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Dynamics of a Worm Propagation Model with Quarantine in Wireless Sensor Networks

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Abstract: We study the attacking behavior of possible worms in Wireless Sensor Network (WSNs). Using epidemic theory, we propose a susceptible-infectious-quarantine-recovered (SIQR) model to describe dynamics of worms propagation with quarantine in the wireless sensor network. Mathematical analysis shows that dynamics of the spread of worms are determined by the threshold R_0. If $R_0 \leq 1$, the worm-free equilibrium is globally asymptotically stable, and if $R_0 > 1$, the worm-endemic equilibrium is globally asymptotically stable. Lyapunov functional method as well as geometric approach are used for proving the global stability of equilibria. A numerical investigation is carried out to confirm the analytical results. As a result of parameter analysis, some effective strategies for eliminating worms are suggested.

Keywords: Wireless sensor networks, basic reproduction number, worm-free equilibrium, worm-endemic equilibrium, global stability

1 Introduction

Wireless sensor networks (WSNs) have attracted much attention and have a wide variety of military and civil applications, such as military target tracking, battlefield surveillance, intrusion detection, disaster recovery, biological detection, ambient assisted living, personal care, seismic sensing, etc [1, 2]. Wireless sensor networks are composed of small-sized, cheap, energy-limited, and multi-functional devices called sensors that are deployed to collect data from an environment or monitor a phenomenon [3]. Each wireless sensor, which is also called sensor node, can sense, measure, and gather information from the environment and, based on some local decision process, it can transmit the sensed data to the user. Since the sensor nodes have limited memory and are typically deployed in difficult-to-access locations, a radio is implemented for wireless communication to transfer the data to a base station (likes a personal handheld device or an access point to a fixed infrastructure) [4]. Because sensor nodes are resource constrained, they generally have weak defense capability and are attractive targets of malware (e.g., virus, worm or trojan). Worms are self-replicating computer virus which can spread through computer networks without any human intervention. Thus, security mechanism that can guarantee sensor nodes against malware is one of great interest to the wireless sensor network community.

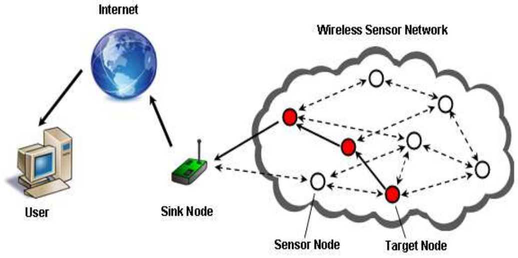

Fig. 1: Sensor network communication structure.

One way to control the spread of worms for the nodes which are highly infected is to be kept in isolation for some time. The word quarantine means to say about the forced isolation or stoppage of interaction with others. When a node is found to be effected, it can immediately be quarantined by the worm detection program. Then we monitor it for a period of time corresponding to the erratic behavior shown by the process. If the node does not show any behavior during the time it's monitored, it is released.

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If it shows the same behavior again and again, it is labeled as a worm-node and will be blocked.

The analysis of malicious objects propagation in networks has been the subject of interest in the field of computer science, mainly following approaches borrowed from biological epidemiology. Models for the transmission of computer viruses based on epidemic models started to be studied by Kephart [5]. He used the epidemic models to find out the rule in computer viruses and paid attention on the topological properties of the network on the spread of viruses. Many authors used the SIR and SEIR model to analyze the behavior of dynamics of computer virus [6–10]. An SAIR model for computer virus, that includes an antidotal population compartment is proposed by Roberto [11]. Moreover, the impact of connection mode in the computer network for the propagation of computer virus is studied in [12–14]. Quarantine is one of the important remedial processes for malware attack in network. Several researches take quarantine as one of important components in the epidemic models [15, 16]. Much attention has recently been study mathematical model on the transmission of worm in wireless sensor network [1, 17]. In [18], Wan et al. studied a iSIRS model of worm propagation in wireless sensor networks with the working state and sleeping state of nodes. Mishra et al. [2, 19] present the model of worms in wireless sensor network with quarantine and maintenance mechanism in the sleep nodes.

In this paper, we consider a proposed model depicting a worm propagation in wireless sensor network with quarantine and I-type infection function, which is real-world mode for malware propagation. The model is given by a system of four differential equations depending on parameters. By using the method of next generation matrix [20], we found a threshold R_0 called basic reproduction number. In general, when $R_0 \leq 1$, the spread dies out and when $R_0 > 1$, the spread persists in the network. If we suppose that the worm-endemic equilibrium also exists for $R_0 < 1$, although it is not true, then the bifurcation occurring in the model can be explained as a transcritical bifurcation. We concentrate our study on the globally stable stability of equilibria. This is obtained by Lyapunov functional method as well as and geometric approach. A numerical investigation is carried out by Mathematica software and AUTO software package [21] confirming theoretical results.

The paper is organized as follows. In the next section, we introduce the structure of the transmission model, equilibria and the basic reproduction number. Section 3 deals with the local stability of equilibria. In section 4, we prove the global stability of equilibria by using Lyapunov functional approach and geometric approach. Some numerical simulations are given in section 5. Finally, section 6 summarizes this work.

2 The model and its basic properties

2.1 The structure of the model

We study a quarantine model of worms transmission in a wireless sensor network. In the network, all nodes are assumed to be in one of four possible states during the propagation:

- **Susceptible state (S):** the nodes in (S) have not been infected by any worm in a WSNs and these nodes are vulnerable to worms.
- **Infectious state (I):** the nodes in (I) have been infected by worms in a WSNs and they may infect some nodes in (S).
- **Quarantine state (Q):** the nodes in (Q) are quarantined from nodes in (I).
- **Recovered state (R):** the nodes in (R) are cleaned of worms, and are temporarily immune and there after again become susceptible towards the possible attack of worms.

Let $S(t)$, $I(t)$, $Q(t)$ and $R(t)$ denote the number of susceptible, infectious, quarantine and recovered nodes at time t, respectively.

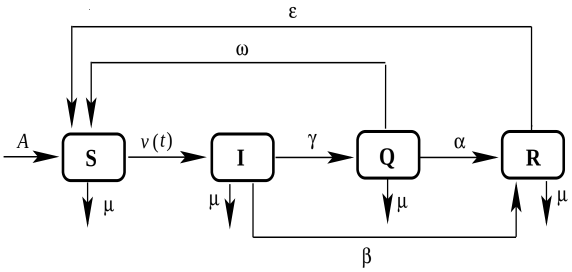

Fig. 2: Schematic diagram for the flow of nodes in the model.

The force of infection, $v(t) = \triangle SI(t)$, is the essential rate at which susceptible nodes become infected which is determined by the virus infection mode. The results based on real-world-network structure show that our network is close to P2G infection mode network, the function $v(t)$ is expected to be I-type infection function, that is $v(t) = \lambda I(t)$ [12, 13]. Since homogenous models are widely used in the study of worm infection, parameters will be considered as constants.

For the modeling purpose, the following hypotheses are imposed (see Fig. 2):

(H1) Nodes out side the network enters the wireless network at rate A. Every node in the state (S), (I), (Q) or (R) leaves the network, without connecting with others node, with rate μ.

(H2) Every susceptible node in the network is transferred to infectious node with probability $v(t) = \lambda I(t)$.

(H3) Due to the detection program, every infectious node is isolated to be a quarantined node with probability γ, a quarantine node is released to recovered node with a rate α, and a recovered node becomes susceptible node with a rate ε.

(H4) Every quarantine node is released to susceptible state, by reinstall the system and other means, with a rate ω. Every infectious node becomes recovered node, by using antivirus program, with a rate β.

The model obtained from these hypotheses is formulated as a system of differential equations:

$$\begin{aligned}\frac{dS}{dt} &= A-\lambda S(t)I(t)+\omega Q(t)+\varepsilon R(t)-\mu S(t)\\ \frac{dI}{dt} &= \lambda S(t)I(t)-(\beta+\gamma+\mu)I(t)\\ \frac{dQ}{dt} &= \gamma I(t)-(\alpha+\mu+\omega)Q(t)\\ \frac{dR}{dt} &= \beta I(t)+\alpha Q(t)-(\varepsilon+\mu)R(t),\end{aligned} \tag{1}$$

Let $N(t)$ be the total number of nodes in WSNs, that is

$$N(t)=S(t)+I(t)+Q(t)+R(t) \quad \forall t.$$

It follows from the system (1) that

$$(S+I+Q+R)'=A-\mu N.$$

Then $\limsup_{t\to\infty}(S+I+Q+R)\le\frac{A}{\mu}$. Therefore, the feasible region for system (1) is

$$\Omega=\{(S,I,Q,R):\ S,I,Q,R>0,S+I+Q+R\le\tfrac{A}{\mu}\}.$$

2.2 Equilibria

To find equilibria, we set the right-hand side of the system (1) equals zero. Then we get two equilibria in the coordinate (S,I,Q,R):

Worm-free equilibrium $\mathrm{P}_0(\frac{A}{\mu},0,0,0)$.

Worm-endemic equilibrium $\mathrm{P}_1(S^*,I^*,Q^*,R^*)$ where $S^*>0$, $I^*>0$, $Q^*>0$, $R^*>0$ and

$$\begin{aligned}S^* &= \frac{\beta+\gamma+\mu}{\lambda},\\ I^* &= \frac{(\varepsilon+\mu)(\alpha+\mu+\omega)G}{\lambda F},\\ Q^* &= \frac{\gamma(\varepsilon+\mu)G}{\lambda F},\\ R^* &= \frac{(\alpha\beta+\alpha\gamma+\beta\mu+\beta\omega)G}{\varepsilon\lambda F},\end{aligned}$$

with

$$G=\lambda A-\mu(\beta+\gamma+\mu), \tag{2}$$

$$F=\mu[\alpha(\beta+\varepsilon+\gamma+\mu)+\beta(\mu+\omega)+(\varepsilon+\mu)(\gamma+\mu+\omega)]. \tag{3}$$

It is seen that the equilibrium P_0 always exists. When $R_0=\frac{\lambda A}{\mu(\beta+\gamma+\mu)}>1$, we have $G>0$. This implies the equilibrium P_1 exists for $R_0>1$.

2.3 Basic reproduction number

Using the method proposed in [20], we determine the basic reproduction number R_0, that is the number of secondary cases which one case would produce in a completely susceptible population. The model (1) always has a worm-free equilibrium $\mathrm{P}_0(\frac{A}{\mu},0,0,0)$. Let $x=(I,Q,S,R)^{\top}$. Then the model (1) can be written as

$$\frac{dx}{dt}=\mathscr{F}(x)-\mathscr{V}(x),$$

where

$$\mathscr{F}(x)=\begin{pmatrix}\lambda SI\\0\\0\\0\end{pmatrix},\ \mathscr{V}(x)=\begin{pmatrix}(\beta+\gamma+\mu)I\\ -\gamma I+(\alpha+\mu+\omega)Q\\ -A+\lambda SI-\omega Q-\varepsilon R+\mu S\\ -\beta I-\alpha Q+(\varepsilon+\mu)R\end{pmatrix}.$$

We can get

$$F=\begin{pmatrix}\frac{\partial\mathscr{F}_1}{\partial I} & \frac{\partial\mathscr{F}_1}{\partial Q}\\ \frac{\partial\mathscr{F}_2}{\partial I} & \frac{\partial\mathscr{F}_2}{\partial Q}\end{pmatrix}_{P_0}=\begin{pmatrix}\frac{\lambda A}{\mu} & 0\\ 0 & 0\end{pmatrix},$$

$$V=\begin{pmatrix}\frac{\partial\mathscr{V}_1}{\partial I} & \frac{\partial\mathscr{V}_1}{\partial Q}\\ \frac{\partial\mathscr{V}_2}{\partial I} & \frac{\partial\mathscr{V}_2}{\partial Q}\end{pmatrix}_{P_0}=\begin{pmatrix}\beta+\gamma+\mu & 0\\ -\gamma & \alpha+\mu+\omega\end{pmatrix},$$

giving

$$V^{-1}=\begin{pmatrix}\frac{1}{\beta+\gamma+\mu} & 0\\ \frac{\gamma}{(\beta+\gamma+\mu)(\alpha+\mu+\omega)} & \frac{1}{\alpha+\mu+\omega}\end{pmatrix}.$$

The next generation matrix for the model (1) is

$$FV^{-1}=\begin{pmatrix}\frac{\lambda A}{\mu(\beta+\gamma+\mu)} & \frac{\gamma}{(\beta+b+\mu)}\\ 0 & 0\end{pmatrix}.$$

The spectral radius of matrix FV^{-1} is $\rho(FV^{-1})=\frac{\lambda A}{\mu(\beta+\gamma+\mu)}$. According to the Theorem 2 in [20], the basic reproduction number of the system (1) is

$$R_0=\rho(FV^{-1})=\frac{\lambda A}{\mu(\beta+\gamma+\mu)}.$$

Note that when $R_0>1$ then $G=\lambda A-\mu(\beta+\gamma+\mu)>0$ and the worm-endemic equilibrium P_1 exists.

3 Local stability and bifurcation of equilibria

3.1 Local stability of the worm-free equilibrium

Theorem 1. P_0 *is locally asymptotically stable if* $R_0<1$. *Whereas,* P_0 *is unstable if* $R_0>1$.

Proof. The Jacobian matrix at P_0 is given by:

$$J_{P_0}=\begin{pmatrix}-\mu & -\frac{\lambda A}{\mu} & \omega & \varepsilon\\ 0 & \frac{\lambda A}{\mu}-(\beta+\gamma+\mu) & 0 & 0\\ 0 & \gamma & -(\alpha+\mu+\omega) & 0\\ 0 & \beta & \alpha & -(\varepsilon+\mu)\end{pmatrix}.$$

Eigenvalues of the above matrix are

$$\lambda_1 = -\mu,\ \ \lambda_2 = -(\varepsilon+\mu),\ \ \lambda_3 = -(\alpha+\mu+\omega),$$
$$\lambda_4 = \frac{\lambda A - \mu(\beta+\gamma+\mu)}{\mu} \equiv \frac{G}{\mu}.$$

Eigenvalues λ_1, λ_2 and λ_3 are always negative. If $R_0 < 1$, then $G < 0$. It implies $\lambda_4 < 0$. Therefore, P_0 is locally asymptotically stable. Whereas, for $R_0 > 1$ then $\lambda_4 > 0$ and P_0 is unstable. ■

3.2 *Local stability of the worm-endemic equilibrium*

The local stability of the worm-endemic equilibrium P_1 is proved by the Routh-Hurwitz criterion.

Theorem 2. *The worm-endemic equilibrium* P_1 *of the system (1) is locally asymptotically stable in* Ω *for* $R_0 > 1$.

Proof. The Jacobian matrix at P_1 is given by:

$$J_{P_1} = \begin{pmatrix} -\lambda I^* - \mu & -\lambda S^* & \omega & \varepsilon \\ \lambda I^* & J_{22}^1 & 0 & 0 \\ 0 & \gamma & -(\alpha+\mu+\omega) & 0 \\ 0 & \beta & \alpha & -(\varepsilon+\mu) \end{pmatrix},$$

where $J_{22}^1 = \lambda S^* - (\beta+\gamma+\mu)$.

The characteristic equation is

$$\lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0,$$

with

$$a_0 = \mu(\varepsilon+\mu)(\alpha+\mu+\omega)[\alpha(\beta+\varepsilon+\gamma+\mu)+\beta(\mu+\omega)+(\varepsilon+\mu)(\gamma+\mu+\omega)]L,$$

$$a_1 = (\varepsilon+\mu)(\alpha+\mu+\omega)[\mu+3\mu^2L+2\mu(\alpha+\beta+\varepsilon+\gamma+\omega)L+(\varepsilon\gamma+\alpha(\beta+\varepsilon+\gamma)+(\beta+\varepsilon)\omega)L],$$

$$a_2 = \alpha(\varepsilon+2\mu)+\varepsilon(2\mu+\omega)+\mu(3\mu+2\omega)+(\varepsilon+\mu)(\alpha+\mu+\omega)(\alpha+\beta+\varepsilon+\gamma+3\mu+\omega)L,$$

$$a_3 = \alpha+\varepsilon+3\mu+\omega+(\varepsilon+\mu)(\alpha+\mu+\omega)L,$$

where $L = \frac{G}{F}$ and G, F are given by equations (2) and (3).

From the Routh-Hurwitz criterion, the worm-endemic equilibrium P_1 is locally stable when

$$a_0 > 0,\ a_1 > 0,\ a_3 > 0 \ \text{ and } \ a_1a_2a_3 - a_1^2 - a_0a_3^2 > 0.$$

It is easy to see that $a_0 > 0$, $a_1 > 0$ and $a_3 > 0$. By using the Mathematica software, the conditions $a_1a_2a_3 - a_1^2 - a_0a_3^2 > 0$ is satisfied for $R_0 > 1$. ■

3.3 *Bifurcation of equilibria*

The change of local stability of the equilibria P_0 and P_1 can be explained by a transcritical bifurcation. In theory bifurcation, transcritical bifurcation is a local bifurcation in which an equilibrium having an eigenvalue whose real part passes through zero. In transcritical bifurcation, an equilibrium exists for all values of a parameter and is never destroyed. Such an equilibrium interchanges its stability with another equilibrium at bifurcation value, where they collide. In our system, the worm-free equilibrium P_0 always exists. It is stable for $R_0 < 1$ and unstable for $R_0 > 1$. The worm-endemic equilibrium P_1 exists for $R_1 > 1$ and it is unstable. If we suppose that P_1 also exists for $R_0 < 1$, although it is not real, then bifurcation in the model (1) can be seen as a form of transcritical bifurcation at $R_0 = 1$.

4 Global stability of equilibria

This section represents the global stability of equilibria in the model.

4.1 *Global stability of the worm-free equilibrium*

We use Lyapunov function method [22, 23] to prove the global stability of the worm-free equilibrium.

Theorem 3. *If* $R_0 \le 1$, *then the worm-free equilibrium* P_0 *of the system is globally asymptotically stable in* Ω.

Proof. We define the global Lyapunov function:

$$W(t) = (S - S_0 - S_0 \ln\frac{S}{S_0}) + I + a_1Q + a_2R.$$

where $S_0 = \frac{A}{\mu}$, $a_1 = \frac{(\alpha+\mu+\omega)\lambda A(1-R_0)}{2\mu R_0(\alpha\gamma+\beta(\alpha+\mu+\omega))} \ge 0$, and $a_2 = \frac{\lambda A(1-R_0)}{2\mu R_0(\alpha\gamma+\beta(\alpha+\mu+\omega))} \ge 0$.

The derivative of $W(t)$ along the solution curves of (1) in $\mathbb{R}_+^4$ is given by the expression:

$$\begin{aligned} W'(t) &= \left(1 - \tfrac{S_0}{S}\right)S' + I' + a_1Q' + a_2R' \\ &= \left(1 - \tfrac{A}{\mu S}\right)(A - \lambda SI + \omega Q + \varepsilon R - \mu S) \\ &\quad + [\lambda SI - (\beta+\gamma+\mu)I] + a_1[\gamma I - (\alpha+\mu+\omega)]Q \\ &\quad + a_2[\beta I + \alpha Q - (\varepsilon+\mu)R] \\ &= \left(1 - \tfrac{A}{\mu S}\right)(A + \omega Q + \varepsilon R - \mu S) \\ &\quad + \left(\tfrac{\lambda A}{\mu} - (\beta+\gamma+\mu) + \gamma a_1 + \beta a_2\right)I \\ &\quad + [-(\alpha+\mu+\omega)a_1 + \alpha a_2]Q - a_2(\varepsilon+\mu)R. \end{aligned}$$

Since $-(\gamma+\mu+\omega)a_1 + \alpha a_2 = 0$ we have

$$\begin{aligned} W'(t) &= \left(1 - \tfrac{A}{\mu S}\right)(A + \omega Q + \varepsilon R - \mu S) \\ &\quad + \left(\tfrac{\lambda A}{\mu} - (\beta+\gamma+\mu) + \gamma a_1 + \beta a_2\right)I \\ &\quad - a_2(\varepsilon+\mu)R. \end{aligned}$$

Moreover, we have

$$\gamma a_1 + \beta a_2 = -\tfrac{1}{2}\tfrac{\lambda A}{\mu R_0}(R_0 - 1),$$

$$\beta + \gamma + \mu = \tfrac{\lambda A}{\mu R_0},$$

$$\mu S \leq A.$$

This implies

$$\tfrac{\lambda A}{\mu} - (\beta + \gamma + \mu) + \gamma a_1 + \beta a_2 = \tfrac{\lambda A}{2\mu R_0}(R_0 - 1) \leq 0,$$

and

$$(1 - \frac{A}{\mu S})(A + \omega Q + \varepsilon R - \mu S) \leq 0.$$

Therefore, $W^{'}(t)$ is negative if $R_0 \leq 1$. Note that, $W^{'}(t) = 0$ if and only if $S = \frac{A}{\mu}$, $I = Q = R = 0$. Hence, the invariant set $\{(S, I, Q, R) \in \Omega : W^{'}(t) = 0\}$ is the singleton $\{P_0\}$, where P_0 is the worm-free equilibrium point. Therefore, by the LaSalle's Invariance Principle [24], P_0 is globally stable in the set Ω when $R_0 \leq 1$. This completes the proof. ■

4.2 Global stability of the worm-endemic equilibrium

In the following, we will discuss the global stability of the worm-endemic equilibrium P_1 as $R_0 > 1$ by using the geometric approach. Firstly, we present some preliminaries on the geometric approach to global dynamics [25].

Consider the autonomous dynamical system:

$$\dot{x} = f(x), \tag{4}$$

where $f : D \to \mathbb{R}^n$, $D \subset \mathbb{R}^n$ open set and simply connected and $f \in C^1(D)$.

Let $\bar{Q}(x)$ be an $\binom{n}{2} \times \binom{n}{2}$ matrix value function that is C^1 on D and consider

$$B = \bar{Q}_f \bar{Q}^{-1} + \bar{Q} J^{[2]} \bar{Q}^{-1},$$

where the matrix $\bar{Q}_f$ is

$$(q_{ij}(x))_f = (\partial q_{ij}(x)/\partial x)^{\top} \cdot f(x) = \nabla q_{ij} \cdot f(x),$$

and $J^{[2]}$ is the second additive compound matrix of the Jacobian matrix J, i.e. $J(x) = Df(x)$. In general, for a $n \times n$ matrix $J = (J_{ij})$, $J^{[2]}$ is a $\binom{n}{2} \times \binom{n}{2}$ matrix and in the case $n = 3$, we have

$$J^{[2]} = \begin{pmatrix} J_{11} + J_{22} & J_{23} & -J_{13} \\ J_{32} & J_{11} + J_{33} & J_{12} \\ -J_{31} & J_{21} & J_{22} + J_{33} \end{pmatrix}$$

Consider the Lozinskiĭ measure μ of B with respect to a vector norm $|\cdot|$ in $\mathbb{R}^{\binom{n}{2}}$ (see [26]), that is

$$\mu(B) = \lim_{h \to 0^+} \frac{|I + hB| - 1}{h}.$$

Define a quantity $\overline{q}_2$ as

$$\overline{q}_2 = \lim_{t \to \infty} \sup \sup_{x_0 \in K} \frac{1}{t} \int_0^t \mu(B(x(s, x_0)))ds.$$

We will apply the following:

Theorem 4. *(see [25]) Assume that D is simply connected, and*

(H_1) There exists a compact absorbing set $K \subset D$,

(H_2) The system (4) has a unique equilibrium $\tilde{x}$ in D,

then the unique equilibrium $\tilde{x}$ of (4) is globally asymptotically stable in D if $\overline{q}_2 < 0$.

Theorem 5. *For $R_0 > 1$, system admits an unique worm-endemic equilibrium P_1 which globally asymptotically stable, provided that $2\varepsilon < \beta + \gamma$.*

Proof. Because $R(t) = N - S(t) - I(t) - Q(t)$, it is sufficient to consider the three-dimensional system:

$$\begin{aligned} \tfrac{dS}{dt} &= A + \varepsilon N - \lambda S(t)I(t) - (\varepsilon + \mu)S(t) - \varepsilon I(t) \\ &\quad + (\omega - \varepsilon)Q(t) \\ \tfrac{dI}{dt} &= \lambda S(t)I(t) - (\beta + \gamma + \mu)I(t) \\ \tfrac{dQ}{dt} &= \gamma I(t) - (\alpha + \mu + \omega)Q(t). \end{aligned} \tag{5}$$

The Jacobian matrix of the system (5) is

$$J = \begin{pmatrix} -\lambda I - (\varepsilon + \mu) & -\lambda S - \varepsilon & \omega - \varepsilon \\ \lambda I & \lambda S - (\beta + \gamma + \mu) & 0 \\ 0 & \gamma & -(\alpha + \mu + \omega) \end{pmatrix}.$$

The associated second compound matrix is given by

$$J^{[2]} = \begin{pmatrix} J_{11}^2 & 0 & \varepsilon - \omega \\ \gamma & J_{22}^2 & -\lambda S - \varepsilon \\ 0 & \lambda I & J_{33}^2 \end{pmatrix},$$

where

$$\begin{aligned} J_{11}^2 &= \lambda S - \lambda I - (\beta + \varepsilon + \gamma + 2\mu), \\ J_{22}^2 &= -\lambda I - (\alpha + \varepsilon + 2\mu + \omega), \\ J_{33}^2 &= \lambda S - (\alpha + \beta + \gamma + 2\mu + \omega). \end{aligned}$$

We set the matrix function $\bar{Q}$ by

$$\bar{Q} = \text{diag}\left\{1, \frac{I}{Q}, \frac{I}{Q}\right\}.$$

Then $\bar{Q}_f \bar{Q}^{-1} = \text{diag}\left\{0, \frac{I^{'}}{I} - \frac{Q^{'}}{Q}, \frac{I^{'}}{I} - \frac{Q^{'}}{Q}\right\}$. We obtain

$$B = \bar{Q}_f \bar{Q}^{-1} + \bar{Q} J^{[2]} \bar{Q}^{-1}$$

$$= \begin{pmatrix} \lambda S - \lambda I - (\beta + \varepsilon + \gamma + 2\mu) & 0 & (\varepsilon - \omega)\frac{Q}{I} \\ \gamma \frac{\varepsilon I}{Q} & b_{22} & -\lambda S - \varepsilon \\ 0 & \lambda I & b_{33} \end{pmatrix},$$

where

$$b_{22} = \tfrac{I^{'}}{I} - \tfrac{Q^{'}}{Q} - \lambda I - (\alpha + \varepsilon + 2\mu + \omega),$$

$b_{33} = \frac{I'}{I} - \frac{Q'}{Q} + \lambda S - (\alpha+\beta+\gamma+2\mu+\omega).$

The matrix B can be written in block form

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$

where

$B_{11} = \lambda S - \lambda I - (\beta+\varepsilon+\gamma+2\mu)$, $B_{12} = \left(0 \;\; (\varepsilon-\omega)\frac{Q}{I} \right)$,

$$B_{21} = \begin{pmatrix} \gamma\frac{I}{Q} \\ 0 \end{pmatrix}, \; B_{22} = \begin{pmatrix} b_{11} & -\lambda S-\varepsilon \\ \lambda I & b_{22} \end{pmatrix},$$

where

$b_{11} = \frac{I'}{I} - \frac{Q'}{Q} - \lambda I - (\alpha+\varepsilon+2\mu+\omega),$

$b_{22} = \frac{I'}{I} - \frac{Q'}{Q} + \lambda S - (\alpha+\beta+\gamma+2\mu+\omega).$

The vector norm $|\cdot|$ in $\mathbb{R}^3$ can be chosen as

$$|(u,v,w)| = \max\{\, |u|, |v|+|w| \,\}.$$

Let μ denote the Lozinskiĩ measure with respect to this norm. Then we can obtain

$$\mu(B) \le \sup\{g_1, g_2\},$$

with

$$g_1 = \mu_1(B_{11}) + |B_{12}|,$$
$$g_2 = \mu_1(B_{22}) + |B_{21}|,$$

where $|B_{12}|$, $|B_{21}|$ are matrix norms with respect to the L^1 vector norm and μ_1 denotes the Lozenskiĩ measure with respect to the L^1 norm. Specifically,

$\mu_1(B_{11}) = \lambda S - \lambda I - (\beta+\varepsilon+\gamma+2\mu),$

$|B_{12}| = \max\left\{ 0, \left|(\varepsilon-\omega)\frac{Q}{I}\right| \right\} = |\varepsilon-\omega|\frac{Q}{I}.$

$|B_{21}| = \gamma\frac{I}{Q},$

$\mu_1(B_{22})$

$= \max\{\frac{I'}{I} - \frac{Q'}{Q} - \lambda I - (\alpha+\varepsilon+2\mu+\omega) + |\lambda I|,$

$\frac{I'}{I} - \frac{Q'}{Q} + \lambda S - (\alpha+\beta+\gamma+2\mu+\omega) + |-\lambda S-\varepsilon|\}.$

$= \frac{I'}{I} - \frac{Q'}{Q} - (\alpha+\varepsilon+2\mu+\omega) + \max\{0, 2\lambda S+2\varepsilon - (\beta+\gamma)\}.$

We have

$$g_1 = \lambda S - \lambda I - (\beta+\varepsilon+\gamma+2\mu) + |\varepsilon-\omega|\frac{Q}{I}.$$

It follows from that $\frac{I'}{I} = \lambda S - (\beta+\gamma+\mu)$, then

$$g_1 = \frac{I'}{I} - (\varepsilon+\mu) - \lambda I + |\varepsilon-\omega|\frac{Q}{I}.$$

Moreover,

$$g_2 = \frac{I'}{I} - \frac{Q'}{Q} - (\alpha+\varepsilon+2\mu+\omega) + \max\{0, 2\lambda S+2\varepsilon-(\beta+\gamma)\} + \gamma\frac{I}{Q}.$$

Based on the equation

$$\frac{Q'}{Q} = \gamma\frac{I}{Q} - (\alpha+\mu+\omega),$$

we obtain

$$g_2 = \frac{I'}{I} - (\varepsilon+\mu) + \max\{0, 2\lambda S+2\varepsilon-(\beta+\gamma)\}.$$

Since (1) is uniformly persistent, there exists $T_1 > 0$ such that $S(t) \le \frac{\beta+\gamma-2\varepsilon}{2\lambda}$ for $t > T_1$. Moreover, there also exists T_2 such that $Q(t) \le \frac{\lambda}{|\varepsilon-\omega|} I^2(t)$ for $t > T_2$. Therefore, for $t > T = \max\{T_1, T_2\}$ we have $-\lambda I + |\varepsilon-\omega|\frac{Q}{I} \le 0$ and $2\lambda S + 2\varepsilon - (\beta+\gamma) \le 0$. This implies

$$g_1 \le \frac{I'}{I} - (\varepsilon+\mu),$$

and

$$g_2 \le \frac{I'}{I} - (\varepsilon+\mu)$$

Therefore

$$\mu(B) \le \frac{I'}{I} - (\varepsilon+\mu).$$

Thus, for $t > T$ we have

$$\frac{1}{t}\int_0^t \mu(B)ds \le \frac{1}{t}\log\frac{I'(t)}{I(t)} + \frac{1}{t}\int_0^T \mu(B)ds - (\varepsilon+\mu)\frac{t-T}{t},$$

which implies $\overline{q}_2 < 0$. This complete the proof. ■

5 Numerical simulation

In this section, we realize a numerical investigation for the system (1) to illustrate the analytic results obtained above. Numerical results are represented in the following figures.

Figure 3 shows time series of solutions of the model as $R_0 \le 1$. For $A = 3$, $\alpha = 0.005$, $\beta = 0.25$, $\varepsilon = 0.02$, $\gamma = 0.001$, $\lambda = 0.25$, $\mu = 0.75$ and $\omega = 0.01$, we have $R_0 = 0.8138 < 1$. In this case, the worm-free equilibrium P_0 is globally asymptotically stable. With the initial condition $(S(0), I(0), Q(0), R(0)) = (1.5, 2.75, 1.15, 0.01)$, the infectious component $I(t)$ of solution tends to 0 as t approaches to $+\infty$. This implies that the spread dies out.

Figure 4 indicates time series of solutions of the model as $R_0 > 1$. For $A = 2$, $\alpha = 0.075$, $\beta = 0.1$, $\varepsilon = 0.1$, $\gamma = 0.75$, $\lambda = 0.32$, $\mu = 0.35$ and $\omega = 0.005$, we have $R_0 = 1.5238 > 1$. In this case, the worm-endemic equilibrium is globally asymptotically stable. With the initial condition $(S(0), I(0), Q(0), R(0)) = (2.5, 3.75, 0.025, 4.0)$, the infectious component $I(t)$ approaches to positive value 0.957 as t tends to $+\infty$. This means that the spread remains in the population.

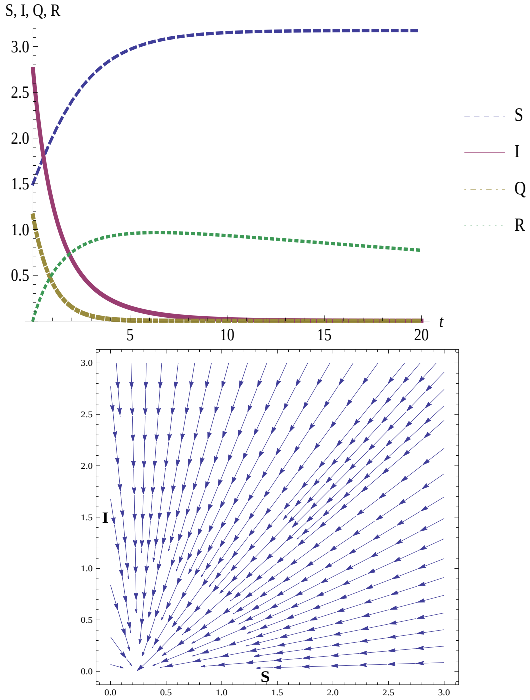

Fig. 3: Time series of solutions of the model and the vector field as $R_0 \leq 1$.

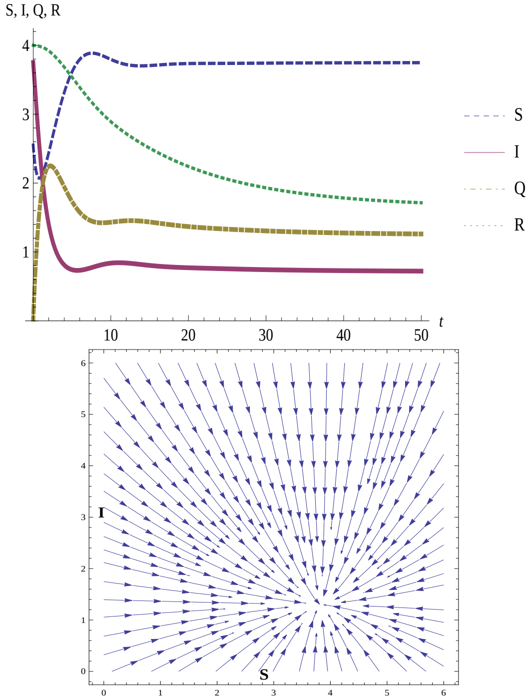

Fig. 4: Time series of solutions of the model and the vector field as $R_0 > 1$.

Using AUTO software package [21], we can detect the transcritical bifurcation in the model. For $A = 1$, $\alpha = 0.35$, $\beta = 0.015$, $\varepsilon = 0.025$, $\gamma = 0.025$, $\mu = 0.19$, $\omega = 0.75$ let λ vary then we get a transcritical bifurcation occurring at the value $\lambda = 0.874$. The bifurcation diagram for this case is given in Fig. 5 (above figure). For $A = 1$, $\alpha = 0.35$, $\beta = 0.025$, $\varepsilon = 0.025$, $\gamma = 0.15$, $\lambda = 0.35$, $\omega = 0.75$, let μ vary then a transcritical bifurcation occurs at $\mu = 0.003$. The bifurcation diagram is given in Fig. 5 (below figure). In the figure 5, the line passing through the solution 1, 2 and 3 is the curve of worm-free equilibrium, and the line containing the solution 4, 2 and 5 is the curve of the worm-endemic equilibrium. The solid line is for stable equilibria and the dashed line is for unstable equilibria. A transcritical bifurcation occurs at the solution 2, corresponding to $R_0 = 1$. The same bifurcation is also obtained for other variable parameters.

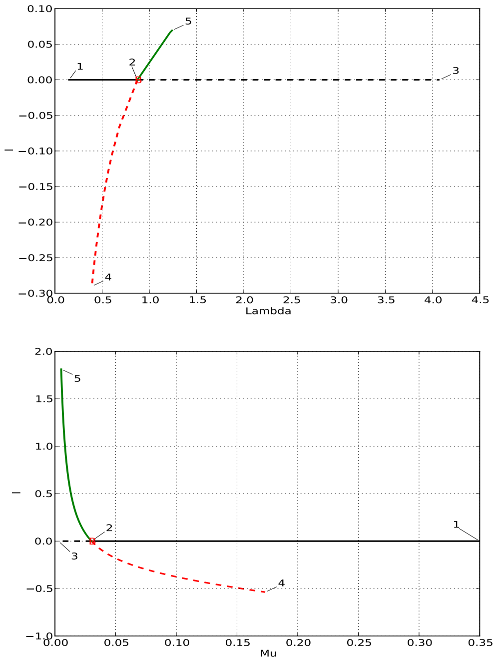

Fig. 5: Bifurcation of the model as λ varies (above figure) and μ varies (below figure).

6 Conclusions

In this paper, a proposed model for propagation of worms in the wireless sensor network, that closing to P2G infection mode network and containing quarantine, is introduced and studied. Theoretical analysis indicates the global stability of equilibria. The basic reproduction number R_0 is the threshold condition that determines the propagation dynamics. When $R_0 \leq 1$, the system has only a worm-free equilibrium P_0 which is globally asymptotically stable. It implies that the spread is extinct eventually. When $R_0 > 1$, the system has a unique worm-endemic equilibrium P_1, which is globally asymptotically stable. This shows that the transmission persists in the network and tends to a positive steady state. The local bifurcation, occurring at $R_0 = 1$, is explained by the transcritical bifurcation. As results indicate that spread of worms is very sensitive to contact parameter λ, and transform parameters β, γ and μ. The propagation will slow down if the value of λ is decreasing, and β, γ and μ are increasing (see Fig. 5). Therefore, we need to develop effective firewall network systems that can prevent the infection of worms (dearesing λ). Moreover, new antivirus programs and advanced network quarantine systems should be constructed from the field of artificial intelligence, allow us to detect, quarantine and remove worms in quickest way (decreasing β, γ, μ).

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