzHugh-Nagumo system is the simplest model to describe pulse propagation in a spatial region. The simplest version of this system is

$$\begin{cases} \partial_t u(x,t) = mu - u^3 - v + L_u \Delta u \\ \\ \partial_t v(x,t) = c(u - av - b) + L_v \Delta v, \end{cases} \tag{1.1}$$

where the system parameters a, b, c, m, L_u, and L_v, are assumed to be positive, and the functions u and v depend on time $t \geq 0$ and the position $x \in \mathbb{R}$ on the domain of interest. The variable u promotes the self-growth of u and, at the same time, the growth of v and can thus be named an activator, while v plays the role of an inhibitor that annuls the growth of u.

In this paper, we introduce a p-adic counterpart of system (1.1). In the new model x runs through the ring of p-adic integers $\mathbb{Z}_p$, here p is a fixed prime number, and t is a real variable. Geometrically, $\mathbb{Z}_p$ is an infinite rooted tree; analytically, $\mathbb{Z}_p$ is a locally compact topological additive group, with a very rich mathematical structure. The system takes the following form

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) = f(u,v) - \left(\boldsymbol{D}_0^{\alpha} - \lambda\right) u(x,t) \\ \\ \frac{\partial v}{\partial t}(x,t) = g(u,v) - d\left(\boldsymbol{D}_0^{\alpha} - \lambda\right) v(x,t), \quad x \in \mathbb{Z}_p, \quad t \geq 0, \end{cases} \tag{1.2}$$

where $\boldsymbol{D}_0^{\alpha} - \lambda$ is the Vladimirov operator on $\mathbb{Z}_p$, and $f(u,v) = \mu u - u^3 - v$, $g(u,v) = \gamma(u - \delta v - \beta)$, where μ, β are real numbers, and γ, δ, d are positive real numbers.

This system admits a natural discretization of the form

$$\begin{cases} \frac{\partial}{\partial t}\left[u_L(I,t)\right]_{I \in G_L} = & \left[\mu u_L(I,t) - u_L^3(I,t) - v_L(I,t)\right]_{I \in G_L} - A_L^{\alpha}\left[u_L(I,t)\right]_{I \in G_L} \\ \\ \frac{\partial}{\partial t}\left[v_L(I,t)\right]_{I \in G_L} = & \left[\gamma\left(u_L(I,t) - \delta v_L(I,t) - \beta\right)\right]_{I \in G_L} - dA_L^{\alpha}\left[v_L(I,t)\right]_{I \in G_L}, \end{cases} \tag{1.3}$$

where G_L is a finite rooted three with L levels, and matrix A_L^{α} is a discretization of operator $\boldsymbol{D}_0^{\alpha} - \lambda$.

We present instability Turing criteria for systems (1.2) and (1.3), see Theorems 4.1, 5.1. The conditions for the existence of Turing patterns for both systems are essentially the same, except for one condition which involves a subset Γ of the eigenvalues of $\boldsymbol{D}_0^{\alpha} - \lambda$, in the case of system (1.2), and a subset Γ_L of the eigenvalues of matrix A_L^{α}, in the case of (1.3). We provide extensive numerical simulations of some systems of type (1.3); in particular, these experiments show that the Turing patterns are traveling waves inside the unit ball $\mathbb{Z}_p$. Our numerical experiments also show that the eigenvalues of matrix A_L^{α} approximate the eigenvalues of $\boldsymbol{D}_0^{\alpha} - \lambda$. We conjecture that the Turing patterns of (1.3) converge, in some sense, to the Turing patterns of (1.2). The results of Digernes and his collaborators on the problem of approximation of spectra of Vladimirov operator $\boldsymbol{D}^{\alpha}$ by matrices of type A_L^{α}, [6]-[8] provide strong support to our conjecture.

Nowadays, the study of Turing patterns on networks is a relevant area. In the 70s, Othmer and Scriven pointed out that Turing instability can occur in network-organized systems [18]-[19]. Since then, reaction-diffusion models on networks has been studied intensively, see, e.g., [2], [3], [5], [11], [14]- [15], [17]-[19], [22], [26]-[27], and the references therein. In particular, Turing patterns of discrete FitzHugh-Nagumo systems have also been studied [4]. In [31]-[32], the last author established the existence of Turing patterns for specific p-adic systems of reaction-diffusion equations, but these papers do not consider the problem of the numerical approximation of the Turing patterns.

The article is organized as follows. In Section 2, we review some basic aspects of the p-adic analysis and fix the notation. In Section 3, we present some basic aspects of the Vladimirov operator, the p-adic heat equation on the unit ball. In Section 4, we introduce our p-adic FitzHugh-Nagumo system, and give a Turing instability criterion, see Theorem 4.1. In Section 5, we study a discrete version of our p-adic FitzHugh-Nagumo system, and and give a Turing instability criterion, see Theorem 5.1. Finally, in Section 6, we provide extesnive numerical simulation for some discrete FitzHugh-Nagumo systems and their Turing patterns.

2 p-Adic Analysis: Essential Ideas

In this section, we collect some basic results on p-adic analysis that we use through the article. For a detailed exposition the reader may consult [1], [13], [23], [25].

2.1 The field of p-adic numbers

Along this article p will denote a prime number. The field of p-adic numbers $\mathbb{Q}_p$ is defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the p-adic norm $|\cdot|_p$, which is defined as

$$|x|_p = \begin{cases} 0 & \text{if} \quad x = 0 \\ \\ p^{-\gamma} & \text{if} \quad x = p^{\gamma}\frac{a}{b}, \end{cases}$$

where a and b are integers coprime with p. The integer $\gamma := ord(x)$, with $ord(0) := +\infty$, is called the *p-adic order of x*.

Any p-adic number $x \neq 0$ has a unique expansion of the form

$$x = p^{ord(x)} \sum_{j=0}^{\infty} x_j p^j,$$

where $x_j \in \{0, \ldots, p-1\}$ and $x_0 \neq 0$. By using this expansion, we define *the fractional part of $x \in \mathbb{Q}_p$*, denoted $\{x\}_p$, as the rational number

$$\{x\}_p = \begin{cases} 0 & \text{if} \quad x = 0 \text{ or } ord(x) \geq 0 \\ \\ p^{ord(x)} \sum_{j=0}^{-ord_p(x)-1} x_j p^j & \text{if} \quad ord(x) < 0. \end{cases}$$

2.2 Basic topology of $\mathbb{Q}_p$

For $r \in \mathbb{Z}$, denote by $B_r(a) = \{x \in \mathbb{Q}_p; |x - a|_p \leq p^r\}$ *the ball of radius p^r with center at $a \in \mathbb{Q}_p$*, and take $B_r(0) := B_r$. We also denote by $S_r(a) = \{x \in \mathbb{Q}_p; |x - a|_p = p^r\}$ *the sphere of radius p^r with center at $a \in \mathbb{Q}_p$*, and take $S_r(0) := S_r$. We notice that $S_0^1 = \mathbb{Z}_p^{\times}$ (the group of units of $\mathbb{Z}_p$). The balls and spheres are both open and closed subsets in $\mathbb{Q}_p$. In addition, two balls in $\mathbb{Q}_p$ are either disjoint or one is contained in the other.

As a topological space $(\mathbb{Q}_p, |\cdot|_p)$ is totally disconnected, i.e., the only connected subsets of $\mathbb{Q}_p$ are the empty set and the points. A subset of $\mathbb{Q}_p$ is compact if and only if it is closed and bounded in $\mathbb{Q}_p$, see e.g., [25, Section 1.3], or [1, Section 1.8]. Since $(\mathbb{Q}_p, +)$ is a locally compact topological group, there exists a Haar measure dx, which is invariant under translations, i.e., $d(x + a) = dx$. If we normalize this measure by the condition $\int_{\mathbb{Z}_p} dx = 1$, then dx is unique.

Notation 1 *We will use $\Omega\left(p^{-r}|x - a|_p\right)$ to denote the characteristic function of the ball $B_r(a)$. For more general sets, we will use the notation 1_A for the characteristic function of set A.*

2.3 The Bruhat-Schwartz space

A complex-valued function φ defined on $\mathbb{Q}_p$ is *called locally constant* if for any $x \in \mathbb{Q}_p$ there exist an integer $l(x) \in \mathbb{Z}$ such that

$$\varphi(x + x') = \varphi(x) \text{ for any } x' \in B_{l(x)}. \tag{2.1}$$

A function $\varphi : \mathbb{Q}_p \to \mathbb{C}$ is called a *Bruhat-Schwartz function (or a test function)* if it is locally constant with compact support. Any test function can be represented as a linear combination, with complex coefficients, of characteristic functions of balls. The $\mathbb{C}$-vector space of Bruhat-Schwartz functions is denoted by $\mathcal{D}(\mathbb{Q}_p)$. We denote by $\mathcal{D}_{\mathbb{R}}(\mathbb{Q}_p)$ the $\mathbb{R}$-vector space of Bruhat-Schwartz functions. For $\varphi \in \mathcal{D}(\mathbb{Q}_p)$, the largest number $l = l(\varphi)$ satisfying (2.1) is called *the exponent of local constancy (or the parameter of constancy) of* φ.

We denote by $\mathcal{C}(\mathbb{Q}_p)$, the $\mathbb{C}$-vector space of continuous functions defined on $\mathbb{Q}_p$, and by $\mathcal{C}_{\mathbb{R}}(\mathbb{Q}_p)$ its real counterpart.

2.4 L^ρ spaces

Given an open subset U of subset of $\mathbb{Q}_p$, and $\rho \in [1, \infty)$, we denote by $L^\rho(U) := L^\rho(U, dx)$, the $\mathbb{C}$-vector space of all the complex-valued functions φ satisfying

$$\|\varphi\|_\rho = \left\{ \int_U |\varphi(x)|^\rho \, dx \right\}^{\frac{1}{\rho}} < \infty.$$

The corresponding $\mathbb{R}$-vector space are denoted as $L^\rho_{\mathbb{R}}(U) = L^\rho_{\mathbb{R}}(U, dx)$, $1 \leq \rho < \infty$. We denote by $\mathcal{D}(U)$ the $\mathbb{C}$-vector space of test functions with supports contained in U, then $\mathcal{D}(U)$ is dense in $L^\rho(U)$, for $1 \leq \rho < \infty$, see e.g., [1, Section 4.3].

2.5 The Fourier transform

Set $\chi_p(y) = \exp(2\pi i\{y\}_p)$ for $y \in \mathbb{Q}_p$. The map $\chi_p(\cdot)$ is an additive character on $\mathbb{Q}_p$, i.e., a continuous map from $(\mathbb{Q}_p, +)$ into S (the unit circle considered as multiplicative group) satisfying $\chi_p(x_0 + x_1) = \chi_p(x_0)\chi_p(x_1)$, $x_0, x_1 \in \mathbb{Q}_p$. The additive characters of $\mathbb{Q}_p$ form an Abelian group which is isomorphic to $(\mathbb{Q}_p, +)$. The isomorphism is given by $\kappa \to \chi_p(\kappa x)$, see, e.g., [1, Section 2.3].

The Fourier transform of $\varphi \in \mathcal{D}(\mathbb{Q}_p)$ is defined as

$$(\mathcal{F}\varphi(\xi) = \int_{\mathbb{Q}_p} \chi_p(\xi x)\varphi(x)dx, \text{ for } \xi \in \mathbb{Q}_p,$$

where dx is the normalized Haar measure on $\mathbb{Q}_p$. We will also use the notation $\mathcal{F}_{x\to\xi}\varphi$ and $\widehat{\varphi}$ for the Fourier transform of φ.

The Fourier transform extends to $L^2(\mathbb{Q}_p)$. If $f \in L^2(\mathbb{Q}_p)$, its Fourier transform is defined as

$$(\mathcal{F}f)(\xi) = \lim_{k\to\infty} \int_{|x|_p \leq p^k} \chi_p(\xi x) f(x) dx, \quad \text{for } \xi \in \mathbb{Q}_p,$$

where the limit is taken in $L^2(\mathbb{Q}_p)$.

2.6 Distributions

The $\mathbb{C}$-vector space $\mathcal{D}'(\mathbb{Q}_p)$ of all continuous linear functionals on $\mathcal{D}(\mathbb{Q}_p)$ is called the *Bruhat-Schwartz space of distributions*. Every linear functional on $\mathcal{D}(\mathbb{Q}_p)$ is continuous, i.e., $\mathcal{D}'(\mathbb{Q}_p)$ agrees with the algebraic dual of $\mathcal{D}(\mathbb{Q}_p)$, see e.g., [25, Chapter 1, VI.3, Lemma]. We denote by $\mathcal{D}'_{\mathbb{R}}(\mathbb{Q}_p)$ the dual space of $\mathcal{D}_{\mathbb{R}}(\mathbb{Q}_p)$.

We endow $\mathcal{D}'(\mathbb{Q}_p)$ with the weak topology, i.e., a sequence $\{T_j\}_{j\in\mathbb{N}}$ in $\mathcal{D}'(\mathbb{Q}_p)$ converges to T if $\lim_{j\to\infty} T_j(\varphi) = T(\varphi)$ for any $\varphi \in \mathcal{D}(\mathbb{Q}_p)$. The map

$$\begin{array}{lll} \mathcal{D}'(\mathbb{Q}_p) \times \mathcal{D}(\mathbb{Q}_p) & \to & \mathbb{C} \\ (T, \varphi) & \to & T(\varphi) \end{array}$$

is a bilinear form, which is continuous in T and φ separately. We call this map the *pairing* between $\mathcal{D}'(\mathbb{Q}_p)$ and $\mathcal{D}(\mathbb{Q}_p)$. From now on we will use (T, φ) instead of $T(\varphi)$.

Every f in L^1_{loc} defines a distribution $f \in \mathcal{D}'(\mathbb{Q}_p)$ by the formula

$$(f, \varphi) = \int_{\mathbb{Q}_p} f(x)\, \varphi(x)\, dx.$$

Notice that for $f \in L^2_{\mathbb{R}}(\mathbb{Q}_p)$, $(f, \varphi) = \langle f, \varphi \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $L^2_{\mathbb{R}}(\mathbb{Q}_p)$.

2.7 The Fourier transform of a distribution

The Fourier transform $\mathcal{F}[T]$ of a distribution $T \in \mathcal{D}'(\mathbb{Q}_p)$ is defined by

$$(\mathcal{F}[T], \varphi) = (T, \mathcal{F}[\varphi]) \text{ for all } \varphi \in \mathcal{D}(\mathbb{Q}_p).$$

The Fourier transform $T \to \mathcal{F}[T]$ is a linear and continuous isomorphism from $\mathcal{D}'(\mathbb{Q}_p)$ onto $\mathcal{D}'(\mathbb{Q}_p)$. Furthermore, $T = \mathcal{F}[\mathcal{F}[T](-\xi)]$.

3 Vladimirov operator and p-adic wavelets

3.1 The p-adic heat equation

For $\alpha > 0$, the Vladimirov operator $\boldsymbol{D}^\alpha$ is defined as

$$\begin{array}{ccc} \mathcal{D}(\mathbb{Q}_p) & \to & L^2(\mathbb{Q}_p) \cap \mathcal{C}(\mathbb{Q}_p) \\ \\ \varphi & \to & \boldsymbol{D}^\alpha \varphi, \end{array}$$

where

$$(\boldsymbol{D}^\alpha \varphi)(x) = \frac{1 - p^\alpha}{1 - p^{-\alpha-1}} \int_{\mathbb{Q}_p} \frac{[\varphi(x-y) - \varphi(x)]}{|y|_p^{\alpha+1}} dy.$$

The p-adic analogue of the heat equation is

$$\frac{\partial u(x,t)}{\partial t} + a\boldsymbol{D}^\alpha u(x,t) = 0, \text{ with } a > 0.$$

The solution of the Cauchy problem attached to the heat equation with initial datum $u(x, 0) = \varphi(x) \in \mathcal{D}(\mathbb{Q}_p)$ is given by

$$u(x,t) = \int_{\mathbb{Q}_p} Z(x-y, t)\, \varphi(y)\, dy,$$

where $Z(x,t)$ is the *p-adic heat kernel* defined as

$$Z(x,t) = \int_{\mathbb{Q}_p} \chi_p(-x\xi)\, e^{-at|\xi|_p^\alpha} d\xi, \tag{3.1}$$

where $\chi_p(-x\xi)$ is the standard additive character of the group $(\mathbb{Q}_p, +)$. The p-adic heat kernel is the transition density function of a Markov stochastic process with space state $\mathbb{Q}_p$, see, e.g., [13], [28].

3.2 The p-adic heat equation on the unit ball

We define the operator $\boldsymbol{D}_0^\alpha$, $\alpha > 0$, by restricting $\boldsymbol{D}^\alpha$ to $\mathcal{D}(\mathbb{Z}_p)$ and considering $(\boldsymbol{D}^\alpha\varphi)(x)$ only for $x \in \mathbb{Z}_p$. The operator $\boldsymbol{D}_0^\alpha$ satisfies

$$\left(\boldsymbol{D}_0^\alpha - \lambda\right)\varphi(x) = \frac{1-p^\alpha}{1-p^{-\alpha-1}} \int\limits_{\mathbb{Z}_p} \frac{\varphi(x-y)-\varphi(x)}{|y|_p^{\alpha+1}} dy, \tag{3.2}$$

for $\varphi \in \mathcal{D}(\mathbb{Z}_p)$, with

$$\lambda = \frac{p-1}{p^{\alpha+1}-1} p^\alpha.$$

Consider the Cauchy problem

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} + \left(\boldsymbol{D}_0^\alpha - \lambda\right) u(x,t) = 0, & x \in \mathbb{Z}_p, \quad t > 0; \\ u(x,0) = \varphi(x), & x \in \mathbb{Z}_p, \end{cases}$$

where $\varphi \in \mathcal{D}(\mathbb{Z}_p)$. The solution of this problem is given by

$$u(x,t) = \int\limits_{\mathbb{Z}_p} Z_0(x-y,t)\varphi(y)\, dy,\ x \in \mathbb{Z}_p,\ t > 0,$$

where

$$Z_0(x,t) := e^{\lambda t} Z(x,t) + c(t),\ x \in \mathbb{Z}_p,\ t > 0,$$

$$c(t) := 1 - (1-p^{-1})e^{\lambda t} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} t^n \frac{1}{1-p^{-n\alpha-1}},$$

and $Z(x,t)$ is given (3.1). The function $Z_0(x,t)$ is non-negative for $x \in \mathbb{Z}_p$, $t > 0$, and

$$\int\limits_{\mathbb{Z}_p} Z_0(x,t) dx = 1,$$

[13]. Furthermore, $Z_0(x,t)$ is the transition density function of a Markov process with space state $\mathbb{Z}_p$.

3.3 p-adic wavelets supported in balls

The set of functions $\{\Psi_{rnj}\}$ defined as

$$\Psi_{rnj}(x) = p^{\frac{-r}{2}} \chi_p\left(p^{-1} j \left(p^r x - n\right)\right) \Omega\left(\left|p^r x - n\right|_p\right), \tag{3.3}$$

where $r \in \mathbb{Z}$, $j \in \{1, \cdots, p-1\}$, and n runs through a fixed set of representatives of $\mathbb{Q}_p/\mathbb{Z}_p$, is an orthonormal basis of $L^2(\mathbb{Q}_p)$ consisting of eigenvectors of operator $\boldsymbol{D}^\alpha$:

$$\boldsymbol{D}^\alpha \Psi_{rnj} = p^{(1-r)\alpha} \Psi_{rnj} \text{ for any } r, n, j,$$

see, e.g., [12, Theorem 3.29], [1, Theorem 9.4.2]. By using this basis, it is possible to construct an orthonormal basis for $L^2(\mathbb{Z}_p)$:

Proposition 1 ([29, Propositions 1, 2]) *The set of functions*

$$\left\{\Omega\left(|x|_p\right)\right\} \bigcup \bigcup_{j \in \{1,\dots,p-1\}} \bigcup_{r \leq 0} \bigcup_{\substack{np^{-r} \in \mathbb{Z}_p \\ n \in \mathbb{Q}_p/\mathbb{Z}_p}} \left\{\Psi_{rnj}(x)\right\} \tag{3.4}$$

is an orthonormal basis of $L^2(\mathbb{Z}_p)$. *Furthermore,*

$$L^2(\mathbb{Z}_p) = \mathbb{C}\Omega\left(|x|_p\right) \bigoplus L_0^2(\mathbb{Z}_p), \tag{3.5}$$

where

$$L_0^2(\mathbb{Z}_p) = \left\{ f \in L^2(\mathbb{Z}_p); \int_{\mathbb{Z}_p} f\ dx = 0 \right\}.$$

Now, by using (3.2), (3.3), (3.4), the functions in (3.4) are eigenfunctions of $\boldsymbol{D}_0^\alpha - \lambda$:

$$\left(\boldsymbol{D}_0^\alpha - \lambda\right)\Psi_{rnj} = p^{(1-r)\alpha}\Psi_{rnj} \tag{3.6}$$

for any $r \leq 0$, $n \in p^r\mathbb{Z}_p \cap \mathbb{Q}_p/\mathbb{Z}_p$, $j \in \{1, \ldots, p-1\}$, and

$$\left(\boldsymbol{D}_0^\alpha - \lambda\right)\Omega\left(|x|_p\right) = \lambda\Omega\left(|x|_p\right), \text{ for } x \in \mathbb{Z}_p. \tag{3.7}$$

3.3.1 An eigenvalue problem in the unit ball

We now consider the following eigenvalue problem:

$$\begin{cases} \left(\boldsymbol{D}_0^\alpha - \lambda\right)\theta(x) = \kappa\theta(x), \quad \kappa \in \mathbb{R} \\ \\ \theta \in L^2_{\mathbb{R}}(\mathbb{Z}_p). \end{cases} \tag{3.8}$$

By using (3.6)-(3.7), the functions $\Psi_{rnj}(x)$ given in (3.4) are complex-valued eigenfunctions of (3.8) with eigenvalues $\kappa \in \left\{p^{(1-r)\alpha}; r \leq 0\right\}$. Therefore

$$p^{\frac{-r}{2}}\cos\left(\left\{p^{r-1}jx - p^{-1}nj\right\}_p\right)\Omega\left(|p^r x - n|_p\right),$$
$$p^{\frac{-r}{2}}\sin\left(\left\{p^{r-1}jx - p^{-1}nj\right\}_p\right)\Omega\left(|p^r x - n|_p\right),$$

with $|p^{-r}n|_p \leq 1$ and $r \leq 0$, $n \in p^r\mathbb{Z}_p \cap \mathbb{Q}_p/\mathbb{Z}_p$, $j \in \{1, \ldots, p-1\}$, are real-valued eigenfunctions of (3.8) with $\kappa = p^{(1-r)\alpha}$. Furthermore, any $f(x) \in L^2_{\mathbb{R}}(\mathbb{Z}_p)$ admits an expansion of the form

$$\begin{aligned} f(x) = &\sum_{rnj} p^{\frac{-r}{2}}\operatorname{Re}(A_{rnj})\cos\left(\left\{p^{r-1}jx - p^{-1}nj\right\}_p\right)\Omega\left(|p^r x - n|_p\right) \\ &- \sum_{rnj} p^{\frac{-r}{2}}\operatorname{Im}(A_{rnj})\sin\left(\left\{p^{r-1}jx - p^{-1}nj\right\}_p\right)\Omega\left(|p^r x - n|_p\right) \\ &+ A_0\Omega\left(|x|_p\right) \end{aligned} \tag{3.9}$$

where

$$\operatorname{Re}(A_{rnj}) = p^{\frac{-r}{2}}\int_{\mathbb{Z}_p} f(x)\cos\left(\left\{p^{r-1}jx - p^{-1}nj\right\}_p\right)\Omega\left(|p^r x - n|_p\right)dx,$$

$$\operatorname{Im}(A_{rnj}) = p^{\frac{-r}{2}}\int_{\mathbb{Z}_p} f(x)\sin\left(\left\{p^{r-1}jx - p^{-1}nj\right\}_p\right)\Omega\left(|p^r x - n|_p\right)dx,$$

and

$$A_0 = \int_{\mathbb{Z}_p} f(x)dx.$$

4 A p-adic FitzHugh-Nagumo system on $\mathbb{Z}_p$

A reaction-diffusion system exhibits diffusion-driven instability, or Turing instability, if the homogeneous steady state is stable to small perturbations in the absence of diffusion but unstable to small spatial perturbations when diffusion is present. The main process driving the spatially inhomogeneous instability is diffusion: the mechanism determines the spatial pattern that evolves. For Turing instability, we require that the system is stable in the absence of diffusion.

From now on, we set $u(x,t), v(x,t) : \mathbb{Z}_p \times [0,\infty) \to \mathbb{R}$. We consider the following FitzHugh-Nagumo system with p-adic diffusion:

$$\begin{cases} u(\cdot,t), v(\cdot,t) \in L^2_{\mathbb{R}}(\mathbb{Z}_p), \quad \text{for } t \geq 0; \\ \\ u(x,0), v(x,0) \in L^2_{\mathbb{R}}(\mathbb{Z}_p), \quad u(x,0), v(x,0),\, x \in \mathbb{Z}_p; \\ \\ \frac{\partial u}{\partial t}(x,t) = f(u,v) - \left(\boldsymbol{D}_0^{\alpha} - \lambda\right) u(x,t), \quad x \in \mathbb{Z}_p,\, t \geq 0; \\ \\ \frac{\partial v}{\partial t}(x,t) = g(u,v) - d\left(\boldsymbol{D}_0^{\alpha} - \lambda\right) v(x,t), \quad x \in \mathbb{Z}_p,\;\; t \geq 0, \end{cases} \tag{4.1}$$

where

$$f(u,v) = \mu u - u^3 - v,\; g(u,v) = \gamma(u - \delta v - \beta), \tag{4.2}$$

and μ, β, $\gamma \neq 0$, $\delta \neq 0$, d are real numbers.

4.1 Homogeneous steady states

We now consider a homogeneous steady state (also called an equilibrium point) of (4.1) which is a positive (u_0, v_0) solution of

$$\begin{cases} \frac{\partial u}{\partial t} = f(u,v), \quad t \geq 0 \\ \\ \frac{\partial v}{\partial t} = g(u,v), \quad t \geq 0. \end{cases} \tag{4.3}$$

The equilibrium points associated with (4.3) are given by

$$\begin{cases} \mu u - u^3 - v = 0 \\ \\ \gamma(u - \delta v - \beta) = 0. \end{cases} \tag{4.4}$$

Using the substitution method on (4.4), we have that (4.4) is equivalent to

$$u^3 + \eta u + \tau = 0, \tag{4.5}$$

where $\eta := \frac{1-\delta\mu}{\delta}$ and $\tau := -\frac{\beta}{\delta}$. Here we use the hypothesis that $\gamma \neq 0$, $\delta \neq 0$. We denote by u_0 a real solution of (4.5). Then (u_0, v_0), with $v_0 = \frac{u_0 - \beta}{\delta}$, is the real equilibrium point of (4.3).

We denote by $\sigma_{\text{eigen}}\left(\boldsymbol{D}_0^{\alpha} - \lambda\right)$ to the set of eigenvalues of $\boldsymbol{D}_0^{\alpha} - \lambda$. We also set

$$\kappa_1 = \frac{1}{2d}\left\{\left(d\left(\mu - 3u_0^2\right) - \gamma\delta\right) - \sqrt{\left(d\left(\mu - 3u_0^2\right) - \gamma\delta\right)^2 - 4d\det(A)}\right\}, \tag{4.6}$$

$$\kappa_2 = \frac{1}{2d}\left\{\left(d\left(\mu - 3u_0^2\right) - \gamma\delta\right) + \sqrt{\left(d\left(\mu - 3u_0^2\right) - \gamma\delta\right)^2 - 4d\det(A)}\right\}, \tag{4.7}$$

where

$$A = \begin{bmatrix} f_u & f_v \\ g_u & g_v \end{bmatrix}_{u=u_0, v=v_0} := \begin{bmatrix} f_{u_0} & f_{v_0} \\ g_{u_0} & g_{v_0} \end{bmatrix}.$$

Notice that A is the Jacobian matrix of the mapping $(u,v) \to (f(u,v), g(u,v))$. A straighforward calculation shows that

$$A = \begin{bmatrix} \mu - 3u_0^2 & -1 \\ \gamma & -\gamma\delta \end{bmatrix}. \tag{4.8}$$

Theorem 4.1 *Consider the reaction-diffusion system (4.1). The steady state (u_0, v_0) is linearly unstable (Turing unstable), if the following conditions hold:*

1. $Tr(A) = \mu - 3u_0^2 - \gamma\delta < 0$;
2. $\det(A) = -\mu\gamma\delta + 3\gamma\delta u_0^2 + \gamma > 0$;
3. $d\left(\mu - 3u_0^2\right) - \gamma\delta > 0$;
4. *The derivatives* $\mu - 3u_0^2$ *and* $-\gamma\delta$ *must have opposite signs;*
5. $\left(d\left(\mu - 3u_0^2\right) - \gamma\delta\right)^2 - 4d\left(-\mu\gamma\delta + 3\gamma\delta u_0^2 + \gamma\right) > 0$;
6. $\Gamma = \{\kappa \in \sigma_{eigen}\left(\boldsymbol{D}_0^\alpha - \lambda\right); \kappa_1 < \kappa < \kappa_2\} \neq \emptyset$.

Furthermore, there are infinitely many unstable eigenmodes, and the Turing pattern has the form (4.9).

Proof. The proof is similar to the one given in [32]-[31]. However, in [32], the Turing pattern is a function from the subspace of $L^2_{\mathbb{R}}(\mathbb{Z}_p)$ consisting of functions with average zero, while here, the pattern is a function from $L^2_{\mathbb{R}}(\mathbb{Z}_p)$. For further details the reader may consult the above mentioned references. The Turing pattern $w(x,t)$ has the form

$$\begin{aligned} w(x,t) \sim & \sum_{\kappa_1<\kappa<\kappa_2} \sum_{r,n} A_{rn} e^{\rho(\kappa)t} \Omega\left(|p^r x - n|_p\right) \\ & + \sum_{\kappa_1<\kappa<\kappa_2} \sum_{r,n,j} A_{rnj} e^{\rho(\kappa)t} p^{-\frac{r}{2}} \cos\left(\left\{p^{-1} j\left(p^r x - n\right)\right\}_p\right) \Omega\left(|p^r x - n|_p\right) \\ & + \sum_{\kappa_1<\kappa<\kappa_2} \sum_{r,n,j} B_{rnj} e^{\rho(\kappa)t} p^{-\frac{r}{2}} \sin\left(\left\{p^{-1} j\left(p^r x - n\right)\right\}_p\right) \Omega\left(|p^r x - n|_p\right), \end{aligned} \tag{4.9}$$

for $t \to \infty$, where $\rho(\kappa)$ are eigenvalues of matrix A depending on $\kappa \in \sigma_{\text{eigen}}\left(\boldsymbol{D}_0^\alpha - \lambda\right)$, with $\text{Re}(\rho(\kappa)) > 0$. ■

5 Discrete FitzHugh-Nagumo systems

5.1 The Spaces $\mathcal{D}_L$

We fix $L \in \mathbb{N} \smallsetminus \{0\}$, and define

$$G_L = \mathbb{Z}_p / p^L \mathbb{Z}_p.$$

Then, G_L is a finite ring, with $\#G_L = p^L$ elements. We set the following set of representatives for the elements of G_L:

$$I = I_0 + I_q p^1 + \ldots + I_{L-1} p^{L-1},$$

where the I_js are p-adic digits, i.e., elements from $\{0, 1, \ldots, p-1\}$. We define $\mathcal{D}_L$ to be the space of test functions φ supported in the unit ball having the form

$$\varphi(x) = p^{\frac{L}{2}} \sum_{I \in G_L} \varphi(I) \Omega\left(p^L |x - I|_p\right), \text{ with } \varphi(I) \in \mathbb{R}. \tag{5.1}$$

Since $\Omega\left(p^L|x - I|_p\right) \Omega\left(p^L|x - J|_p\right) = 0$ if $I \neq J$, the set

$$\left\{p^{\frac{L}{2}} \Omega\left(p^L |x - I|_p\right); I \in G_{L,M}\right\}$$

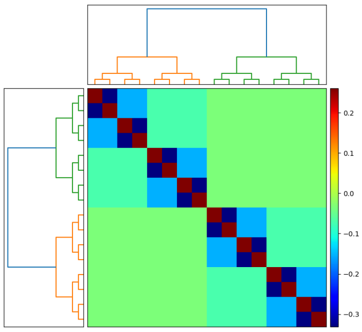

Figure 1: The heat map for matrix A_L^α; $p=2, L=4, \alpha=0.1$.

is an orthonormal basis for $\mathcal{D}_L$. Then, by using that

$$\|\varphi\|_{L^2} = \sqrt{p^L \sum_{I\in G_L} |\varphi(I)|^2 \int_{\mathbb{Z}_p} \Omega\left(p^L|x-I|_p\right) dx} = \sqrt{\sum_{I\in G_L} |\varphi(I)|^2},$$

we have

$$(\mathcal{D}_L, \|\cdot\|_{L^2}) \simeq \left(\mathbb{R}^{\#G_L}, |\cdot|_{\mathbb{R}}\right) \text{ as Hilbert spaces,}$$

where $|\cdot|_{\mathbb{R}}$ denotes the usual norm of $\mathbb{R}^{\#G_L}$.

5.2 Discretization of the operator $\boldsymbol{D}_0^\alpha - \lambda$

A natural discretization of $\boldsymbol{D}_0^\alpha - \lambda$ is obtained by taking its restriction to $\mathcal{D}_L$. We denote this restriction by $\boldsymbol{D}_L^\alpha - \lambda$. Since $\mathcal{D}_L$ is a finite vector space, $\boldsymbol{D}_L^\alpha - \lambda$ is represented by a matrix $A_L^\alpha = \left[A_{K,I}^\alpha\right]_{K,I\in G_L}$, where

$$A_{K,I}^\alpha = \begin{cases} p^{-\frac{L}{2}} \frac{1-p^\alpha}{1-p^{-\alpha-1}} \frac{1}{|K-I|_p^{\alpha+1}} & \text{if} \quad K \neq I; \\ \\ -p^{-\frac{L}{2}} \frac{1-p^\alpha}{1-p^{-\alpha-1}} \sum_{K\neq I} \frac{1}{|K-I|_p^{\alpha+1}} - \lambda & \text{if} \quad K = I, \end{cases} \tag{5.2}$$

see [32].

5.3 Discretization of the p-adic Turing System (4.1)

A discretization of the Turing system (4.1) is obtained by approximating the functions $u(x,t)$, $v(x,t)$ as

$$u_L(x,t) = \sum_{I\in G_L} u_L(I,t)\Omega\left(p^L|x-I|_p\right)$$

and

$$v_L(x,t) = \sum_{I\in G_L} v_L(I,t)\Omega\left(p^L|x-I|_p\right),$$

where $u_L(I,\cdot)$, $v_L(I,\cdot) \in C^1([0,T])$ for some fixed positive T. We set

$$u_L(x,t) = [u_L(I,t)]_{I\in G_L}\,,\;\; v_L(x,t) = [v_L(I,t)]_{I\in G_L}\,.$$

Notice that

$$\begin{aligned} f &\left(\sum_{I\in G_L} u_L(I,t)\Omega\left(p^L|x-I|_p\right), \sum_{J\in G_L} u_L(J,t)\Omega\left(p^L|x-J|_p\right)\right) \\ &= \sum_{I\in G_L} f\left(u_L(I,t), v_L(I,t)\right)\Omega\left(p^L|x-I|_p\right) \\ &= \sum_{I\in G_L} \left\{\mu u_L(I,t) - u_L^3(I,t) - u_L(I,t)\right\}\Omega\left(p^L|x-I|_p\right). \end{aligned}$$

A similar formula holds for function g. Then, using (4.2), the discretization of the p-adic Turing system (4.1) has the form:

$$\begin{cases} \frac{\partial}{\partial t}\left[u_L(I,t)\right]_{I\in G_L} = & \left[\mu u_L(I,t) - u_L^3(I,t) - v_L(I,t)\right]_{I\in G_L} - A_L^\alpha\left[u_L(I,t)\right]_{I\in G_L} \\ \frac{\partial}{\partial t}\left[v_L(I,t)\right]_{I\in G_L} = & \left[\gamma\left(u_L(I,t) - \delta v_L(I,t) - \beta\right)\right]_{I\in G_L} - dA_L^\alpha\left[v_L(I,t)\right]_{I\in G_L}, \end{cases} \tag{5.3}$$

where $A_L^\alpha = \left[A_{K,I}^\alpha\right]_{K,I\in G_L}$.

We now rewrite system (5.3) in a matrix form. We denote by $diag\,(a_I; I\in G_L)$, a diagonal matrix of size $\#G_L \times \#G_L$. Now, by using (5.2) and (5.3), we have

$$\frac{\partial}{\partial t}\begin{bmatrix} [u_L(I,t)]_{I\in G_L} \\ [v_L(I,t)]_{I\in G_L} \end{bmatrix} = \tag{5.4}$$

$$\begin{bmatrix} diag\left(f\left(u_L(I,t), v_L(I,t)\right); I\in G_L\right) & 0_{\#G_L\times\#G_L} \\ 0_{\#G_L\times\#G_L} & diag\left(g\left(u_L(I,t), v_L(I,t)\right); I\in G_L\right) \end{bmatrix}$$

$$-\begin{bmatrix} I_{\#G_L\times\#G_L} & 0_{\#G_L\times\#G_L} \\ 0_{\#G_L\times\#G_L} & dI_{\#G_L\times\#G_L} \end{bmatrix}\begin{bmatrix} A_L^\alpha & 0_{\#G_L\times\#G_L} \\ 0_{\#G_L\times\#G_L} & A_L^\alpha \end{bmatrix}\begin{bmatrix} [u_L(I,t)]_{I\in G_L} \\ [v_L(I,t)]_{I\in G_L} \end{bmatrix},$$

where $0_{\#G_L\times\#G_L}$ denotes a matriz of size $\#G_L \times \#G_L$ with all its entries equal to zero, and $I_{\#G_L\times\#G_L}$ denotes the identity matrix of size $\#G_L \times \#G_L$.

5.4 Discrete homogeneous steady states

We study the equilibrium points of the system

$$\frac{\partial}{\partial t}\begin{bmatrix} [u_L(I,t)]_{I\in G_L} \\ [v_L(I,t)]_{I\in G_L} \end{bmatrix} = \tag{5.5}$$

$$\begin{bmatrix} diag\left(f\left(u_L(I,t), v_L(I,t)\right); I\in G_L\right) & 0_{\#G_L\times\#G_L} \\ 0_{\#G_L\times\#G_L} & diag\left(g\left(u_L(I,t), v_L(I,t)\right); I\in G_L\right) \end{bmatrix}.$$

The equilibrium points are the solutions of the following system of algebraic equations:

$$\begin{cases} f\left(u_L(I), v_L(I)\right) = 0 \\ g\left(u_L(I), v_L(I)\right) = 0, \end{cases} \tag{5.6}$$

where $I \in G_L$. Notice that if $f(u_0, v_0) = g(u_0, v_0) = 0$, then $u_L(I) = u_0$, $v_L(I) = v_0$ is a solution of (5.6) for any $I \in G_L$.

Take $\eta = \frac{1-\delta\mu}{\delta}$ and $\tau = -\frac{\beta}{\delta}$, as before. Then,

$$\begin{bmatrix} [u_0]_{I \in G_L} \\ \\ [v_0]_{I \in G_L} \end{bmatrix} \tag{5.7}$$

is one equilibrium point.

5.5 The Jacobian matrix

We now consider the following polynomial mapping:

$$\begin{array}{ccc} \mathbb{R}^{2\#G_L} & \to & \mathbb{R}^{2\#G_L} \\ \\ \begin{bmatrix} [u_L(I)]_{I \in G_L} \\ \\ [v_L(I)]_{I \in G_L} \end{bmatrix} & \to & \begin{bmatrix} [f(u_L(I), v_L(I))]_{I \in G_L} \\ \\ [g(u_L(I), v_L(I))]_{I \in G_L} \end{bmatrix}. \end{array} \tag{5.8}$$

We denote by $\nabla f(u_0, v_0)$, the 1×2 matrix $\begin{bmatrix} \frac{\partial f(u_0,v_0)}{\partial u} & \frac{\partial f(u_0,v_0)}{\partial v} \end{bmatrix}$, and by

$$diag\left(\nabla f(u_0, v_0); I \in G_L\right),$$

the block diagonal matrix

$$\begin{bmatrix} \nabla f(u_0, v_0) & & 0 \\ & \ddots & \\ 0 & & \nabla f(u_0, v_0) \end{bmatrix}$$

of size $\#G_L \times 2\#G_L$. In a similar form, we define the block diagonal matrix

$$diag\left(\nabla g(u_0, v_0); I \in G_L\right).$$

The Jacobian matrix $\mathcal{A}$ of mapping (5.8) at the equilibrium point (5.7) is the $2\#G_L \times 2\#G_L$ matrix

$$\mathcal{A} = \begin{bmatrix} \nabla f(u_0, v_0) & & 0 \\ & \ddots & \\ 0 & & \nabla f(u_0, v_0) \\ \nabla g(u_0, v_0) & & 0 \\ & \ddots & \\ 0 & & \nabla g(u_0, v_0) \end{bmatrix} = \begin{bmatrix} diag\left(\nabla f(u_0, v_0); I \in G_L\right) \\ \\ diag\left(\nabla g(u_0, v_0); I \in G_L\right) \end{bmatrix}.$$

We now set

$$A = \begin{bmatrix} \frac{\partial f(u_0,v_0)}{\partial u} & \frac{\partial f(u_0,v_0)}{\partial v} \\ \frac{\partial g(u_0,v_0)}{\partial u} & \frac{\partial g(u_0,v_0)}{\partial v} \end{bmatrix} = \begin{bmatrix} \nabla f(u_0, v_0) \\ \nabla g(u_0, v_0) \end{bmatrix}$$

as before, and by a finite sequence of swapings of rows, matrix $\mathcal{A}$ can be writen as

$$\mathcal{A}' = \begin{bmatrix} A & & 0 \\ & \ddots & \\ 0 & & A \end{bmatrix}, \tag{5.9}$$

which is a $\#G_L \times \#G_L$ block matrix.

We denote by $\sigma\left(A_L^{\alpha}\right)$ the spectrum of A, and use the κ_1, κ_2 defined in (4.6)-(4.7).

Theorem 5.1 *Let us consider the reaction-diffusion system (5.4). The discrete steady state* $\begin{bmatrix} [u_0]_{I\in G_L} \\ \\ [v_0]_{I\in G_L} \end{bmatrix}$ *is linearly unstable (Turing unstable), if the following conditions hold:*

1. $Tr(A) = \mu - 3u_0^2 - \gamma\delta < 0$;
2. $\det(A) = -\mu\gamma\delta + 3\gamma\delta u_0^2 + \gamma > 0$;
3. $d\left(\mu - 3u_0^2\right) - \gamma\delta > 0$;
4. *The derivatives* $\mu - 3u_0^2$ *and* $-\gamma\delta$ *must have opposite signs;*
5. $\left(d\left(\mu - 3u_0^2\right) - \gamma\delta\right)^2 - 4d\left(-\mu\gamma\delta + 3\gamma\delta u_0^2 + \gamma\right) > 0$;
6. $\Gamma_L = \{\kappa_L \in \sigma\left(A_L^\alpha\right); \kappa_1 < \kappa_L < \kappa_2\} \neq \emptyset$.

Furthermore, the Turing pattern has the form (5.19).

Proof. We first linearize sytem (5.4) about the steady state (5.7). Set

$$\begin{bmatrix} \left[w_L^{(1)}(I,t)\right]_{I\in G_L} \\ \\ \left[w_L^{(2)}(I,t)\right]_{I\in G_L} \end{bmatrix} := \begin{bmatrix} [u_L(I,t) - u_0]_{I\in G_L} \\ \\ [v_L(I,t) - v_0]_{I\in G_L} \end{bmatrix}.$$

Then the linear approximation is

$$\begin{bmatrix} \left[w_L^{(1)}(I,t)\right]_{I\in G_L} \\ \\ \left[w_L^{(2)}(I,t)\right]_{I\in G_L} \end{bmatrix} = \mathcal{A} \begin{bmatrix} \left[w_L^{(1)}(I,t)\right]_{I\in G_L} \\ \\ \left[w_L^{(2)}(I,t)\right]_{I\in G_L} \end{bmatrix}.$$

The equilibrium point

$$\begin{bmatrix} [0]_{I\in G_L} \\ \\ [0]_{I\in G_L} \end{bmatrix} \tag{5.10}$$

is linearly stable, if the eigenvalues of $\mathcal{A}$ have negative real parts. By a suitable sequence of swapings of the rows of $\mathcal{A}$, we have

$$\det\left(\mathcal{A} - \rho I\right) = \pm\det\left(\mathcal{A}' - \rho I\right) = \pm\det\left(A - \rho I\right)^{\#G_L}.$$

Then the eigenvalues of $\mathcal{A}$ are exactly the eigenvalues of A counted with multiplicity G_L:

$$\det\left(A - \rho I\right) = \det\begin{bmatrix} \mu - 3u_0^2 - \rho & -1 \\ \gamma & -\lambda\delta - \rho \end{bmatrix} = \rho^2 - \rho Tr(A) + \det(A) = 0. \tag{5.11}$$

Then

$$\rho_{1,2} = \frac{\pm\sqrt{\left(\mu - 3u_0^2 - \gamma\delta\right)^2 - 4\left(-\mu\gamma\delta + 3\gamma\delta u_0^2 + \gamma\right)}}{2} + \frac{\mu - 3u_0^2 - \gamma\delta}{2}.$$

The condition $\mathrm{Re}(\rho_{1,2}) < 0$ is guaranteed, if the trace and the determinant of matrix A satisfy

$$Tr(A) < 0,\ \det(A) > 0. \tag{5.12}$$

Now, we linearize the entire reaction-ultradiffusion system close to the steady state (5.10):

$$\frac{\partial}{\partial t}\left[\begin{array}{c}\left[w_L^{(1)}(I,t)\right]_{I\in G_L}\\ \\ \left[w_L^{(2)}(I,t)\right]_{I\in G_L}\end{array}\right]=(\mathcal{A}-D_L\mathcal{A}_L^{\alpha})\left[\begin{array}{c}\left[w_L^{(1)}(I,t)\right]_{I\in G_L}\\ \\ \left[w_L^{(2)}(I,t)\right]_{I\in G_L}\end{array}\right], \tag{5.13}$$

where

$$D_L:=\left[\begin{array}{cc}I_{\#G_L\times\#G_L} & 0_{\#G_L\times\#G_L}\\ \\ 0_{\#G_L\times\#G_L} & dI_{\#G_L\times\#G_L}\end{array}\right],\ \mathcal{A}_L^{\alpha}:=\left[\begin{array}{cc}A_L^{\alpha} & 0_{\#G_L\times\#G_L}\\ \\ 0_{\#G_L\times\#G_L} & A_L^{\alpha}\end{array}\right].$$

The matrices A_L^{α}, $\mathcal{A}_L^{\alpha}$ are real symmetric, and consequently they are diagonalizable. Then, there exists a basis $\{\boldsymbol{e}_\kappa\}$ of $\mathbb{R}^{\#G_L}$ such that

$$A_L^{\alpha}\boldsymbol{e}_\kappa=\kappa\boldsymbol{e}_\kappa,$$

where $\kappa=\kappa(L)$. Then

$$\mathcal{A}_L^{\alpha}\left[\begin{array}{c}\boldsymbol{e}_\kappa\\ \boldsymbol{e}_\kappa\end{array}\right]=\kappa\left[\begin{array}{c}\boldsymbol{e}_\kappa\\ \boldsymbol{e}_\kappa\end{array}\right].$$

We now look for a solution of system (5.13) of the form

$$\left[\begin{array}{c}\left[w_L^{(1)}(I,t)\right]_{I\in G_L}\\ \\ \left[w_L^{(2)}(I,t)\right]_{I\in G_L}\end{array}\right],$$

where $w_L^{(j)}(I,t)=\sum_{\kappa,\rho}C_{\kappa,\rho}e^{\rho t}\boldsymbol{e}_\kappa$, where $\rho=\rho(j,I,L)$, $\kappa=\kappa(j,I,L)$. The function $e^{\rho t}\boldsymbol{e}_\kappa$ is a non-trivial solution of (5.13), if ρ satisfies

$$\det(\rho I-\mathcal{A}+\kappa D_L)=0. \tag{5.14}$$

By a finite sequence of swapings of rows, we have

$$\begin{aligned}\det(\rho I-\mathcal{A}+\kappa D_L)&=\pm\det\left[\begin{array}{ccc}\rho I_{2\times 2}-A+\kappa D & & 0\\ & \ddots & \\ 0 & & \rho I_{2\times 2}-A+\kappa D\end{array}\right]\\ &=\pm\det\left(\rho I_{2\times 2}-A+\kappa D\right)^{\#G_L}\\ &=\rho^2+[\kappa(1+d)-Tr(A)]\rho+h(\kappa)=0, \qquad (5.15)\end{aligned}$$

where

$$D=\left[\begin{array}{cc}1 & 0\\ 0 & d\end{array}\right],$$

and

$$h(\kappa):=d\kappa^2-\kappa\left(d\left(\mu-3u_0^2\right)-\gamma\delta\right)+\det(A). \tag{5.16}$$

Since $\kappa=0$ is not an eigenvalue of the matrix A_L^{α}, the conditions (5.11) and (5.15) are independent. For that the steady state to be unstable for spatial perturbations, we need that $\mathrm{Re}(\rho(\kappa))>0$, for some $\kappa\neq 0$, this can happen either if the coefficient of ρ in (5.15) is negative or if $h(\kappa)<0$, for some $\kappa\neq 0$ in (5.16). For being $Tr(A)<0$ of the conditions (5.12) and the coefficient of ρ in (5.15) is $\kappa(1+d)-Tr(A)$, which is positive, so the only way that $\mathrm{Re}(\rho(\kappa))$ can be positive is if $h(\kappa)<0$ for some $\kappa\neq 0$. As $\det(A)>0$ of (5.12), in order for $h(\kappa)$ to be negative, it is necessary that $d\left(\mu-3u_0^2\right)-\gamma\delta>0$. Now, since $Tr(A)=\mu-3u_0^2-\gamma\delta<0$, necessarily $d\neq 1$

and $\mu - 3u_0^2$ and $-\gamma\delta$ must have opposite signs. Thus, we have that an additional requirement to (5.12) is that $d \neq 1$. This is a necessary, but not sufficient, condition for that $\mathrm{Re}(\rho(\kappa)) > 0$. For that $h(\kappa)$ to be negative for some non zero κ, the minimum h_{min} of $h(\kappa)$ must be negative. Using elementary calculations, we show that

$$h_{\min} = \det(A) - \frac{\left(d\left(\mu - 3u_0^2\right) - \gamma\delta\right)^2}{4d},$$

and the minimum is reached at

$$k_{min} = \frac{d\left(\mu - 3u_0^2\right) - \gamma\delta}{2d}. \tag{5.17}$$

Therefore, the condition $h(\kappa) < 0$ for some $\kappa \neq 0$ is

$$\frac{\left(d(\mu - 3u_0^2) - \gamma\delta\right)^2}{4d} > \det(A).$$

A bifurcation occurs when $h_{min} = 0$, this happens when the condition

$$\det(A) = \frac{\left(d\left(\mu - 3u_0^2\right) - \gamma\delta\right)^2}{4d},$$

is verifiesd. This condition defines a critical diffusion d_c, which is given as an appropriate root of

$$\left(\mu - 3u_0^2\right)^2 d_c^2 + 2\left(-2\gamma + \mu\gamma\delta - 3\gamma\delta u_0^2\right) d_c + \gamma^2\delta^2 = 0.$$

The model (5.4) for $d > d_c$ exhibits Turing instability, while for $d < d_c$ it does not. Note that $d_c > 1$. A critical 'wavenumber' is obtained using (5.17)

$$\kappa_c = \frac{d_c\left(\mu - 3u_0^2\right) - \gamma\delta}{2d_c} = \sqrt{\frac{\det(A)}{d_c}}. \tag{5.18}$$

When $d > d_c$, there is a range of number of unstable positive waves $\kappa_1 < \kappa < \kappa_2$, where κ_1, κ_2 are the zeros of $h(\kappa) = 0$, see (4.6)-(4.7). We call to function $\rho(\kappa)$ the dispersion relation. We note that, within the unstable range, $\mathrm{Re}(\rho(\kappa)) > 0$ has a maximum for the wavenumber $\kappa_{min}^{(0)}$ obtained from (5.17) with $d > d_c$. Then as t it increases, the behavior of $\begin{bmatrix} \left[w_L^{(1)}(I,t)\right]_{I\in G_L} \\ \\ \left[w_L^{(2)}(I,t)\right]_{I\in G_L} \end{bmatrix}$ is controlled by the dominant mode, that is, those $e^{\rho(\kappa)t}\begin{bmatrix} \boldsymbol{e}_\kappa \\ \boldsymbol{e}_\kappa \end{bmatrix}$ with $\mathrm{Re}(\rho(\kappa)) > 0$, since the other modes go to zero exponentially. We recall that $\kappa = \kappa(L)$. For this reason, we use the notation $\kappa = \kappa_L$. Wit this notation,

$$w_L^{(j)}(I,t) \sim \sum_{\kappa_1 < \kappa_L < \kappa_2} A_\kappa(j, I)\, e^{\rho(\kappa_L)t}\boldsymbol{e}_\kappa, \text{ for } t \to \infty, \tag{5.19}$$

where $j = 1, 2$. ■

Digernes and his collaborators have studies extensively the problem of approximation of spectra of Vladimirov operator $\boldsymbol{D}^\alpha$ by matrices of type A_L^α, [6]-[8]. By using the fact that the eigenvalues $\varsigma \neq \lambda$ and eigenfuntions Ψ_{rnj} of $\boldsymbol{D}_0^\alpha - \lambda$ are also eigenvalues and eigenfunctions of $\boldsymbol{D}^\alpha$, and Theorem 4.1 in [7], one concludes that for L sufficiently large, the eigenvalues of matrix A_L^α approximate the eigenvalues $\varsigma \neq \lambda$ of $\boldsymbol{D}_0^\alpha - \lambda$, in a symbolic form $\Gamma_L \approx \Gamma \smallsetminus \{\lambda\}$.

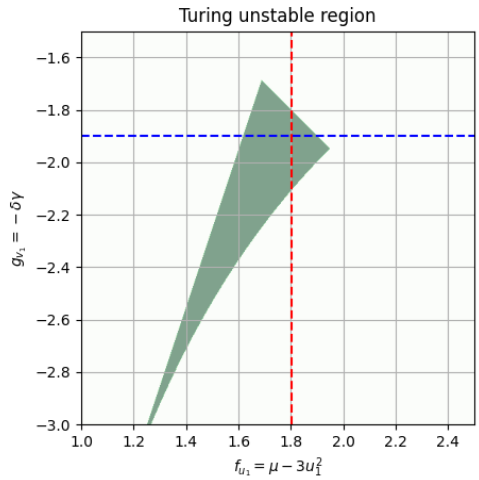

Figure 2: All the points in the green region of the $(f_{u_1} g_{v_1})$-plane, which satisfies the conditions $(1)-(5)$ of Theorem 5.1. The parameters are $p = 2, \mu = 2.66, \beta = 0.1, \delta = 0.5, \gamma = 3.8,\ d = 3,\ L = 9$, and $(u_1, v_1) = (-0.535, -1.270)$.

6 Numerical approximations of Turing patterns

This section presents numerical approximations of Turing patterns associated with specific p-adic FitzHugh-Nagumo systems. By suitable choosing of the parameters ($\mu, \gamma, \delta, \beta, d$, with $d > 1$), we find a region where the conditions (1)-(5) of Theorem 5.1 are satisfied. Then we solve numerically the system of ODEs (5.4). Finally, we give various visualizations of the solutions intending to show several aspects of the Turing patterns. To construct a region (called the Turing unstable region), we use an (f_{u_1}, g_{v_1}) plane, i.e., we set

$$x = f_{u_1} = \mu - 3u_1^2, \quad y = g_{v_1} = -\gamma\delta.$$

Figure 2 shows a Turing unstable region associated with a steady state of system (5.4). The parameters ($\mu, \gamma, \delta, \beta, d$, with $d > 1$) that give rise to green points in Figure 2 correspond to some Turing pattern.

The last condition in Theorem 5.1 is shown in the left part of Figure 3. More precisely, the eigenvalues of matrix A_L^α between the dotted lines (which represent the values κ_1, κ_2) satisfy condition (6) in Theorem 5.1. The right part of Figure 3 shows the eigenvalues of operator $\boldsymbol{D}_0^\alpha - \lambda$, see Section 3.3. For L sufficiently large, the eigenvalues of A_L^α approach to the ones of $\boldsymbol{D}_0^\alpha - \lambda$, such it was discussed at the end of Section 5.

Figures 4 and 5 show the Turing patterns, which are solutions of the Cauchy problem associated with system (5.4), with an initial datum close to (u_1, v_1), for t sufficiently large. Figure 4 shows that the activator $u_L(I, t)$ behaves a wave for t large, while the inhibitor $v_L(I, t)$ has a negligible oscillation. This fact is shown very clearly in Figure 6.

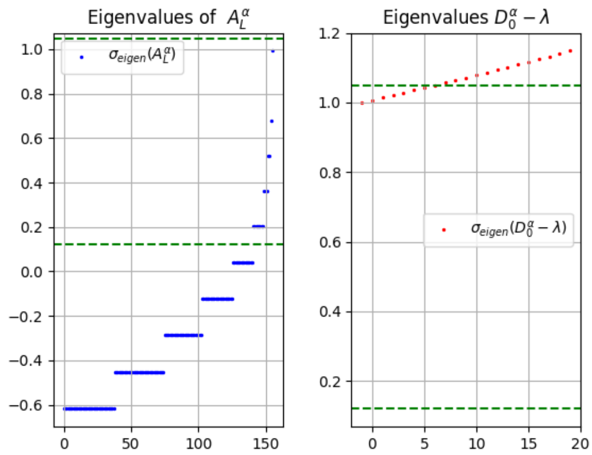

Figure 3: The left part of the figure shows the first 150 eigenvalues of the matrix A_L^α, which is a discretization of the Vladimirov operator $\boldsymbol{D}_0^\alpha - \lambda$. The right part of the figure shows the first 20 eigenvalues of $\boldsymbol{D}_0^\alpha - \lambda$. Notice that eigenvalue λ is very close to 1.

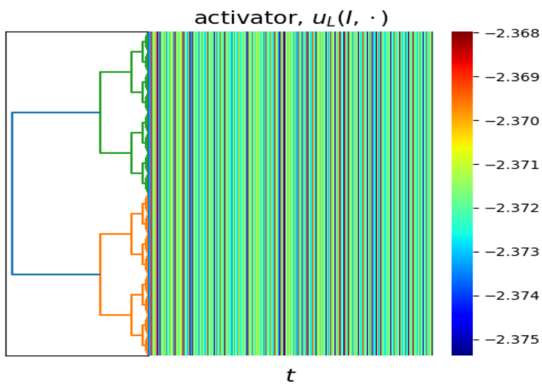

Figure 4: The activator states $u_L(I, \cdot)$ for $2000 < t < 10000$, and $L = 9$. The vertical scale runs through the points of tree G_9.

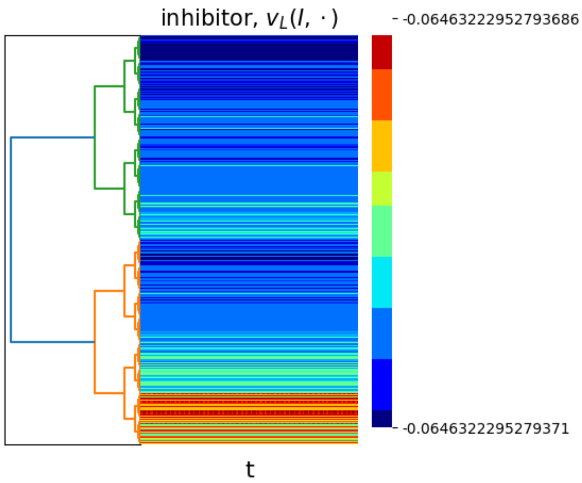

Figure 5: The inhibitor states $u_L(I, \cdot)$ for $2000 < t < 10000$, and $L = 9$. The vertical scale runs through the points of tree G_9. The oscillation of the values of the inhibitor states is negligible.

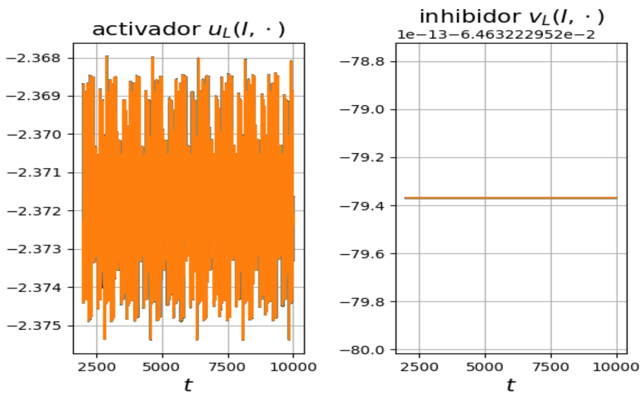

Figure 6: This figure shows the evolution of all the states of the system (5.4) for time $2000 < t < 10000$. At time $t = 0$, the initial datum for the Cauchy problem is $(\widetilde{u}_1, \widetilde{v}_1)$, where $\widetilde{u}_1$ is sample of a Gaussian variable with mean u_1 and variance 0.1, and $\widetilde{v}_1$ is sample of a Gaussian variable with mean v_1 and variance 0.1. For any initial state $(\widetilde{u}_1, \widetilde{v}_1)$, the system (5.4) develops the Turing pattern showed in this figure. In the right figure, the vertical scale takes values in the interval $\left[1 \times 1^{-13}, 6463222952 \times 1^{-2}\right]$.

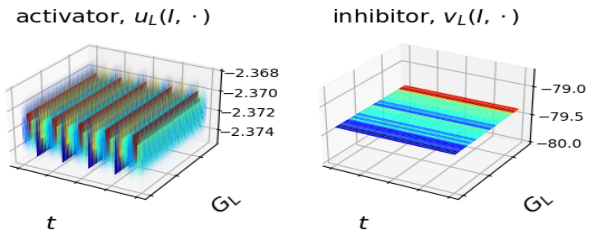

Figure 7: This figure is a 3D version of Figure 6. It shows the evolution of all the states of the system (5.4) for time $2000 < t < 10000$. The initial datum for the Cauchy problem chose as in Figure 6. The Turing pattern is a traveling wave.

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