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Quantifying Diversification Effects of A Portfolio Using the Generalised Extreme Value Distribution- Archimedean Gumbel Copula Model

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Abstract: This paper uses the Generalized Extreme Value Distribution - Archimedean Gumbel copula modelling approach to quantify diversification effects in a bivariate portfolio of financial asset returns. This paper estimates Value at Risk (VaR) and Expected Shortfall (ES) of a portfolio consisting of the South African Industrial and Financial Indices using Monte-Carlo simulation. Results show that the portfolio risks are smaller than the sum of the individual component risks, indicating diversification benefits for investors. This approach is valuable for assessing, preparing, and mitigating risks in investment decisions, particularly for international investors considering cross-market diversification.

Keywords: Archimedean Gumbel copula, Diversification effects, Dependence structure, Generalised Extreme Value Distribution, Monte Carlo simulation.

Abbreviations

VaR - Value-at-Risk
IFM - Inference Function for Margins
SAR/(ZAR) - South African Rand
ES - Expected Shortfall
BM- Block Maxima
GBP - British Pound
GFC - Global Financial Crisis
EUR - European Union euro
CVaR - Conditional Value-at-Risk
ALSI - All Share Index
EVT - Extreme Value Theory
KLSE - Malaysian futures markets
USD - United States Dollar
DCE - Dalian Commodity Exchange
ADF - Augmented Dickey Fuller
J520 - South African Industrial Index
MLE - Maximum Likelihood Estimation
J580 - South African Financial Index
BM - Block Maxima
GPD - Generalised Pareto Distribution
GEVD - Generalised Extreme Value Distribution
BRICS - Brazil, Russia, India, China and South Africa
SGX-DT - Singapore Exchange Derivatives Trading Limited
G7 - Canada, France, Germany, Italy, Japan, the United Kingdom and the United States

1 Introduction

In the world of investment, the subject of building a portfolio is still one of the frequently discussed subjects and unquestionably vital for investors and practitioners. Understanding and forecasting the portfolio risk is worth investigating since there are diversification benefits to be harvested. In finance, multivariate analysis is used mainly to model risk for the joint large losses, which may lead to disaster in the event of say, a stock market crash. According to [1] investors and practitioners need to know how strongly stock markets are interconnected in order to make the appropriate financial decisions as they face difficult financial situations in choosing how much of their assets to invest in each sub-market. This paper extends studies by [2,3,4] who used bivariate portfolios to estimate bivariate portfolio risk, however this paper

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goes further in estimating diversification effects/benefits for the portfolio under consideration. According to [5] diversification benefits of international investing are on the decrease since there is higher dependence and correlation between Global stock markets (which mainly excluded developing countries' stock markets). The developing countries' markets are less dependent on major Global stock markets. The benefit of diversification is minimised when there is positive correlation between the assets, making the risk of the portfolio equivalent to that of any individual equity or even greater [6]. Efforts have been made to explore interdependence among stock returns at the industry/sector level [7]. The South African Industrial and Financial sectors are used in this paper to show such diversification benefits. The main metrics of risk measure are the Value at Risk (VaR) and Expected Shortfall (ES). [8] stated that investors and practitioners commonly use the VaR metric as a risk measure, although it is not sub-additive, and hence an incoherent risk measure. Practically VaR can be sub-additive in certain situations [9], and hence consistent with the diversification concept of modern portfolio theory. This property of sub-additivity is a mathematical description of diversification which illustrates the many benefits of portfolio diversification [6] to be estimated and quantified in this paper and is discussed in Section 3.5. [10] proposed to apply a combination of the heavy-tailed EVT statistical distributions and copula functions to overcome the limitations of using the Normal distribution approach to estimating VaR and ES.

This paper will quantify diversification effects of a portfolio consisting of the South African Industrial Index(J520) and the South African Financial Index (J580) using the Generalised Extreme Value Distribution (GEVD)-Archimedean Gumbel copula modelling approach. The financial returns are fitted separately to the GEVD as an estimate of the marginal distributions to both tails (gains and losses). The dependence structure (co-movement) is explained by the Archimedean Gumbel copula, and hence the construction of the joint probability distribution to the two financial risks, allowing for heavy-tailedness, asymmetry and nonlinearity in the financial returns distribution. This paper is confined to a bivariate case using an equally weighted portfolio, although in principle the ideas discussed can be applied to higher dimension models. Often univariate risks of component assets and portfolio risks are forecasted before quantifying diversification effects when portfolios are formed.

1.1 Statement of the problem

In finance, one of the challenging problems is managing risk, specifically forecasting portfolio risk, used in turn to quantify diversification effects. Extreme events that have catastrophic consequences do not only happen in isolation, but are made worse as a result of inter-connectedness/interaction/dependence of risky events. The Global Financial Crisis (GFC) of 2007-2008 exposed how interconnected the financial systems have become globally [1]. Each event or risk alone can be devastating, but together the risk level is compounded. Therefore the ability to quantify diversification effects of compound multivariate extreme values is important to allow for assessment, preparation and risk mitigation, for better investment decisions. If the portfolio risks are not estimated correctly, the incorrect estimates would result in failure to properly diversify risk with resultant potential disastrous consequences. The main reason investors seek diversification is to reduce the portfolio risk inherent in investing in risky assets. This paper will estimate diverification effects/benefits of a bivariate portfolio consisting of two financial assets, viz; J520 and J580 indices

1.2 Justification of the Study

The rapid increase in Globalisation of information and capital has created dependencies among the stock markets and industrial sectors [11]. To estimate dependence in risks is crucial for portfolio diversification aims. If there is low correlation between two markets, a portfolio comprising of financial assets from those markets might be used to create a diversified portfolio. The use of the GEVD-Archimedean Gumbel copula modelling approach between financial assets provides a more efficient method to estimate portfolio risk and the subsequent diversification effects. The copula approach, circumvents the limitations of multivariate dependence analysis by defining dependence structures that are determined by the properties of the data [12]. [3] found that univariate and multivariate models based on traditional Normal distribution-based models produced much less accurate risk estimations in the majority of cases.

1.3 Objective of the Study

The main objective of this paper is to estimate portfolio risk in order to quantify diversification effects of investing in a portfolio consisting of two financial Indices: the J520 and the J580, using the GEVD-Archimedean Gumbel copula model and Monte - Carlo simulation of an equally weighted portfolio. The specific objectives are:

–To fit the GEVD marginals to the two Indices returns (separately for losses and gains).
–Estimate univariate VaR and ES using the GEVD model.
–To determine which bivariate copula is to be fitted to the bivariate distribution (in this case it was found to be the Archimedean Gumbel copula which was fitted to the GEVD marginal distributions)

– Estimate the portfolio risks using GEVD-Archimedean Gumbel copula model and to interpret the risk measures associated with the portfolio.
– To quantify the diversification effects thereof.

Given the limited empirical investigations of the dependence structure (co-movements) of the South African sector Indices, extreme correlations and diversification effects at a sector level, the bivariate portfolio consisting of the South Africa Industrial and Financial Sectors and Indices is considered in this paper. To the best of our knowledge, no such application has been done to the South African assets, more specifically to the mentioned assets. The paper is organised as follows: section 2: Literature Review, section 3: Research Methodology, section 4: Empirical results and discussion and section 5: Conclusions and future possible research.

2 Literature Review

The Extreme Value Theory (EVT) - copula model offers to investors and practitioners, a powerful tool to model the portfolio risk and diversification effects/benefits between the different financial assets and are preferable to the traditional linear correlation-based methods [3]. According to literature, the common marginal distributions used in the presence of extremes, are the GEVD and GPD marginals [12].

[13] used the GEVD marginals to investigate the tail behaviour of the palm oil futures markets. The copula model was used estimate the dependence structure and the joint probability distribution between the returns of palm oil futures markets. The Gumbel-copula and Husler Reiss- copula were used to estimate this dependence structure. The results reveal that the KLSE and SGX-DT are highly dependent. The findings also show that there is no dependence between KLSE and DCE, SGX-DT and DCE. These findings are beneficial to investors and practitioners who wish to get involved the in trading of palm oil commercially, whilst minimising their portfolio investment risk.

The VaR of an investment portfolio consisting of IBOVESPA (Brazil) and MERVAL (Argentina) Indices was estimated using the EVT-Archimedean copula model [14]. The GPD marginals were used to describe the tails of the left tail (large losses). The Monte-Carlo simulation method was used to quantify portfolio VaR and the findings were compared with other traditional methods. The EVT-Archimedean copula model outperformed other traditional methods. The GPD-Gumbel copula model produced the better model for the losses. The model can also be applied to model the largest gains. The results illustrate that the EVT-copula model is a robust method which can be very useful in estimating the VaR of a portfolio in the presence of extremes in the data.

[4] forecasted the portfolio VaR of currency exchange rates using an EVT- copula model. The data set included exchange rates between the US dollar (USD), British pound sterling (GBP), Euro (EUR), and South African Rand (SAR) and the Kenyan shilling (KES). The GPD was used to characterise the tails of the distribution of the returns. A copula was used to construct the dependence structure/ the joint probability distribution among the currencies returns. The Monte Carlo simulation of an equally weighted portfolio of four currency exchange rates was used to estimate the portfolio VaR. The results showed that the Student-t-copula as the most suitable copula to construct the dependence structure/joint probability distributions of the currency exchange rates.

[15] estimated diversification effects of financial assets using the GPD and a Student-t-copula. Financial returns data from the United States of America banks was used. The GPD was used to model the tails of the financial return distributions of the data from the banks. The dependence structure (co-movements) was modelled using a Student-t-copula since the main interest was the joint extreme values. The univariate VaR and ES were estimated, including the portfolio VaR and ES. These risk measures were used to estimate the diversification effects. The results showed that U.S. diversification effects range widely from 20% to 70%.

[16] used the GPD- copula approach to estimate the multivariate portfolio risk of natural gas. The ARMA-GARCH was applied to fit natural gas return series. The GPD was used as the marginal distribution. The Gaussian and Student-t-copula were used to estimate the dependence structure of the natural gas portfolio. The researchers used the Mote-Carlo simulation method to estimate VaR and CVaR. The results revealed that the portfolio risk estimated from the Student-t-copula were larger and more appropriate than those from the Gaussian copula.

[17] used a Skewed-Student-t-distribution and a copula to model diversification effects for Japanese Stock Indices and bonds. The researcher quantified diversification effects using the Archmedian Gumbel copula functions, Student-t- distribution and the Gaussian copula. The researcher used high quantiles to estimate the VaR and ES. For the Japanese stock Indices and bonds, the Skewed-Student-t-distribution was used as the marginal distribution. The results indicate that the dependence structure increased estimates for portfolio risk and thus reducing diversification benefits.

[18] applied the EVT-copula model to estimate the dependence structure of a portfolio of stock market returns for the G7, BRICS and other 14 emerging stock markets. The GPD was used to characterise the tails of the lower and upper tails in order to forecast the portfolio risk. The returns were simulated to estimate portfolio risk. The EVT-copula model results were more accurate than historical simulation.

There are many other studies in literature which include [19,20,21,22,23,24,25,26] that have applied copula functions to describe the multivariate dependence structure between stock markets with the different statistical distributions as marginals to estimate portfolio risk for the returns. This paper uses the GEVD marginal distributions which requires the Block Maxima (BM) method in modelling the data. The BM method has the advantage of producing less auto-correlated data and is hence adopted in this paper. This paper also differs from the other studies discussed as it uses an unconditional approach which involves the direct application of GEVD as the marginal distribution to model extreme risk in the context of South African asset returns. This paper will provide investors and practitioners a model framework that will allow for portfolio risk and diversification effects to be estimated more accurately, whilst providing information on the South African assets.

3 Methodology

This paper uses the GEVD for modelling the marginal returns distributions and the Archimedean Gumbel copula for modelling the dependence structure (co-movement) of the bivariate financial asset returns. In general, it is argued that the EVT-copula performs better than the traditional Normal distribution-based models in estimating portfolio VaR [12]. According to [27] copula functions provide alternative interpretations of non-linear relationships between associated random variables or their marginals. Monte Carlo simulation is used to estimate portfolio risk, which in turn , is then used to quantify diversification effects. According to [28] it is challenging to estimate the joint distribution of risk components, but the copula approach offers a method for separating the marginal behaviour from the dependence structure.

3.1 Marginal distributions

Copulas are the functions that describe the dependence structure between random variables and couple say, the GEVD marginal distributions of these variables into their joint distribution function. Given the descriptive statistics, the conclusion is that the data is heavy-tailed, and hence the GEVD was deemed suitable for characterising the tails of the marginal distributions.

Generalised Extreme Value Distribution (GEVD) The GEVD is presented by [29] and [30] as follows;

$$G_{\xi,\mu,\sigma}(x) = \exp\left(-\left[1+\xi\left(\frac{x-\mu}{\sigma}\right)\right]^{\frac{-1}{\xi}}\right) \quad \text{if} \quad \xi \neq 0 \tag{1}$$

$$G_{\xi,\mu,\sigma}(x) = exp[-exp(\frac{(x-\mu)}{\sigma})] \quad \text{if} \quad \xi = 0, \tag{2}$$

where, $\sigma > 0$ and $1+\xi(\frac{(x-\mu)}{\sigma}) > 0$, *mu* is the location parameter, σ is the scale parameter and ξ is the shape parameter.

When $\xi > 0$, G becomes the heavy-tailed Fretchet class distribution. When $\xi < 0$, G becomes a short-tailed Weibull class distribution. $\xi = 0$ gives G as a light-tailed Gumbel class distribution. Practically, to select the data to use for parameter estimation, the monthly data is first divided into non-overlapping blocks (eg quarterly blocks), and the maximum in each block is identified. The maxima are then used in parameter estimation.

The Maximum Likelihood Estimate (MLE) of the unknown parameters μ, σ, and ξ is optimised from the logarithmic likelihood of the GEVD model with respect to the parameters [31]. The equation of logarithmic likelihood of the model with n observations is represented as follows:

$$I_{\xi,\mu,\sigma}(x) = -n\ln\sigma\left(1+\frac{1}{\xi}\sum_{i=1}^{n}\ln\left(1+\xi\left(\frac{x_i-\mu}{\sigma}\right)\right)-\sum_{i=1}^{n}\ln\left(1+\xi\left(\frac{x_i-\mu}{\sigma}\right)\right)^{\frac{-1}{\xi}}\right) \tag{3}$$

Where $1+\xi\left(\frac{x_i-\mu}{\sigma}\right) > 0$ for $i = 1,........n$.

Differentiating the log of the likelihood of the GEVD model with respect to the parameters will result in the estimated parameters when the equations are equated to zero. Numerical methods are often used to find solutions do the equations, as the solutions are not in closed form.

3.2 The concept of Archimedean copula model

The bivariate Archimedean copula is expressed as follows:

$$C(\mu_1,\mu_2 = \phi^{-1}(\phi(\mu_1)+\phi(\mu_2)), \tag{4}$$

Where $\phi(x))$ is the generator function with a convex decreasing function defined in [0,1], satisfying $\phi(1) = 0$ and $\lim_{t\to 0}(\phi(t)) = 1$. Where $\phi^{-1}(x)$ is an inverse function. The function ϕ represents various types of Archimedean copulas, including the Archimedean Gumbel copula which is adopted in this paper. In this section, the basic properties for the Archimedean copulas are presented. The Archimedean copulas are used widely in practice because:

- –They are simple to formulate.
- –Most of the parametric families of copulas belong to the Archimedean copula
- –They give various types of dependence structure
- –They use straightforward simple closed form expressions
- –They reduce multivariate copulas to single univariate functions
- –They are very useful in empirical modelling

In many financial applications, there is a strong upper tail dependence between extreme/maximum losses compared to extreme/maximum gains [32]. The Archimedean Gumbel copula is an asymmetric copula that exhibits upper tail dependence in the upper corner

[33]. The Archimedean Gumbel copula is defined by [34] and [35] as:

$$C^{G_\mu}(\mu_1,\mu_2,\theta) = \exp\left(-\left[(-\log\mu_1)^\theta + (-\log\mu_2)^\theta\right]^{\frac{1}{\theta}}\right) \quad (5)$$

Where, parameter $\theta \in [1,\infty)$, μ_1 and μ_2 are say, GEVD uniform cumulative marginal distributions of the J520 and J580 returns respectively.

The Archimedean Gumbel parameter θ and Kendall's tau (τ) are related by the following function:"

$$\tau = \frac{\theta - 1}{\theta} = 1 - \theta^{-1}, \quad (6)$$

Where, θ is a parameter for dependence. The Archimedean Gumbel copula's upper (λ_U) and lower (λ_L) tail dependence are estimated by the following functions: $\lambda_U = 2 - 2^{-\theta}$ and $\lambda_L = 0$.

3.3 Parameter Estimation

The MLE approach is used to quantify the parameters for the GEVD marginal distributions. The Inference Function for Margins (IFM) method is used estimate the copula parameters.

3.4 Estimation of Risk

X and Y are, let's say, two financial asset returns, and a risk measure $\rho()$ is coherent if the following four axioms are satisfied, according to [36]:

Axiom 1: Monotonicity

$$\rho(Y) \geq \rho(X), \quad \text{if} \quad X \leq Y \quad (7)$$

A higher predicted loss necessitates holding more capital.

Axiom 2: Sub-additivity

$$\rho(X+Y) \leq \rho(X) + \rho(Y) \quad (8)$$

The level of risk will not increase when two or more risk factors are combined or merged. The diversification benefits in a portfolio are captured by this axiom. A portfolio's risk should be less than (or equal to) the combined risk of its constituent securities [6]. This is the mathematical description of diversification which is applied in this paper.

Axiom 3: Homogeneity
For any number $k > 0$, $\rho(\alpha X) = k\rho(X)$ where, k is a constant positive amount; If we say, $k = 2$, then doubling the size of the loss situation, will double the risk.

Axiom 4: Translation Invariance

$$\rho(X+Y) \leq \rho(X) - k, \quad (9)$$

for any value of k which is a constant.

The capital needed to lessen the effects of the loss rises by the same amount if we increase the observed loss by a certain amount.

The risk measures adopted are associated with gains and losses for the GEVD-Archimedean Gumbel copula model are estimated. The estimation of the portfolio VaR/ES is done using the Monte Carlo simulation of an equally weighted portfolio. According to [37], VaR is common risk measure that is of interest to investors and practitioners as it enables them to evaluate the portfolio risks, and thus allowing them to mitigate against any potential losses.

3.5 Monte-Carlo Simulation method

A five -step estimation procedure [4,14,16] is followed when fitting the GEVD- Archimedean Gumbel copula model in this paper to arrive at the portfolio results:

Step 1: Fit the GEVD to the quarterly BM of the losses and gains separately using the two separate log return series, and thus arrive at univariate parameter estimates for the J520 and J580 returns (losses and gains) (see Table 4).

Step 2: Determine which bivariate copula is to be fitted to the bivariate return series for the gains and losses using a plot of the transformed variables μ_1 and μ_2 (see Figure 5 and Figure 6).

Step 3: The transformed return series pair should be fitted to the copula. Utilize the Inference Function for Margins (IFM) estimation method to calculate the dependence parameter. Again the gains and losses for the J520 and J580 returns are handled separately for the two series.

Step 4: Simulate N uniform random numbers (N = 5000 in this case) using the determined copula parameters representing the joint uniform cumulative distribution of the portfolio. Use the inverse quantile function of the distribution to transform the uniform random variable to the original scales of the log returns. Use the average of the input parameters (weighted) as the new parameters of the portfolio (see Table 5).

Step **Step 5:** two Indices were assumed to be equal, however this is optional and they can be adjusted freely [38]. The VaR and the ES risk measures are estimated by the Monte-Carlo simulation method and then they are used to estimate diversification effects.

3.6 Diversification effects

According to [15,17,39,40] the difference between the portfolio risk (diversified value) and the simple sum of individual component risks (undiversified values) expressed as a percentage of simple sum of individual component risks (undiversified value) is a measure of the diversification effects. This paper uses a diversification effects formula which is consistent with past studies.

The diversification effects formula is given as follows:

$$Diversification\ \ effects = \frac{Simple\ \ Sum\ \ VaR - Aggregate\ \ VaR}{Simple\ \ Sum\ \ VaR} \times 100\% \tag{10}$$

$$Diversification\ \ effects = \frac{Simple\ \ Sum\ \ ES - Aggregate\ \ ES}{Simple\ \ Sum\ \ ES} \times 100\% \tag{11}$$

Where, $Simple\ \ sum\ \ VaR$: it total sum from the addition of the VaR values of the two risk factors used in this paper and is greater than the portfolio VaR (i.e $Simple\ \ Sum\ \ VaR = VaR_1 + VaR_2 > VaR_{portifolio}$) $Simple\ \ sum\ \ ES$: it total sum from the addition of the ES values of the two risk factors used in this paper and is greater than the portfolio ES (i.e $Simple\ \ Sum\ \ ES = ES_1 + ES_2 > ES_{portifolio}$) $Aggregate\ \ VaR$: the portfolio VaR of the two risky factors $VaR_{portifolio}$ $Aggregate\ \ ES$: the portfolio ES of the two risky factors = $ES_{portifolio}$ The formula 10 and 11 will be used to estimate the diversification effects of a portfolio for VaR and ES.

3.7 Tests for stationarity, heteroscedasticity and autocorrelation

Table 1: Test for stationarity, heteroscedasticity and autocorrelation.

Test	Method
Stationarity	The Augmented Dickey Fuller (ADF) test (a unit root or non-stationary test) is used to test for stationarity [41] in J520 and J580 return series.
Heteroscedasticity	To test for the presence of Arch effect, the Lagrange Multiplier (LM) Test is used to test for the presence of heteroscedasticity [42], in residuals of J520 and J580 return series.
Auto-correlation	The Ljung-Box test is used to test for autocorrelation [43] of each of the J520 and J580 returns series.

Table 1 describes the tests for, heteroscedasticity, stationarity and autocorrelation adopted in this paper.

4 Research findings and discussion

This paper applied the GEVD-Archimedean Gumbel copula model to the monthly J520 and J580 returns over the period 1995 to 2018 in analysing a bivariate portfolio and the diversification effects thereof.

4.1 Software used and Research Data

The following packages were used for data analysis in the R programming environment: actuar, Copula, fCopula, QRM, Mass, evir, eva, fExtremes, and extRemes.

This paper makes use of secondary data of the South African stock market that was taken, with permission, from the website iress expert at https://expert.inetbfa.com. The analysis involved the use of the J520 and the J580 returns gains/loss distributions (data spanning the years: 1995-2018). These Indices are calculated from values of stocks in the Industrial and Financial sector companies listed on the South African stock market, and represents the performance of those specific industries within the stock market. The monthly logarithmic returns for both Indices are calculated as follows:

$$X_t = \ln \frac{M_t}{M_{t-1}} \tag{12}$$

where, x_t represents the monthly log returns at month in t, M_t is the monthly Index value at month t and ln- the natural logarithm.

4.2 Descriptive Statistics

The descriptive statistics for the monthly J520 and the monthly J580 Indices return series are given in Table 2.

Table 2: Descriptive statistics..

	Industrial Index (J520)	Financial Index(J580)
Observations	271	271
Minimum	-0.140273	-0.216516
Maximum	0.328471	0.511949
Mean	-0.009366	-0.008353
Median	-0.010478	-0.010155
Variance	0.003302	0.003651
Skewness	1.016932	2.194276
Kurtosis	4.420852	19.439520

In Table 2, the maximum and minimum values for the two indices are quite far apart from the mean, suggesting the presence of some extreme returns. The mean returns for the two log return distributions are small and near zero, showing that the trend in the return distribution is not significant. The skewness is positive for both Indices returns, which also suggests that the extreme values are present in the return series. The data exhibits excess kurtosis, which implies that the return distributions are fat-tailed and leptokurtic. The various attributes of returns distributions, such as skewness and high kurtosis are present which allows one to conclude that both Indices as heavy-tailed. The two financial assets are heavy-tailed arguing for the use of the GEVD marginal distributions in this paper.

[44] stated that the sub-Indices of the South Africa's All Share Index (ALSI) may not be informationally efficient, although the ALSI is weakly efficient. This may allow investor s to make excess profits when invested in the sub-Indices which are not informationally efficient. This implies that the sub-Indices returns for the J520 and J580 Indices returns may be modelled using extreme value distributions such as the GEVD. This confirms the results of descriptive statistics which revealed that the data sets are heavy-tailed. A loss distribution of the returns distribution will have gains on the left tail and losses on the right tail.

4.3 Tests for Stationarity, Heteroscedasticity and Auto-correlation

The return series data are checked for stationarity, heteroscedasticity and auto-correlation in order to confirm certain properties in the use of the statistical methodologies.

Testing for stationarity
The ADF test was used to investigate whether the monthly J520 and J580 Indices returns are stationary series. A p-value of greater than 0.05 confirmed the returns series data to be stationary.

Testing for heteroscedasticity
The Arch (LM) test is applied to test for heteroscedasticity in the J520 and J580 Indices returns series. The Arch (LM) test checks for the presence of ARCH effects. There were no ARCH effects in the J520 returns: χ^2 = 8.37, degrees of freedom = 12, p-value = 0.76 and the J580 returns: $\chi^2 = 6.24$, df = 12, p-value = 0.90. The p-values are greater than 0.05 for both the return series data, which led to the acceptance of H0 (there are no Arch effects).

Testing for auto-correlation
The Box-Ljung test is used to test for auto-correlation in the monthly J520 and J580 Indices returns series. The results gave a p-value ¿ 0.05 for each of the returns series data, indicating weak evidence against the null hypothesis, so we fail to reject the null hypothesis of no auto-correlation. This means that the returns distribution can be considered independently distributed. Therefore, applying the EVT to the return series is appropriate as each series is independently and identically distributed. The two series however, may be dependent to each other as discussed in the next section.

4.4 Analyzing the gains and losses using the GEVD model

The gains and losses are separated and modelled separately. for each of the series. Losses are mainly negative returns multiplied by negative one to make them positive. The GEVD is fitted to the two log return series data points for both the upper tail (gains) and lower tail (losses) of the distribution using the Block Maxima (BM) method.

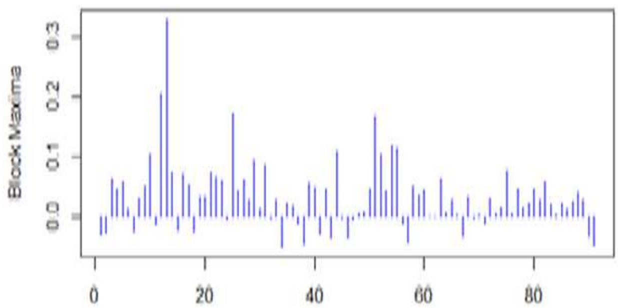

Fig. 1: The quarterly block maxima for J520 losses.

To use the GEVD, the data is put into quarterly blocks, and the maximum in each block is selected. Figure 1 and Figure 2 shows how the block maxima for losses are determined for the two indices. The data points obtained above from the BM method are fitted to the GEVD and used to estimate the model parameters. The gains maxima are selected similarly.

4.5 Model diagnostics for the quarterly block minima and maxima.

The diagnostic plots of the minima (losses) of the quarterly blocks are shown in Figures 3 and Figure 4 to assess the GEVD model's goodness of fit. The conclusion is that the GEVD model offers a strong model fit for the data at the tails of the distribution. This is supported by the Return Level plot, density plots, P-P plot , and the Q-Q plot which does not deviate from the straight line. The diagnostic plots

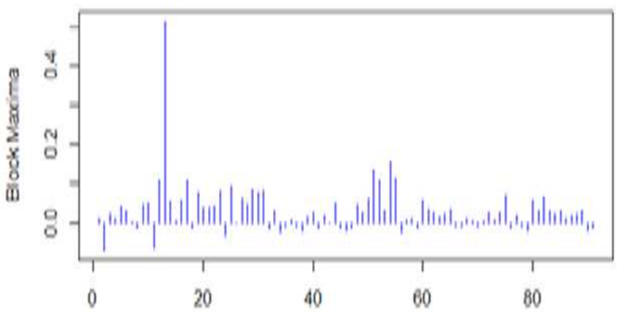

Fig. 2: The quarterly block maxima for the J580 losses.

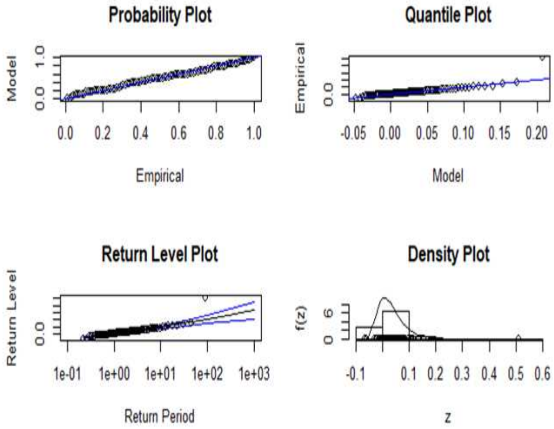

Fig. 4: Diagnostic plots for the quarterly block minima of the return distribution of the J580 losses.

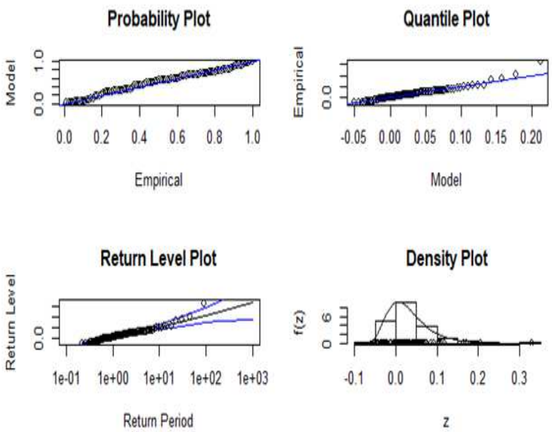

Fig. 3: Diagnostic plots for the quarterly block minima of the return distribution of the J520 losses.

Table 3: GEVD Model Parameters

Asset	Shape ξ	Scale (σ)	Location (μ)
Left / Upper Tail			
J520	-0.2915	0.0392	0.0384
J580	-0.0673	0.0362	0.0327
Average	-0.1794	0.0377	0.0356
Right / Lower Tail			
J520	0.0535	0.0400	0.0077
J580	0.0582	0.0385	0.0082
Average	0.0559	0.0393	0.0080

Table 4: Average Parameters for Inverse Copula Distribution.

Parameter	Shape (ξ)	Scale σ	Location (μ)
Left Tail/Gains	-0.1794	0.0362	0.0356
Right Tail/Losses	0.0559	0.0393	0.0080

support the fitted model at the tails of the J520 and J580 return distributions. The diagnostic plots for the maxima returns were done and are also in favour of the fitted model but are not presented in the paper.

The parameters obtained from fitting the GEVD model to the two financial asset return distributions are estimated in Table 3. The gains have a negative shape parameter, indicating that they are bounded. The losses have a positive shape parameter, a sign that heavy losses are indeed possible. These parameters are used to estimate the univariate VaR and ES) for the individual asset returns.

In Table 4 the average parameters for the two financial assets are calculated and used in the simulation of the J520 and J580 marginals which are used in the determination of inverse distribution of the Archimedean Gumbel copula.

4.6 Selection of appropriate copula function for bivariate analysis

The scatterplots are used as one of the methods for choosing the right copula function for fitting to the financial returns series data. The scatterplots are plotted in order to determine the type of copula to fit to the bivariate gains and the bivariate losses. In Figure 5, the bivariate series for gains seems to have an increasing pattern and convergence in upper tail, hence dependence could be modelled with the Archimedean Gumbel copula. The

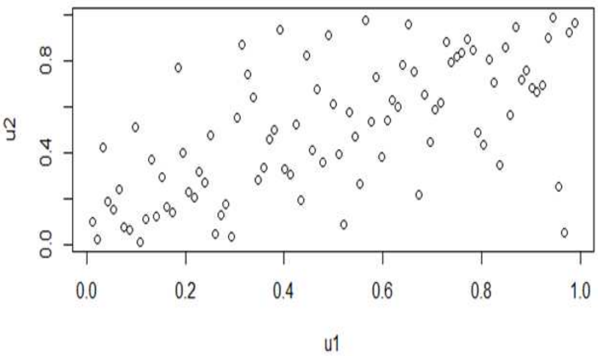

Fig. 5: Scatterplot of GEVD marginal distributions μ_1 and μ_2 for the bivariate gains

Archimedean Gumbel copula is thus proposed, and is able to capture dependency in extreme values.

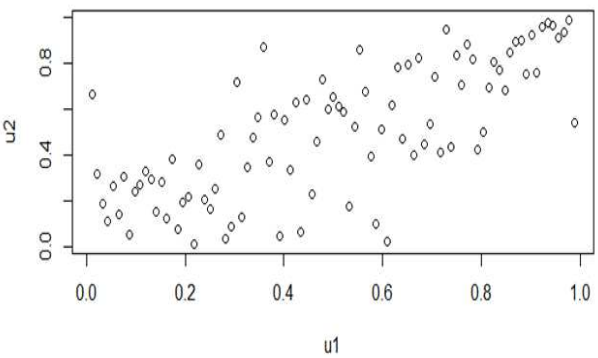

Fig. 6: Scatterplot of GEVD marginal distributions μ_1 and μ_2 for the bivariate losses

In Figure 6, The bivariate series shows an increasing pattern with an upper tail dependence, so this could again be modelled using the Archimedean Gumbel copula for the losses. The losses follow a Archimedean Gumbel copula in dependency since there is evidence of strong upper tail dependence and no distinct strong lower tail dependence.

4.7 Kendall?s tau and the copula (upper tail dependence) parameter measures

A test for the degree to which non-Normal returns data are dependent, is the Kendall's tau. A statistic called Kendall's tau is used to quantify the ordinal relationship between two measured quantities. Kendal's tau values are $\hat{\tau}$ =0.5921856 for the losses and $\hat{\tau}$ = 0.4554335 for the gains show that there is a positive correlation between the variables but not very strong (Table 6).

Table 5: Kendall's tau and copula parameters.

Copula	$\hat{\tau}$	$\hat{\theta}$	Upper $\hat{\lambda_U}$	Lower $\hat{\lambda_L}$
Gains	0.455	1.836	0.643	0
Losses	0.592	2.452	0.671	0

In Table 5, the estimated Archimedean Gumbel copula parameters (θ) for the losses and gains are $\hat{\theta}$=2.452 and $\hat{\theta}$ =1.832 respectively . The parameters imply the presence of tail dependence in the losses and gains. The two stock indices have dependence in the extremities for both the losses and gains analysed separately. The large losses and large gains from the two stock indices have greater probability to co-move together concurrently [32]. Expressed in other words, the two stock markets indices may tend to rise and fall together during periods of economic recessions and economic booms. The upper tail dependence measures for the gains and losses estimated using Archimedean Gumbel copula are $\hat{\lambda}_U$ is 0.6425575 and $\hat{\lambda}_U$ is 0.6713104 respectively. The large gains and losses from the two stock indices have a greater probability to co-move together concurrently. Tail dependence measures indicate the degree of extreme co-movements of large gains and losses in the stock markets, which allows investors and practitioners to quantify portfolio risk and quantify diversification effects.

4.8 Archimedean Gumbel copula density and contour plots

The Archimedean copulas allows modelling multivariate dependence with a one parameter to estimate the strength of the dependence.

In Figure 7, the density plots for the Archimedean Gumbel copula for the gains and losses are given.

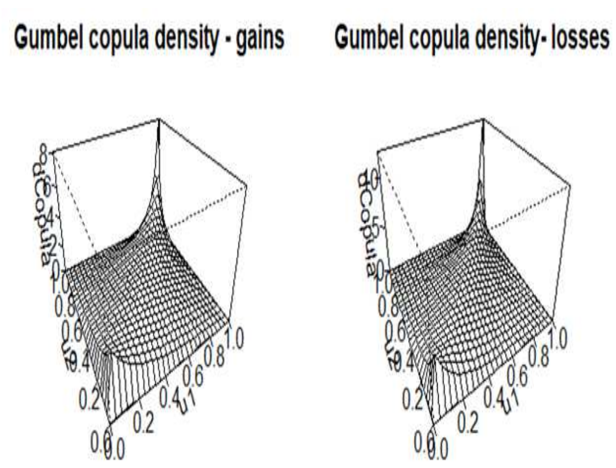

Fig. 7: Density plots of the joint distribution for the Gumbel copula (θ=1.836) for the gains and Gumbel copula (θ=2.439) for the losses.

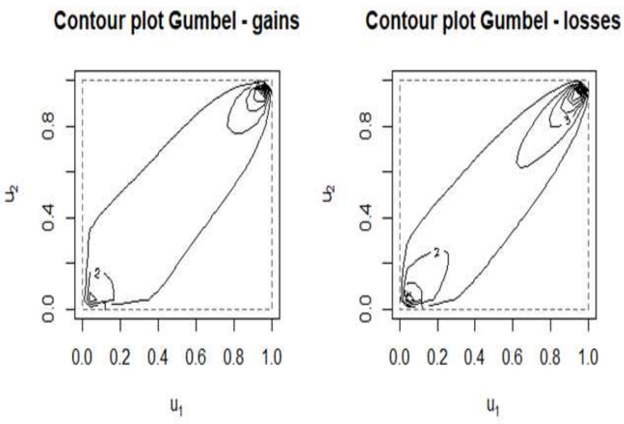

Fig. 8: Contour plots of the joint distribution for the Gumbel copula (θ=1.836) for the gains and Gumbel copula (θ=2.439) for the losses.

In Figure 8, the contour plots using the estimated parameters confirm the presence of the upper tail dependence for both the gains and losses respectively. The Archimedean Gumbel copula is characterised by the presence of the upper tail dependence as can be concluded from the density and contour plots respectively.

4.9 Estimation of univariate risk measures

The quarterly losses and gains are fitted to the GEVD to model the tails of the distributions to obtain the model parameters used to estimate VaR and ES.

Table 6: Estimates of univariate risk measures.

	J520		J580	
Alpha	VaR	ES	VaR	ES
Measures of Risk (Left Tail / Gains)				
0.950	0.1163	0.1642	0.1302	0.1759
0.990	0.1377	0.1887	0.1759	0.2258
0.995	0.1442	0.1956	0.1940	0.2445
Measures of Risk (Right Tail / Losses)				
0.950	0.1365	0.1614	0.1330	0.1578
0.990	0.2164	0.2470	0.2112	0.2426
0.995	0.2527	0.2843	0.2469	0.2783

In Table 6, the univariate VaR and ES of the J520 and J580 Indices returns are given. The J580 gains are riskier than the J520 gains since the risk measures are greater. The J520 losses are riskier than the J580 losses since the risk measures for the former are bigger. These risk measures are also used to determine the diversification effects of the portfolio.

4.10 Estimation of portfolio risk using the GEVD-Archimedean copula model.

The portfolio risk is estimated using the GEVD-Archimedean Gumbel copula model. Using the Monte-Carlo simulation of an evenly weighted portfolio, the portfolio VaR and ES are forecasted.

Table 7: Estimates of portfolio risk using the Monte-Carlo simulation method

Copula	Marginals	Portfolio VaR			Portfolio ES		
		95%	99%	99.5%	95%	99%	99.5%
Left Tail of loss distribution / Gains							
Gumbel	GEVD	0.1211	0.1534	0.1644	0.1409	0.1692	0.1802
Left Tail of loss distribution / Gains							
Gumbel	GEVD	0.1307	0.2138	0.2488	0.1839	0.2741	0.3483

Table 8: Kendall's tau and copula parameters.

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Losses	0.592	2.452	0.671	0

In Table 5, the estimated Archimedean Gumbel copula parameters (θ) for the losses and gains are

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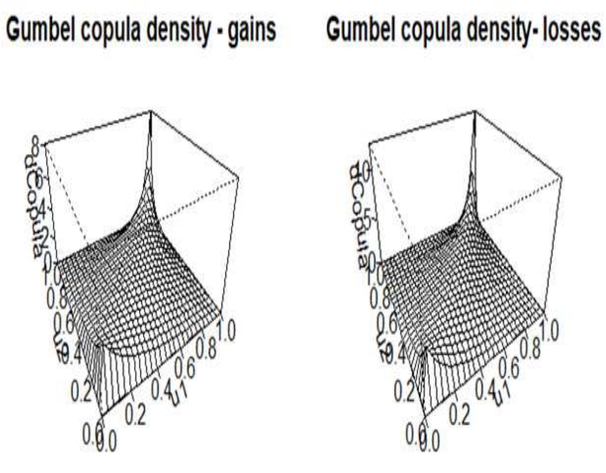

Fig. 9: Density plots of the joint distribution for the Gumbel copula (θ=1.836) for the gains and Gumbel copula (θ=2.439) for the losses.

In Figure 7, the density plots for the Archimedean Gumbel copula for the gains and losses are given.

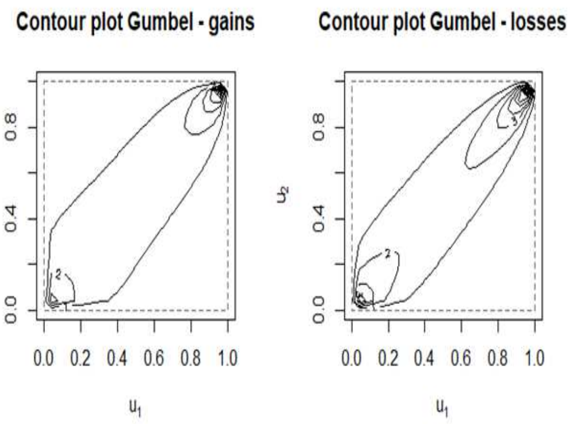

Fig. 10: Contour plots of the joint distribution for the Gumbel copula (θ=1.836) for the gains and Gumbel copula (θ=2.439) for the losses.

In Figure 8, the contour plots using the estimated parameters confirm the presence of the upper tail dependence for both the gains and losses respectively. The Archimedean Gumbel copula is characterised by the presence of the upper tail dependence as can be concluded from the density and contour plots respectively.

4.12 Estimation of univariate risk measures

The quarterly losses and gains are fitted to the GEVD to model the tails of the distributions to obtain the model parameters used to estimate VaR and ES.

Table 9: Estimates of univariate risk measures.

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0.990	0.1377	0.1887	0.1759	0.2258
0.995	0.1442	0.1956	0.1940	0.2445
Measures of Risk (Right Tail / Losses)				
0.950	0.1365	0.1614	0.1330	0.1578
0.990	0.2164	0.2470	0.2112	0.2426
0.995	0.2527	0.2843	0.2469	0.2783

In Table 6, the univariate VaR and ES of the J520 and J580 Indices returns are given. The J580 gains are riskier than the J520 gains since the risk measures are greater. The J520 losses are riskier than the J580 losses since the risk measures for the former are bigger. These risk measures are also used to determine the diversification effects of the portfolio.

4.13 Estimation of portfolio risk using the GEVD-Archimedean copula model.

The portfolio risk is estimated using the GEVD-Archimedean Gumbel copula model. Using the Monte-Carlo simulation of an evenly weighted portfolio, the portfolio VaR and ES are forecasted.

For the gains, with a 95 % level of confidence, the GEVD- Archimedean Gumbel copula model gives portfolio VaR and ES estimates of 12.11% (0.1211) and 14.09% (0.1409) respectively (Table 7). The results are interpreted as follows: the expected market gains will not go above 12.11% (0.1211) at this level of confidence level, if it goes beyond, it will average 14.09% (0.1409) at the same confidence level. The interpretation is the same for the losses. The estimated portfolio risk in a bivariate setting can be used to account for the diversification effects.

4.14 Estimation of diversification effects

The main reason investors seek diversification is to reduce the portfolio risk inherent in investing in risky assets. The degree of risk diversification is determined by how much the portfolio risk deviates from the sum of its individual component risks [15,17,39]. In this section diversification effects are determined.

In Table 8 at 95 level of confidence for the gains, and using VaR, the portfolio incurs diversification effects of 50.87%. For the losses, at 95% level of confidence, the portfolio incurs diversification effects of 51.50%. There is a reduction of risk in the portfolio when compared to the risk of the simple sum of single securities which implies that there are diversification benefits to be harvested.

In Table 9 at 95% level of confidence, for the gains, and using ES, the portfolio incurs diversification effects of 58.57%. For the losses at 95% level of confidence, the portfolio incurs diversification effects of 42.41%. The ES is a coherent risk measure and the diversification benefits for gains and losses are more in line with each other. These findings have significant implications for investors' decisions on diversification that are made to lower risk exposure. For VaR and ES the diversification effects imply that there is a trade-off for less gains for the portfolio whilst there is protection against large loss. These results imply that there is reduction in the portfolio risk.

4.15 Discussion

The GEVD marginals and Archimedean Gumbel copula are used to model the joint and individual behavior of the financial returns data of the Indices. The GEVD is used as the marginal distribution because the two datasets were found to be heavy-tailed. The Archimedean Gumbel copula was applied because of its key characteristic in bringing the separate marginal distributions to form the joint distribution of the returns [12]. The EVT-copula model is a robust method which can be very useful in estimating the VaR of a portfolio in the presence of extremes in the data from any stock market [14]. The Archimedean Gumbel copula is well suited to combining an extreme value marginal distributions with other extreme value distributions in estimating portfolio risk and diversification effects. [45] confirm that Archimedean copula functions are useful for modelling bivariate distributions in finance, and this can be extended to include the use of other or different extreme value distribution marginals depending on the dataset. The diversification effects estimated are consistent with results estimated by [15,17,40] when analysing the results which can go up to 60%. The implication is that, for those who wish to invest in the South African stock market, such diversification effects will cushion against large losses. The limitation of GEVD-Archimedean Gumbel copula model is that, it uses maxima data only, it ignores the rest of the data and thus is not efficient in data usage. The form of the dependence is assumed to be in the upper tail only and not in the lower tail.

5 Concluding Remarks and future possible research

5.1 Conclusions

This paper estimated portfolio VaR and ES in order to quantify diversification benefits of two financial assets, viz: J520 and the J580 Indices using the GEVD-Archimedean Gumbel copula model. The portfolio diversification results point to a reduction in losses for investors holding the portfolio. The average mean return of the portfolio remains the same and stable when compared to individual returns as the risk is mitigated. The model can be used as a case for reducing exposure to risk by diversification of risk for the same expected returns. This will be appealing to the risk averse investor wishing to avoid making extreme losses when invested in a single. There is also a reduction in size of potential gain when invested in the portfolio. According to [46], accurate risk evaluation can result in better trading and investment decisions. The information is important to local and international investors who wish to include developing countries' stock containing financial assets from the South African Industrial and Financial markets.

5.2 Future possible research

Future research will be in the application of the two-parameter copula to increase flexibility in modelling

Table 10: Estimates of portfolio risk using the Monte-Carlo simulation method

Copula	Marginals	Portfolio VaR			Portfolio ES		
		95%	99%	99.5%	95%	99%	99.5%
Left Tail of loss distribution / Gains							
Gumbel	GEVD	0.1211	0.1534	0.1644	0.1409	0.1692	0.1802
Left Tail of loss distribution / Gains							
Gumbel	GEVD	0.1307	0.2138	0.2488	0.1839	0.2741	0.3483

Table 11: Estimates of portfolio risk using the Monte-Carlo simulation method

Copula	Marginals	Portfolio VaR			Portfolio ES		
		95%	99%	99.5%	95%	99%	99.5%
Left Tail / Gains							
Gumbel	GEVD	0.1211	0.1534	0.1644	0.1409	0.1692	0.1802
Right Tail / Losses							
Gumbel	GEVD	0.1307	0.2138	0.2488	0.1839	0.2741	0.3483

Table 12: Estimates of diversification effects for ES.

Alpha	ES for J520	ES for J580	Simple Sum for ES	ES for portfolio	Diversification effects
Archimedean Gumbel Copula - Gains					
95	0.1642	0.1759	0.3401	0.1409	58.57%
99	0.1887	0.2258	0.4145	0.1693	59.16%
99.5	0.1956	0.2445	0.4401	0.1802	59.05%
Archimedean Gumbel Copula - Losses					
95	0.1615	0.1578	0.3193	0.1839	42.41%
99	0.2470	0.2426	0.4896	0.2741	44.02%
99.5	0.2843	0.2783	0.5626	0.3183	43.42%

the nature of the dependence and explore the generalisation of this model.

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