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Two Reliable Efficient Methods for Solving Time-Fractional Coupled Klein-Gordon-Schrdinger Equations

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Abstract: In this paper, homotopy perturbation method and homotopy perturbation transform method have been implemented for solving time fractional coupled Klein-Gordon-Schrdinger equations. We first applied homotopy perturbation method for solving time fractional coupled Klein-Gordon-Schrdinger equations, which does not require a small parameter in the equations. Then we presented an algorithm of the homotopy perturbation transform method to solve coupled Klein-Gordon-Schrdinger equations. This paper establishs the effectiveness of the homotopy perturbation transformation method in solving fractional coupled Klein-Gordon-Schrdinger equations over homotopy Perturbation method. Here we obtain the solutions of fractional coupled Klein-Gordon-Schrodinger equations, which are obtained by replacing the time derivatives with a fractional derivatives of order $\alpha \in (1,2]$, $\beta \in (0,1]$ respectively. The results obtained by homotopy perturbation transform method are numerically and graphically compared with homotopy Perturbation method in order to exhibit the efficiency of the homotopy perturbation transformation method. The fractional derivatives here are described in Caputo sense.

Keywords: Homotopy perturbation method (HPM), Caputo Fractional Derivative, fractional coupled Klein-Gordon-Schrdinger(K-G-S) equation, Homotopy perturbation transform method (HPTM)

1 Introduction

In this paper, the time-fractional coupled nonlinear Klein-Gordon-Schrodinger equation is considered in the following form:

$$D_t^{\alpha} u - u_{xx} + u - |v|^2 = 0 \quad (1)$$

$$i D_t^{\beta} v + v_{xx} + uv = 0 \quad (2)$$

where $v(x,t)$represents a complex scalar nucleon field, $u(x,t)$ a real scalar meson field and $i = \sqrt{-1}$. They describe a system of conserved scalar nucleons interacting with neutral scalar meson coupled with Yukawa interaction [1]. Here α, β are the parameters standing for the order of the fractional derivatives which satisfy m-1 $< \alpha \leq m$, n -1 $<\beta \leq n$ where $m = 2$, $n = 1$ and $t > 0$. When $\alpha = 2$ and $\beta = 1$, the fractional equation reduces to the classical coupled K-G-S equation.
Fan et al [2] have been proposed an algebraic method to obtain the explicit exact solutions for coupled K-G-S equations. Recently, the Jacobi elliptic function expansion method has been applied to obtain the solitary wave solutions for coupled K-G-S equations [3]. Xia et al [4] has applied homogenous balance principle to obtain the exact solitary wave solutions of the K-G-S equations. Hioe [5] has obtained periodic solitary waves for two coupled nonlinear Klein–Gordon and Schrdinger equations. Bao and Yang [6] have presented efficient, unconditionally stable and accurate numerical methods for approximations of KGS equations.Recently, Naber [7] has constructed the time fractional Schrdinger equation which is solved for a free particle and for a potential well. In [8], fractional nonlinear Schrdinger equation has been solved by Adomian decomposition method. The modified decomposition method for the solution of integer order classical coupled K-G-S equation has been applied by Saha Ray [9].

In this paper HPM [10,11] and HPTM [12,13,14] have been applied for solving fractional coupled K-G-S equations which play an important role in quantum

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physics. The HPM does not depend upon a small parameter in the equation. On the other hand, HPTM is a combined form of the Laplace transform method with the homotopy perturbation method. The above methods find the solution without any discretization or restrictive assumptions and avoid the round-off errors.

This paper is organized as follows. In Section 2 some basic definitions of fractional calculus theory are given. In Section 3 and 4, the solution procedure and results of the HPM and HPTM are given respectively; we present the numerical simulations of proposed methods with error analysis Section 5 and Section 6 respectively. The conclusions are drawn in Section 7.

2 Mathematical preliminaries of fractional calculus

The fractional calculus involves different definitions of the fractional operators such as Riemann–Liouville fractional derivative, Caputo derivative, Riesz derivative and Grunwald–Letnikov fractional derivative. The fractional calculus has gained considerable importance during the past decades mainly due to its applications in diverse fields of science and engineering. For the purpose of this paper the Caputo definition of fractional derivative will be used, with regard to the advantage of Caputo approach that the initial conditions for fractional differential equations with Caputo derivatives take on the traditional form as for integer-order differential equations.

Definition 2.1.

A real function $f(t)$, t >0, is said to be in the space C_μ; $\mu \in \Re$ if there exists a real number $p(>\mu)$, such that $f(t)=t^p f_1(t)$, where $f_1(t)\in C[0,\infty)$, and it is said to be in the space $C_\mu^m, f^{(m)} \in C_m, m \in N$.

The Riemann–Liouville fractional integral operator is defined as follows:

Definition 2.2.

The most frequently encountered definition of an integral of fractional order is the Riemann–Liouville integral, in which the fractional integral of order $\alpha(>0)$ is defined as [15,16]

$$J_t^\alpha f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)\,d\tau, t>0, \alpha\in\Re^+ \tag{3}$$

$$J_t^0 f(t)=f(t)$$

where $\Re^+$ is the set of positive real numbers.

Properties of the operator J^α can be found in [14,15, 16] and we mention only the following: for $f\in C_\mu, \mu\geq 1$, $\alpha,\beta\geq 0$ and $\gamma\geq 1$, we have

1.$J^\alpha J^\beta f(t)=J^{\alpha+\beta}f(t)$
2.$J^\alpha J^\beta f(t)=J^\beta J^\alpha f(t)$
3.$J^\alpha t^\gamma=\frac{\Gamma(\gamma+1)\,t^{\gamma+\alpha}}{\Gamma(\alpha+\gamma+1)}$

Lemma 2.1 If $m-1<\alpha\leq m, m\in N$ and $f\in C_\mu^m, \quad \mu\geq 1$, then $D^\alpha J^\alpha f(t)=f(t)$ and

$$J^\alpha D^\alpha f(t)=f(t)-\sum_{i=0}^{m-1} f^{(i)}(0^+)\frac{t^i}{i!}, \quad t>0$$

Definition 2.3. (Caputo fractional derivative) The fractional derivative introduced by Caputo, is called Caputo Fractional Derivative. The fractional derivative of $f(t)$ in the Caputo sense is defined by [15,16]

$$D_t^\alpha f(t)=J^{m-\alpha}D^m f(t)=\frac{1}{\Gamma(m)}\int_0^t (t-\tau)^{m-\alpha-1} f^{(m)}(\tau)\,d\tau \tag{4}$$

for $m-1<\alpha\leq m$, $m\in N$, $t>0$, $f\in C_{-1}^m$

Definition 2.4.

For m to be the smallest integer that exceeds α, the caputo time-fractional derivative operator of order $\alpha\in\Re^+$is defined as

$$D_t^\alpha f(t)=J_t^{m-\alpha}f(t)= \begin{cases} \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1}\frac{d^m}{d\tau^m}f(\tau)d\tau & \text{if } m-1<\alpha<m, m\in N \\ \frac{d^m}{d\tau^m}f(\tau) & \text{if } \alpha=m, \quad m\in N \end{cases} \tag{5}$$

For the Caputo derivative we have following properties

1.$D^\alpha k=0$, (k is a constant)
2.$D^\alpha t^\gamma=\frac{\Gamma(\gamma+1)\,t^{\gamma-\alpha}}{\Gamma(\gamma-\alpha+1)} \quad \gamma>\alpha-1$

3 Basic idea of Homotopy Perturbation Method (HPM)

In this section to illustrate the basic idea of HPM [10,11], we consider the following nonlinear differential equation

$$A(u)-f(r)=0, r\in\Omega, \tag{6}$$

with the boundary condition

$$B\left(u,\frac{\partial u}{\partial n}\right)=0, r\in\Gamma, \tag{7}$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is known as analytical function, Γ is the boundary of domain Ω and $\frac{\partial}{\partial n}$ denotes differentiation along the normal drawn outwards from Ω .

A can be divided into two parts which are linear L and nonlinear N. Therefore eq.(7) can be rewritten as follows

$$L(u)+N(u)-f(r)=0. \tag{8}$$

We construct a homotopy of eq. (6)$v(r,p):\Omega\times[0,1]\rightarrow\Re$ which satisfies

$$H(v,p)=(1-p)[L(v)-L(u_0)]+p[A(v)-f(r)]=0,\quad p\in[0,1],r\in\Omega \quad (9)$$

which is equivalent to

$$H(v,p)=L(v)-L(u_0)+pL(u_0)+p[N(v)-f(r)]=0 \quad (10)$$

where $p\in[0,1]$ is an embedding parameter and u_0 is an initial approximation of eq.(6) which satisfies the boundary conditions. It follows from eqs. (9) and (10) that

$$H(v,0)=L(v)-L(u_0)=0,H(v,1)=A(v)-f(r)=0. \quad (11)$$

In topology $L(v)-L(u_0)$, $A(v)-f(r)$ are called homotopy. We assume the solution of eq. (10) can be written as a power series in p, as following

$$v=v_0+pv_1+p^2v_2+.... \quad (12)$$

The approximate solution of eq. (6) can be obtained as

$$u=\lim_{p\to1}v=v_0+v_1+v_2+.... \quad (13)$$

The converges of the series (13) has been discussed in [6, 7].

3.1 Implementation of the HPM method

We first consider the application of HPM for the solution of fractional coupled K-G-S eqs. (1) and (2) with given initial conditions

$$\begin{cases}u(x,0)=6B^2\sec h^2(Bx),\\ u_t(x,0)=-12B^2c\sec h^2(Bx)\tanh(Bx),\\ v(x,0)=3B\sec h^2(Bx)e^{idx}.\end{cases} \quad (14)$$

where $B(\geq 1/2)$, c and d are arbitrary constants with $c=\frac{\sqrt{4B^2-1}}{2}$, $d=-\frac{c}{2B}$ for fractional coupled K-G-S eqs.(1) and (2). Applying the Riemann–Liouville integral to the both sides of eqs. (1) and (2) respectively, we get

$$J_t^\alpha D_t^\alpha u=J_t^\alpha(u_{xx}-u+|v|^2) \quad (15)$$

$$J_t^\beta D_t^\beta v=iJ_t^\beta(v_{xx}+uv) \quad (16)$$

After simplification of eqs. (15) and (16), we get

$$u(x,t)=u(x,0)+t\,u_t(x,0)+J_t^\alpha(u_{xx}-u+|v|^2) \quad (17)$$

$$v(x,t)=v(x,0)+iJ_t^\beta(v_{xx}+uv) \quad (18)$$

By homotopy perturbation method, we will construct the homotopy of eqs. (17) and (18) as

$$u(x,t)=p\left(u(x,0)+t\,u_t(x,0)+J_t^\alpha\left(u_{xx}-u+|v|^2\right)\right) \quad (19)$$

$$v(x,t)=p(v(x,0)+iJ_t^\beta(v_{xx}+uv)) \quad (20)$$

By substituting, $u(x,t)=\sum_{n=0}^\infty p^nu_n(x,t)$ and $v(x,t)=\sum_{n=0}^\infty p^nv_n(x,t)$ in eqs. (19) and (20) respectively, we get

$$\sum_{n=0}^\infty p^nu_n(x,t)= p\left(u(x,0)+tu_t(x,0)+J_t^\alpha\left(\left(\sum_{n=0}^\infty p^nu_n(x,t)\right)_{xx}\right.\right. \left.\left.-\sum_{n=0}^\infty p^nu_n(x,t)+\left(\left(\sum_{n=0}^\infty p^nv_n(x,t)\right)\left(\sum_{n=0}^\infty p^n\bar{v}_n(x,t)\right)\right)\right)\right) \quad (21)$$

$$\sum_{n=0}^\infty p^nv_n(x,t)=p\left(v(x,0)+iJ_t^\beta\left(\left(\sum_{n=0}^\infty p^nv_n(x,t)\right)_{xx}\right)\right) +p\left(\left(\sum_{n=0}^\infty p^nu_n(x,t)\right)\left(\sum_{n=0}^\infty p^nv_n(x,t)\right)\right) \quad (22)$$

Comparing the coefficients of like powers in p for both sides of eqs. (21) and (22), we have the following system of fractional differential equations.

Coefficients of

$$p^0:u_0=0 \quad (23)$$

$$v_0=0 \quad (24)$$

Coefficients of

$$p:u_1=u(x,0)+t\,u_t(x,0)+J_t^\alpha\left(\frac{\partial^2u_0}{\partial x^2}-u_0+v_0\bar{v}_0\right) \quad (25)$$

$$v_1=v(x,0)+iJ_t^\beta\left(\frac{\partial^2v_0}{\partial x^2}+u_0v_0\right) \quad (26)$$

Coefficients of

$$p^2:u_2=J_t^\alpha\left(\frac{\partial^2u_1}{\partial x^2}-u_1+v_0\bar{v}_1+v_1\bar{v}_0\right) \quad (27)$$

$$v_2=iJ_t^\beta\left(\frac{\partial^2v_1}{\partial x^2}+u_0v_1+v_0u_1\right) \quad (28)$$

Coefficients of

$$p^3:u_3=J_t^\alpha\left(\frac{\partial^2u_2}{\partial x^2}-u_2+v_1\bar{v}_1++v_2\bar{v}_0+v_0\bar{v}_2\right) \quad (29)$$

$$v_3 = iJ_t^{\beta}\left(\frac{\partial^2 v_2}{\partial x^2} + u_1 v_1 + v_2 u_0 + v_0 u_2\right) \quad (30)$$

By considering the initial conditions of eq. (6) in eqs.(23) to (30) and solving them, we obtain

$$\begin{aligned}
u_0(x,t) &= 0\\
v_0(x,t) &= 0\\
u_1(x,t) &= 6B^2 \sec h^2(Bx) - 12B^2 c \sec h^2(Bx)\tanh(Bx)\\
v_1(x,t) &= 3B \sec h^2(Bx)\, e^{idx}
\end{aligned}$$

$$\begin{aligned}
u_2(x,t) &= 6B^2 \sec^2(Bx)\frac{t^{\alpha}}{\Gamma(\alpha+1)}\\
&\left(-1 - 2B^2 \sec h^2(Bx) + 4B^2 \tanh^2(Bx)\right)\\
&+ 6B^2 \sec^2(Bx)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} 16B^2 c\, \sec h^2(Bx)\tanh^2(Bx)\\
&- 6B^2 \sec^2(Bx)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\\
&2c\tanh(Bx)\left(-1 + 4B^2 \tanh^2(Bx)\right)
\end{aligned}$$

$$\begin{aligned}
v_2(x,t) &= -i3B \sec h^2(Bx)\, e^{idx}\frac{t^{\beta}}{\Gamma(\beta+1)}\\
&\left(2B^2 \sec h^2(Bx) + \left(d + 2iB\tanh^2(Bx)\right)^2\right)
\end{aligned}$$

$$\begin{aligned}
u_3(x,t) &= 3B^2 \sec h^2(Bx)\,(-4c\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\tanh(Bx)\\
&(136B^4 \sec h^4(Bx) + (1 - 4B^2 \tanh^2(Bx))^2\\
&+ \sec h^2(Bx)\,(16B^2 - 208B^4 \tanh^2(Bx))\\
&+ \frac{3t^{\alpha}}{\Gamma(\alpha+1)}\sec h^2(Bx) + \frac{2t^{2\alpha}}{\Gamma(2\alpha+1)}\\
&(16B^4 \sec h^4(Bx) + (1 - 4B^2 \tanh^2(Bx))^2
\end{aligned}$$

$$\begin{aligned}
v_3(x,t) &= -i3B \sec h^2(Bx)\, e^{idx}(-6B^2 \sec h^2(Bx)\\
&\left(\frac{-t^{\beta}}{\Gamma(\beta+1)} + \frac{2ct^{\beta+1}\tanh(Bx)}{\Gamma(\beta+2)}\right)\\
&+ \frac{it^{2\beta}}{\Gamma(2\beta+1)}(16B^4 \sec h^4(Bx) + \left(d + 2iB\tanh^2(Bx)\right)^4\\
&+ 4B^2 \sec h^2(Bx)\,(3d^2 + 16iBd\tanh(Bx) - 22B^2 \tanh^2(Bx))))
\end{aligned}$$

and so on. In this manner the other components of the Homotopy series can be easily obtained by which $u(x,t)$ and $v(x,t)$ can be evaluated in a series form as

$$\begin{aligned}
u(x,t) &= u_0(x,t) + u_1(x,t) + u_2(x,t)\\
&= 6B^2 \sec h^2(Bx) - 12B^2 c \sec h^2(Bx)\tanh(Bx)\\
&6B^2 \sec^2(Bx)\frac{t^{\alpha}}{\Gamma(\alpha+1)}\\
&\left(-1 - 2B^2 \sec h^2(Bx) + 4B^2 \tanh^2(Bx)\right)\\
&+ 6B^2 \sec^2(Bx)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} 16B^2 c\, \sec h^2(Bx)\tanh^2(Bx)\\
&- 6B^2 \sec^2(Bx)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\\
&2c\tanh(Bx)\left(-1 + 4B^2 \tanh^2(Bx)\right) + \ldots
\end{aligned} \quad (31)$$

$$\begin{aligned}
v(x,t) &= v_0(x,t) + v_1(x,t) + v_2(x,t)\\
&= 3B \sec h^2(Bx)\, e^{idx}\\
&- i3B \sec h^2(Bx)\, e^{idx}\frac{t^{\beta}}{\Gamma(\beta+1)}\\
&\left(2B^2 \sec h^2(Bx) + \left(d + 2iB\tanh^2(Bx)\right)^2\right)\\
&- i3B \sec h^2(Bx)\, e^{idx}(-6B^2 \sec h^2(Bx) + \ldots
\end{aligned} \quad (32)$$

4 Homotopy Perturbation Transform Method (HPTM)

To illustrate the basic idea of HPTM [12,13,14], we consider a general fractional nonlinear nonhomogeneous partial differential equation with the initial conditions of the form

$$D_t^{\alpha} u(x,t) + Ru(x,t) + Nu(x,t) = f(x,t) \quad (33)$$

$$u(x,0) = h(x),\, u_t(x,0) = g(x) \quad (34)$$

where $D_t^{\alpha} u(x,t)$ is the Caputo fractional derivative of the function $u(x,t)$, R is the linear differential operator, N represents the general nonlinear differential operator and $f(x,t)$ is the source term. Taking the Laplace transform on both sides of eq. (33), we get

$$L[D_t^{\alpha} u(x,t)] + L[Ru(x,t)] + L[Nu(x,t)] = L[f(x,t)] \quad (35)$$

Using the property of the Laplace transform, we have

$$\begin{aligned}
L[u(x,t)] &= \frac{h(x)}{s} + \frac{g(x)}{s^2} + \frac{1}{s^{\alpha}}L[f(x,t)]\\
&- \frac{1}{s^{\alpha}}L[Ru(x,t)] - \frac{1}{s^{\alpha}}L[Nu(x,t)]
\end{aligned} \quad (36)$$

Operating with the Laplace inverse on both sides of eq. (36) yields

$$u(x,t) = G(x,t) - L^{-1}\left[\frac{1}{s^{\alpha}}L[Ru(x,t) + Nu(x,t)]\right] \quad (37)$$

where $G(x,t) = \frac{h(x)}{s} + \frac{g(x)}{s^2} + \frac{1}{s^\alpha} L[f(x,t)]$ represents the term arising from the source term and the prescribed initial conditions. By homotopy perturbation method, we will construct the homotopy of equation eq.(37) as

$$u(x,t) = G(x,t) - p\left(L^{-1}\left[\frac{1}{s^\alpha} L[Ru(x,t) + Nu(x,t)]\right]\right) \tag{38}$$

By substituting, $u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t)$ and $v(x,t) = \sum_{n=0}^{\infty} p^n v_n(x,t)$ in eq. (38) and subsequently equating the like powers of p, we can finally obtain the analytical approximate solution $u(x,t)$ in truncated series as

$$u(x,t) = \lim_{N\to\infty} \sum_{n=0}^{N} u_n(x,t). \tag{39}$$

4.1 Application of HPTM method

In this section, we apply HPTM to eqs. (1) and (2) with considering the initial conditions (14)

Applying Laplace transform on the both sides of eqs. (1) and (2) respectively, we get

$$L[u(x,t)] = \frac{u(x,0)}{s} + \frac{u_t(x,0)}{s^2} + \frac{1}{s^\alpha} L[u_{xx}] - \frac{1}{s^\alpha} L[u] + \frac{1}{s^\alpha} L\left[|v|^2\right] \tag{40}$$

$$L[v(x,t)] = \frac{v(x,0)}{s} + i\frac{1}{s^\beta} L[v_{xx}] + i\frac{1}{s^\beta} L[uv] \tag{41}$$

Then applying inverse Laplace transform on both sides of eqs. (40) and (41) respectively, we get

$$u(x,t) = u(x,0) + t\,u_t(x,0) + L^{-1}\left[\frac{1}{s^\alpha} L\left[u_{xx} - u + |v|^2\right]\right] \tag{42}$$

$$v(x,t) = v(x,0) + iL^{-1}\left[\frac{1}{s^\beta} L[v_{xx} + uv]\right] \tag{43}$$

By homotopy perturbation method, we will construct the homotopy of eqs. (42) and (43) as

$$u(x,t) = u(x,0) + tu_t(x,0) + p\left(L^{-1}\left[\frac{1}{s^\alpha} L\left[u_{xx} - u + |v|^2\right]\right]\right) \tag{44}$$

$$v(x,t) = v(x,0) + i\,p\left(L^{-1}\left[\frac{1}{s^\beta} L[v_{xx} + uv]\right]\right) \tag{45}$$

By substituting $u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t)$ and $v(x,t) = \sum_{n=0}^{\infty} p^n v_n(x,t)$ in eqs. (44) and (45), we get

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = u(x,0) + tu_t(x,0) + p\left(L^{-1}\left(\frac{1}{s^\alpha} L\left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right)_{xx}\right) - L^{-1}\sum_{n=0}^{\infty} p^n u_n(x,t) + L^{-1}\left(\left(\sum_{n=0}^{\infty} p^n v_n(x,t)\right)\left(\sum_{n=0}^{\infty} p^n \bar{v}_n(x,t)\right)\right)\right) \tag{46}$$

$$\sum_{n=0}^{\infty} p^n v_n(x,t) = v(x,0) + ip\left(L^{-1}\left[\frac{1}{s^\alpha} L\left(\left(\sum_{n=0}^{\infty} p^n v_n(x,t)\right)_{xx} + \left(\left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right)\left(\sum_{n=0}^{\infty} p^n v_n(x,t)\right)\right)\right)\right]\right) \tag{47}$$

Comparing the coefficients of like powers in p for eqs. (44) and (45), we have the following system of fractional differential equations

Coefficients of

$$p^0 : u_0 = u(x,0) + t\,u_t(x,0) \tag{48}$$

$$v_0 = v(x,0) \tag{49}$$

Coefficients of

$$p : u_1 = L^{-1}\left[\frac{1}{s^\alpha} L\left[\left(\frac{\partial^2 u_0}{\partial x^2} - u_0 + v_0\bar{v}_0\right)\right]\right] \tag{50}$$

$$v_1 = iL^{-1}\left[\frac{1}{s^\beta} L\left[\left(\frac{\partial^2 v_0}{\partial x^2} + u_0 v_0\right)\right]\right] \tag{51}$$

Coefficients of

$$p^2 : u_2 = L^{-1}\left[\frac{1}{s^\alpha} L\left[\left(\frac{\partial^2 u_1}{\partial x^2} - u_1 + v_0\bar{v}_1 + + v_1\bar{v}_0\right)\right]\right] \tag{52}$$

$$v_2 = iL^{-1}\left[\frac{1}{s^\beta} L\left[\left(\frac{\partial^2 v_1}{\partial x^2} + u_0 v_1 + v_0 u_1\right)\right]\right] \tag{53}$$

Coefficients of

$$p^3 : u_3 = L^{-1}\left[\frac{1}{s^\alpha} L\left[\left(\frac{\partial^2 u_2}{\partial x^2} - u_2 + v_1\bar{v}_1 + + v_2\bar{v}_0 + v_0\bar{v}_2\right)\right]\right] \tag{54}$$

$$v_3 = iL^{-1}\left[\frac{1}{s^\beta} L\left[\left(\frac{\partial^2 v_2}{\partial x^2} + u_1 v_1 + v_2 u_0 + v_0 u_2\right)\right]\right] \tag{55}$$

and and by putting the initial conditions in eq. (6) and (45) into eqs. (48) and (55) and solving them, we obtain

$$u_0(x,t) = 6B^2 \sec h^2(Bx) - 12B^2 c \sec h^2(Bx)\tanh(Bx)$$

$$v_0(x,t) = 3B \sec h^2(Bx)\, e^{idx}$$

$$u_1(x,t) = 3B^2 \sec^2(Bx)\, t^\alpha \left(\frac{\left(-1+4B^2\right)\left(-2+Cosh(2Bx)\sec h^2(Bx)\right)}{\Gamma(\alpha+1)} + \frac{4ct\tanh(Bx)}{\Gamma(\alpha+2)}\left(1+8B^2\sec h^2(Bx) - 4B^2\tanh^2(Bx)\right)\right)$$

$$v_1(x,t) = -3iB\sec h^2(Bx)\,e^{idx}\frac{t^{\beta}}{\Gamma(\beta+1)\Gamma(\beta+2)}\left(12B^2ct\,\Gamma(\beta+1)\sec h^2(Bx)\tanh(Bx)\right.$$
$$\left.+\Gamma(\beta+2)\left(-4B^2+d^2+4iBd\tanh(Bx)\right)\right)$$

$$u_2(x,t) = \frac{3}{4}B^2\sec^6(Bx)\,t^{\alpha}\left(\frac{48Bd\sinh(2Bx)}{\Gamma(1+\alpha+\beta)}\right.$$
$$+\frac{t^{\alpha}}{\Gamma(2\alpha+2)}\left(3\left(-1-40B^2+176B^4\right)(1+2\alpha)\right.$$
$$-2\left(1-56B^2+208B^4\right)(1+2\alpha)\cosh(2Bx)$$
$$+\left(1-4B^2\right)^2(1+2\alpha)\cosh(4Bx)$$
$$2ct\left(\left(-2-80B^2+928B^4\right)\sinh(2Bx)\right.$$
$$\left.\left.\left.-\left(1-4B^2\right)^2\sinh(4Bx)-2880B^4\tanh(Bx)\right)\right)\right)$$

$$v_2(x,t) = -\frac{1}{\Gamma(\beta+1)\Gamma(\beta+2)\Gamma(2+\alpha+\beta)\Gamma(3+2\beta)}$$
$$\left[3B\sec h^2(Bx)\,e^{idx}t^{\beta}\left(144B^4c^2t^{\beta+2}\;\Gamma(\beta+1)\right.\right.$$
$$\Gamma(\beta+3)\,\Gamma(2+\alpha+\beta)\sec h^4(Bx)\tanh^2(Bx)$$
$$+3iB^2t^{\alpha}\,\Gamma(\beta+1)\Gamma(\beta+2)\,\Gamma(3+2\beta)\sec h^2(Bx)$$
$$((-1+4B^2)(1+\alpha+\beta)(-2+3\sec h^2(Bx)$$
$$-4ct(1-4B^2+12B^2\sec h^2(Bx))\tanh(Bx))$$
$$+2t^{\beta}\,\Gamma(\beta+2)^2\,\Gamma(2+\alpha+\beta)(24B^3ct$$
$$\sec h^4(Bx)(-id(7+2\beta)+12B\tanh(Bx)$$
$$+(1+2\beta)(16B^4-24B^2d^2+d^4$$
$$-8iBd\left(4B^2-d^2\right)\tanh(Bx))$$
$$+12B^2\sec h^2(Bx)(2d(d+2d\beta+2iBct(3+\beta))$$
$$+(cd^2t(2+\beta)-4B^2ct(5+\beta)$$
$$\left.\left.+2iB(d+2d\beta))\tanh(Bx))))\right)\right]$$

$$u(x,t) = u_0(x,t)+u_1(x,t)+u_2(x,t)$$
$$= 6B^2\sec h^2(Bx) - 12B^2c\sec h^2(Bx)\tanh(Bx)$$
$$+3B^2\sec^2(Bx)\,t^{\alpha}$$
$$\left(\frac{\left(-1+4B^2\right)\left(-2+Cosh(2Bx)\sec h^2(Bx)\right)}{\Gamma(\alpha+1)}\right.$$
$$\left.+\frac{4ct\tanh(Bx)}{\Gamma(\alpha+2)}\left(1+8B^2\sec h^2(Bx)-4B^2\tanh^2(Bx)\right)\right)$$
$$+\frac{3}{4}B^2\sec^6(Bx)\;t^{\alpha}\left(\frac{48Bd\sinh(2Bx)}{\Gamma(1+\alpha+\beta)}+\ldots \qquad (56)\right.$$

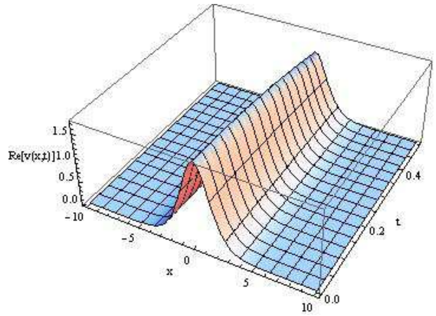

Fig. 1: The HPTM method solution for $u(x,t)$

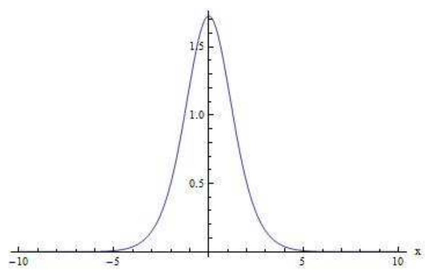

Fig. 2: corresponding solution for $u(x,t)$ when $t = 0$

$$v(x,t) = v_0(x,t)+v_1(x,t)+v_2(x,t)$$
$$= 3B\sec h^2(Bx)\,e^{idx}$$
$$-3iB\sec h^2(Bx)\,e^{idx}\frac{t^{\beta}}{\Gamma(\beta+1)\Gamma(\beta+2)}$$
$$\left(12B^2ct\,\Gamma(\beta+1)\sec h^2(Bx)\tanh(Bx)\right.$$
$$\left.+\Gamma(\beta+2)\left(-4B^2+d^2+4iBd\tanh(Bx)\right)\right)+\ldots \qquad (57)$$

5 Numerical results and discussion

In the present numerical computation we have assumed B = 0.575.

5.1 The numerical simulations for HPTM method

In this present numerical experiment, eqs. (56) and (57)have been used to draw the graphs as shown in Figs. 1-12 respectively. The numerical solutions of coupled K–G–S eqs. (1)and (2)have been shown in Figs. 1–12 with the help of five-term approximations for the homotopy series solutions of $u(x,t)$and $v(x,t)$respectively.
Case 1: when $\alpha = 2,\ \beta = 1$ (Classical order)
Case 2: when $\alpha = 1.75,\ \beta = 0.75$ (Fractional order)
Case 3: when $\alpha = 1.5,\ \beta = 0.5$ (Fractional order)

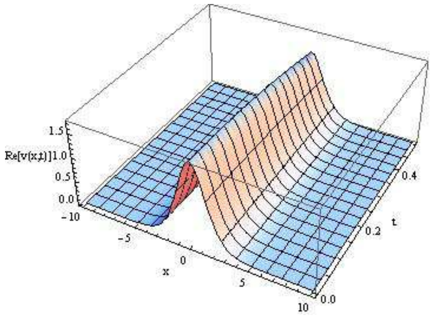

Fig. 3: The HPTM method solution for $Re\,(v\,(x,t))$

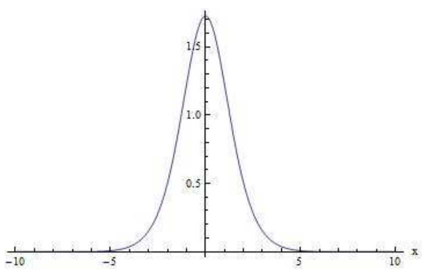

Fig. 4: corresponding solution for $Re\,(v\,(x,t))$ when $t = 0$

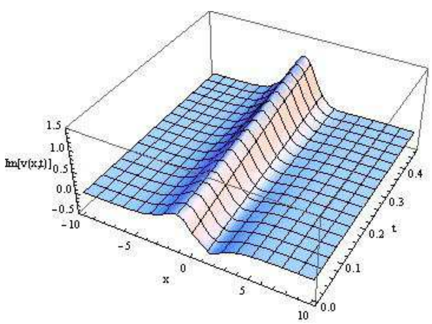

Fig. 5: The HPTM method solution for $Im\,(v\,(x,t))$

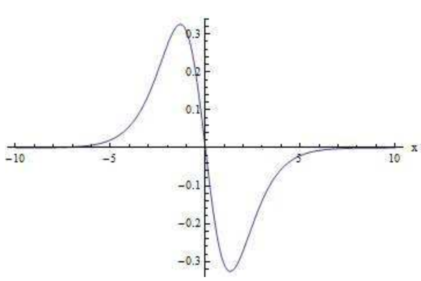

Fig. 6: The HPTM method solution for $Im\,(v\,(x,t))$

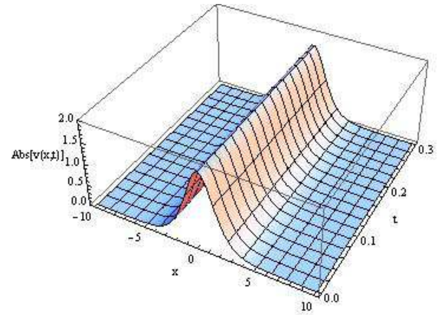

Fig. 7: The HPTM method solution for $Abs\,(v\,(x,t))$

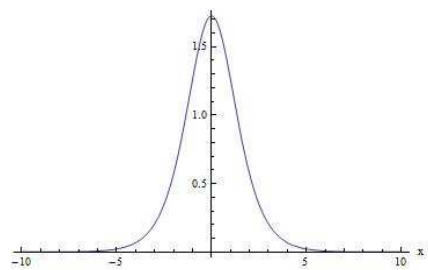

Fig. 8: corresponding solution for $Abs\,(v\,(x,t))$ when $t = 0$

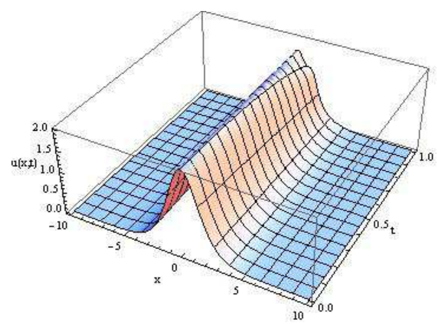

Fig. 9: The HPTM method solution for $u\,(x,t)$

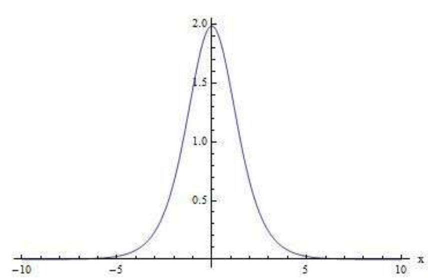

Fig. 10: corresponding solution for $u\,(x,t)$ when $t = 0$.

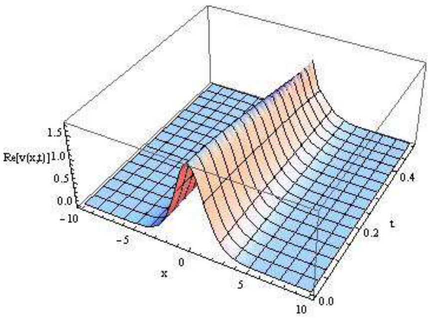

Fig. 11: The HPTM method solution for $Re(v(x,t))$

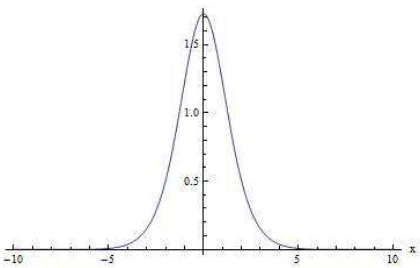

Fig. 12: corresponding solution for $Re(v(x,t))$ when $t = 0$

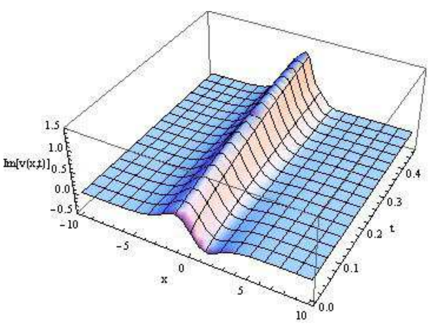

Fig. 13: The HPTM method solution for $Im(v(x,t))$

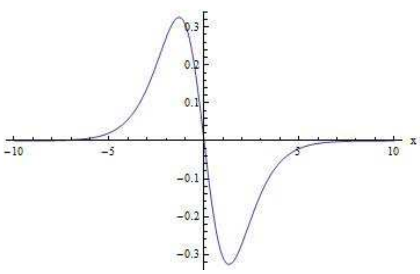

Fig. 14: orresponding solution for $Im(v(x,t))$ when $t = 0$

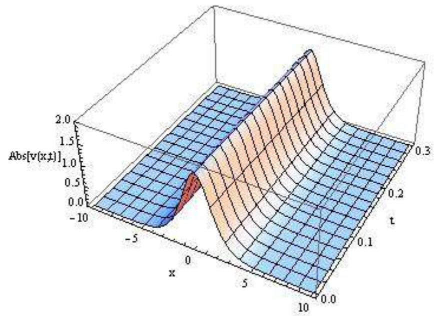

Fig. 15: The HPTM method solution for $Abs(v(x,t))$

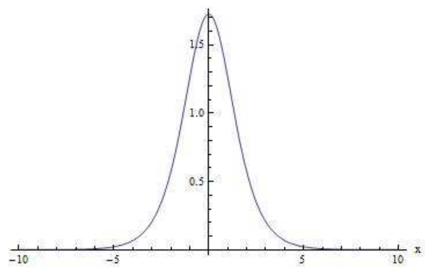

Fig. 16: corresponding solution for $Abs(v(x,t))$ when $t = 0$

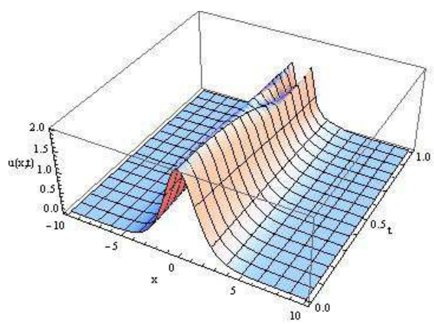

Fig. 17: The HPTM method solution for $u(x,t)$

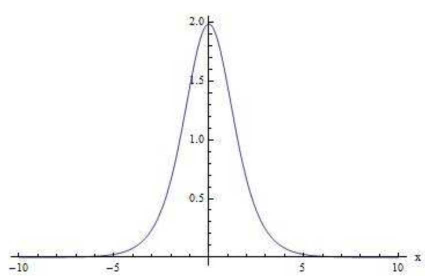

Fig. 18: corresponding solution for $u(x,t)$ when $t = 0$

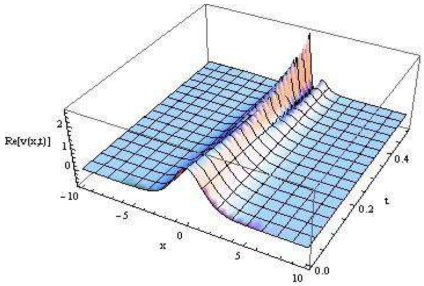

Fig. 19: The HPTM method solution for $Re\left(v\left(x,t\right)\right)$

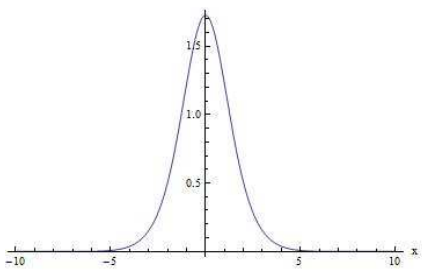

Fig. 20: corresponding solution for $Re\left(v\left(x,t\right)\right)$ when $t=0$

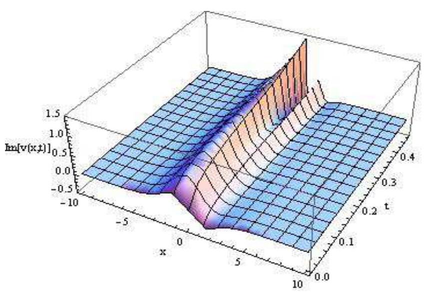

Fig. 21: The HPTM method solution for $Im\left(v\left(x,t\right)\right)$

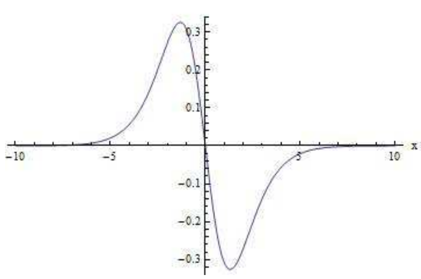

Fig. 22: corresponding solution for $Im\left(v\left(x,t\right)\right)$ when $t=0$

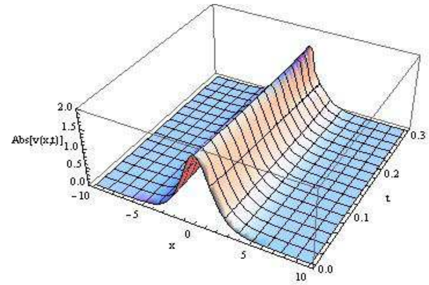

Fig. 23: The HPTM method solution for $Abs\left(v\left(x,t\right)\right)$

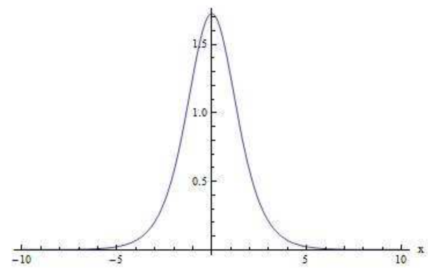

Fig. 24: corresponding solution for $Abs\left(v\left(x,t\right)\right)$ when $t=0$

In Figs. 1-8, the HPTM approximate solutions of $u(x, t)$ and $v(x, t)$ are plotted for the intervals $-10 \leq x \leq 10$ and $0 \leq t \leq 1$ for the classical order that is for $\alpha = 2$ and $\beta = 1$. In Figs. 9-24, the HPTM approximate solutions of $u(x, t)$ and $v(x, t)$ are plotted for the same intervals for the fractional order that is for $\alpha = 1.75$, $\beta = 0.75$ and $\alpha = 1.5$, $\beta = 0.5$respectively. As value of α decreases from 2 to 1.5 and β decreases from 1 to 0.5, the distribution of $u(x, t)$ bifurcated into two waves. Similarly when α decreases from 2 to 1.5 and β decreases from 1 to 0.5 , the distribution of $Re\left(v\left(x,t\right)\right)$ and $Im\left(v\left(x,t\right)\right)$ bifurcated into three waves.

5.2 The numerical simulations for absolute errors in HPM and HPTM solutions

In this section, we present Figs.25-28 citing the numerical simulations for comparison of absolute errors in solutions of $u\left(x,t\right)$ and $v\left(x,t\right)$ obtained by HPM and HPTM at $x = 0.3$.

Although both the methods are reliable and efficient but Figs. 25-28 assure plausibility to consider HPTM provides more accurate solutions than HPM solutions for coupled fractional K-G-S equation.

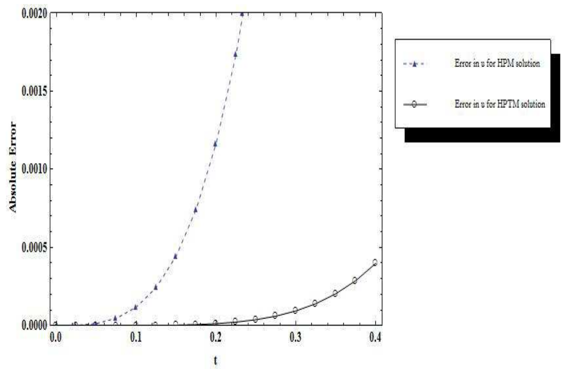

Fig. 25: Graphical comparison of absolute errors in the solution of $Re\left(v\left(x,t\right)\right)$ for (a) HPM and (b) HPTM, when $\alpha = 2$, $\beta = 1$

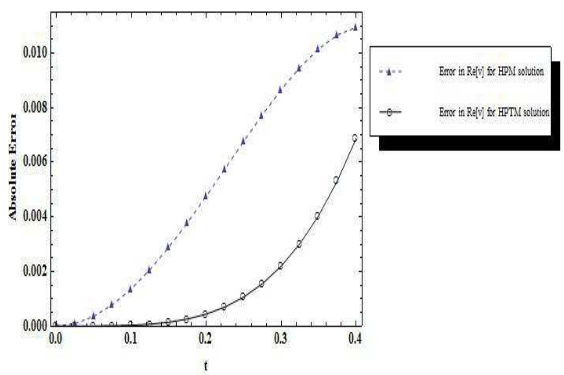

Fig. 26: Graphical comparison of absolute errors in the solution of $u\left(x,t\right)$ for (a) HPM and (b) HPTM, when $\alpha = 2$, $\beta = 1$

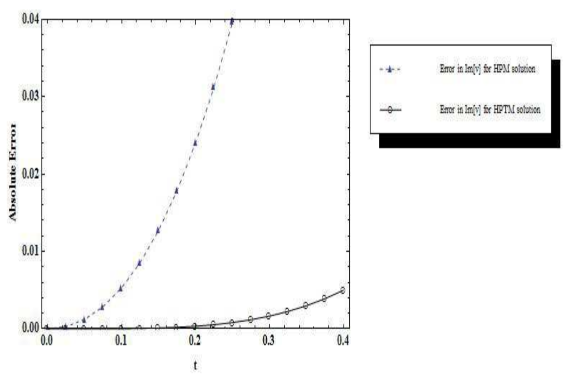

Fig. 27: Graphical comparison of absolute errors in the solution of $Im\left(v\left(x,t\right)\right)$ for (a) HPM and (b) HPTM, when $\alpha = 2$, $\beta = 1$

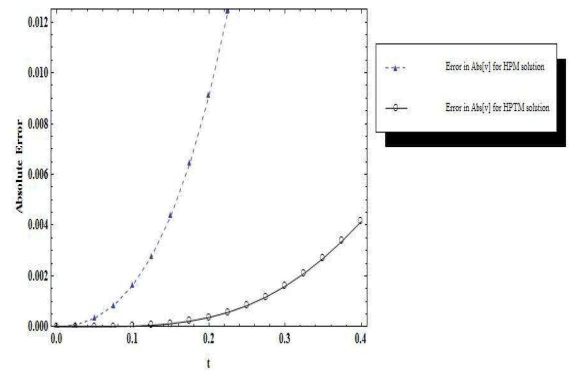

Fig. 28: Graphical comparison of absolute errors in the solution of $Abs\left(v\left(x,t\right)\right)$ for (a) HPM and (b) HPTM, when $\alpha = 2$, $\beta = 1$

6 Comparison of HPM and HPTM solutions with regard to exact solutions

In case of integer order, exact solutions of eqs. (1) and (2) are given by [6]

$$u\left(x,t\right) = 6B^2 \sec h^2 \left(Bx + ct\right) \tag{58}$$

$$v\left(x,t\right) = 3B \sec h^2 \left(Bx + ct\right)\; e^{i\left(dx + \left(4B^2 - d^2\right)t\right)} \tag{59}$$

where $B(\geq 1/2)$, c and $d$are arbitrary constants with $c = \frac{\sqrt{4B^2-1}}{2}$, $d = -\frac{c}{2B}$ for the fractional coupled K-G-S eqs.(1) and (2).
In this present analysis, we present the comparison for the solutions of HPM with HPTM. Here we demonstrate the absolute and relative errors by taking different values of t with respect to some fixed value of x.

In case of $\alpha = 2$ and $\beta = 1$, Table 1 represent comparison of absolute errors four term HPM and HPTM solutions with respect to exact solutions when $x = 1.5$and Table 2 represent comparison of relative errors four term HPM and HPTM solutions with respect to exact solutions when $x = 1.5$. Comparison results in Tables 1 and 2 exhibit that there is a good agreement between HPM and HPTM solutions with exact solutions. Although HPTM provides more accurate solutions in compared to HPM solutions which can be easily verified from Tables 1 and 2.

7 Conclusion

In this paper, the homotopy perturbation method (HPM) and homotopy perturbation transform method (HPTM) have been applied for finding the solutions for fractional coupled K-G-S equations with initial conditions. The approximate solutions of the equations have been calculated by using HPM method without any need of

Table 1: Comparison of absolute errors between four term HPM and HPTM solutions with regard to Exact solutions for different values of t respectively when $x = 1.5$in case of $\alpha = 2$ and $\beta = 1$

t	$x = 1.5$							
	Absolute error between Exact solution and four term HPM solution				Absolute error between Exact solution and four term HPTM solution			
	$u(x,t)$	$Re(v(x,t))$	$Im(v(x,t))$	$Abs(v(x,t))$	$u(x,t)$	$Re(v(x,t))$	$Im(v(x,t))$	$Abs(v(x,t))$
0.2	2.464E-5	1.192E-2	1.002E-2	1.3071E-2	4.107E-8	2.09E-4	1.693E-4	1.877E-4
0.4	3.79E-4	2.597E-2	5.46E-2	2.05E-2	4.56E-6	3.358E-3	2.926E-3	3.716E-3
0.6	1.734E-3	8.084E-3	1.585E-1	3.487E-2	5.157E-5	1.701E-2	1.581E-2	2.16E-2
0.8	4.599E-3	7.651E-2	3.496E-1	1.999E-1	2.645E-4	5.37E-2	5.295E-2	7.4637E-2
1.0	8.418E-3	2.619E-1	6.594E-1	3.403E-1	8.99E-4	1.31E-1	1.362E-1	1.8816E-1

Table 2: Comparison of Relative errors between four term HPM and HPTM solutions with regard to Exact solutions for different values of t respectively when $x = 1.5$in case of $\alpha = 2$and $\beta = 1$

t	$x = 1.5$							
	Relative error between Exact solution and four term HPM solution				Relative error between Exact solution and four term HPTM solution			
	$u(x,t)$	$Re(v(x,t))$	$Im(v(x,t))$	$Abs(v(x,t))$	$u(x,t)$	$Re(v(x,t))$	$Im(v(x,t))$	$Abs(v(x,t))$
0.2	8.05E-4	7.17E-2	2.43E-2	7.05E-2	4.33E-8	2.57E-4	1.76E-3	2.29E-4
0.4	6.31E-3	3.03E-1	1.01E-1	3.009E-1	5.08E-6	4.51E-3	2.91E-2	4.94E-3
0.6	2.06E-2	7.67E-1	2.22E-2	7.009E-1	5.82E-5	2.66E-2	6.08E-2	3.14E-2
0.8	4.63E-2	6.52E-1	3.36E-1	1.27	2.79E-4	1.06E-1	1.40E-1	1.18E-1
1.0	8.35E-2	3.462	5.05E-1	2.03	7.65E-4	3.61E-1	3.01E-1	3.24E-1

transformation techniques and linearization of the equations whereas by using HPTM method, we use transformation method to solve fractional coupled K-G-S equations. Additionally, HPTM method does not need any discretization method to have numerical solutions. This method thus eliminates the difficulties and massive computation work. The above perturbation methods are straight forward, without restrictive assumptions and the components of the series solution can be easily computed using any mathematical symbolic software. Moreover, these methods do not change the problem into a convenient one for the use of linear theory. It is obvious to see that the HPTM is more accurate, easy and efficient technique for solving fractional coupled K-G-S equations.

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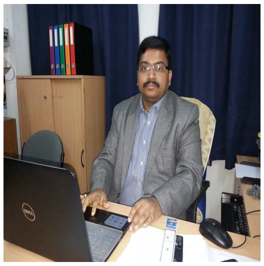
References

[1] H. Liang, Applied Mathematics and Computation **238**, 475-484 (2014).
[2] A. Darwish, E. G. Fan, Chaos Soliton and Fractals **20**, 609-17 (2004).
[3] S. Liu, Z. Fu, S. Liu, Z. Wang, Phys Lett A **323**, 415-420 (2004).
[4] J. Xia, S. Han, M. Wang, Appl. Math. Mech. **23**, 52-57 (2002).
[5] F. T. Hioe, J Phys A: Math Gen **36**, 7307-7330 (2003).
[6] W. Bao, L.Yang, J. Comp. Phys. **225**, 1863-1893 (2007).
[7] M. Naber, J. Math. Phys. **45**, 3339-3352 (2004).
[8] S.Z. Rida, H.M. El-Sherbiny, A.A.M. Arafa, Physics Letters A **372**, 553-558 (2008).
[9] S. Saha Ray, Communications in Nonlinear Science and Numerical Simulation **13**, 1311-1317 (2008).
[10] J.H. He, Comput. Meth. Appl. Mech. Eng. **178**, 257-262 (1999).
[11] J.H. He, Int. J. Mod. Phys. B **20**, 1141-1199 (2006).
[12] J. Singh, D. Kumar, Journal of Fractional Calculus and Applications **4**, 290-302 (2013).
[13] Y. Khan, Q. Wu, Computer and mathematics with applications **61**, 1963-1967 (2011).
[14] Y. Liu, Abstract and Applied Analysis **75289**, 14 pages (2012).

[15] I. Podlubny, Fractional differential Equation, Academic Press, New York, 1999.
[16] S. G. Samko, A. A. Kilbas, O. I. Marichev, Fractional Integrals and derivatives: Theory and Applications, Texts in Applied Mathematics, Taylor and Francis, London, 2002.



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