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Matrix Completion Problems for the Positiveness and Contraction Through Graphs

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MATRIX COMPLETION PROBLEMS FOR THE POSITI-VENESS AND CONTRACTION THROUGH GRAPHS

A Thesis

by

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VENESS AND CONTRACTION THROUGH GRAPHS

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ABSTRACT

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In this work, we study contractive and positive real symmetric matrix completion problems which are motivated in part by studies on sparce (or dense) matrices for weighted sparse recovery problems and rating matrices with rating density in recommender systems. Matrix completions problems also have many applications in probability and statistics, chemistry, numerical analysis (e.g. optimization), electrical engineering, and geophysics. In this paper we seek to connect the contractive and positive completion property to a graph theoretic property. We then answer whether the graphs of real symmetric matrices having loops at every vertex have the contractive completion property if and only if the graph of said matrix is chordal. If this is not true, we characterize all graphs of real symmetric matrices having the contractive completion property.

DEDICATION

This thesis is dedicated to my father who has supported me financially throughout my academic career. I'm forever grateful for your support and sacrifices made so that I could pursue my passions.

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I would like to extend my sincere thanks to Dr. J. Yoon, for your willingness to undertake a first-semester master's student to take on this captivating endeavor. I appreciate you introducing me to the fascinating and important field of matrix completion problems and their graphs. I'm grateful to have had the opportunity to learn from you as my professor, thesis advisor, and mentor.

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CHAPTER I

INTRODUCTION

The purpose of this study is to consider the contractive and positive real symmetric matrix completion problems motivated in part by studies on sparse (or dense) matrices for weighted sparse recovery problems and rating matrices with rating density in recommender systems. Matrix completion problems have been studied by G. Arsene and A. Gheondea, by C. Davis, W. Kahan and H. Weinberger, by C. Foiaş and A. Frazho (using Redheffer products), by S. Parrott, and by Y. L. Shmul'yan and R. N. Yanovskaya; a solution is also implicit in the work of W. Arveson. A matrix completion problem has many applications in probability and statistics (e.g. entropy methods for missing data), chemistry (e.g. the molecular conformation problem), numerical analysis (e.g. optimization), electrical engineering (e.g. data transmission, coding and image enhancement) and geophysics (seismic reconstruction problems). In recent years, graphs and digraphs have been used very effectively to study matrix completion problems. The question that any partial positive definite matrix specifying a pattern can be completed to a positive definite matrix was studied through the use of graph theoretic techniques.

A *partial matrix* is a square array in which some entries are defined (or specified) and others are not. A *completion* of a partial matrix is a choice of values for the unspecified entries. A *matrix completion problem* asks whether a partial matrix has a completion of a specific type (or a pattern, see the detailed definition given below), such as a positive definite matrix. If a partial matrix of a specific type has a completion, then we say that the partial matrix has *the specific type completion property* (or it is *soluble*). A partial matrix of a specific type is called *well-posed* if every completely determined submatrix of it is of the specific type. Let A be an $m \times n$ matrix. We call A a *contractive* matrix or a *contraction* if the operator norm of A does not exceed 1. The *contractive symmetric completion problem* (CSCP) asks which partial contractive symmetric matrices have a symmetric contractive completion.

In linear algebra, an *inner product space* is a vector space with an additional structure called an inner product. A simple example is the real numbers with the standard multiplication as the *inner product*: $\langle x,y\rangle := xy$, where $x,y \in R$. More generally, the real 2-space R^2 with the dot product is an inner product space, an example of a Euclidean 2-space:

$$\langle x,y\rangle := x_1y_1 + x_2y_2,$$

where $\overrightarrow{x} = (x_1,x_2)$, and $\overrightarrow{y} = (y_1,y_2)$.

We say that A is *positive semidefinite* $(A \geq 0)$ if $\langle A\overrightarrow{x},\overrightarrow{x}\rangle \geq 0$ for all $\overrightarrow{x} \in R^2$. We say that A is *positive* $(A > 0)$ if $\langle A\overrightarrow{x},\overrightarrow{x}\rangle > 0$ for all $\overrightarrow{x} \in R^2$.

Let $M_n(C)$ be a collection of square $n \times n$ complex matrices. For a matrix $A \in M_n(C)$, we say that A is *symmetric* if $A^t = A$. We define the *interchange of rows (resp. column)* of A to be the interchange of one row (resp. column) of the matrix A with another row of the matrix A. A *principal minor* of $A \in M_n(C)$ is the determinant of a submatrix of A that is obtained by deleting some (or none) of its rows as well as the corresponding columns. For matrices $A,B \in M_n(C)$, we let $A \circ B$ denote their *Schur product (*also called *Hadamard product)* , where $(A \circ B)_{i,j} := (A)_{i,j}\,(B)_{i,j}$ for $1 \leq i,j \leq n$. The following result is well known: If $A \geq 0$ and $B \geq 0$, then $A \circ B \geq 0$ and $\det(A \circ B) \geq (\det A)(\det B)$.

We recall the notion of graphs. We will denote by $G = (V,E) = (V(G),E(G))$ a finite (undirected) graph. The set $V(G)$ of *vertices* is finite, and the set $E(G)$ of *edges* is a subset of the set $\{\{i,j\} : i,j \in V(G)\}$. We allow that E may contain loops, i.e., i may equal j for an edge $\{i,j\} \in E$. Two vertices connected by an edge are said to be *adjacent.* Notice that two vertices may be connected by more than one edge, a vertex need not be connected to any other vertex, and a vertex may be connected to itself (a loop). A *walk* in a graph G is a finite or infinite sequence of edges which joins a sequence of vertices. A *trail* is a walk in which all edges are distinct. A *path* is a trail in which all vertices are distinct. The *order* of G is the number of vertices of G. A *subgraph* of the graph G is a graph $H = (V(H), E(H))$, where $V(H)$ is a subset of $V(G)$ and $E(H)$ is a subset of $E(G)$ (note that $\{i, j\} \in E(H)$ requires $i, j \in V(H)$ since H is a graph). Let $A_n = [a_{ij}]$ be a symmetric $n \times n$ matrix. The nonzero-graph $G(A_n) = (V_n, E)$ of A_n is the graph having as vertex set $V_n = \{1, \cdots, n\}$ and as edge set $E = \{\{i, j\} : i, j \in V_n\}$ with the property that an (undirected) edge $\{i, j\}$ occurs in $G(A_n)$ if and only if the entries a_{ij} and a_{ji} of A_n are specified. Define a *partial graph* $[G(A_n)]$ as a subgraph of $G(A_n)$, where $\{i, j\} \in E$ if and only if $\{j, i\} \in E$ (so a_{ij} in a partial matrix of A_n is defined if and only if a_{ji} is). A *clique* is a subset $C \in V$ having the property that $\{i, j\} \in E$ for all $i, j \in C$. A *cycle* in G is a sequence of pairwise distinct vertices $\gamma = \{v_1, v_2, \cdots, v_s\}$ having the property that $\{v_1, v_2\}, \{v_2, v_3\}, \cdots, \{v_{s-1}, v_s\}, \{v_s, v_1\} \in E$, where $v_i \in V$ and $i \in \{1, \cdots, s\}$, where s is referred to as the *length* of the cycle. A *chord* of the cycle γ is an edge $\{v_i, v_j\} \in E$, where $1 \leq i < j \leq s$, $\{i, j\} \neq \{1, s\}$, and $|i - j| \geq 2$.

On this note, we also study the CSCP using a graph theoretic tool. Naturally, some questions arise about the CSCP using a graph theoretic tool.

For $n \in N$ with $n \geq 2$, let

$$\begin{aligned} S_n &= S_n(a_1, a_2, \cdots, a_n; x_1, \cdots, x_{\frac{(n-1)n}{2}}) \\ &:= \begin{pmatrix} a_1 & x_1 & x_2 & \cdots & x_{\frac{(n-2)(n-1)}{2}+1} \\ x_1 & a_2 & x_3 & \cdots & x_{\frac{(n-2)(n-1)}{2}+2} \\ x_2 & x_3 & \ddots & \ddots & \vdots \\ \vdots & \cdots & \ddots & a_{n-1} & x_{\frac{(n-1)n}{2}} \\ x_{\frac{(n-2)(n-1)}{2}+1} & x_{\frac{(n-2)(n-1)}{2}+2} & \cdots & x_{\frac{(n-1)n}{2}} & a_n \end{pmatrix} \end{aligned} \tag{1.1}$$

be real symmetric matrices, where the diagonal entries $a = \{a_i\}_{i=1}^n \subset R$ are specified and unknown variables $x_1, \cdots, x_{\frac{(n-1)n}{2}}$ are to be determined. We consider some specified values $b = \{b_{k_i}\}_{i=1}^{\ell} \subset R$ with $|b_{k_i}| \leq 1$ and $\ell \leq \frac{(n-1)n}{2}$. After allotting some specified values $\{b_{k_i}\}_{i=1}^{\ell}$ for the unknown variables $x = \{x_{k_i}\}_{i=1}^{\ell}$ in (1.1), the new partial matrix for S_n is denoted by

$$S_n(a;b,x) := S_n(a_1,\cdots,a_n;x_1,\cdots,x_{k_1-1},b_{k_1},x_{k_1+1},\cdots,x_{k_i-1},b_{k_i},x_{k_i+1},\cdots,x_{\frac{n(n-1)}{2}}).$$

We say that $G(S_n(a;b,x))$ is *soluble* if $S_n(a;b,x)$ is soluble.

The following result is well-known in the paper “Positive definite completions of partial Hermitian matrices, *Linear Algebra Appl.* 58(1984) 109-124 by R. Grone, C.R. Johnson, E.M. Sa, and H. Wolkowicz":

Theorem: A graph having a loop at every vertex has the positive definite completion property if and only if it is chordal (any cycle of length ≥ 4 has a chord).

For $a \in C$, we see that $|a| \leq 1 \Longleftrightarrow 1-\bar{a}a \geq 0$. Similarly, for any $n \times n$ matrix A, we can observe that

$$\|A\| \leq 1 \Longleftrightarrow I - A^*A \geq 0, \tag{1.2}$$

where I is the $n \times n$ identity matrix and A^* is the conjugate transpose of A (see Lemma 13). We expect that $I - A^*A$ does not have the same structure as A. However, motivated by the above Theorem and (1.2), in this note, we try to connect the contractive and positive completion property to a graph theoretic property as follows:

Problem A

(i) Is it true that the graph $G(S_n)$ $(n \geq 4)$ of a real symmetric matrix S_n with n vertices having a loop at every vertex has the contractive completion property if and only if it is chordal?

If (i) is not true, then we consider:

(ii) Let $n \geq 1$. Characterize all graphs $G(S_n)$ of S_n having the contractive completion property.

The following is an example of the nonzero-graph $G(A_5) = (V_5, E)$ of A_5 :

$$A_5 = \begin{pmatrix} 1 & x_1 & x_2 & -\frac{1}{6} & x_4 \\ x_1 & \frac{1}{2} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} \\ x_2 & \frac{1}{6} & \frac{1}{3} & x_3 & x_5 \\ -\frac{1}{6} & \frac{1}{7} & x_3 & \frac{1}{4} & x_6 \\ x_4 & \frac{1}{8} & x_5 & x_6 & \frac{1}{5} \end{pmatrix}$$

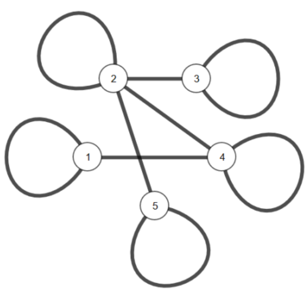

Figure 1.1: Graph $G(A_5)$

CHAPTER II

MAIN RESULTS

We first recall Sylvester's Criterion for Positive Semidefinite Matrices: A matrix is positive semidefinite if and only if all of its principal minors are non-negative.

Recall: For $a \in C$, we see that

$$|a| \leq 1 \Longleftrightarrow a\overline{a} \leq 1 \Longleftrightarrow \begin{pmatrix} 1 & a \\ \overline{a} & 1 \end{pmatrix} \geq 0. \tag{2.1}$$

Similarly, for any $n \times n$ matrix A, we can observe that:

Theorem 1: For any $n \times n$ matrix A_n, we have

$$\|A_n\| \leq 1 \Longleftrightarrow \begin{pmatrix} I_n & A_n \\ A_n^* & I_n \end{pmatrix} \geq 0 \Longleftrightarrow I_n - A_n^* A_n \geq 0, \tag{2.2}$$

where I_n is the $n \times n$ identity matrix and A_n^* is the conjugate transpose of A_n.

Proof of Theorem 1: We prove it using Mathematical Induction.

If $n = 1$, then by (2.1), (2.2) is true.

For $n = 2$, we let λ_1 and λ_2 be the eigenvalues of A_2. Since $\|A_2\| \leq 1$, $\lambda_1^2, \lambda_2^2 \leq 1$. By (2.1), we have that for all $1 \leq i \leq 2$

$$\begin{pmatrix} 1 & \lambda_i \\ \lambda_i & 1 \end{pmatrix} \geq 0$$

Hence we get that

$$E_2 := \begin{pmatrix} 1 & \lambda_1 & 0 & 0 \\ \lambda_1 & 1 & 0 & 0 \\ 0 & 0 & 1 & \lambda_2 \\ 0 & 0 & \lambda_2 & 1 \end{pmatrix} \geq 0$$

$$\Longleftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \lambda_1 & 0 & 0 \\ \lambda_1 & 1 & 0 & 0 \\ 0 & 0 & 1 & \lambda_2 \\ 0 & 0 & \lambda_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \geq 0$$

$$\Longleftrightarrow \begin{pmatrix} I_2 & S_2 \\ S_2 & I_2 \end{pmatrix} \geq 0, \text{ where } S_2 := \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \text{ and } \lambda_1 \geq \lambda_2.$$

Let U_2 and V_2 be unitary matrices such that $U_2^* U_2 = V_2^* V_2 = I_2$. Note that

$$\begin{pmatrix} I_2 & S_2 \\ S_2 & I_2 \end{pmatrix} \geq 0 \Longleftrightarrow \begin{pmatrix} U_2 & 0 \\ 0 & V_2^* \end{pmatrix} \begin{pmatrix} I_2 & S_2 \\ S_2 & I_2 \end{pmatrix} \begin{pmatrix} U_2^* & 0 \\ 0 & V_2 \end{pmatrix} \geq 0$$

$$\Longleftrightarrow \begin{pmatrix} I_2 & U_2 S_2 V_2 \\ V_2^* S_2 U_2^* & I_2 \end{pmatrix} \geq 0$$

$$\Longleftrightarrow \begin{pmatrix} I_2 & A_2 \\ A_2^* & I_2 \end{pmatrix} \geq 0, \text{ where } A_2 := U_2 S_2 V_2.$$

$$\Longleftrightarrow I_2 - A_2^* A_2 \geq 0.$$

Suppose that $n = k$ is true, that is,

$$\|A_k\| \leq 1 \Longleftrightarrow I_k - A_k^* A_k \geq 0.$$

Now, for $n=k+1$, we let $\lambda_1,\ldots,\lambda_{k+1}$ be the eigenvalues of A_{k+1} with $\lambda_1^2,\ldots,\lambda_{k+1}^2 \le 1$. By (2.1) again, we have

$$\lambda_1^2,\ldots,\lambda_{k+1}^2 \le 1 \Longleftrightarrow \begin{pmatrix} 1 & \lambda_i \\ \lambda_i & 1 \end{pmatrix} \ge 0 \text{ for all } 1 \le i \le k+1$$

$$\Longleftrightarrow E_{k+1} := \begin{pmatrix} \begin{pmatrix} 1 & \lambda_1 \\ \lambda_1 & 1 \end{pmatrix} & 0 & \cdots & 0 \\ 0 & \begin{pmatrix} 1 & \lambda_2 \\ \lambda_2 & 1 \end{pmatrix} & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & 0 & \begin{pmatrix} 1 & \lambda_{k+1} \\ \lambda_{k+1} & 1 \end{pmatrix} \end{pmatrix} \ge 0.$$

By changing the rows and columns of E_{k+1}, that is, we multiply suitable elementary matrices to the front and back of E_{k+1}, simultaneously, we obtain the matrix $\begin{pmatrix} I_{k+1} & S_{k+1} \\ S_{k+1} & I_{k+1} \end{pmatrix} \ge 0$, where $S_{k+1} := \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_{k+1} \end{pmatrix} = S_k \oplus \lambda_{k+1}$ and $\lambda_i \ge \lambda_j$ with $j \ge i$. Let $U_{k+1} = U_k \oplus u_{k+1}$ and $V_{k+1} = V_k \oplus v_{k+1}$ be unitary matrices such that $U_{k+1}^* U_{k+1} = V_{k+1}^* V_{k+1} = I_{k+1} = I_k \oplus 1$ with $u_{k+1}^* u_{k+1} = v_{k+1}^* v_{k+1} = 1$. Observe that

$$\begin{pmatrix} I_{k+1} & S_{k+1} \\ S_{k+1} & I_{k+1} \end{pmatrix} \ge 0$$

$$\Longleftrightarrow \begin{pmatrix} U_{k+1} & 0 \\ 0 & V_{k+1}^* \end{pmatrix} \begin{pmatrix} I_{k+1} & S_{k+1} \\ S_{k+1} & I_{k+1} \end{pmatrix} \begin{pmatrix} U_{k+1}^* & 0 \\ 0 & V_{k+1} \end{pmatrix} \ge 0$$

and

$$\begin{pmatrix} I_{k+1} & S_{k+1} \\ S_{k+1} & I_{k+1} \end{pmatrix} \geq 0$$

$$\Longleftrightarrow \begin{pmatrix} U_k \oplus u_{k+1} & 0 \\ 0 & V_k^* \oplus v_{k+1}^* \end{pmatrix} \begin{pmatrix} I_k \oplus 1 & S_k \oplus \lambda_{k+1} \\ S_k \oplus \lambda_{k+1} & I_k \oplus 1 \end{pmatrix} \begin{pmatrix} U_k^* \oplus u_{k+1}^* & 0 \\ 0 & V_k \oplus v_{k+1} \end{pmatrix} \geq 0$$

$$\Longleftrightarrow \begin{pmatrix} U_k U_k^* \oplus u_{k+1} u_{k+1}^* & U_k S_k V_k \oplus u_{k+1} \lambda_{k+1} v_{k+1} \\ V_k^* S_k U_k^* \oplus v_{k+1}^* \lambda_{k+1} u_{k+1}^* & V_k^* V_k \oplus v_{k+1}^* v_{k+1} \end{pmatrix} \geq 0$$

$$\Longleftrightarrow \begin{pmatrix} I_{k+1} & U_{k+1} S_{k+1} V_{k+1} \\ V_{k+1}^* S_{k+1} U_{k+1}^* & I_{k+1} \end{pmatrix} \geq 0$$

$$\Longleftrightarrow \begin{pmatrix} I_{k+1} & A_{k+1} \\ A_{k+1}^* & I_{k+1} \end{pmatrix} \geq 0, \text{ where } A_{k+1} := U_{k+1} S_{k+1} V_{k+1}$$

$$\Longleftrightarrow I_{k+1} - A_{k+1}^* A_{k+1} \geq 0.$$

Therefore, $\|A_n\| \leq 1 \Longleftrightarrow I_n - A_n^* A_n \geq 0$ is true for all $n \in N$.

In view of Theorem 1, the following problem is of interest:

Question: Is it true that the graph $G(S_n)$ of a real symmetric matrix S_n with n vertices having a loop at every vertex has the contractive completion property if and only if it is chordal?

Theorem 2 For $n \geq 2$, S_n has the positive completion property if and only if $\prod_{i=1}^{n} a_i > 0$.

Proof of Theorem 2 $(\Longrightarrow)$: We prove it using Mathematical Induction.

For $n = 2$, we let $a_1 a_2 = 0$. Without loss of generality, we let $a_1 = 0$. Then $S_2 = \begin{pmatrix} 0 & x_1 \\ x_1 & a_2 \end{pmatrix} \geq 0$. If $x_1 \neq 0$, then $\det S_2 = -x_1^2 < 0$ and S_2 is not positive and it is a contradiction to our assumption. Thus, $x_1 = 0$ and $S_2 \notin M_2(C)$ which drives a contradiction. Therefore the case $a_1 a_2 = 0$ cannot occur.

Let $a_1 a_2 < 0$. Then either $a_1 < 0$ or $a_2 < 0$ but not both. By Sylvester's Criterion for

Positive Semidefinite Matrices, the conditions $a_1 \geq 0$ and $a_2 \geq 0$ must be satisfied. Therefore, $a_1a_2 < 0$ cannot occur. Therefore, by the above two arguments, we have that $a_1a_2 > 0$.

For $n = 3$, we let $a_1a_2a_3 = 0$. Without loss of generality, we let $a_1 = 0$. By a similar argument given above, the case $a_1a_2a_3 = 0$ cannot occur. Similary, by Sylvester's Criterion for Positive Semidefinite Matrices and the above case, $a_1a_2a_3 < 0$ cannot occur. Therefore, by the above two arguments, we have that $a_1a_2a_3 > 0$.

Using Mathematical Induction, we assume that $n = k$ is true.

For $n = k+1$ we have the matrix

$$S_{k+1} = \begin{pmatrix} a_1 & x_1 & x_2 & \cdots & x_{\frac{(k-1)k}{2}+1} \\ x_1 & a_2 & x_3 & \cdots & x_{\frac{(k-1)k}{2}+2} \\ x_2 & x_3 & \ddots & \ddots & \vdots \\ \vdots & \cdots & \ddots & a_k & x_{\frac{k(k+1)}{2}} \\ x_{\frac{(k-1)k}{2}+1} & x_{\frac{(k-1)k}{2}+2} & \cdots & x_{\frac{k(k+1)}{2}} & a_{k+1} \end{pmatrix}.$$

If $\prod_{i=1}^{k+1} a_i \leq 0$, then we will get a contradiction.

Case 1: Suppose $\prod_{i=1}^{k+1} a_i = 0$. Since we know that $\prod_{i=1}^{k} a_i > 0$, the diagonal entry a_{k+1} must equal zero. By the Nested Determinants Test (or Choleski's Algorithm) property (iv) we have $x_1 = x_1 = \ldots = x_{\frac{(k-1)k}{2}+1} = 0$ since by assumption $S_{k+1} \geq 0$. But this is a contradiction since our S_{k+1} matrix is now our $k \times k$ matrix S_k. Therefore, suppose $\prod_{i=1}^{k+1} a_i = 0$ cannot occur.

Case 2: Suppose $\prod_{i=1}^{k+1} a_i < 0$. If $\prod_{i=1}^{k+1} a_i < 0$, then the entry $a_{k+1} < 0$ since we know that $\prod_{i=1}^{k} a_i > 0$. But, by Sylvester's Criterion for Positive Semi-Definite matrices, all principal minors must be non-negative and the diagonal entries $a_1, a_2, \ldots, a_{k+1}$ are principal minors of our S_{k+1} matrix. Thus, we have a contradiction since $\prod_{i=1}^{k+1} a_i < 0$.

Hence by Cases 1,2, $\prod_{i=1}^{k+1} a_i > 0$. Therefore, if S_n has the positive completion property, then $\prod_{i=1}^{n} a_i > 0$, as desired.

$(\Longleftarrow)$: Suppose that $\prod_{i=1}^{k+1} a_i > 0$. Let $x_1 = x_2 = \ldots = x_{\frac{k(k+1)}{2}} = 0$. Then $S_{k+1} \geq 0$, so for

$n \geq 2$, S_n has the positive completion property.

Theorem 3 For $n \geq 2$, consider S_n with $\prod_{i=1}^{n} a_i > 0$ and unknown x_i, where $1 \leq i \leq \frac{k(k+1)}{2}$. Then S_n has the positive completion property with $x_1 = x_2 = \ldots = x_{\frac{k(k+1)}{2}} = 0$.

Proof of Theorem 3 It is clear from Theorem 2 that S_n has the positive completion property, because $\prod_{i=1}^{n} a_i > 0$. If $x_1 = x_2 = \ldots = x_{\frac{k(k+1)}{2}} = 0$, then by Sylvester's Criterion for Positive Semi-Definite matrices, S_n has the positive completion property.

We observe that the following example gives a negative answer for Problem A, because the graph $G(S_4)$ of S_4 is chordal.

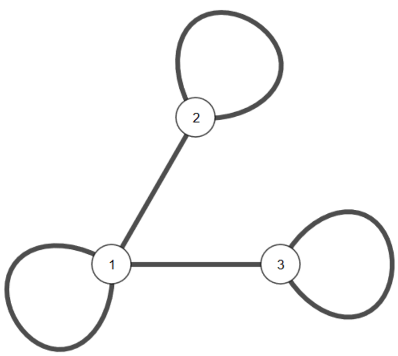

Figure 2.1: Graph $G(L_3)$ in Example 4.

Example 4 Let $L_3 := S_3(\frac{1}{2}, \frac{1}{2}, -\frac{7499}{15000}; \frac{1}{100}, \frac{\sqrt{7499}}{100}, x) = \begin{pmatrix} \frac{1}{2} & \frac{1}{100} & \frac{\sqrt{7499}}{100} \\ \frac{1}{100} & \frac{1}{2} & x \\ \frac{\sqrt{7499}}{100} & x & -\frac{7499}{15000} \end{pmatrix} \in M_3(R)$. Then L_3 is well-posed, but not soluble for any $x \in R$.

Proof of Example 4

Claim 1: L_3 is well-posed.

Proof of Claim 1: The absolute values of the diagonal entries of L_3 are less than 1. So it suffices to show that the following two submatrices are contractive.

$$L_3[1] := \begin{pmatrix} \frac{1}{2} & \frac{1}{100} \\ \frac{1}{100} & \frac{1}{2} \end{pmatrix} \text{ and } L_3[2] := \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{7499}}{100} \\ \frac{\sqrt{7499}}{100} & -\frac{7499}{15000} \end{pmatrix}$$

Since $P(L_3[1]) := I - (L_3[1])^* L_3[1] = \begin{pmatrix} \frac{7499}{10000} & -\frac{1}{100} \\ -\frac{1}{100} & \frac{7499}{10000} \end{pmatrix}$ and $\det(P(L_3[1])) = \frac{56225001}{100000000}$, by the Nested Determinants Test (or Choleski's Algorithm) in Lemma 14, $P(L_3[1])$ is positive semi-definite and so $L_3[1]$ is contractive by Lemma 13.

Similarly, since $P(L_3[2]) = \begin{pmatrix} \frac{1}{10000} & -\frac{\sqrt{7499}}{1500000} \\ -\frac{\sqrt{7499}}{1500000} & \frac{37499}{225000000} \end{pmatrix}$ and $\det(P(L_3[2])) = \frac{1}{75000000}$, by the Nested Determinants Test (or Choleski's Algorithm) again, $L_3[2]$ is contractive. Therefore, we prove **Claim 1**.

Claim 2: L_3 is not soluble for any $x \in R$.

Proof of Claim 2: Suppose that L_3 is soluble for some $x_0 \in R$. Then by Lemma 13, $P(L_3)$ is positive. Note that

$$P(L_3) = \begin{pmatrix} 0 & -\frac{\sqrt{7499}}{100}x_0 - \frac{1}{100} & -\frac{1}{100}x_0 - \frac{\sqrt{7499}}{1500000} \\ -\frac{\sqrt{7499}}{100}x_0 - \frac{1}{100} & \frac{7499}{10000} - x_0^2 & -\frac{1}{15000}x_0 - \frac{\sqrt{7499}}{10000} \\ -\frac{1}{100}x_0 - \frac{\sqrt{7499}}{1500000} & -\frac{1}{15000}x_0 - \frac{\sqrt{7499}}{10000} & \frac{37499}{225000000} - x_0^2 \end{pmatrix}.$$

By Lemma 14 (iii), the positivity of $P(L_3)$ implies that

$$-\frac{\sqrt{7499}}{100}x_0 - \frac{1}{100} = -\frac{1}{100}x_0 - \frac{\sqrt{7499}}{1500000} = 0$$

and

$$\begin{pmatrix} \frac{7499}{10000} - x_0^2 & -\frac{1}{15000}x_0 - \frac{\sqrt{7499}}{10000} \\ -\frac{1}{15000}x_0 - \frac{\sqrt{7499}}{10000} & \frac{37499}{225000000} - x_0^2 \end{pmatrix} \geq 0,$$

i.e.,

$$x_0 = -\frac{1}{\sqrt{7499}},\ x_0 = -\frac{\sqrt{7499}}{15000},\ \text{and } -0.008165 \leq x_0 \leq 0.008165,\ \text{simultaneously.}$$

This is a contradiction. So L_3 is not soluble for any $x \in R$. Therefore, by **Claims 1** and **2**, we have the desired results.

Remark 5 Let $K_3 := S_3(\frac{1}{2}, \frac{1}{2}, -\frac{7499}{15000}; \frac{1}{100}, x, \frac{\sqrt{7499}}{100}) = \begin{pmatrix} \frac{1}{2} & \frac{1}{100} & x \\ \frac{1}{100} & \frac{1}{2} & \frac{\sqrt{7499}}{100} \\ x & \frac{\sqrt{7499}}{100} & -\frac{7499}{15000} \end{pmatrix} \in M_3(R)$.
Then, we observe that K_3 and L_3 are essentially equivalent to each other (where essentially equivalent means that there exists a permutation matrix P_π such that $K_3 = P_\pi^{-1} L_3 P_\pi$, so K_3 is not soluble for any $x \in R$.

We can observe that the following example gives a negative answer for Problem A, because the graph $G(S_4)$ of S_4 is chordal.

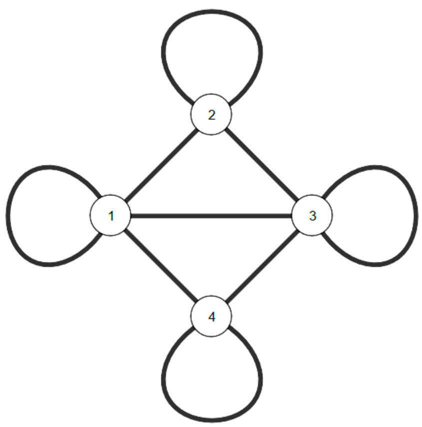

Figure 2.2: Graph $G(L_4)$ in Example 6.

Example 6 Let $L_4 := S_4(1, \frac{1}{2}, \frac{1}{2}, -\frac{7499}{15000}; 0, 0, 0, \frac{1}{100}, x, \frac{\sqrt{7499}}{100}) \in M_4(R)$, where

$$L_4 = \begin{pmatrix} 1 & 0 \\ 0 & (K_3) \end{pmatrix}.$$

Then L_4 is well-posed, but not soluble for any $x \in R$.

Proof of Example 6

Claim 1: S_4 is well-posed.

Proof of Claim 1:

The absolute values of the diagonal entries are less than 1. So it suffices to show that the 2×2 and 3×3 submatrices are contractive. They are as the following:

$$\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -\frac{7499}{15000} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & \frac{1}{100} \\ \frac{1}{100} & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{7499}}{100} \\ \frac{\sqrt{7499}}{100} & -\frac{7499}{15000} \end{pmatrix},$$
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{100} \\ 0 & \frac{1}{100} & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{\sqrt{7499}}{100} \\ 0 & \frac{\sqrt{7499}}{100} & -\frac{7499}{15000} \end{pmatrix}.$$

By the proof in Example 4, they are all contractive. So L_4 is well-posed.

Claim 2: L_4 is not soluble for any $x \in R$.

Proof of Claim 2:

By Examples 4, it is clear.

Thus, by **Claims 1** and **2,** L_4 is not soluble for any $x \in R$.

Next we have:

Theorem 7

Consider S_n $(n = 1,2,3)$ and some specified values $b_i \in R$ with $|b_i| \leq 1$ and $i \in \left\{1,2,\cdots,\frac{(n-1)n}{2}\right\}$. If S_n is well-posed, then we have:

a) If $n = 1,2$, then S_n is soluble.

b) If $G(S_3)$ is one of the two graphs (iii) and (iv) in Figure 2.3, then S_3 is always soluble.

c) If $G(S_3)$ is (v) in Figure 2.3, then S_3 can't be soluble for any values of x_i's.

Proof of Theorem 7

a) is clear.

For b) and c), we note that there are only seven cases, that is,

$$S_3(a_1,a_2,a_3;x_1,x_2,x_3),\ S_3(a_1,a_2,a_3;b_1,x_2,x_3),\ S_3(a_1,a_2,a_3;x_1,b_2,x_3),$$
$$S_3(a_1,a_2,a_3;x_1,x_2,b_3),\ S_3(a_1,a_2,a_3;b_1,b_2,x_3),\ S_3(a_1,a_2,a_3;b_1,x_2,b_3),$$
$$\text{and } S_3(a_1,a_2,a_3;x_1,b_2,b_3).$$

Now, we can see that

$$S_3(a_1,a_2,a_3;b_1,x_2,x_3),\ S_3(a_1,a_2,a_3;x_1,b_2,x_3), \text{ and } S_3(a_1,a_2,a_3;x_1,x_2,b_3) \tag{2.3}$$

are essentially equivalent and

$$S_3(a_1,a_2,a_3;b_1,b_2,x_3),\ S_3(a_1,a_2,a_3;b_1,x_2,b_3), \text{ and } S_3(a_1,a_2,a_3;x_1,b_2,b_3). \tag{2.4}$$

are essentially equivalent. Therefore, by Lemma 15, it suffices to consider three cases as follows:

$$S_3(a_1,a_2,a_3;x_1,x_2,x_3),\ S_3(a_1,a_2,a_3;b_1,x_2,x_3), \text{ and } S_3(a_1,a_2,a_3;b_1,b_2,x_3).$$

If $G(S_3)$ is one of the two graphs (iii) and (iv) in Figure 2.3, then S_3 is one of the following:

$$S_3(a_1,a_2,a_3;x_1,x_2,x_3) \text{ and } S_3(a_1,a_2,a_3;b_1,x_2,x_3).$$

If we put $x_i = 0$ $(i = 1,2,3)$, they all are contractive, so that S_3 is soluble.

If $G(S_3)$ is (v) in Figure 2.3. It follows from Example 4 and Remark 5 that $S_3(a_1,a_2,a_3;x_1,b_2,b_3)$ can't be soluble.

Therefore, we complete our proof.

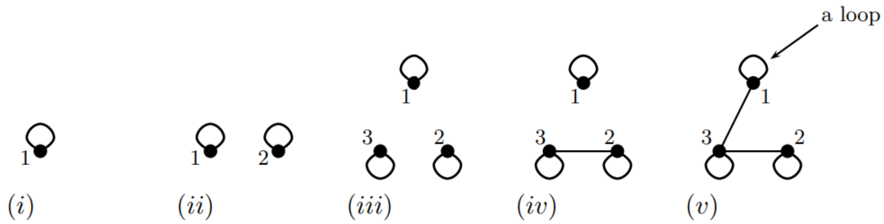

Figure 2.3: Graphs $G(S_n)$ of S_n $(n = 1, 2, 3)$

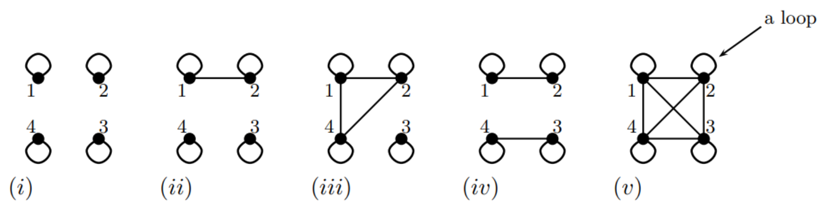

Figure 2.4: Graphs $G(S_4)$ of S_4 which are soluble in Theorem 8

Theorem 8

Consider $S_4 \in M_4(R)$ (or $G(S_4)$) and some specified values $b_i \in R$ with $|b_i| \leq 1$, where $i \in \{1, 2, \cdots, 6\}$. Let $G(S_4)$ be well-posed. Then, $G(S_4)$ is the disjoint union of completely connected components if and only if S_4 is soluble.

Proof of Theorem 8

$(\Longrightarrow)$

If $G(S_4)$ is the disjoint union of completely connected components, then $G(S_4)$ is one of the five graphs in Figure 2.4.

If the $G(S_4)$ is one of the four shapes as in Figure 2.4 (i)-(iv). Then S_4 of $G(S_4)$ is one of the following four types:

$$S_4(a_1, a_2, a_3, a_4; x_1, \cdots, x_6),\ S_4(a_1, a_2, a_3, a_4; b_1, x_2, \cdots, x_6),$$
$$S_4(a_1, a_2, a_3, a_4; b_1, b_2, x_3 \cdots, x_6),\ \text{or}\ S_4(a_1, a_2, a_3, a_4; x_1, x_2, x_3, b_4, b_5, b_6).$$

We let $x_i = 0$ for all unspecified $\{x_i\}$ in the above four types' matrices S_4. Then, by well-posedness, S_4 is a contraction, i.e., it is soluble.

If the graph of $G(S_4)$ is the shape as in Figure 2.4 (v). Then the S_4 of $G(S_4)$ is

$$S_4(a_1, a_2, a_3, a_4; b_1, b_2, b_3, b_4, b_5, b_6)$$

and it is clearly a contraction by well-posedness.

Therefore, if $G(S_4)$ is the disjoint union of completely connected components, then S_4 is soluble.

($\Longleftarrow$) Using the contrapositive, we will show that if $G(S_4)$ is not the disjoint union of completely connected components, then S_4 is not soluble. If $G(S_4)$ is not a disjoint union of completely connected components, then $G(S_4)$ doesn't belong to the graphs in Figure 2.4, so that $G(S_4)$ is one of graphs in Figures 2.5, 2.6, and 2.7.

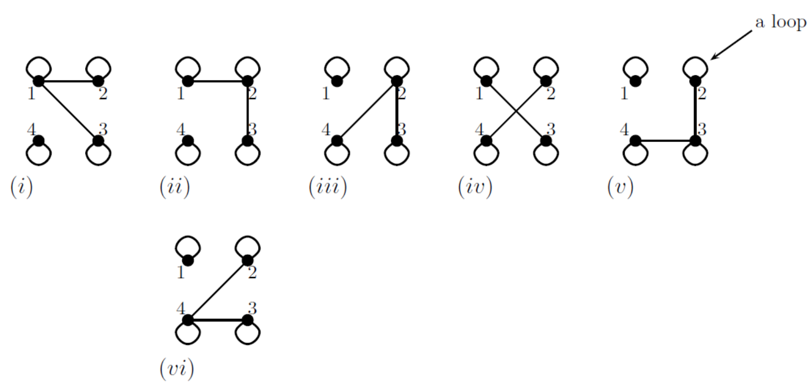

Figure 2.5: Graphs $G(S_4)$ of S_4 which are in **Case 1** in Theorem 8

Case 1: If $G(S_4)$ is a shape as in Figure 2.5, then S_4 is a matrix whose four $\{x_i\}$ are unspecified.

Subcase 1: By Lemma 15, we first observe that the graphs (i), (iii) and (vi) in Figure 2.5 are essentially equivalent. Let $G(S_4)$ be the shape as in Figure 2.5 (i). By Lemma 15, the matrix

S_4 of $G(S_4)$ is essentially equivalent to

$$N_1 = S_4(a_1, a_2, a_3, a_4; b_1, b_2, x_3, \cdots, x_6) = \begin{pmatrix} a_1 & b_1 & b_2 & x_3 \\ b_1 & a_2 & x_4 & x_5 \\ b_2 & x_4 & a_3 & x_6 \\ x_3 & x_5 & x_6 & a_4 \end{pmatrix}$$

whose submatrix $\begin{pmatrix} a_1 & b_1 & b_2 \\ b_1 & a_2 & x_4 \\ b_2 & x_4 & a_3 \end{pmatrix}$ may not be soluble by Theorem 7 (c). Therefore, N_1 is not soluble.

Subcase 2: By Lemma 15, we also observe that graphs (ii) and (v) in Figure 2.5 are essentially equivalent. Let $G(S_4)$ be the shape as in Figure 2.5 (ii). By Lemma 15, the matrix S_4 of $G(S_4)$ is essentially equivalent to

$$N_2 = S_4(a_1, a_2, a_3, a_4; b_1, b_4) = \begin{pmatrix} a_1 & b_1 & x_2 & x_3 \\ b_1 & a_2 & b_4 & x_5 \\ x_2 & b_4 & a_3 & x_6 \\ x_3 & x_5 & x_6 & a_4 \end{pmatrix}$$

whose submatrix $\begin{pmatrix} a_1 & b_1 & x_2 \\ b_1 & a_2 & b_4 \\ x_2 & b_4 & a_3 \end{pmatrix}$ may not be soluble by Theorem 7 (c). Therefore, N_2 is not soluble.

Subcase 3: Let the graphs of $G(S_4)$ be the shape as in Figure 2.5 (iv). By Lemma 15, the matrix S_4 of $G(S_4)$ is essentially equivalent to

$$N_3 = S_4(a_1, a_2, a_3, a_4; b_2, b_4) = \begin{pmatrix} a_1 & x_1 & b_2 & x_3 \\ x_1 & a_2 & b_4 & x_5 \\ b_2 & b_4 & a_3 & x_6 \\ x_3 & x_5 & x_6 & a_4 \end{pmatrix}$$

whose submatrix $\begin{pmatrix} a_1 & x_1 & b_2 \\ x_1 & a_2 & b_4 \\ b_2 & b_4 & a_3 \end{pmatrix}$ may not be soluble by Theorem 7 (c). Therefore, N_3 is not soluble.

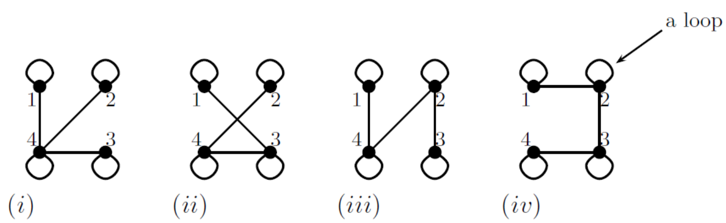

Figure 2.6: Graphs $G(S_4)$ of S_4 which are in **Case 2** in Theorem 8

Case 2: If $G(S_4)$ is a shape as in Figure 2.6, then S_4 is a matrix whose three $\{x_i\}$ are unspecified. **Subcase 4:** Let the graphs of $G(S_4)$ be the shape as in Figure 2.6 (i). By Lemma 15, the matrix S_4 of $G(S_4)$ is essentially equivalent to

$$N_4 = S_4(a_1, a_2, a_3, a_4; b_3, b_5, b_6) = \begin{pmatrix} a_1 & x_1 & x_2 & b_3 \\ x_1 & a_2 & x_4 & b_5 \\ x_2 & x_4 & a_3 & b_6 \\ b_3 & b_5 & b_6 & a_4 \end{pmatrix}$$

whose submatrix $\begin{pmatrix} a_2 & x_4 & b_5 \\ x_4 & a_3 & b_6 \\ b_5 & b_6 & a_4 \end{pmatrix}$ may not be soluble by Theorem 7 (c). Therefore, N_4 is not soluble.

Subcase 5: By Lemma 15, we note that the graphs (ii) and (iv) in Figure 2.6 are essentially equivalent. Let $G(S_4)$ be the shape as in Figure 2.6 (ii). By Lemma 15, the matrix S_4 of $G(S_4)$ is essentially equivalent to

$$N_5 = S_4(a_1, a_2, a_3, a_4; b_2, b_5, b_6) = \begin{pmatrix} a_1 & x_1 & b_2 & x_3 \\ x_1 & a_2 & x_4 & b_5 \\ b_2 & x_4 & a_3 & b_6 \\ x_3 & b_5 & b_6 & a_4 \end{pmatrix}$$

whose submatrix $\begin{pmatrix} a_2 & x_4 & b_5 \\ x_4 & a_3 & b_6 \\ b_5 & b_6 & a_4 \end{pmatrix}$ may not be soluble by Theorem 7 (c). Therefore, N_5 is not soluble.

Subcase 6: Let $G(S_4)$ be the shape as in Figure 2.6 (iii). By Lemma 15, the matrix S_4 of $G(S_4)$ is essentially equivalent to

$$N_6 = S_4(a_1, a_2, a_3, a_4; b_3, b_4, b_5) = \begin{pmatrix} a_1 & x_1 & x_2 & b_3 \\ x_1 & a_2 & b_4 & b_5 \\ x_2 & b_4 & a_3 & x_6 \\ b_3 & b_5 & x_6 & a_4 \end{pmatrix}$$

whose submatrix $\begin{pmatrix} a_2 & b_4 & b_5 \\ b_4 & a_3 & x_6 \\ b_5 & x_6 & a_4 \end{pmatrix}$ may not be soluble by Theorem 7 (c). Therefore, N_6 is not soluble.

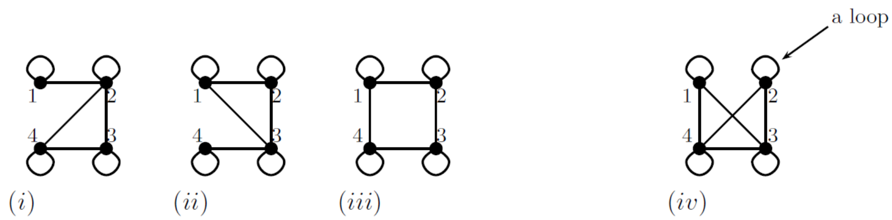

Figure 2.7: Graphs $G(S_4)$ of S_4 which are in **Cases 3,4** in Theorem 8

Case 3: If the graphs of $G(S_4)$ is a shape as in Figure 2.7 (i), (ii), (iii), then S_4 is a matrix whose two $\{x_i\}$ are unspecified.

Subcase 7: By Lemma 15, we note that the graphs (i) and (ii) in Figure 2.7 are essentially equivalent. Let $G(S_4)$ be the shape as in Figure 2.7 (i). By Lemma 15, the matrix S_4 of $G(S_4)$ is essentially equivalent to

$$N_7 = S_4(a_1, a_2, a_3, a_4; b_1, b_4, b_5, b_6) = \begin{pmatrix} a_1 & b_1 & x_2 & x_3 \\ b_1 & a_2 & b_4 & b_5 \\ x_2 & b_4 & a_3 & b_6 \\ x_3 & b_5 & b_6 & a_4 \end{pmatrix}$$

whose submatrix $\begin{pmatrix} a_1 & b_1 & x_2 \\ b_1 & a_2 & b_4 \\ x_2 & b_4 & a_3 \end{pmatrix}$ may not be soluble by Theorem 7 (c). Therefore, N_7 is not soluble.

Subcase 8: Let $G(S_4)$ be the shape as in Figure 2.7 (iii). By Lemma 15, the matrix S_4 of

$G(S_4)$ is essentially equivalent to

$$N_8 = S_4(a_1,a_2,a_3,a_4;b_1,b_3,b_4,b_6) = \begin{pmatrix} a_1 & b_1 & x_2 & b_3 \\ b_1 & a_2 & b_4 & x_5 \\ x_2 & b_4 & a_3 & b_6 \\ b_3 & x_5 & b_6 & a_4 \end{pmatrix}$$

whose submatrix $\begin{pmatrix} a_1 & b_1 & x_2 \\ b_1 & a_2 & b_4 \\ x_2 & b_4 & a_3 \end{pmatrix}$ may not be soluble by Theorem 7 (c). Therefore, N_8 is not soluble.

Case 4: Let $G(S_4)$ be the shape as in Figure 2.7 (iv). Then S_4 is a matrix whose one x_i is unspecified. By Lemma 15, the matrix S_4 of $G(S_4)$ is essentially equivalent to

$$N_9 = S_4(a_1,a_2,a_3,a_4;b_2,b_3,b_4,b_5,b_6) = \begin{pmatrix} a_1 & x_1 & b_2 & b_3 \\ x_1 & a_2 & b_4 & b_5 \\ b_2 & b_4 & a_3 & b_6 \\ b_3 & b_5 & b_6 & a_4 \end{pmatrix}$$

whose submatrix $\begin{pmatrix} a_1 & x_1 & b_2 \\ x_1 & a_2 & b_4 \\ b_2 & b_4 & a_3 \end{pmatrix}$ may not be soluble by Theorem 7 (c). Therefore, N_9 is not soluble.

Case 5: Let $G(S_4)$ be the shape as in Figure 2.4 (v). Then, all x_i's are specified. In this case, $N_5 = S_4(a_1,a_2,a_3,a_4;b_1,b_2,b_3,b_4,b_5,b_6)$ is clearly a contraction by well-posedness.

Therefore, by **Cases 1-5**, if S_4 is soluble, then $G(S_4)$ is the disjoint union of completely connected components, as desired.

End of the proof of Theorem 8

Theorem 9

Consider S_5 (resp. $G(S_5)$) and some specified values $b_i \in R$ with $|b_i| \leq 1$, where $i \in \{1, 2, \cdots, 10\}$. Let $G(S_5)$ be well-posed. Then,

(a) $G(S_5)$ is the disjoint union of completely connected components if and only if S_5 is soluble.

(b) If the graph of the partial matrix S_5 is not in Figure 2.8, then it can't be soluble for any values of x_i's.

Proof of Theorem 9

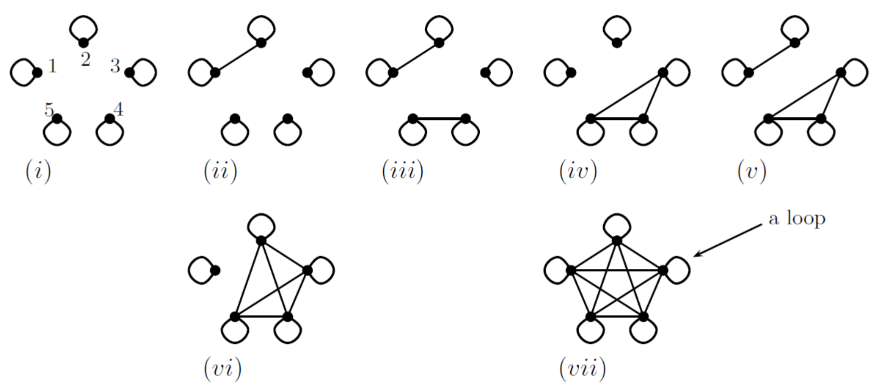

Figure 2.8: Graphs $G(S_5)$ of S_5 which are soluble in Theorem 9

($\Longrightarrow$) If $G(S_5)$ is the disjoint union of completely connected components, then $G(S_5)$ is one of the seven graphs in Figure 2.8.

Case 1: Let $G(S_5)$ be one of the graphs in Figure 2.8 (i), (ii), (iii), (iv), (v), and (vi). Then choose $x_i = 0$ for all $\{x_i\}$ and by well-posedness, S_5 is soluble.

Case 2: Let the graph $G(S_5)$ be like the graph in Figure 2.8 (vii). In this case, S_5 is clearly a contraction by well-posedness.

($\Longleftarrow$) We prove it by the contrapositive. If $G(S_5)$ is not a disjoint union of completely connected components, then $G(S_5)$ doesn't belong to the graphs in Figure 2.8.

Case 3: Only eight of the x_i's are unspecified in the matrix S_5: Since $G(S_5)$ is not a disjoint union of completely connected components, the graph $G(S_5)$ of the partial matrix S_5 is the second graph in Figure 2.9 (i) given below. By Theorem 7, we note that the submatrix corresponding to the second graph (in the dashed part with the lightgray color) in Figure 2.9 (i) is not soluble. Thus, S_5 whose graph is the second one in Figure 2.9 (i) is not soluble.

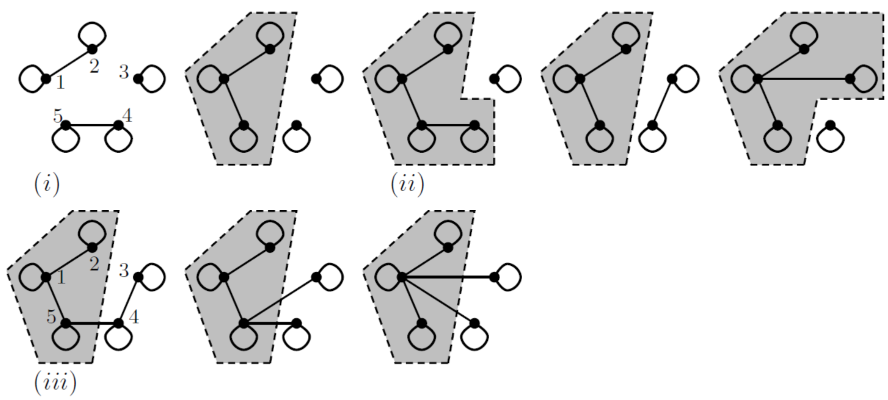

Figure 2.9: Graphs $G(S_5)$ of S_5 which are in **Cases 3,4,5** in Theorem 9

Case 4: Only seven of the x_i's are unspecified in the matrix S_5: The graph $G(S_5)$ of the partial matrix S_5 is one of the graphs in Figure 2.9 (ii) given above. By Theorem 8, the submatrix corresponding to the first graph (in the dashed part with the lightgray color) in Figure 2.9 (ii) is not soluble.

By Theorem 7, the submatrix corresponding to the second graph (in the dashed part with the lightgray color) in Figure 2.9 (ii) is not soluble.

By Theorem 8, the submatrix corresponding to the third graph (in the dashed part with the lightgray color) in Figure 2.9 (ii) is not soluble. Thus, S_5 whose graph is one of those in Figure 2.9 (ii) is not soluble.

Case 5: Only six of the x_i's are unspecified in the matrix S_5: The graph $G(S_5)$ of the partial matrix S_5 is one of the graphs in Figure 2.9 (iii) given above. With similar arguments used in **Case 4**, S_5 whose graph is one of those in Figure 2.9 (iii) is not soluble.

Case 6: Only five of the x_i's are unspecified in the matrix S_5:

The graph $G(S_5)$ of the partial matrix S_5 is one of the graphs in Figure 2.10 given below. By Theorem 7 or Theorem 8, since the subgraph corresponding to one of the graphs $G(S_5)$ (in the dashed part with the lightgray color) in Figure 2.10 is not soluble, S_5 whose graph is one of those in Figure 2.10 (i) is not soluble.

Case 7 Only four of the x_i's are unspecified in the matrix S_5:

The graph $G(S_5)$ of the partial matrix S_5 is one of the graphs in Figure 2.10 (ii).

Since $G(S_5)$ is not a disjoint union of completely connected components, $G(S_5)$ is the first, second, or fourth one in Figure 2.10. For the first, second, and fourth graphs of $G(S_5)$, the subgraph corresponding to one of the graphs of $G(S_5)$ (in the dashed part with the lightgray color) in Figure 2.10 is not soluble by Theorem 7 or Theorem 8. Hence, S_5 whose graph is one of the first, second, and fourth graphs in Figure 2.10 is not soluble.

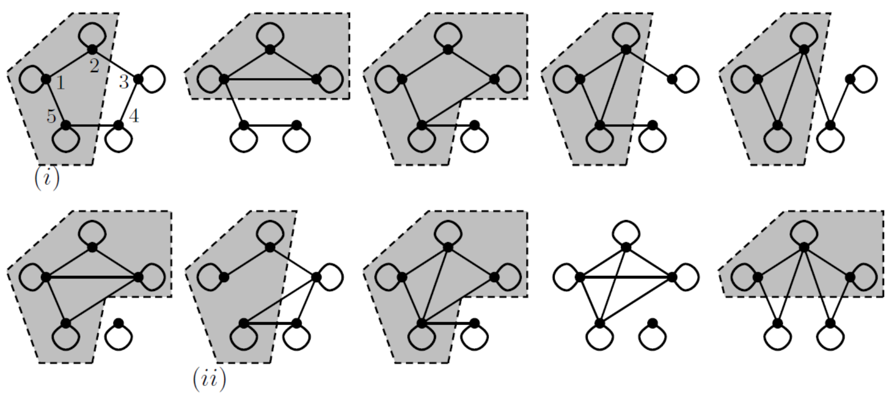

Figure 2.10: Graphs $G(S_5)$ of S_5 which are in **Cases 6,7** in Theorem 9

Case 8: Only three of the x_i's are unspecified in the matrix S_5:

The graph $G(S_5)$ of the partial matrix S_5 is one of the graphs in Figure 2.11 (i) given below. With the same arguments in **Case 7**, S_5 whose graph is one of the graphs in Figure 2.11 (i) is not soluble.

Case 9: Only two of the x_i's are unspecified in the matrix S_5:

The graph $G(S_5)$ of the partial matrix S_5 is one of the graphs in Figure 2.11 (ii) given above. Similarly, S_5 whose graph is one of the graphs in Figure 2.11 (ii) is not soluble.

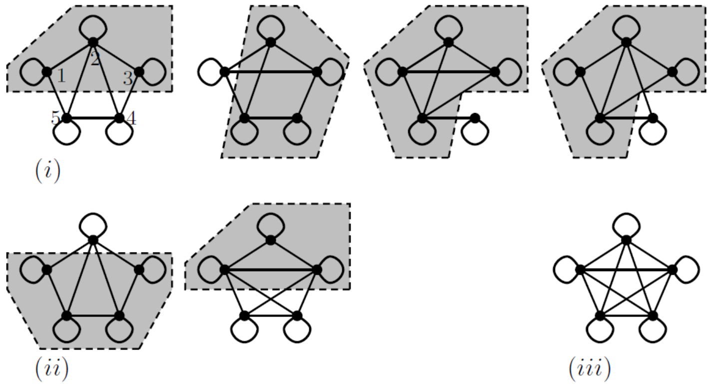

Figure 2.11: Graphs $G(S_5)$ of S_5 which are in **Cases 8,9,10** in Theorem 9

Case 10: Only one of the x_i's is unspecified in the matrix S_5:

In this case, all matrices of the graphs $G(S_5)$ are equivalent to $S_5(a_1, \cdots, a_5; x_1, b_2, b_3, \cdots, b_9, b_{10})$. By Theorem 8, since the submatrix $S_4(a_1, \cdots, a_4; x_1, b_2, b_3, b_5, b_6, b_8)$ is not soluble, S_5 which has only one unspecified x_i is not soluble. Therefore, by **Cases 3-10**, if S_5 is soluble, then $G(S_5)$ is the disjoint union of completely connected components, as desired.

End of the proof of Theorem 9

In view of Theorems 7, 8, and 9, we have:

Theorem 10

Consider S_n and some specified values $b_i \in R$ with $|b_i| \leq 1$ and $i \in \left\{1, 2, \cdots, \frac{(n-1)n}{2}\right\}$. Let $G(S_n)$ be well-posed. For $1 \leq n \leq 5$, the $G(S_n)$ is the disjoint union of completely connected components if and only if S_n is soluble.

Remark 11

We consider $L_5 := \begin{pmatrix} 1 & 0 \\ 0 & (L_4) \end{pmatrix} \in M_5(R)$, where L_4 is as in Example 6. Then L_5 is well-posed, but not soluble for any $x \in R$. Thus, we need to investigate Problem A (ii) for $G(S_5)$, i.e., characterize all graphs $G(S_5)$ having the contractive completion property. However, in comparison with S_n $(1 \leq n \leq 4)$, the possible cases of the well-posed partial matrices of S_5 are more than 1000, because

$$\sum_{n=1}^{10} \begin{pmatrix} 10 \\ n \end{pmatrix} = 1023.$$

We expect that we can solve the matrix completion problem for S_n, $n \geq 2$, if we use the graph theoretic method used in Theorems 8 and 9.

In view of Theorems 8 and 9, we have:

Conjecture 12

For $n \in N$, consider S_n and some specified values $b_i \in R$ with $|b_i| \leq 1$ and $i \in \left\{1, 2, \cdots, \frac{(n-1)n}{2}\right\}$. Let $G(S_n)$ be well-posed for some $x_i = b_i$. If the $G(S_n)$ is the disjoint union of completely connected components, then S_n is always soluble.

CHAPTER III

APPENDIX

For the reader's convenience, in this section, we gather several well known auxiliary results which are needed for the proofs of the main results in this article.

Lemma 13 For $n \times n$ matrix M is a contraction if and only if the matrix

$$P(M) := I - MM^* \tag{3.1}$$

is positive semi-definite (in symbols, $P(M) \geq 0$), where I is the identity matrix and M^* is the adjoint of M.

Recall the following version of the Nested Determinants Test (or Choleski's Algorithm).

Lemma 14 Assume

$$P := \left(p_{ij}\right)_{i,j=1}^{n} := \begin{pmatrix} u & \mathbf{t} \\ \mathbf{t}^* & P_0 \end{pmatrix},$$

where P_0 is an $(n-1) \times (n-1)$ matrix, $\mathbf{t}$ is a row vector, and u is a real number.

(i) If P_0 is invertible, then $\det P = \det P_0 (u - \mathbf{t} P_0^{-1} \mathbf{t}^*)$.

(ii) If P_0 is invertible and positive, then $P \geq 0 \Longleftrightarrow (u - \mathbf{t} P_0^{-1} \mathbf{t}^*) \geq 0 \Longleftrightarrow \det P \geq 0$.

(iii) If $u > 0$ then $P \geq 0 \Longleftrightarrow P_0 - \mathbf{t}^* u^{-1} \mathbf{t} \geq 0$.

(iv) If $P \geq 0$ and $p_{ii} = 0$ for some i, $1 \leq i \leq n$, then $p_{ij} = p_{ji} = 0$ for all $j = 1, \cdots, n$.

Lemma 15 For $A \in M_n(R)$ and any permutation matrix P_π, A is a contraction if and only if $P_\pi^{-1} A P_\pi$ is a contraction.

CHAPTER IV

FUTURE WORK AND OPEN QUESTIONS

We will investigate **Conjecture 12** and update it as follows:

For $n \in N$, consider S_n and some specified values $b_i \in R$ with $|b_i| \leq 1$ and $i \in \left\{1, 2, \cdots, \frac{(n-1)n}{2}\right\}$. Let $G(S_n)$ be well-posed for some $x_i = b_i$. Then we have that $G(S_n)$ is the disjoint union of completely connected components if and only if S_n is always soluble.

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BIOGRAPHICAL SKETCH

Louis Christopher (louiscchristopher@gmail.com) graduated with his Associate of Arts from Tarrant County College, then transferred to Chadron State College where he graduated Summa Cum Laude with his Bachelor of Science in Mathematics with a minor in applied statistics. While at Chadron State College Louis became a member of Kappa Mu Epsilon Mathematics Honor Society. In August 2023, Louis graduated with a Master of Science in Mathematics with a concentration in applied mathematics. During Louis' time at the University of Texas Rio Grande Valley he was a National Science Foundation Scholar where he chose to research matrix completion problems and their graphs. This was a natural fit as he has always been fascinated by the power of matrix algebra. Outside of his direct academics, Louis has been working as a Mathematics Instructional Associate at Tarrant County College. There he has helped hundreds of students understand math from basic algebra to statistics and differential equations. Louis enjoys being outdoors walking his dog Buddy, mountain biking, and riding motocross along with indoor activities such climbing, ice skating, and dancing.