Recovery of Multiple Parameters in Subdiffusion from One Lateral Boundary Measurement*

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Abstract

This work is concerned with numerically recovering multiple parameters simultaneously in the subdiffusion model from one single lateral measurement on a part of the boundary, while in an incompletely known medium. We prove that the boundary measurement corresponding to a fairly general boundary excitation uniquely determines the order of the fractional derivative and the polygonal support of the diffusion coefficient, without knowing either the initial condition or the source. The uniqueness analysis further inspires the development of a robust numerical algorithm for recovering the fractional order and diffusion coefficient. The proposed algorithm combines small-time asymptotic expansion, analytic continuation of the solution and the level set method. We present extensive numerical experiments to illustrate the feasibility of the simultaneous recovery. In addition, we discuss the uniqueness of recovering general diffusion and potential coefficients from one single partial boundary measurement, when the boundary excitation is more specialized.

Key words: subdiffusion, lateral boundary measurement, discontinuous diffusivity, unknown medium, level set method

1 Introduction

This work is concerned with an inverse problem of simultaneously recovering multiple parameters in a subdiffusion model from one single lateral boundary measurement in a partly unknown medium. Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be an open bounded domain with a Lipschitz and piecewise $C^{1,1}$ boundary and $T > 0$ be a fixed final time. Consider the following subdiffusion problem for the function u:

$$\begin{cases} \partial_t^\alpha u + \mathcal{A}u = f & \text{in } \Omega \times (0, T], \\ a\partial_\nu u = g & \text{on } \partial\Omega \times (0, T], \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $u_0 \in L^2(\Omega)$ and (time-independent) $f \in L^2(\Omega)$ are unknown initial and source data, and ν denotes the unit outward normal vector to the boundary $\partial\Omega$. The elliptic operator $\mathcal{A}$ is defined by

$$\mathcal{A}u(x) := -\nabla \cdot (a(x)\nabla u(x)), \quad x \in \overline{\Omega}.$$

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Without loss of generality, the diffusion coefficient a is assumed to be piecewise constant:

$$a(x) = 1 + \mu\,\chi_D(x), \tag{1.2}$$

where $\mu > -1$ is a nonzero unknown constant, D is an unknown convex polyhedron in Ω satisfying $\mathrm{diam}(D) < \mathrm{dist}(D, \partial\Omega)$ and χ_D denotes the characteristic function of D. In the model (1.1), $\partial_t^\alpha u$ denotes the Djrbashian-Caputo fractional derivative in time t of order $\alpha \in (0,1)$ defined by ([32, p. 92] or [20, Section 2.3])

$$\partial_t^\alpha u(t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} u'(s)\,\mathrm{d}s.$$

The model (1.1) has attracted a lot of recent attention, due to its excellent capability to describe anomalous diffusion phenomena observed in many engineering and physical applications. The list of successful applications is long and still fast growing, e.g., ion transport in column experiments [15], protein diffusion within cells [14] and contaminant transport in underground water [33]. See the reviews [40, 39] for the derivation of relevant mathematical models and diverse applications. The model (1.1) differs considerably from the normal diffusion model due to the presence of the nonlocal operator $\partial_t^\alpha u$: it has limited smoothing property in space and slow asymptotic decay at large time [34, 20].

In this paper, we study mathematical and numerical aspects of an inverse problem of recovering the diffusion coefficient a and fractional order α from a single lateral boundary measurement of the solution, without the knowledge of the initial data u_0 and source f. The Neumann data g is taken to be separable:

$$g(x,t) = \psi(t)\eta(x), \tag{1.3}$$

where $0 \not\equiv \eta \in H^{\frac{1}{2}}(\partial\Omega)$ satisfies the compatibility condition $\int_{\partial\Omega} \eta \mathrm{d}S = 0$ and $\psi \in C^1(\mathbb{R}_+)$ satisfies

$$\psi(t) = \begin{cases} 0, & t < T_0, \\ 1, & t > T_1, \end{cases} \tag{1.4}$$

with $0 < T_0 < T_1 < T$. The measurement data $h(x,t) = u(x,t)$ is taken on a part of the boundary $\Gamma_0 \subset \partial\Omega$. Note that the inverse problem involves missing data (u_0 and f), whereas the available data is only on a partial boundary. Thus, it is both mathematically and numerically very challenging, due to not only the severe ill-posed nature and high degree of nonlinearity but also the unknown forward map from the parameters a and α to the data $h(x,t)$.

The mathematical study on inverse problems for time-fractional models is of relatively recent origin, starting from the pioneering work [8] (see [24, 36, 37] for overviews) and there are several existing works on recovering a space-dependent potential or diffusion coefficient from lateral Cauchy data [44, 45, 50, 28, 25, 30]. Rundell and Yamamoto [44] showed that the lateral Cauchy data can uniquely determine the spectral data when $u_0 \equiv f \equiv 0$, and proved the unique determination of the potential using Gel'fand-Levitan theory. They also numerically studied the singular value spectrum of the linearized forward map, showing the severe ill-posed nature of the problem. Later, they [45] relaxed the regularity condition on the boundary excitation $g(t)$ in a suitable Sobolev space. Recently, Jing and Yamamoto [28] proved the identifiability of multiple parameters (including order, spatially dependent potential, initial value and Robin coefficients in the boundary condition) in a time-fractional subdiffusion model with a zero boundary condition and source, excited by a nontrivial initial condition from the lateral Cauchy data at both end points; see also [27]. Jin and Zhou [25] studied the unique recovery of the potential, fractional order and either initial data or source from the lateral Cauchy data, when the boundary excitation is judiciously chosen. All these interesting works are concerned with the one-dimensional setting due to their essential use of the inverse Sturm-Liouville theory. Wei et al [51] numerically investigated the recovery of the zeroth-order coefficient and fractional order in a time-fractional reaction-diffusion-wave equation from lateral boundary data. A direct extension of these theoretical works to the multi-dimensional case is challenging since the Gel'fand-Levitan theory is no longer applicable. Kian et al. [31] provided the first results for the multi-dimensional case, including the uniqueness for identifying two spatially distributed parameters in the subdiffusion model from one single lateral observation with a specially designed excitation Dirichlet input; see also [17] for a related study on determining the manifold from one measurement corresponding to a specialized source. Kian [30] studied also the issue of simultaneous recovery of these parameters along with the order and initial data using a similar choice of the boundary data. However, in the works [31, 30], the excitation data, which plays the role of infinity measurements, is numerically inconvenient to realize, if not impossible at all; see Remark 3.3 and the appendix for further discussions. These considerations motivate the current work, i.e., to design robust numerical algorithm for recovering multiple parameters from a single partial boundary measurement for multi-dimensional subdiffusion with a computable excitation Neumann data, in the presence of a partly unknown medium.

In this work, we make the following contributions to the mathematical analysis and numerics of the concerned inverse problem. First, we examine the feasibility to recover multiple parameters. We show that if the coefficient a is piecewise constant as defined in (1.2), then one single boundary measurement can uniquely determine the coefficient a and fractional order α, even though the initial data u_0 and source f are unknown. Note that the exciting Neumann data g given in (1.3) is easy to realize and hence allows the numerical recovery. The proof relies on the asymptotic behavior of Mittag-Leffler functions, analyticity in time of the solution, and the uniqueness of the inverse conductivity problem (for elliptic problems) from one boundary measurement. In particular, the subdomain D can be either a convex polygon / polyhedron or a disc / ball, cf. Theorem 3.2 and Remark 3.2. This analysis strategy follows a well-established procedure in the community, and roughly it consists of two steps. (1) Using the time-analyticity, the uniqueness for the original inverse problem is reduced to the one for an inverse problem for the corresponding time-independent elliptic equation; (2) The reduction can be done by the Laplace transform or considering the asymptotics. Both strategies of reductions are well known. For example, the former way is used for an Dirichlet-to-Neumann map for the inverse coefficient problem for a multi-term time-fractional diffusion equation [35], while the latter way is used for the Dirichlet-to-Neumann map for the inverse parabolic problem [18, Section 4, Chapter 9]. Second, the uniqueness analysis lends itself to the development of a robust numerical algorithm: we develop a three-step recovery algorithm for identifying the piecewise constant coefficient a and the fractional order α: (i) use the asymptotic behavior of the solution of problem (1.1) near $t = 0$ to recover α; (ii) use analytic continuation to extract the solution of problem (1.1) with zero f and u_0; (iii) use the level set method to recover the shape of subdomain D. To the best of our knowledge, this is the first work on the numerical recovery of a (piecewise constant) diffusion coefficient in the context of multi-dimensional subdiffusion model with missing initial and source data. Last, we present extensive numerical experiments to illustrate the feasibility of the approach. We refer interested readers to [46, 43] for some numerical studies for identifying a piecewise constant source from the boundary measurement.

The rest of the paper is organized as follows. In Section 2 we describe preliminary results on the model, especially time analyticity of the data. Then in Section 3 we give the uniqueness result in case of piecewise constant a, and in Section 4 we develop a recovery algorithm based on the level set method. We present extensive numerical experiments to illustrate the feasibility of recovering multiple parameters in Section 5. In an appendix, we discuss the possibility of recovering two coefficients from one boundary measurement induced by a specialized boundary excitation. Throughout, the notation $(\cdot, \cdot)$ denotes the standard $L^2(\Omega)$ inner product, and $\langle \cdot, \cdot \rangle$ the $L^2(\partial\Omega)$ inner product. For a Banach space B, $C^\omega(T, \infty; B)$ denotes the set of functions valued in B and analytic in $t \in (T, \infty)$. The notation c, with or without a subscript, denotes a generic constant which may change at each occurrence, but it is always independent of the concerned quantities.

2 Preliminaries

In this section, we present preliminary analytical results. Let A be the $L^2(\Omega)$ realization of the elliptic operator $\mathcal{A}$, with a domain $\mathrm{Dom}(A) := \{v \in L^2(\Omega) : \mathcal{A}v \in L^2(\Omega), \partial_\nu v|_{\partial\Omega} = 0\}$. Let $\{\lambda_\ell\}_{\ell \geq 1}$ be a strictly increasing sequence of eigenvalues of A, and denote the multiplicity of λ_ℓ by m_ℓ and $\{\varphi_{\ell,k}\}_{k=1}^{m_\ell}$ an $L^2(\Omega)$

orthonormal basis of $\ker(A-\lambda_\ell)$. That is, for any $\ell\in\mathbb{N}$, $k=1,\ldots,m_\ell$:

$$\begin{cases}\mathcal{A}\varphi_{\ell,k}=\lambda_\ell\varphi_{\ell,k} & \text{in }\Omega,\\ a\partial_\nu\varphi_{\ell,k}=0 & \text{on }\partial\Omega.\end{cases}\tag{2.1}$$

The eigenvalues $\{\lambda_\ell\}_{\ell=1}^\infty$ are nonnegative, and the eigenfunctions $\{\varphi_{\ell,k}:k=1,\ldots,m_\ell\}_{\ell=1}^\infty$ form a complete orthonormal basis of $L^2(\Omega)$. Note that $\lambda_1=0$ (and has multiplicity 1) and the corresponding eigenfunction $\varphi_1=|\Omega|^{-\frac12}$ is constant valued, where $|E|$ denotes the Lebesgue measure of a set E. Due to the piecewise constancy of the coefficient a, $\varphi_{\ell,k}$ is smooth in D and $\Omega\setminus\overline{D}$. Moreover, it satisfies the following transmission condition on the interface ∂D:

$$\varphi_{\ell,k}|_-=\varphi_{\ell,k}|_+\quad\text{and}\quad\partial_n\varphi_{\ell,k}|_-=(1+\mu)\partial_n\varphi_{\ell,k}|_+\quad\text{on }\partial D,\tag{2.2}$$

where $\varphi_{\ell,k}|_+$ and $\varphi_{\ell,k}|_-$ denote the limits from D and $\Omega\setminus\overline{D}$ to the interface ∂D, respectively, and $\partial_n\varphi_{\ell,k}|_\pm$ denotes the derivative with respect to the unit outer normal vector n on ∂D. Then we define the fractional power A^s $(s\geq 0)$ via functional calculus by

$$A^s v:=\sum_{\ell=1}^\infty\lambda_\ell^s\sum_{k=1}^{m_\ell}(v,\varphi_{\ell,k})\varphi_{\ell,k},$$

with a domain $\mathrm{Dom}(A^s)=\{v\in L^2(\Omega):A^s v\in L^2(\Omega)\}$, and the associated graph norm

$$\|v\|_{\mathrm{Dom}(A^s)}=\Big(\sum_{\ell=1}^\infty\lambda_\ell^{2s}\sum_{k=1}^{m_\ell}(v,\varphi_{\ell,k})^2\Big)^{\frac12}.$$

We use extensively the Mittag-Leffler function $E_{\alpha,\beta}(z)$ defined by ([32, pp. 40-45], [20, Section 3.1])

$$E_{\alpha,\beta}(z)=\sum_{k=0}^\infty\frac{z^k}{\Gamma(k\alpha+\beta)},\quad\forall z\in\mathbb{C}.$$

The function $E_{\alpha,\beta}(z)$ generalizes the exponential function e^z. The following decay estimate of $E_{\alpha,\beta}(z)$ is crucial in the analysis below; See e.g., [32, eq. (1.8.28), p. 43] and [20, Theorem 3.2] for the proof.

Lemma 2.1. *Let $\alpha\in(0,2)$, $\beta\in\mathbb{R}$, $\varphi\in(\frac{\alpha}{2}\pi,\min(\pi,\alpha\pi))$ and $N\in\mathbb{N}$. Then for $\varphi\leq|\arg z|\leq\pi$ with $|z|\to\infty$, there holds*

$$E_{\alpha,\beta}(z)=-\sum_{k=1}^N\frac{z^{-k}}{\Gamma(\beta-\alpha k)}+O(|z|^{-N-1}).$$

By linearity, we may split the solution u of problem (1.1) into $u=u_i+u_b$, with u_i and u_b solving

$$\begin{cases}\partial_t^\alpha u_i+\mathcal{A}u_i=f & \text{in }\Omega\times(0,T],\\ a\partial_\nu u_i=0 & \text{on }\partial\Omega\times(0,T],\\ u_i(0)=u_0 & \text{in }\Omega\end{cases}\quad\text{and}\quad\begin{cases}\partial_t^\alpha u_b+\mathcal{A}u_b=0 & \text{in }\Omega\times(0,T],\\ a\partial_\nu u_b=g & \text{on }\partial\Omega\times(0,T],\\ u_b(0)=0 & \text{in }\Omega,\end{cases}\tag{2.3}$$

respectively. The following result gives the representations of u_i and u_b.

Proposition 2.1. *Let $u_0,f\in L^2(\Omega)$. Then there exist unique solutions $u_i,u_b\in L^2(0,T;H^1(\Omega))$ that can be respectively represented by*

$$\begin{aligned}u_i(t)&=(u_0,\varphi_1)\varphi_1+\frac{(f,\varphi_1)\varphi_1t^\alpha}{\Gamma(1+\alpha)}\\&\quad+\sum_{\ell=2}^\infty\sum_{k=1}^{m_\ell}\left(\left[(u_0,\varphi_{\ell,k})-\lambda_\ell^{-1}(f,\varphi_{\ell,k})\right]E_{\alpha,1}(-\lambda_\ell t^\alpha)+\lambda_\ell^{-1}(f,\varphi_{\ell,k})\right)\varphi_{\ell,k},\end{aligned}$$

$$u_b(t) = \sum_{\ell=1}^{\infty} \sum_{k=1}^{m_\ell} \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_{\ell,k}(t-s)^\alpha) \langle g(s), \varphi_{\ell,k} \rangle \, \mathrm{d}s \, \varphi_{\ell,k}.$$

Hence, the solution u to problem (1.1) can be represented as

$$u(t) = \rho_0 + \rho_1 t^\alpha + \sum_{\ell=2}^{\infty} E_{\alpha,1}(-\lambda_\ell t^\alpha)\rho_\ell + \sum_{\ell=1}^{\infty} \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_\ell (t-s)^\alpha) \sum_{k=1}^{m_\ell} \langle g(s), \varphi_{\ell,k} \rangle \, \mathrm{d}s \, \varphi_{\ell,k},$$

with ρ_ℓ given by

$$\rho_\ell := \begin{cases} (u_0, \varphi_1)\varphi_1 + \displaystyle\sum_{\ell=2}^{\infty} \sum_{k=1}^{m_\ell} \lambda_\ell^{-1} (f, \varphi_{\ell,k}) \varphi_{\ell,k}, & \ell = 0, \\ \dfrac{(f, \varphi_1)}{\Gamma(1+\alpha)} \varphi_1, & \ell = 1, \\ \displaystyle\sum_{k=1}^{m_\ell} \left[(u_0, \varphi_{\ell,k}) - \lambda_\ell^{-1}(f, \varphi_{\ell,k}) \right] \varphi_{\ell,k}, & \ell = 2, 3, \ldots. \end{cases} \tag{2.4}$$

Proof. The representations follow from the standard separation of variables technique ([47], [20, Section 6.2]). The piecewise constancy of the diffusivity a requires special care due to a lack of global regularity. By multiplying the governing equation of u_i by $\varphi_{\ell,k}$ and then integrating over Ω, we get

$$\partial_t^\alpha (u_i(t), \varphi_{\ell,k}) + (\mathcal{A} u_i(t), \varphi_{\ell,k}) = (f, \varphi_{\ell,k}).$$

Integrating by parts twice and using the transmission condition (2.2) for $\varphi_{\ell,k}$ (and u_i) on ∂D gives

$$\begin{aligned} (\mathcal{A} u_i(t), \varphi_{\ell,k}) &= -\int_{\Omega \setminus \overline{D}} \nabla \cdot (\nabla u_i) \, \varphi_{\ell,k} \, \mathrm{d}x - \int_D \nabla \cdot ((1+\mu)\nabla u_i) \, \varphi_{\ell,k} \, \mathrm{d}x \\ &= -\int_{\partial\Omega} (\nabla u_i \cdot \nu) \, \varphi_{\ell,k} \, \mathrm{d}S - \int_{\partial D} (\nabla u_i \cdot n_-) \, \varphi_{\ell,k}|_- \, \mathrm{d}S + \int_{\Omega \setminus \overline{D}} \nabla u_i \cdot \nabla \varphi_{\ell,k} \, \mathrm{d}x \\ &\quad - \int_{\partial D} (1+\mu)(\nabla u_i \cdot n_+) \, \varphi_{\ell,k}|_+ \, \mathrm{d}S + \int_D (1+\mu) \nabla u_i \cdot \nabla \varphi_{\ell,k} \, \mathrm{d}x \\ &= \int_{\Omega \setminus \overline{D}} \nabla u_i \cdot \nabla \varphi_{\ell,k} \, \mathrm{d}x + \int_D (1+\mu) \nabla u_i \cdot \nabla \varphi_{\ell,k} \, \mathrm{d}x \\ &= \int_{\partial\Omega} (\nabla \varphi_{\ell,k} \cdot \nu) \, u_i \, \mathrm{d}S + \int_{\partial D} (\nabla \varphi_{\ell,k} \cdot n_-) \, u_i|_- \, \mathrm{d}S - \int_{\Omega \setminus \overline{D}} \nabla \cdot (\nabla \varphi_{\ell,k}) \, u_i \, \mathrm{d}x \\ &\quad + \int_{\partial D} (1+\mu)(\nabla \varphi_{\ell,k} \cdot n_+) \, u_i|_+ \, \mathrm{d}S - \int_D \nabla \cdot ((1+\mu)\nabla \varphi_{\ell,k}) \, u_i \, \mathrm{d}x \\ &= (u_i, \mathcal{A}\varphi_{\ell,k}) = \lambda_\ell (u_i, \varphi_{\ell,k}). \end{aligned}$$

Hence, the scalar function $u_i^{\ell,k}(t) := (u_i(t), \varphi_{\ell,k})$ satisfies the following initial value problem for a fractional ordinary differential equation:

$$(\partial_t^\alpha + \lambda_\ell) u_i^{\ell,k}(t) = f_{\ell,k} := (f, \varphi_{\ell,k}) \qquad \text{for } 0 < t \leq T, \quad \text{with } u_i^{\ell,k}(0) = u_0^{\ell,k} := (u_0, \varphi_{\ell,k}).$$

Then $u_i^{\ell,k}(t)$ is given by [20, Proposition 4.5]

$$u_i^{\ell,k}(t) = u_0^{\ell,k} E_{\alpha,1}(-\lambda_\ell t^\alpha) + f_{\ell,k} \int_0^t s^{\alpha-1} E_{\alpha,\alpha}(-\lambda_\ell s^\alpha) \, \mathrm{d}s.$$

Note that $u_i^1 = u_0^1 + \frac{1}{\Gamma(1+\alpha)} f_1 t^\alpha$. Now using the identity

$$\frac{\mathrm{d}}{\mathrm{d}t} E_{\alpha,1}(-\lambda t^\alpha) = -\lambda \, t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha), \tag{2.5}$$

we have for $\ell \geq 2$ and $k = 1, \ldots, m_\ell$ that

$$\begin{aligned} u_i^{\ell,k}(t) &= u_0^{\ell,k} E_{\alpha,1}(-\lambda_\ell t^\alpha) + \lambda_\ell^{-1} \left[1 - E_{\alpha,1}(-\lambda_\ell t^\alpha)\right] f_{\ell,k} \\ &= \left(u_0^{\ell,k} - \lambda_\ell^{-1} f_{\ell,k}\right) E_{\alpha,1}(-\lambda_\ell t^\alpha) + \lambda_\ell^{-1} f_{\ell,k}. \end{aligned}$$

This gives the representation of u_i. Similarly, multiplying the governing equation for u_b by $\varphi_{\ell,k}$ and integrating over Ω give $\partial_t^\alpha (u_b(t), \varphi_{\ell,k}) + (\mathcal{A} u_b(t), \varphi_{\ell,k}) = 0$. Repeating the argument yields that $u_b^{\ell,k}(t) := (u_b(t), \varphi_{\ell,k})$ satisfies

$$(\partial_t^\alpha + \lambda_\ell) u_b^{\ell,k}(t) = \langle g(t), \varphi_{\ell,k} \rangle \quad \text{for } 0 < t \leq T, \qquad \text{with} \;\; u_b^{\ell,k}(0) = 0.$$

The solution $u_b^{\ell,k}(t)$ is given by [20, Proposition 4.5]

$$u_b^{\ell,k}(t) = \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_\ell (t-s)^\alpha) \langle g(s), \varphi_{\ell,k} \rangle \, \mathrm{d}s \, \varphi_{\ell,k}.$$

Thus the desired assertion follows. The representation of the solution u to problem (1.1) follows directly from that of u_b and u_i, and the identity (2.5). $\square$

Next we show properties of the boundary data h. This is achieved by first proving related properties of u and then applying the trace theorem. Below we study the analyticity of

$$\begin{aligned} u_i(t) &= \rho_0 + \rho_1 t^\alpha + \sum_{\ell=2}^\infty E_{\alpha,1}(-\lambda_\ell t^\alpha) \rho_\ell, \\ u_b(t) &= \sum_{\ell=1}^\infty \sum_{k=1}^{m_\ell} \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_\ell (t-s)^\alpha) \langle g(s), \varphi_{\ell,k} \rangle \, \mathrm{d}s \, \varphi_{\ell,k}. \end{aligned}$$

Since our focus is the trace on $\partial\Omega$, we only study u on the subdomain $\Omega \setminus \overline{D}$. Recall that for a Banach space B, the notation $C^\omega(T, \infty; B)$ denotes the set of functions valued in B and analytic in $t \in (T, \infty)$.

Proposition 2.2. *Let $D' \supset D$ be a small neighborhood of D with a smooth boundary and denote $\Omega' = \Omega \setminus \overline{D'}$. For $u_0 \in L^2(\Omega)$, $f \in L^2(\Omega)$ and g as in (1.3), the following statements hold.*

(i) *$u_i \in C^\omega(0, \infty; H^2(\Omega'))$ and $u_b \in C^\omega(T_1 + \varepsilon, \infty; H^2(\Omega'))$ for arbitrarily fixed $\varepsilon > 0$.*

(ii) *The Laplace transforms $\widehat{u}_i(z)$ and $\widehat{u}_b(z)$ of u_i and u_b in t exist for all $\Re(z) > 0$ and are respectively given by*

$$\widehat{u}_i(z) = z^{-1} \rho_0 + \Gamma(\alpha+1) z^{-\alpha-1} \rho_1 + \sum_{\ell=2}^\infty \frac{\rho_\ell z^{\alpha-1}}{z^\alpha + \lambda_\ell} \quad \text{and} \quad \widehat{u}_b(z) = \sum_{\ell=1}^\infty \sum_{k=1}^{m_\ell} \frac{\langle \widehat{g}(z), \varphi_{\ell,k} \rangle \varphi_{\ell,k}}{z^\alpha + \lambda_\ell}.$$

Proof. Throughout this proof, let $\varepsilon > 0$ be arbitrarily fixed. Since $\lambda_1 = 0$, by Lemma 2.1, there exist constants $c > 0$ and $\theta \in (0, \frac{\pi}{2})$ such that for any $z \in \Sigma_\theta := \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| \leq \theta\}$, we have

$$\begin{aligned} \|u_i(z)\|^2_{\mathrm{Dom}(A)} &= \sum_{n=1}^\infty \lambda_n^2 \sum_{j=1}^{m_n} \left(\varphi_{n,j}, \rho_0 + \rho_1 z^\alpha + \sum_{\ell=2}^\infty E_{\alpha,1}(-\lambda_\ell z^\alpha) \rho_\ell \right)^2 \\ &= \sum_{n=2}^\infty \lambda_n^2 \sum_{j=1}^{m_n} \left(\varphi_{n,j}, \sum_{\ell=2}^\infty \sum_{k=1}^{m_\ell} \left\{ E_{\alpha,1}(-\lambda_\ell z^\alpha) \left[(u_0, \varphi_{\ell,k}) - \lambda_\ell^{-1}(f, \varphi_{\ell,k}) \right] + \lambda_\ell^{-1}(f, \varphi_{\ell,k}) \right\} \varphi_{\ell,k} \right)^2 \\ &\leq c \sum_{n=2}^\infty \lambda_n^2 E_{\alpha,1}(-\lambda_n z^\alpha)^2 \sum_{j=1}^{m_n} \left\{ (u_0, \varphi_{n,j})^2 + \lambda_n^{-2}(f, \varphi_{n,j})^2 \right\} + c \sum_{n=1}^\infty \sum_{j=1}^{m_n} (f, \varphi_{n,j})^2 \end{aligned}$$

$$
\begin{aligned}
&\leq c|z|^{-2\alpha} \sum_{n=2}^{\infty} \sum_{j=1}^{m_n} \left\{ (u_0, \varphi_{n,j})^2 + \lambda_n^{-2} (f, \varphi_{n,j})^2 \right\} + c\|f\|_{L^2(\Omega)}^2 \\
&\leq c|z|^{-2\alpha} \left(\|u_0\|_{L^2(\Omega)}^2 + \|f\|_{L^2(\Omega)}^2 \right) + c\|f\|_{L^2(\Omega)}^2 .
\end{aligned}
$$

Since $u_0, f \in L^2(\Omega)$, $\|u_i(z)\|_{\mathrm{Dom}(A)}^2$ is uniformly bounded for $z \in \Sigma_\theta$. Since $E_{\alpha,1}(-\lambda_n z^\alpha)$ is analytic in $z \in \Sigma_\theta$ and the series converges uniformly in any compact subset of Σ_θ, $u_i(t)$ is analytic in $t \in (0, \infty)$ as a $\mathrm{Dom}(A)$-valued function, i.e., $u_i \in C^\omega(0, \infty; \mathrm{Dom}(A))$. By Sobolev embedding, $u_i \in C^\omega(0, \infty; H^2(\Omega'))$.

Next we prove the analyticity of u_b. By the choice $g(x,t) = \eta(x)\psi(t)$ in (1.3) and integration by parts, for $t > T_1$, $u_b^1(t) := (u_b(t), \varphi_1)$ is given by

$$
\begin{aligned}
u_b^1(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \langle g(s), \varphi_1 \rangle \, \mathrm{d}s = \frac{\langle \eta, \varphi_1 \rangle}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \psi(s) \, \mathrm{d}s \\
&= \frac{\langle \eta, \varphi_1 \rangle}{\alpha \, \Gamma(\alpha)} \left[-(t-s)^\alpha \psi(s)|_{s=0}^{s=t} + \int_0^t (t-s)^\alpha \psi'(s) \, \mathrm{d}s \right] \\
&= \frac{\langle \eta, \varphi_1 \rangle}{\Gamma(\alpha+1)} \int_{T_0}^{T_1} (t-s)^\alpha \psi'(s) \, \mathrm{d}s,
\end{aligned}
$$

where the last step follows from the condition on ψ in (1.4). Thus the time-analyticity of $u_b^1(t)\varphi_1$ for $t \in (T_1 + \varepsilon, \infty)$ follows. Next, again by integration by parts, (1.3)–(1.4) and the identity (2.5), for $t > T_1$, $u_b^{\ell,k}(t) := (u_b(t), \varphi_{\ell,k})$ with $\ell \geq 2$, $k = 1, \ldots, m_\ell$ can be written as

$$
\begin{aligned}
u_b^{\ell,k}(t) &= \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_\ell (t-s)^\alpha) \langle g(s), \varphi_{\ell,k} \rangle \, \mathrm{d}s \\
&= \int_0^t \frac{\langle g(s), \varphi_{\ell,k} \rangle}{\lambda_\ell} \frac{\mathrm{d}}{\mathrm{d}s} E_{\alpha,1}(-\lambda_\ell (t-s)^\alpha) \, \mathrm{d}s \\
&= \lambda_\ell^{-1} \Big[\langle g(s), \varphi_{\ell,k} \rangle E_{\alpha,1}(-\lambda_\ell (t-s)^\alpha) \Big]_{s=0}^{s=t} - \frac{\langle \eta, \varphi_{\ell,k} \rangle}{\lambda_\ell} \int_0^t E_{\alpha,1}(-\lambda_\ell (t-s)^\alpha) \psi'(s) \, \mathrm{d}s \\
&= \frac{\langle \eta, \varphi_{\ell,k} \rangle}{\lambda_\ell} \psi(t) - \frac{\langle \eta, \varphi_{\ell,k} \rangle}{\lambda_\ell} \int_{T_0}^{T_1} E_{\alpha,1}(-\lambda_\ell (t-s)^\alpha) \psi'(s) \, \mathrm{d}s =: u_{b,1}^{\ell,k}(t) + u_{b,2}^{\ell,k}(t).
\end{aligned}
$$

Since $\psi(t) = 1$ for $t > T_1$, we see that $u_{b,1}^{\ell,k}(t)$ is a constant for $t > T_1$. Next we consider the following boundary value problem

$$
\mathcal{A}U = 0 \ \text{ in } \Omega, \quad \text{with} \quad a\partial_\nu U = \eta \ \text{ on } \partial\Omega. \tag{2.6}
$$

The compatibility condition $\langle \eta, 1 \rangle = 0$ implies that there exist solutions to problem (2.6). We take an arbitrary solution U. Since a is piecewise constant and $\eta \in H^{\frac{1}{2}}(\partial\Omega)$, we know that $U \in H^1(\Omega)$ and its restriction $U|_{\Omega'} \in H^2(\Omega')$. Integrating by parts twice yields

$$
\langle \eta, \varphi_{\ell,k} \rangle = \lambda_\ell (U, \varphi_{\ell,k}).
$$

Similar to the argument for Proposition 2.1, from the transmission condition (2.2), we deduce

$$
\sum_{\ell=2}^{\infty} \sum_{k=1}^{m_\ell} u_{b,1}^{\ell,k}(t) \varphi_{\ell,k} = \sum_{\ell=2}^{\infty} \sum_{k=1}^{m_\ell} \psi(t) (U, \varphi_{\ell,k}) \varphi_{\ell,k},
$$

which is analytic in $t \in (T_1 + \varepsilon, \infty)$ since it is constant in time and $U \in L^2(\Omega)$. Moreover, by the standard elliptic regularity theory,

$$
\sum_{\ell=2}^{\infty} \sum_{k=1}^{m_\ell} u_{b,1}^{\ell,k} \varphi_{\ell,k} \in C^\omega(T_1 + \varepsilon, \infty; H^2(\Omega')).
$$

Recall Young's inequality for convolution, i.e., $\|f * g\|_{L^r(\mathbb{R})} \leq \|f\|_{L^p(\mathbb{R})}\|g\|_{L^q(\mathbb{R})}$ for $p, q, r \geq 1$ with $p^{-1} + q^{-1} = r^{-1} + 1$ and any $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$. Then by Young's inequality, Lemma 2.1 and the regularity estimate $\sum_{\ell=2}^{\infty} \lambda_\ell^{-2} \sum_{k=1}^{m_\ell} \langle \eta, \varphi_{\ell,k}\rangle^2 \leq \|U\|_{L^2(\Omega)} < \infty$, we deduce

$$\begin{aligned}
\left\| \sum_{\ell=2}^{\infty} \sum_{k=1}^{m_\ell} u_{b,2}^{\ell,k}(z)\varphi_{\ell,k} \right\|_{\mathrm{Dom}(A)}^2 &= \sum_{n=1}^{\infty} \lambda_n^2 \sum_{j=1}^{m_n} \left(\varphi_{n,j}, \sum_{\ell=2}^{\infty} \sum_{k=1}^{m_\ell} u_{b,2}^{\ell,k}(z)\varphi_{\ell,k} \right)^2 \\
&= \sum_{n=2}^{\infty} \lambda_n^2 \sum_{j=1}^{m_n} \left(\frac{\langle \eta, \varphi_{n,j}\rangle}{\lambda_n} \int_{T_0}^{T_1} E_{\alpha,1}(-\lambda_n(z-s)^\alpha)\psi'(s)\,\mathrm{d}s \right)^2 \\
&\leq \sum_{n=2}^{\infty} \sum_{j=1}^{m_n} \langle \eta, \varphi_{n,j}\rangle^2 \left(\frac{c}{\lambda_n |z - T_1|^\alpha} \int_{T_0}^{T_1} |\psi'(s)|\,\mathrm{d}s \right)^2 \\
&\leq \left(\frac{c\|\psi\|_{W^{1,\infty}(\mathbb{R}_+)}}{|z-T_1|^\alpha} \right)^2 \sum_{n=2}^{\infty} \lambda_n^{-2} \sum_{j=1}^{m_n} |\langle \eta, \varphi_{n,j}\rangle|^2 \leq \frac{c}{|z - T_1|^{2\alpha}}.
\end{aligned}$$

Since $u_{b,2}^{\ell,k}(t)$ is analytic in $(T_1 + \varepsilon, \infty)$ and the series $\sum_{\ell=2}^{\infty} \sum_{k=1}^{m_\ell} u_{b,2}^{\ell,k}(z)\varphi_{\ell,k}$ converges uniformly in $\mathrm{Dom}(A)$ for $z \in T_1 + \varepsilon + \Sigma_\theta$, it belongs to $C^\omega(T_1+\varepsilon, \infty; \mathrm{Dom}(A))$, and hence $u_b \in C^\omega(T_1+\varepsilon, \infty; H^2(\Omega'))$. This proves part (i).

The argument for part (i) implies that the series converges uniformly in $\mathrm{Dom}(A)$ for $t \in (0, \infty)$, and

$$\|\mathrm{e}^{-tz} u_i(t)\|_{\mathrm{Dom}(A)} \leq c\,\mathrm{e}^{-t\,\Re(z)}(t^{-\alpha} + 1), \quad t > 0.$$

The function $\mathrm{e}^{-t\,\Re(z)}(t^{-\alpha}+1)$ is integrable in t over $(0, \infty)$ for any fixed z with $\Re(z) > 0$. By Lebesgue's dominated convergence theorem and taking Laplace transform termwise, we obtain

$$\widehat{u}_i(z) = z^{-1}\rho_0 + \Gamma(\alpha+1)z^{-\alpha-1}\rho_1 + \sum_{\ell=2}^{\infty} \frac{\rho_\ell z^{\alpha-1}}{z^\alpha + \lambda_\ell}, \quad \forall\, \Re(z) > 0.$$

The argument for part (i) also implies

$$\|\mathrm{e}^{-tz} u_b(t)\|_{\mathrm{Dom}(A)} \leq c\,\mathrm{e}^{-t\,\Re(z)}|t - T_1|^{-\alpha}, \quad t > 0.$$

Then termwise Laplace transform and Lebesgue's dominated convergence theorem complete the proof of the proposition. □

Thus, u_i and u_b are analytic in time and have $H^2(\Omega')$ regularity. Since $\partial\Omega$ is Lipschitz and piecewise $C^{1,1}$, their traces on $\partial\Omega$ are well defined. The next result is direct from the trace theorem and Sobolev embedding theorem. Here, we use x and y denote the variables in Ω and on $\partial\Omega$, respectively.

Corollary 2.1. *Let the assumptions in Proposition 2.2 hold. Then the data $h = u|_{\Gamma_0 \times (0,T)}$ to problem* (1.1) *can be represented by*

$$\begin{aligned}
h(t) = &\underbrace{\rho_0 + \rho_1 t^\alpha + \sum_{\ell=2}^{\infty} E_{\alpha,1}(-\lambda_\ell t^\alpha)\rho_\ell}_{=:h_i(t)} \\
&+ \underbrace{\sum_{\ell=1}^{\infty} \sum_{k=1}^{m_\ell} \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_\ell(t-s)^\alpha)\langle g(s), \varphi_{\ell,k}\rangle\,\mathrm{d}s\,\varphi_{\ell,k}}_{=:h_b(t)}.
\end{aligned}$$

Moreover, h_i and h_b satisfy the following properties.

(i) $h_i \in C^\omega(0, \infty; L^2(\Gamma_0))$ and $h_b \in C^\omega(T_1 + \varepsilon, \infty; L^2(\Gamma_0))$ for arbitrarily fixed $\varepsilon > 0$.

(ii) The Laplace transforms $\widehat{h}_i(z)$ and $\widehat{h}_b(z)$ of h_i and h_b in t exist for all $\Re(z) > 0$ and are given by

$$\widehat{h}_i(z) = z^{-1}\rho_0 + \Gamma(\alpha+1)z^{-\alpha-1}\rho_1 + \sum_{\ell=2}^{\infty} \frac{\rho_\ell z^{\alpha-1}}{z^\alpha + \lambda_\ell},$$

$$\widehat{h}_b(z) = \sum_{\ell=1}^{\infty}\sum_{k=1}^{m_\ell} \frac{\langle \widehat{g}(z), \varphi_{\ell,k}\rangle \varphi_{\ell,k}}{z^\alpha + \lambda_\ell}.$$

Remark 2.1. *The analysis of Theorem 3.1 crucially exploits the analyticity of the measurement $h_i(t)$ in time, which relies on condition (1.4), i.e., $\psi(t) \equiv 0$ for $t \in [0, T_0]$. The condition $\psi(t) \equiv 1$ for $t \geq T_1$ for some $T_1 < T$ from (1.4) ensures the time analyticity of $h_b(t)$ for $t > T_1 + \varepsilon$, which is needed for Theorem 3.2. It should be interpreted as analytically extending the observation $h_b(t)$ by analytically extending $\psi(t)$, both from (T_1, T) to (T_1, ∞). Alternative conditions on $\psi(t)$ ensuring the time analyticity of $h_b(t)$ for $t > T_1 + \varepsilon$, e.g., $\psi(t)$ vanishes identically on (T_1, T), would also be sufficient for Theorem 3.2.*

3 Uniqueness

Now we establish a uniqueness result for recovering the fractional order α and piecewise constant a. The proof proceeds in two steps: First we show the uniqueness of the order α from the observation, despite that the initial condition u_0 and source f are unknown. Then we show the uniqueness of a. The key observation is that the contributions from u_0 and f can be extracted explicitly. Since the Dirichlet data is only available on a sub-boundary Γ_0, we view ρ_k as a $L^2(\Gamma_0)$-valued function. The notation $\mathbb{K}$ denotes the set $\{k \in \mathbb{N} : \rho_k \not\equiv 0 \text{ in } L^2(\Gamma_0)\}$, i.e., the support of the sequence $(\rho_0, \rho_1, \dots)$ in $L^2(\Gamma_0)$ sense, similarly, $\widetilde{\mathbb{K}} = \{k \in \mathbb{N} : \widetilde{\rho}_k \not\equiv 0 \text{ in } L^2(\Gamma_0)\}$, and $\mathbb{N}^* = \mathbb{N} \setminus \{1\}$. Below we denote by $\mathfrak{A}$ the admissible set of conductivities, i.e.,

$$\mathfrak{A} = \{1 + \mu\chi_D(x) : \mu > -1 \text{ and } D \subset \Omega \text{ is a convex polygon}\}.$$

Theorem 3.1. *Let $\alpha, \widetilde{\alpha} \in (0,1)$, $(a, f, u_0), (\widetilde{a}, \widetilde{f}, \widetilde{u}_0) \in \mathfrak{A} \times L^2(\Omega) \times L^2(\Omega)$, and fix g as (1.3) with $\psi(t)$ satisfying condition (1.4). Let h and $\widetilde{h}$ be the corresponding Dirichlet observations. Then for some $\sigma > 0$, the condition $h = \widetilde{h}$ on $\Gamma_0 \times [T_0 - \sigma, T_0]$ implies $\alpha = \widetilde{\alpha}$, $\rho_0 = \widetilde{\rho}_0$ and $\{(\rho_k, \lambda_k)\}_{k\in\mathbb{K}} = \{(\widetilde{\rho}_k, \widetilde{\lambda}_k)\}_{k\in\widetilde{\mathbb{K}}}$ if $\mathbb{K}, \widetilde{\mathbb{K}} \neq \emptyset$.*

Proof. By the definition of g, we have $g(y,t) \equiv 0$ for $y \in \partial\Omega$, $t \in [0, T_0]$. Then by Corollary 2.1, $h(y,t)$ admits a Dirichlet representation

$$h(y,t) = \rho_0(y) + \rho_1(y)t^\alpha + \sum_{k\in\mathbb{K}\cap\mathbb{N}^*} \rho_k(y) E_{\alpha,1}(-\lambda_k t^\alpha).$$

By Corollary 2.1(i), $h(t)$ is analytic as an $L^2(\partial\Omega)$-valued function in $t > 0$. By analytic continuation, the condition $h(t) = \widetilde{h}(t)$ for $t \in [T_0 - \sigma, T_0]$ implies that $h(t) = \widetilde{h}(t)$ in $L^2(\Gamma_0)$ for all $t > 0$, i.e.,

$$\rho_0(y) + \rho_1(y)t^\alpha + \sum_{k\in\mathbb{K}\cap\mathbb{N}^*} \rho_k(y) E_{\alpha,1}(-\lambda_k t^\alpha) = \widetilde{\rho}_0(y) + \widetilde{\rho}_1(y)t^{\widetilde{\alpha}} + \sum_{k\in\widetilde{\mathbb{K}}\cap\mathbb{N}^*} \widetilde{\rho}_k(y) E_{\widetilde{\alpha},1}(-\widetilde{\lambda}_k t^{\widetilde{\alpha}}).$$

From the decay property of $E_{\alpha,1}(-\eta)$ (see Lemma 2.1), we derive $\rho_0(y) + \rho_1(y)t^\alpha = \widetilde{\rho}_0(y) + \widetilde{\rho}_1(y)t^{\widetilde{\alpha}}$, indicating $\rho_0 = \widetilde{\rho}_0$ and $\rho_1 = \widetilde{\rho}_1$. Moreover, we have $\alpha = \widetilde{\alpha}$ if $1 \in \mathbb{K}$. If $1 \notin \mathbb{K}$ and $1 \notin \widetilde{\mathbb{K}}$, i.e., $\rho_1 = \widetilde{\rho}_1 = 0$, then

$$\sum_{k\in\mathbb{K}\cap\mathbb{N}^*} \rho_k(y) E_{\alpha,1}(-\lambda_k t^\alpha) = \sum_{k\in\widetilde{\mathbb{K}}\cap\mathbb{N}^*} \widetilde{\rho}_k(y) E_{\widetilde{\alpha},1}(-\widetilde{\lambda}_k t^{\widetilde{\alpha}}) \quad \text{on } \Gamma_0 \times (0,\infty).$$

Proposition 2.1(ii) and Laplace transform give

$$\sum_{k\in\mathbb{K}\cap\mathbb{N}^*} \frac{\rho_k(y) z^{\alpha-1}}{z^\alpha + \lambda_k} = \sum_{k\in\widetilde{\mathbb{K}}\cap\mathbb{N}^*} \frac{\widetilde{\rho}_k(y) z^{\widetilde{\alpha}-1}}{z^{\widetilde{\alpha}} + \widetilde{\lambda}_k}.$$

Assuming that $\alpha > \widetilde{\alpha}$, dividing both sides by $z^{\widetilde{\alpha}-1}$ and setting $\zeta := z^\alpha$, we have

$$\sum_{k\in\mathbb{K}\cap\mathbb{N}^*} \frac{\rho_k(y)\zeta^{1-\frac{\widetilde{\alpha}}{\alpha}}}{\zeta+\lambda_k} = \sum_{k\in\widetilde{\mathbb{K}}\cap\mathbb{N}^*} \frac{\widetilde{\rho}_k(y)}{\zeta^{\frac{\widetilde{\alpha}}{\alpha}}+\widetilde{\lambda}_k}.$$

Upon noting $\mathbb{K}\neq\emptyset$, choosing an arbitrary $k_0\in\mathbb{K}$ and rearranging terms, we derive

$$\rho_{k_0}(y)\zeta^{1-\frac{\widetilde{\alpha}}{\alpha}} = \left(\sum_{k\in\widetilde{\mathbb{K}}\cap\mathbb{N}^*} \frac{\widetilde{\rho}_k(y)}{\zeta^{\frac{\widetilde{\alpha}}{\alpha}}+\widetilde{\lambda}_k} - \sum_{k\in\mathbb{K}\cap\mathbb{N}^*\setminus\{k_0\}} \frac{\rho_k(y)\zeta^{1-\frac{\widetilde{\alpha}}{\alpha}}}{\zeta+\lambda_k}\right)(\zeta+\lambda_{k_0}).$$

Letting $\zeta\to-\lambda_{k_0}$ and noting $\alpha>\widetilde{\alpha}$, the right hand side tends to zero (since all $\widetilde{\lambda}_k$ are positive, and $\arg((-\lambda_{k_0})^{\frac{\widetilde{\alpha}}{\alpha}}) = \frac{\widetilde{\alpha}\pi}{\alpha}\in(0,\pi)$) and hence $\rho_{k_0}\equiv 0$ in $L^2(\Gamma_0)$, which contradicts the assumption $k_0\in\mathbb{K}$. Thus, we deduce $\alpha\le\widetilde{\alpha}$. The same argument yields $\alpha\ge\widetilde{\alpha}$, so $\alpha=\widetilde{\alpha}$. These discussions thus yield

$$\sum_{k\in\mathbb{K}\cap\mathbb{N}^*} \frac{\rho_k(y)}{\zeta+\lambda_k} = \sum_{k\in\widetilde{\mathbb{K}}\cap\mathbb{N}^*} \frac{\widetilde{\rho}_k(y)}{\zeta+\widetilde{\lambda}_k}. \tag{3.1}$$

Note that both sides of the identity (3.1) are $L^2(\Gamma_0)$-valued functions in ζ. Next we show both converge uniformly in any compact subset in $\mathbb{C}\setminus(\{-\lambda_k\}_{k\in\mathbb{K}\cap\mathbb{N}^*}\cup\{-\widetilde{\lambda}_k\}_{k\in\widetilde{\mathbb{K}}\cap\mathbb{N}^*})$ and are analytic in $\mathbb{C}\setminus(\{-\lambda_k\}_{k\in\mathbb{K}\cap\mathbb{N}^*}\cup\{-\widetilde{\lambda}_k\}_{k\in\widetilde{\mathbb{K}}\cap\mathbb{N}^*})$. Indeed, since $u_0,f\in L^2(\Omega)$, for all ζ in any compact subset of $\mathbb{C}\setminus(\{-\lambda_k\}_{k\in\mathbb{K}\cap\mathbb{N}^*}\cup\{-\widetilde{\lambda}_k\}_{k\in\widetilde{\mathbb{K}}\cap\mathbb{N}^*})$, we have

$$\begin{aligned}\left\|\sum_{k\in\mathbb{K}\cap\mathbb{N}^*}\frac{\rho_k}{\zeta+\lambda_k}\right\|^2_{\mathrm{Dom}(A)} &\le c\sum_{\ell\in\mathbb{N}^*}\lambda_\ell^2\frac{|(u_0,\varphi_\ell)|^2+\lambda_\ell^{-2}|(f,\varphi_\ell)|^2}{|\zeta+\lambda_\ell|^2}\\ \le& c\sum_{\ell\in\mathbb{N}^*}\left(|(u_0,\varphi_\ell)|^2+\lambda_\ell^{-2}|(f,\varphi_\ell)|^2\right)<\infty.\end{aligned}$$

Hence, by the trace theorem, the identity (3.1) holds for all $\zeta\in\mathbb{C}\setminus(\{-\lambda_k\}_{k\in\mathbb{K}\cap\mathbb{N}^*}\cup\{-\widetilde{\lambda}_k\}_{k\in\widetilde{\mathbb{K}}\cap\mathbb{N}^*})$. Assume that $\lambda_j\notin\{\widetilde{\lambda}_k\}_{k\in\widetilde{\mathbb{K}}\cap\mathbb{N}^*}$ for some $j\in\mathbb{K}\cap\mathbb{N}^*$. Then we can choose a small circle C_j centered at $-\lambda_j$ which does not contain $\{-\widetilde{\lambda}_k\}_{k\in\widetilde{\mathbb{K}}\cap\mathbb{N}^*}$. Integrating on C_j and applying the Cauchy theorem give $2\pi\sqrt{-1}\,\rho_j/\lambda_j=0$, which contradicts the assumption $\rho_j\not\equiv0$ in $L^2(\Gamma_0)$. Hence, $\lambda_j\in\{\widetilde{\lambda}_k\}_{k\in\widetilde{\mathbb{K}}\cap\mathbb{N}^*}$ for every $j\in\mathbb{K}\cap\mathbb{N}^*$. Likewise, $\widetilde{\lambda}_j\in\{\lambda_k\}_{k\in\mathbb{K}\cap\mathbb{N}^*}$ for every $j\in\widetilde{\mathbb{K}}\cap\mathbb{N}^*$, and hence $\{\lambda_k\}_{k\in\mathbb{K}\cap\mathbb{N}^*}=\{\widetilde{\lambda}_k\}_{k\in\widetilde{\mathbb{K}}\cap\mathbb{N}^*}$. From (3.1), we obtain

$$\sum_{k\in\mathbb{K}\cap\mathbb{N}^*}\frac{\rho_k(y)-\widetilde{\rho}_k(y)}{\zeta+\lambda_k}=0,\quad\forall\zeta\in\mathbb{C}\setminus\{-\lambda_k\}_{k\in\mathbb{K}\cap\mathbb{N}^*}.$$

Varying $j\in\mathbb{K}\cap\mathbb{N}^*$ and integrating over C_j, we obtain $2\pi\sqrt{-1}\,(\rho_j-\widetilde{\rho}_j)/\lambda_j=0$, which directly implies $\rho_j=\widetilde{\rho}_j$ in $L^2(\Gamma_0)$. This completes the proof of the theorem. □

Remark 3.1. *The condition $\mathbb{K}\neq\emptyset$ holds whenever the following condition is valid $(f,\varphi_1)\neq0$ or $(u_0,\varphi_{\ell,k})-\lambda_\ell^{-1}(f,\varphi_{\ell,k})\neq0$, $k=1,\ldots,m_\ell$, $\ell=2,3,\ldots$. Note that the condition $(f,\varphi_1)\neq0$ does not rely on the unknown parameter a, and can be easily guaranteed.*

The next result gives the uniqueness of recovering the diffusion coefficient a from the lateral boundary observation.

Theorem 3.2. *Let condition* (1.4) *be fulfilled, and let (a,f,u_0), $(\widetilde{a},\widetilde{f},\widetilde{u}_0)\in\mathfrak{A}\times L^2(\Omega)\times L^2(\Omega)$, and fix g as* (1.3). *Let h and $\widetilde{h}$ be the corresponding Dirichlet data. Then for any $\sigma\in(0,T_0]$, the condition $h=\widetilde{h}$ on $\Gamma_0\times[T_0-\sigma,T]$ implies $a=\widetilde{a}$.*

Proof. In view of the linearity of problem (1.1), we can decompose the data $h(t)$ into

$$h(t) = h_i(t) + h_b(t), \quad t \in (0, T],$$

with $h_i(t)$ and $h_b(t)$ given by

$$\begin{aligned} h_i(t) &= \rho_0 + \rho_1 t^\alpha + \sum_{k\in\mathbb{K}\cap\mathbb{N}^*} \rho_k E_{\alpha,1}(-\lambda_k t^\alpha), \\ h_b(t) &= \sum_{\ell=1}^{\infty} \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_\ell (t-s)^\alpha) \sum_{k=1}^{m_\ell} \langle g(s), \varphi_{\ell,k}\rangle \, \mathrm{d}s \, \varphi_{\ell,k}, \end{aligned}$$

which solve problem (1.1) with $g \equiv 0$ and $f = u_0 \equiv 0$, respectively. By the choice of g in (1.3), the interval $[0, T]$ can be divided into two subintervals: $(0, T_0]$ and $[T_0, T]$. For $t \in (0, T_0)$, $\psi(t) \equiv 0$, Theorem 3.1 implies that $\{(\rho_k, \lambda_k)\}_{k\in\mathbb{K}} = \{(\widetilde{\rho}_k, \widetilde{\lambda}_k)\}_{k\in\widetilde{\mathbb{K}}}$ and $\alpha = \widetilde{\alpha}$, from which we deduce $h_i(t) = \widetilde{h}_i(t)$ for all $t > 0$. For $t \in [T_0, T]$, this and the condition $h(t) = \widetilde{h}(t)$ imply $h_b(t) = \widetilde{h}_b(t)$ in $L^2(\Gamma_0)$, and hence

$$\begin{aligned} &\sum_{\ell=1}^{\infty} \int_{T_0}^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_\ell (t-s)^\alpha) \sum_{k=1}^{m_\ell} \langle g(s), \varphi_{\ell,k}\rangle \, \mathrm{d}s \, \varphi_{\ell,k} \\ =&\sum_{\ell=1}^{\infty} \int_{T_0}^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\widetilde{\lambda}_\ell (t-s)^\alpha) \sum_{k=1}^{\widetilde{m}_\ell} \langle g(s), \widetilde{\varphi}_{\ell,k}\rangle \, \mathrm{d}s \, \widetilde{\varphi}_{\ell,k}, \quad t \in [T_0, T]. \end{aligned}$$

By the analyticity in Corollary 2.1, the above identity holds for $t \in [T_0, \infty)$. Thus applying Laplace transform on both side gives

$$\sum_{\ell=2}^{\infty} \frac{\sum_{k=1}^{m_\ell} \langle \widehat{g}(z), \varphi_{\ell,k}\rangle \varphi_{\ell,k}}{z^\alpha + \lambda_\ell} = \sum_{\ell=2}^{\infty} \frac{\sum_{k=1}^{\widetilde{m}_\ell} \langle \widehat{g}(z), \widetilde{\varphi}_{\ell,k}\rangle \widetilde{\varphi}_{\ell,k}}{z^\alpha + \widetilde{\lambda}_\ell}, \quad \forall \Re(z) > 0. \tag{3.2}$$

Since $\lambda_1 = \widetilde{\lambda}_1 = 0$ and $\varphi_1 = \widetilde{\varphi}_1 = |\Omega|^{-\frac{1}{2}}$, the index in (3.2) starts with $\ell = 2$. Below we repeat the argument for Theorem 3.1. First we show that both sides of (3.2) are analytic with $\zeta = z^\alpha$ in any compact subset of $\mathbb{C} \setminus \{-\lambda_\ell, -\widetilde{\lambda}_\ell\}_{\ell\geq 2}$. Let $U \in \mathrm{Dom}(A^{\frac{1}{4}+\varepsilon})$ be a solution of problem (2.6), for all ζ in a compact subset of $\mathbb{C} \setminus \{-\lambda_\ell, -\widetilde{\lambda}_\ell\}_{\ell\geq 2}$, we have

$$\begin{aligned} &\left\| \sum_{\ell=2}^{\infty} \frac{\sum_{k=1}^{m_\ell} \langle \widehat{g}(\zeta^{\frac{1}{\alpha}}), \varphi_{\ell,k}\rangle \varphi_{\ell,k}}{\zeta + \lambda_\ell} \right\|^2_{\mathrm{Dom}(A^{\frac{1}{4}+\varepsilon})} \leq c \sum_{\ell=2}^{\infty} \lambda_\ell^{\frac{1}{2}+2\varepsilon} \sum_{k=1}^{m_\ell} \left| \frac{\langle \eta, \varphi_{\ell,k}\rangle}{\zeta + \lambda_\ell} \right|^2 \\ =&c \sum_{\ell=1}^{\infty} \lambda_\ell^{\frac{1}{2}+2\varepsilon} \sum_{k=1}^{m_\ell} \left| \frac{\lambda_\ell (U, \varphi_{\ell,k})}{\zeta + \lambda_\ell} \right|^2 \leq c \|U\|^2_{\mathrm{Dom}(A^{\frac{1}{4}+\varepsilon})} < \infty. \end{aligned}$$

Since each term of the series is a $\mathrm{Dom}(A^{\frac{1}{4}+\varepsilon})$-valued function analytic in ζ and converges uniformly in ζ, by the trace theorem, we obtain that both sides of (3.2) are $L^2(\partial\Omega)$-valued functions analytic in $\zeta \in \mathbb{C} \setminus \{-\lambda_\ell, -\widetilde{\lambda}_\ell\}_{\ell\geq 2}$. Since $\lambda_\ell, \widetilde{\lambda}_\ell > 0$ for $\ell \geq 2$, we may take $\zeta \to 0$ in (3.2) and obtain

$$\sum_{\ell=2}^{\infty} \frac{\sum_{k=1}^{m_\ell} \langle \widehat{g}(0), \varphi_{\ell,k}\rangle \varphi_{\ell,k}}{\lambda_\ell} = \sum_{\ell=2}^{\infty} \frac{\sum_{k=1}^{\widetilde{m}_\ell} \langle \widehat{g}(0), \widetilde{\varphi}_{\ell,k}\rangle \widetilde{\varphi}_{\ell,k}}{\widetilde{\lambda}_\ell}. \tag{3.3}$$

Hence, $w = \widetilde{w}$ on Γ_0, where w and $\widetilde{w}$ are the Dirichlet boundary data with a and $\widetilde{a}$ in the elliptic problem

$$\begin{cases} -\nabla \cdot (a\nabla w) = 0 & \text{in } \Omega, \\ a\partial_\nu w = \widehat{g}(0) & \text{on } \partial\Omega \end{cases} \tag{3.4}$$

with the compatibility condition $\int_\Omega w\,\mathrm{d}x = 0$. Indeed, the solution w of (3.4) can be represented as

$$w = \sum_{\ell=2}^{\infty}\sum_{k=1}^{m_\ell}(w,\varphi_{\ell,k})\varphi_{\ell,k} = \sum_{\ell=2}^{\infty}\sum_{k=1}^{m_\ell}\lambda_\ell^{-1}\langle\widehat{g}(0),\varphi_{\ell,k}\rangle\varphi_{\ell,k},$$

where the first equality follows from the compatibility condition $\int_\Omega w\,\mathrm{d}x = 0$ and the second is due to integration by part. By the choice of g in (1.3), the elliptic problem (3.4) is uniquely solvable. Then from [13, Theorem 1.1], we deduce that $D = \widetilde{D}$ is uniquely determined by the input $\widehat{g}(0) = \widehat{\psi}(0)\eta$. Indeed, Friedman and Isakov [13] proved the unique determination of the convex polygon D for the case $\mu \equiv 1$, based on extending the solution w harmonically across a vertex of D and leading a contradiction. The proof does not depend on the knowledge of the parameter μ and hence it is also applicable here. Once D is determined, it suffices to show the uniqueness of the scalar μ. Suppose $\mu \le \widetilde{\mu}$, i.e., $a \le \widetilde{a}$ in D and $a \equiv \widetilde{a} \equiv 1$ outside D. Thus w and $\widetilde{w}$ are harmonic functions near $\partial\Omega$ with identical Cauchy data on Γ_0, we conclude $w = \widetilde{w}$ near $\partial\Omega$. By multiplying both sides of the governing equation in (3.4) with w, integrating over the domain Ω and applying Green's formula, we have

$$0 = \int_\Omega -\nabla\cdot(a\nabla w)w\,\mathrm{d}x = \int_\Omega a|\nabla w|^2\,\mathrm{d}x - \int_{\partial\Omega} w\,\partial_\nu w\,\mathrm{d}S,$$

i.e.,

$$\int_\Omega a|\nabla w|^2\,\mathrm{d}x = \int_{\partial\Omega} w\,\partial_\nu w\,\mathrm{d}S.$$

Now since w and $\widetilde{w}$ have identical Cauchy data on the boundary $\partial\Omega$, we have $\int_{\partial\Omega} w\,\partial_\nu w\,\mathrm{d}S = \int_{\partial\Omega}\widetilde{w}\,\partial_\nu\widetilde{w}\,\mathrm{d}S$, and consequently

$$\int_\Omega a|\nabla w|^2\,\mathrm{d}x = \int_\Omega \widetilde{a}|\nabla\widetilde{w}|^2\,\mathrm{d}x.$$

This identity and the inequality $\widetilde{a} \ge a$ a.e. in Ω imply

$$\int_\Omega a|\nabla w|^2\,\mathrm{d}x \ge \int_\Omega a|\nabla\widetilde{w}|^2\,\mathrm{d}x,$$

which immediately implies

$$\frac{1}{2}\int_\Omega a|\nabla w|^2\mathrm{d}x - \int_{\partial\Omega} w\widehat{g}(0)\,\mathrm{d}S \ge \frac{1}{2}\int_\Omega a|\nabla\widetilde{w}|^2\,\mathrm{d}x - \int_{\partial\Omega}\widetilde{w}\widehat{g}(0)\,\mathrm{d}S.$$

By the Dirichlet principle [10], w is the minimizer of the energy integral, and hence $w = \widetilde{w}$ and $a = \widetilde{a}$. □

Remark 3.2. *Note that the uniqueness of the inclusion D in [13] relies on the assumption D being a convex polygon with* $\mathrm{diam}(D) < \mathrm{dist}(D,\partial\Omega)$. *Alessandrini [1] removed the diameter assumption for a specialized choice of the boundary data. The works [49, 29] proved the unique determination of D when D is a disc or ball. For general shapes, even for ellipses or ellipsoids, this inverse problem appears still open. Note that in the uniqueness proof, the key is the reduction of the problem to the elliptic case, with a nonzero Neumann boundary condition. In particular, the result will not hold if the temporal component ψ vanishes identically over the interval $[0,T]$, i.e., condition* (1.4) *does not hold.*

Remark 3.3. *If the diffusion coefficient a is not piecewise constant, it is also possible to show the unique recovery if the boundary excitation data g is specially designed. For example, consider problem* (1.1) *with a more general elliptic operator*

$$\mathcal{A}u(x) := -\nabla\cdot(a(x)\nabla u(x)) + q(x)u(x), \quad x \in \overline{\Omega}. \tag{3.5}$$

Here $a \in C^2(\overline{\Omega})$ and $q \in L^\infty(\Omega)$ with $a > 0$ in $\overline{\Omega}$ and $q \ge 0$ in Ω, and the Neumann data g is constructed as follows. First, we choose sub-boundaries Γ_1 and Γ_2 such that $\Gamma_1 \cup \Gamma_2 = \partial\Omega$ and $\Gamma_1 \cap \Gamma_2 \ne \emptyset$.

Let $\chi \in C^\infty(\partial\Omega)$ be a cut-off function with $\mathrm{supp}(\chi) = \Gamma_1$ and $\chi \equiv 1$ on Γ_1', with $\Gamma_1' \subset \Gamma_1$ such that $\Gamma_1' \cup \Gamma_2 = \partial\Omega$, $\Gamma_1' \cap \Gamma_2 \neq \emptyset$; see Fig. 1 for an illustration of the geometry in the two-dimensional case. Now we fix $0 \le T_0 < T_1 < T$ and choose a strictly increasing sequence $\{t_k\}_{k=0}^\infty$ such that $t_0 = T_0$ and $\lim_{k\to\infty} t_k = T_1$. Consider a sequence $\{p_k\}_{k=1}^\infty \subset \mathbb{R}_+$ and a sequence $\{\psi_k\}_{k=1}^\infty \subset C^\infty([0,\infty);\overline{\mathbb{R}_+})$ such that

$$\psi_k = \begin{cases} 0 & \text{on } [0, t_{2k-1}], \\ p_k & \text{on } [t_{2k}, \infty). \end{cases}$$

Then we fix $\{b_k\}_{k=0}^\infty \subset \mathbb{R}_+$ such that $\sum_{k=1}^\infty b_k \|\psi_k\|_{W^{2,\infty}(\mathbb{R}_+)} < \infty$, and define the Neumann data g by

$$g(y,t) := \sum_{k=1}^\infty g_k(y,t) = \chi \sum_{k=1}^\infty b_k \psi_k(t) \eta_k(y), \tag{3.6}$$

where the set $\{\eta_k\}_{k=1}^\infty$ is chosen to be dense in $H^{\frac12}(\partial\Omega)$ and $\|\eta_k\|_{H^{\frac12}(\partial\Omega)} = 1$. Note that the Neumann data g defined in (3.6) plays the role of infinity measurements [5, 6], and hence the unique recovery of the fractional order α and both a and q from one boundary measurement. We provide a detailed proof in the appendix for completeness. See also some related discussions in [31, 30] with different problem settings. However, this choice of g is impossible to numerically realize in practice, due to the need to numerically represent infinitesimally small quantities.

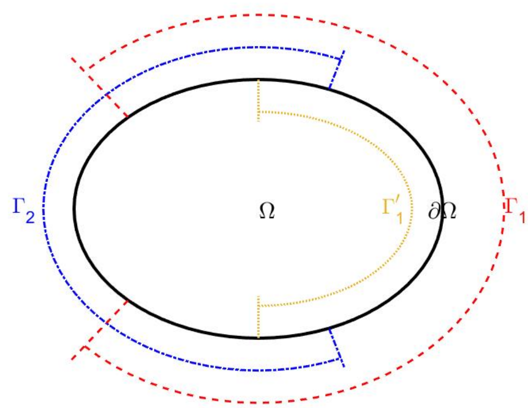

Figure 1: The schematic illustration of the sub-boundaries Γ_1', Γ_1 and Γ_2 of the boundary $\partial\Omega$.

4 Reconstruction algorithm

In this section, we derive an algorithm for recovering the fractional order α and the coefficient a, directly inspired by the uniqueness proof. We divide the recovery procedure into three steps:

(i) use the asymptotic behavior of the solution of problem (1.1) near $t = 0$ to recover α;

(ii) use analytic extension to extract the solution of problem (1.1) with zero f and u_0;

(iii) use the level set method [42] to recover the shape of the unknown medium $D \subset \Omega$.

First, we give an asymptotics of the Dirichlet data $h(t)$ of problem (1.1). The result is direct from the representation and properties of $E_{\alpha,1}(z)$ near $z = 0$ and the trace theorem.

Proposition 4.1. *If $u_0 \in \mathrm{Dom}(A^{1+\frac{s}{2}})$ and $f \in \mathrm{Dom}(A^{\frac{s}{2}})$ with $s > 1$. Let $h = u|_{\partial\Omega\times(0,T)}$ be the Dirichlet trace of the solution to problem (1.1) with g given as (1.3), then the following asymptotic holds:*

$$h(y,t) = u_0(y) + (\mathcal{A}u_0 - f)(y)t^\alpha + O(t^{2\alpha}) \quad \text{as } t \to 0^+.$$

In view of Proposition 4.1, for any fixed $y_0 \in \partial\Omega$, the asymptotic behavior of $h(y_0, t)$ as $t \to 0^+$ allows recovering the order α. This can be achieved by minimizing the following objective in α, c_0 and c_1:

$$J(\alpha, c_0, c_1) = \|c_0 + c_1 t^\alpha - h(y_0, t)\|^2_{L^2(0,t_0)}, \tag{4.1}$$

for some small $t_0 > 0$. Note that it is important to take t_0 sufficiently small so that higher-order terms can indeed be neglected. The idea of using asymptotics for order recovery was employed in [16, 21, 22].

When recovering the diffusion coefficient a, we need to deal with the unknown functions u_0 and f. This poses significant computational challenges since standard regularized reconstruction procedures [12] require a fully known forward operator. To overcome the challenge, we appeal to Theorem 3.2: u_0 and f only contribute to $h_i(t)$ which is fully determined by $\{\lambda_\ell, \rho_\ell\}_{\ell\in\mathbb{K}}$. Indeed, by Theorem 3.1, $\{\lambda_\ell, \rho_\ell\}_{\ell\in\mathbb{K}}$ can be uniquely determined by $h(t)$, $t \in [0, T_0]$. Hence in theory we can extend $h(t) = h_i(t)$ from $t \in [0, T_0]$ to $t \in [0, T]$, by means of analytic continuation, to approximate the Dirichlet data of (1.1) with $g \equiv 0$ and given u_0 and f. In practice, we look for approximations of the form

$$h(t) \approx \frac{p_0 + p_1 t + \cdots + p_r t^r}{q_0 + q_1 t + \cdots + q_r t^r} := h_r(t), \quad t \in [0, T],$$

where $r \in \mathbb{N}$ is the polynomial order. This choice is motivated by the observation that Mittag-Leffler functions can be well approximated by rational polynomials [2, 38, 11]. The approximation h_r can be constructed efficiently by the AAA algorithm [41]. Now, we can get the Dirichlet data of problem (1.1) with a given g and $u_0 = f \equiv 0$, by defining the reduced data

$$\overline{h}(t) := \begin{cases} 0, & t \in [0, T_0], \\ h(t) - h_r(t), & t \in [T_0, T]. \end{cases}$$

Below we use the reduced data $\overline{h}$ to recover a piecewise constant a. Parameter identification for the subdiffusion model is commonly carried out by minimizing a suitable penalized objective. Since a is piecewise constant, it suffices to recover the interface between different media. The level set method can effectively capture the interface in an elliptic problem [48, 19, 3, 9], which we extend to the time-fractional model (1.1) below. Specifically, we consider a slightly more general setting where the inclusion $D \subset \Omega$ has a diffusivity value a_1 and the background $\Omega \setminus D$ has a diffusivity value a_2, with possibly unknown a_1 and a_2. That is, the diffusion coefficient a is represented as

$$a(x) = a_1 H(\phi(x)) + a_2(1 - H(\phi(x))) \quad \text{in } \Omega, \tag{4.2}$$

where $H(x)$ and $\phi(x)$ denote the Heaviside function and level set function (a signed distance function):

$$H(x) := \begin{cases} 1, & x \geq 0, \\ 0, & x < 0, \end{cases} \quad \text{and} \quad \phi(x) := \begin{cases} d(x, \partial D), & x \in D, \\ -d(x, \partial D), & x \in \Omega \setminus \overline{D}, \end{cases}$$

respectively. Then ϕ satisfies $D = \{x \in \Omega : \phi(x) > 0\}$, $\Omega \setminus \overline{D} = \{x \in \Omega : \phi(x) < 0\}$ and $\partial D = \{x \in \Omega : \phi(x) = 0\}$. To find the values a_1 and a_2 and the interface ∂D, we minimize the following functional

$$J(\phi, a_1, a_2) = \frac{1}{2}\|u(a) - \overline{h}\|^2_{L^2(0,T;L^2(\Gamma_0))} + \beta \int_\Omega |\nabla a|, \tag{4.3}$$

where $u(a)$ is the solution to problem (2.3), and $\beta > 0$ is the penalty parameter. The total variation term $\int_\Omega |\nabla a|$ is to stabilize the inverse problem, which is defined by

$$\int_\Omega |\nabla a| := \sup_{\varphi \in (C_0(\overline{\Omega}))^d, |\varphi| \leq 1} \int_\Omega a \nabla \cdot \varphi \, \mathrm{d}x,$$

where $|\cdot|$ denotes the Euclidean norm. Then we apply the standard gradient descent method to minimize problem (4.3). The next result gives the gradient of J. The notations $J^{1-\alpha}_{T_-}$ and $D^\alpha_{T_-}$ denote the backward Riemann-Liouville integral and derivative, defined respectively by [20, Sections 2.2 and 2.3]

$$J^{1-\alpha}_{T_-} v(t) := \frac{1}{\Gamma(1-\alpha)} \int_t^T (s - t)^{-\alpha} v(s) \, \mathrm{d}s,$$

$$D_{T-}^{\alpha} v(t) := -\frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_t^T (s-t)^{-\alpha} v(s)\,\mathrm{d}s.$$

Proposition 4.2. *The derivative* $\frac{\mathrm{d}}{\mathrm{d}a} J$ *is formally given by*

$$\frac{\mathrm{d}}{\mathrm{d}a} J(a) = -\int_0^T \nabla u \cdot \nabla v \,\mathrm{d}t - \beta \nabla \cdot \left(\frac{\nabla a}{|\nabla a|} \right),$$

where $v = v(x,t;a)$ *solves the adjoint problem*

$$\begin{cases} D_{T-}^{\alpha} v - \nabla \cdot (a \nabla v) = 0 & \text{in } \Omega \times [0,T), \\ a \partial_\nu v = (u - \overline{h}) \chi_{\Gamma_0} & \text{on } \partial\Omega \times [0,T), \\ J_{T-}^{1-\alpha} v(\cdot, T) = 0 & \text{in } \Omega. \end{cases} \tag{4.4}$$

Proof. We write $J(a) = J_1(a) + J_2(a)$, with $J_1(a) = \frac{1}{2} \| u(a) - \overline{h} \|^2_{L^2(0,T;L^2(\Gamma_0))}$ and $J_2(a) = \beta \int_\Omega |\nabla a|$. For the term J_1, the directional derivative along b is

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} J_1(a + \varepsilon b) = \int_0^T \int_{\Gamma_0} (u(a) - \overline{h}) u'(a)[b] \,\mathrm{d}S\mathrm{d}t,$$

where $u'(a)[b]$ is the directional derivative with respect to a in the direction b. Let $\widetilde{a} = a + \varepsilon b$ and $\widetilde{u}$ solves problem (2.3) with the coefficient $\widetilde{a}$. Then $w := u'(a)[b] = \lim_{\varepsilon \to 0} \varepsilon^{-1} (\widetilde{u} - u)$. Upon subtracting the equations for $\widetilde{u}$ and u and then taking limits, we get

$$\begin{cases} \partial_t^\alpha w - \nabla \cdot (a \nabla w) = \nabla \cdot (b \nabla u) & \text{in } \Omega \times (0,T], \\ a \partial_\nu w = -b \partial_\nu u & \text{in } \partial\Omega \times (0,T], \\ w(0) = 0 & \text{in } \Omega. \end{cases}$$

Multiplying the equation for w with any $\psi \in L^2(0,T;H^1(\Omega))$ and integrating over $\Omega \times (0,T)$ give

$$\int_0^T \int_\Omega (\psi \,\partial_t^\alpha w + a \nabla w \cdot \nabla \psi) \,\mathrm{d}x\mathrm{d}t = -\int_0^T \int_\Omega b \nabla u \cdot \nabla \psi \,\mathrm{d}x\mathrm{d}t. \tag{4.5}$$

Let v be the solution of problem (4.4). Multiplying the governing equation for v with a test function ψ and integrating over $\Omega \times (0,T)$ give

$$\int_0^T \int_\Omega \left(\psi \, D_{T-}^\alpha v + a \nabla v \cdot \nabla \psi \right) \mathrm{d}x\mathrm{d}t = \int_0^T \int_{\Gamma_0} (u - \overline{h}) \psi \,\mathrm{d}S\mathrm{d}t. \tag{4.6}$$

Note that the following integration by parts formula for fractional derivatives:

$$\int_0^T v \,\partial_t^\alpha w \,\mathrm{d}t = \left[w \, J_{T-}^{1-\alpha} v \right]_{t=0}^{t=T} + \int_0^T w \, D_{T-}^\alpha v \,\mathrm{d}t = \int_0^T w \, D_{T-}^\alpha v \,\mathrm{d}t \tag{4.7}$$

(for suitably smooth v and w with $w(0) = 0$ and $J_{T-}^{1-\alpha} v = 0$). Now by choosing $\psi = v$ in (4.5), $\psi = w$ in (4.6) and applying (4.7), we obtain

$$-\int_0^T \int_\Omega b \nabla u \cdot \nabla v \,\mathrm{d}x\mathrm{d}t = \int_0^T \int_{\Gamma_0} (u - \overline{h}) w \,\mathrm{d}S\mathrm{d}t,$$

implying $\frac{\mathrm{d}}{\mathrm{d}a} J_1(a) = -\int_0^T \nabla u \cdot \nabla v \mathrm{d}t$. For the term J_2, the directional derivative along b is

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} \int_\Omega |\nabla (a + \varepsilon b)| \,\mathrm{d}x = \int_\Omega \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} \left(|\nabla (a + \varepsilon b)|^2 \right)^{1/2} \mathrm{d}x \\ = &\int_\Omega \left(|\nabla (a + \varepsilon b)|^2 \right)^{-\frac{1}{2}} \Big|_{\varepsilon=0} \nabla a \cdot \nabla b \,\mathrm{d}x = \int_\Omega \frac{\nabla a}{|\nabla a|} \cdot \nabla b \,\mathrm{d}x, \end{aligned}$$

and hence we have $\frac{\mathrm{d}}{\mathrm{d}a} J_2(a) = -\beta \nabla \cdot (\frac{\nabla a}{|\nabla a|})$. ☐

By the chain rule, the derivatives of J with respect to a_1, a_2 and ϕ are given by

$$\begin{aligned}
\frac{\partial J}{\partial \phi} &= \frac{\mathrm{d}J}{\mathrm{d}a}\frac{\partial a}{\partial \phi} = \frac{\mathrm{d}J}{\mathrm{d}a}(a_1 - a_2)\delta(\phi),\\
\frac{\partial J}{\partial a_1} &= \int_\Omega \frac{\mathrm{d}J}{\mathrm{d}a}\frac{\partial a}{\partial a_1}\,\mathrm{d}x = \int_\Omega \frac{\mathrm{d}J}{\mathrm{d}a}H(\phi)\,\mathrm{d}x,\\
\frac{\partial J}{\partial a_2} &= \int_\Omega \frac{\mathrm{d}J}{\mathrm{d}a}\frac{\partial a}{\partial a_2}\,\mathrm{d}x = \int_\Omega \frac{\mathrm{d}J}{\mathrm{d}a}(1 - H(\phi))\,\mathrm{d}x,
\end{aligned}$$

where δ is the Dirac delta function. Hence the iterative scheme for updating a_1, a_2 and ϕ reads

$$\phi^{k+1} = \phi^k - \gamma^k \frac{\partial J}{\partial \phi}(\phi^k, a_1^k, a_2^k) \quad \text{and} \quad a_j^{k+1} = a_j^k - \gamma_j^k \frac{\partial J}{\partial a_j}(\phi^{k+1}, a_1^k, a_2^k), \quad j = 1, 2.$$

The step sizes γ^k and γ_j^k can be either fixed or obtained by means of line search. The implementation of the method requires some care. First, we approximate the delta function $\delta(x)$ and Heaviside function $H(x)$ by

$$\delta_\varepsilon(x) = \frac{\varepsilon}{\pi(x^2+\varepsilon^2)} \quad \text{and} \quad H_\varepsilon(x) = \frac{1}{\pi}\arctan\left(\frac{x}{\varepsilon}\right) + \frac{1}{2},$$

respectively, with $\varepsilon > 0$ of order of the mesh size [7, 9]. Second, during the iteration, the new iterate of ϕ may fail to be a signed distance function. Although one is only interested in $\mathrm{sign}(\phi)$, it is undesirable for $|\phi|$ to get too large near the interface. Thus we reset ϕ to a signed distance function whenever ϕ changes by more than 10% in the relative $L^2(\Omega)$-norm. The resetting procedure is to find the steady solution of the following equation [42, 9]:

$$\partial_t d + \mathrm{sign}(d)(|\nabla d| - 1) = 0, \quad \text{with } d(0) = \phi.$$

5 Numerical Experiments and Discussions

Now we present numerical results for reconstructing the fractional order α and piecewise constant diffusion coefficient a, with unknown u_0 and f. In all experiments, the domain Ω is taken to be the unit square $\Omega = (0,1)^2$, and the final time $T = 1$. We divide the domain Ω into uniform squares with a length $h = 0.02$ and then divide along the diagonals of each square. We discretize the time interval $[0, T]$ into uniform subintervals with a time step size $\tau = 0.01$. All direct and adjoint problems are solved by standard continuous piecewise linear Galerkin finite element method in space and backward Euler convolution quadrature in time (see e.g.,[23] and [26, Chapters 2 and 3]). Below we investigate the following four cases:

(i) D is a disc with radius $\frac{1}{3}$, centered at $(\frac{1}{2}, \frac{1}{2})$,

(ii) D is a square with length $\frac{1}{2}$, centered at $(\frac{1}{2}, \frac{1}{2})$,

(iii) D is a concave polygon, and

(iv) D is two discs with radius $\frac{1}{5}$, centered at $(\frac{1}{4}, \frac{1}{2})$ and $(\frac{3}{4}, \frac{1}{2})$, respectively.

Throughout, the unknown initial condition u_0 and source f are fixed as

$$u_0(x_1, x_2) = x_1^2 x_2^2 (1 - x_1)^2 (1 - x_2)^2 \quad \text{and} \quad f(x_1, x_2) = 1 + x_1 + z_2,$$

respectively. Meanwhile, we fix the exact fractional order $\alpha^\dagger = 0.8$ and the diffusion coefficient $a^\dagger = 10 - 9\chi_D$, i.e. $a_1 = 1$, $a_2 = 10$. Unless otherwise stated, the Neumann excitation g is taken as $g(y,t) = \eta(y)\chi_{[0.5,1]}(t)$, where η is the cosine function with a frequency 2π on each edge for cases (i)–(iii) and is constant 1 for case (iv). We set g on $\partial\Omega \times [0, T]$, and take the measurement h on $\partial\Omega \times [0, T]$.

Table 1: The recovered order α based on least-squares fitting.

(a) case (i)

$t_0\backslash\alpha$	0.3000	0.5000	0.8000
1e-3	0.2402	0.5289	0.8353
1e-4	0.2516	0.5244	0.8795
1e-5	0.2649	0.4994	0.8006
1e-6	0.2712	0.4637	0.7978
1e-7	0.2665	0.5267	0.8019
1e-8	0.2558	0.4913	0.7989
1e-9	0.2744	0.4925	0.7999

(b) case (ii)

$t_0\backslash\alpha$	0.3000	0.5000	0.8000
1e-3	0.2380	0.5243	0.8350
1e-4	0.2479	0.5239	0.8797
1e-5	0.2612	0.5022	0.7803
1e-6	0.2695	0.5182	0.7977
1e-7	0.2662	0.5279	0.8019
1e-8	0.2562	0.4914	0.7989
1e-9	0.2741	0.4925	0.7999

(c) case (iii)

$t_0\backslash\alpha$	0.3000	0.5000	0.8000
1e-3	0.2383	0.5214	0.8485
1e-4	0.2480	0.5198	0.8821
1e-5	0.2600	0.5098	0.8005
1e-6	0.2667	0.5213	0.7977
1e-7	0.2634	0.5273	0.8019
1e-8	0.2654	0.4913	0.7990
1e-9	0.2718	0.4925	0.7999

(d) case (iv)

$t_0\backslash\alpha$	0.3000	0.5000	0.8000
1e-3	0.2384	0.5247	0.8436
1e-4	0.2486	0.5221	0.8816
1e-5	0.2617	0.5033	0.8005
1e-6	0.2692	0.5178	0.7977
1e-7	0.2650	0.5273	0.8019
1e-8	0.2703	0.4913	0.7989
1e-9	0.2740	0.4925	0.7999

First, we show the numerical recovery of the fractional order α for three different values, i.e., 0.3, 0.5 and 0.8. In view of Proposition 4.1, it suffices to fix one point $y_0 \in \partial\Omega$ (which is fixed at the origin $y_0 = (0,0)$ below) and to minimize problem (4.1), for which we use the L-BFGS-B with constraint $\alpha \in [0,1]$ [4]. The recovered orders are presented in Table 1. Note that the least-squares functional has many local minima. Hence, the algorithm requires a good initial guess to get a correct value for α. It is observed that the reconstruction is more accurate when $t_0 \to 0^+$, since the high order terms are then indeed negligible. Also, for a fixed interval $(0, t_0)$, due to the asymptotic behavior, we have slightly better approximations when the true order α is large. However, this does not influence much the reconstruction results for cases (i)–(iv), since the coefficient a is constant near origin.

Now we apply analytic continuation to extend the observed data h by a rational function h_r from the interval $[0, 0.5]$ to $[0, 1]$, using the AAA algorithm [41] with degree $r = 4$. This step is essential for dealing with missing data u_0 and f: subtracting h_r from h yields the reduced data $\overline{h}$ for a given g and $u_0 = f \equiv 0$, which is then used in recovering a. Fig. 2 shows the $L^2(\partial\Omega)$ error between h_r and the exact data h_0 which is obtained by solving (1.1) with given g and vanishing u_0 and f. Note that higher order rational approximations can reduce the error over the interval $[0, 0.5]$, but it tends to lead to larger errors in the interval $[0.5, 1]$. The approach is numerically sensitive to the presence of data noise, reflecting the well-known severe ill-posed nature of analytic continuation.

Finally, we present recovery results for the piecewise constant coefficient a, or equivalently, the shape D. The exact value is 1 inside the inclusion D and 10 outside, unless otherwise stated. We use the standard gradient descent method to minimize problem (4.3). Unless otherwise stated, we fix the step sizes $\gamma^k \equiv 1$, $\gamma_1^k \equiv 0$, $\gamma_2^k \equiv 0$, i.e., fixing the values inside and outside the inclusion D. The regularization parameter β is chosen to be 10^{-8}, and the coefficients a_1 and a_2 are set to $a_1 = 0.9$ and $a_2 = 10$. The results are summarized in Figs. 3-9, where dashed lines denote the recovered interfaces.

Fig. 3 shows the result for case (i), when the initial guesses are a small circle but with two different centers. In either case, the algorithm can successfully reconstruct the exact circle after 10000 iterations. For case (ii), the exact interface is a square, again with the initial guess being small circles inside the square, cf. Fig. 4. The algorithm accurately recovers the four edges of the square. However, due to the non-smoothness, the corners are much more challenging to reconstruct and hence less accurately

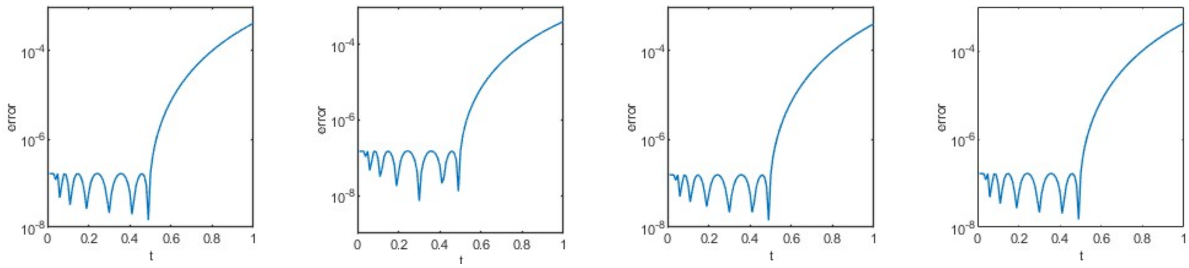

Figure 2: The $L^2(\partial\Omega)$-error between the analytic continuation h_r and true data h_0 for cases (i)–(iv).

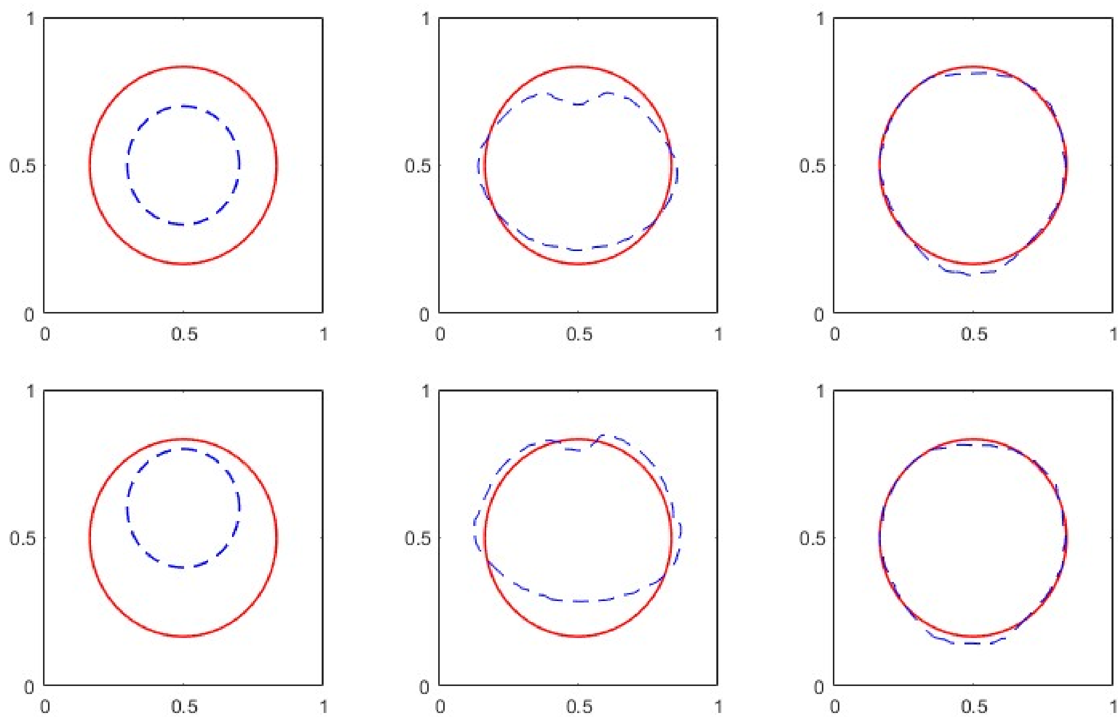

Figure 3: The reconstructions of the interface for case (i) at iteration 0, 100 and 10000 from left to right, with two different initial guesses.

resolved. These results indicate that the method does converge with a reasonable initial guess, but it may take many iterations to yield satisfactory reconstructions. Fig. 5 shows the results for case (iii) for which the exact interface is a concave polygon, which is much more challenging to resolve. Nonetheless, the algorithm can still recover the overall shape of the interface. The reconstruction around the concave part has lower accuracy. To the best of our knowledge, the unique determination of a concave polygonal inclusion (in an elliptic equation) is still open. Fig. 6 shows the results for case (iv) which contains two discs as the exact interface. The initial guess is two small discs near the boundary $\partial\Omega$. Note that in this case, we choose the boundary data $\eta \equiv 1$ in order to strengthen the effect of inhomogeneity. The final reconstruction is very satisfactory.

Fig. 7 shows a variant of case (ii), with the initial interface being two disjoint discs. It is observed that the two discs first merge into one concave contour, and then it evolves slowly to resolve the square. This shows one distinct feature of the level set method, i.e., it allows topological changes. Due to the complex evolution, the algorithm takes many more iterations to reach convergence (i.e., 30000 iterations

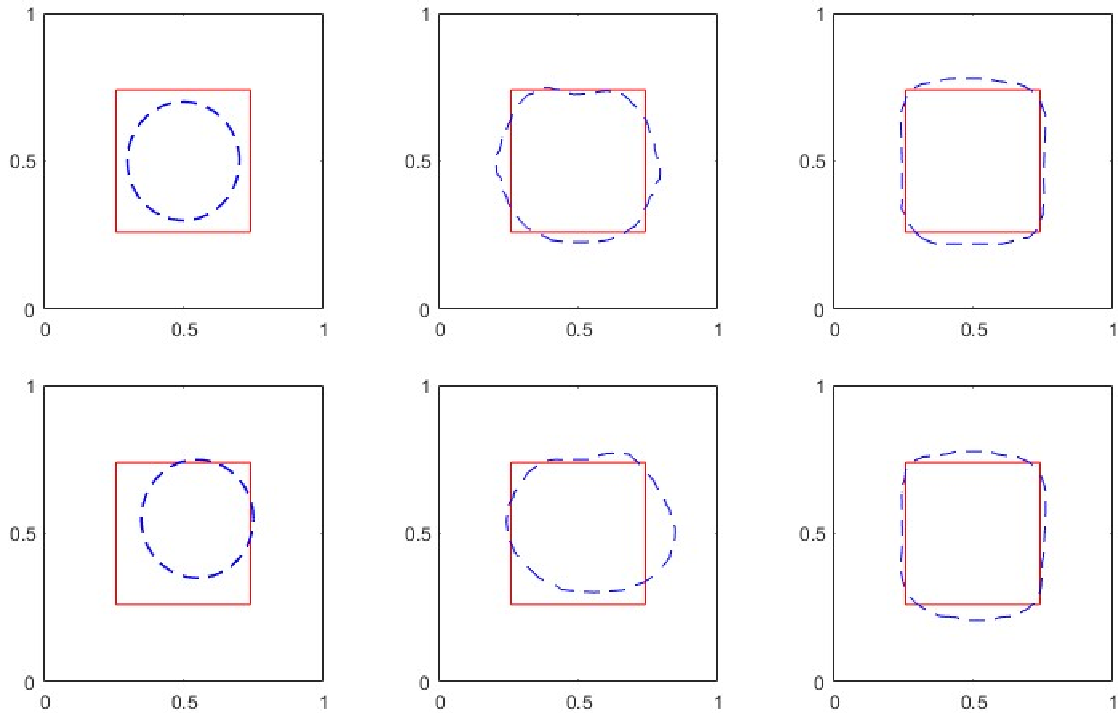

Figure 4: The reconstructions of the interface for case (ii) with different initial guesses at iteration 0, 100 and 8000 from left to right.

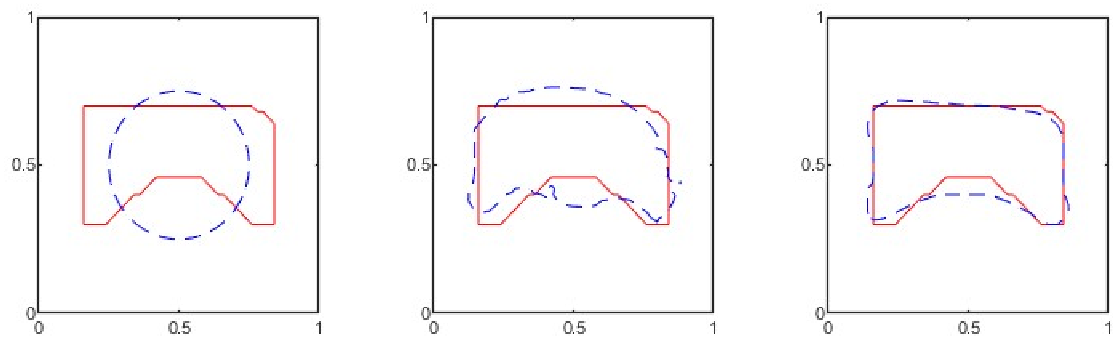

Figure 5: The reconstructions of the interface for case (iii) at iteration 0, 100 and 8000 from left to right.

versus 8000 iterations in case (ii)).

Fig. 8 shows a case which aims at simultaneously recovering the interface and the diffusivity value inside the inclusion, for which the exact interface is a square and the exact values of a_1 and a_2 are 1 and 10, respectively. In the experiment, we take two different initial guesses. The initial value of a_1 for both cases is $a_1 = 1.2$, and we take the step sizes $\gamma^k \equiv 1$, $\gamma_1^k \equiv 10$ and $\gamma_2^k \equiv 0$. The recovered value a_1 is 0.92 for the first row and 0.89 for the second row. It is observed that for both cases, one can roughly recover the interface. These experiments clearly indicate that the level set method can accurately recover the interface D. However, it generally takes many iterations to obtain satisfactory results. This is attributed partly to topological changes and the presence of nonsmooth points, and partly to the direct gradient flow formulation. Indeed, one observes from Proposition 4.2 that the gradient field for updating the level set

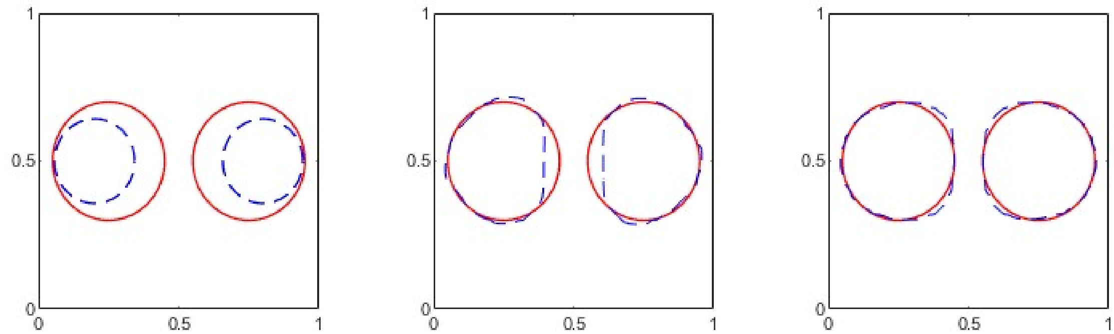

Figure 6: The reconstructions of the interface for case (iii) at iteration 0, 1000 and 15000 from left to right.

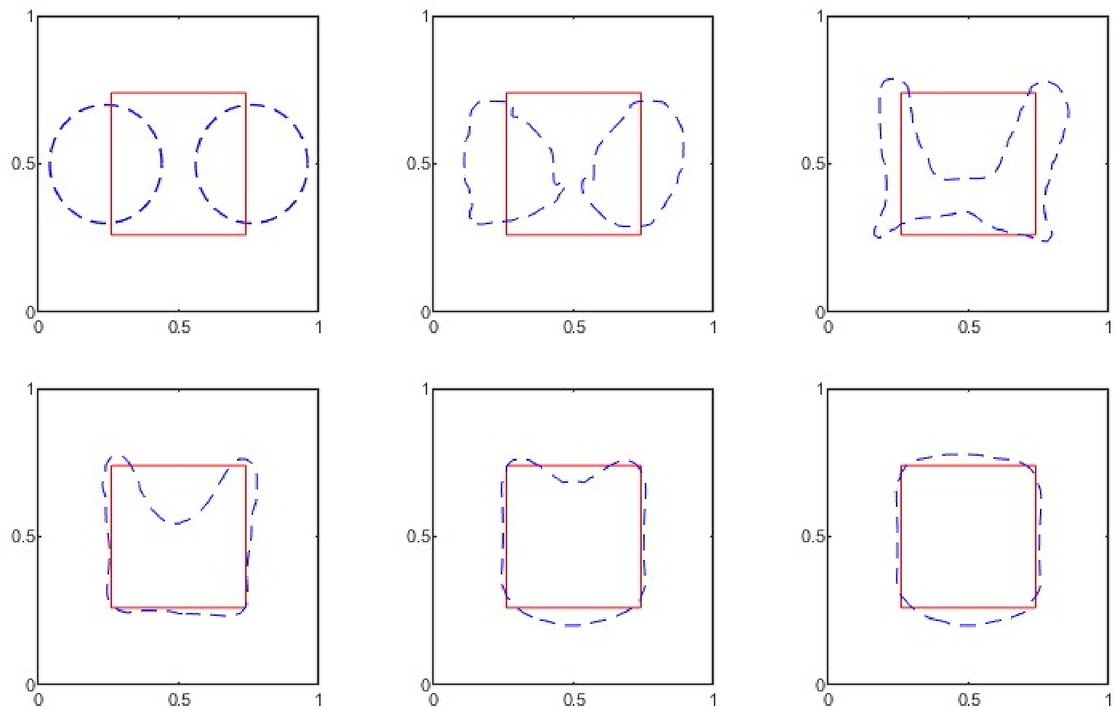

Figure 7: The reconstruction of the interface for case (ii) with a different initial guess, at different iterations 0, 100, 1000, 10000, 20000 and 30000 (from left to right).

function is actually not very smooth, which hinders the rapid evolution of the interface. Hence, there is an imperative need to accelerate the method, especially via suitable preconditioning and post-processing [19].

Last, Fig. 9 shows reconstruction results with noisy data. Due to the instability of analytic continuation for noisy data, we use boundary data corresponding to zero u_0, f as our measurement and only focus on reconstructing a. That is, we denote h^* the solution of problem (1.1) with $u_0 \equiv 0$ and $f \equiv 0$ which plays the role of $\overline{h}$. The noisy measurement h^δ is generated by

$$h^\delta(y,t) = h^*(y,t) + \varepsilon \|h^*\|_{L^\infty(\partial\Omega\times[0,1])}\xi(y,t),$$

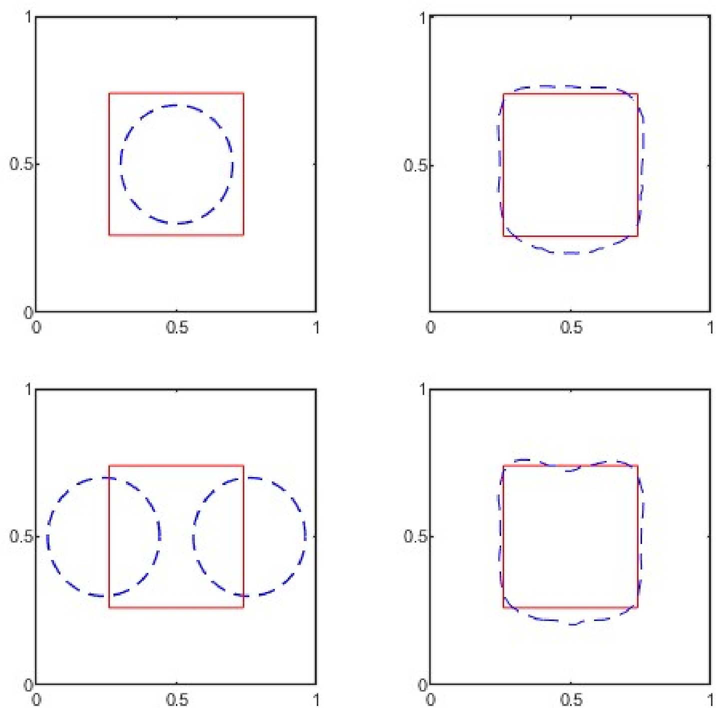

Figure 8: The reconstructions for case (ii) with a non-fixed diffusivity value a_1. Left column: initial guess. Right column: recovered interface.

where $\varepsilon > 0$ denotes the relative noise level, and ξ follows the standard Gaussian distribution. We take the exact interface as a concave polygon and the initial guess is a circle; see the left panel in Fig. 5. We consider two different noise levels and three different input boundary data. The first and second rows in Fig. 9 are for 1% and 5% noise, obtained with a regularization parameter $\beta = 1 \times 10^{-7}$ and $\beta = 5 \times 10^{-7}$, respectively. We consider three input Neumann data g_1, g_2 and g_3: $g_1 = g$ (i.e., identical as before), and g_2 and g_3 are given by

$$
\begin{aligned}
&g_2(x,t) = \eta_1(x)\chi_{[0.25,1]}(t) + \eta_2(x)\chi_{[0.5,1]}(t) + \eta_3(x)\chi_{[0.75,1]}(t),\\
&g_3(x,t) = \eta_1(x)\chi_{[1/6,1]}(t) + \eta_2(x)\chi_{[2/6,1]}(t) + \eta_3(x)\chi_{[3/6,1]}(t) + \eta_4(x)\chi_{[4/6,1]}(t) + \eta_5(x)\chi_{[5/6,1]}(t),
\end{aligned}
$$

where η_n $(n = 1, \ldots, 5)$ is a cosine function with frequency $2n\pi$ on each edge. The inputs g_2 and g_3 contain higher frequency information and are designed to examine the influence of boundary excitation on the reconstruction. Fig. 9 shows that with the knowledge of h^*, the method for recovering the interface is largely stable with respect to the presence of data noise. With more frequencies in the input excitation, the reconstruction results would improve slightly. This agrees with the observation that the concave shape contains more high-frequency information.

6 Concluding remarks

In this work have studied a challenging inverse problem of recovering multiple coefficients from one single boundary measurement, in a partially unknown medium, due to the formal under-determined nature of the problem. We have presented two uniqueness results, i.e., recovering the order and the piecewise constant diffusion coefficient from a fairly general Neumann input data and recovering the order and two distributed parameters from a fairly specialized Neumann input data (in the appendix). For the former, we have also developed a practical reconstruction algorithm based on asymptotic expansion,

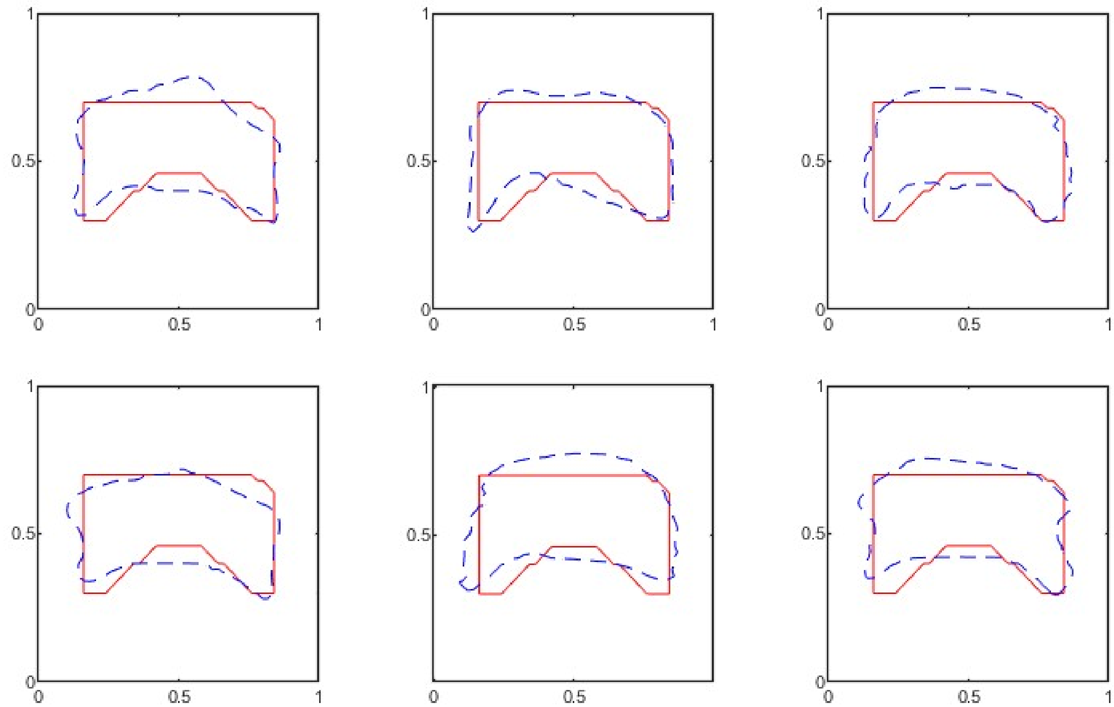

Figure 9: The reconstruction for case (iii) with noisy data and different boundary excitations g_1, g_2 and g_3 (from left to right). The top and bottom rows are for noise levels 1% and 5%.

analytic continuation and level set method, which is inspired by the uniqueness proof, and have presented extensive numerical experiments to showcase the feasibility of the approach.

There remain many important issues to be resolved. Numerically, the overall algorithmic pipeline works well for exact data. However, analytic continuation with rational functions is sensitive with respect to the presence of data noise. Thus it is of much interest to develop one-shot reconstruction algorithms. The main challenge lies in unknown medium properties, i.e., missing data, which precludes a direct application of many standard regularization techniques. It is of much interest to develop alternative approaches for problems with missing data. The level set method does give excellent reconstructions, but it may take many iterations to reach convergence. The acceleration of the method, e.g., via preconditioning, is imperative. Theoretically the specialized Neumann input is very powerful. However, the numerical realization is very challenging. It would also be interesting to develop alternative numerically feasible yet more informative excitations for recovering more general coefficients than polygonal inclusions.

A Recovery of two general coefficients

In this appendix, we discuss the unique recovery of general coefficients mentioned in Remark 3.3. In this setting, we have $g \in C^2(\overline{\mathbb{R}}_+; H^{\frac{1}{2}}(\partial\Omega))$ with support in $\Gamma_1 \times \overline{\mathbb{R}}_+$. Moreover, g is piecewise constant in time t and $g \equiv 0$ when $t \leq T_0$, g is constant when $t \geq T_1$. The proof relies on the representation of the data h, similar to Corollary 2.1 and hence we omit the proof. Note that g is a space-time dependent series. We may write $h = h_i + \sum_{k=1}^{\infty} h_{b,k}$ to distinguish the contributions from u_0 and f, and g (with $g_k(t) := \chi b_k \psi_k(t)\eta_k$)

$$h_i(t) := \rho_0 + \rho_1 t^\alpha + \sum_{n=2}^{\infty} \rho_n E_{\alpha,1}(-\lambda_n t^\alpha),$$

$$h_{b,k}(t) := \sum_{n=1}^{\infty}\sum_{j=1}^{m_n}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-s)^\alpha)\langle g_k(s),\varphi_{n,j}\rangle\,\mathrm{d}s\,\varphi_{n,j}.$$

Proposition A.1. *For $u_0, f \in L^2(\Omega)$ and $\eta_k \in H^{\frac{1}{2}}(\partial\Omega)$, the data $h = u|_{\partial\Omega\times(0,T)}$ to problem* (1.1) *can be represented by*

$$\begin{aligned}h(t) =&\rho_0 + \rho_1 t^\alpha + \sum_{n=2}^{\infty}\rho_n E_{\alpha,1}(-\lambda_n t^\alpha)\\ &+ \sum_{n=1}^{\infty}\sum_{j=1}^{m_n}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-s)^\alpha)\langle g(s),\varphi_{n,j}\rangle\,\mathrm{d}s\,\varphi_{n,j},\end{aligned}$$

with ρ_n defined in (2.4)*. Moreover, the following statements hold.*

(i) $h_i \in C^\omega(0,\infty;L^2(\partial\Omega))$ and $h_{b,k} \in C^\omega(t_{2k}+\varepsilon,\infty;L^2(\partial\Omega))$ with an arbitrarily fixed $\varepsilon > 0$.

(ii) The Laplace transforms $\widehat{h}_i(z)$ and $\widehat{h}_{b,k}(z)$ of h_i and $h_{b,k}$ in t exist and are given by

$$\begin{aligned}\widehat{h}_i(z) &= \rho_0 z^{-1} + \Gamma(1+\alpha)\rho_1 z^{-1-\alpha} + \sum_{n=2}^{\infty}\frac{\rho_k z^{\alpha-1}}{z^\alpha+\lambda_k},\\ \widehat{h}_{b,k}(z) &= \sum_{n=1}^{\infty}\sum_{j=1}^{m_n}\frac{\langle\widehat{g}_k(z),\varphi_{n,j}\rangle\varphi_{n,j}}{z^\alpha+\lambda_n}.\end{aligned}$$

Now we can state the main result of this part. First, we uniquely determine the fractional order α using the data near $t = T_0$, and then use the special boundary excitation g to determine the coefficients a and q. The proof of part (i) is identical with that for Theorem 3.1, and hence omitted. The unique determination of a and q is proved below.

Theorem A.1. *Let $\alpha, \widetilde{\alpha} \in (0,1)$, $(a,q,f,u_0), (\widetilde{a},\widetilde{q},\widetilde{f},\widetilde{u}_0) \in C^2(\overline{\Omega}) \times L^\infty(\Omega) \times L^2(\Omega) \times L^2(\Omega)$ and fix g as* (3.6)*. Let h and $\widetilde{h}$ be the corresponding Dirichlet data, and let $\sigma \in (0,T_0]$ be fixed.*

(i) The condition $h = \widetilde{h}$ on $\Gamma_2 \times [T_0-\sigma,T_0]$ implies $\alpha = \widetilde{\alpha}$, $\rho_0 = \widetilde{\rho}_0$ and $\{(\rho_k,\lambda_k)\}_{k\in\mathbb{K}} = \{(\widetilde{\rho}_k,\widetilde{\lambda}_k)\}_{k\in\widetilde{\mathbb{K}}}$, if $\mathbb{K},\widetilde{\mathbb{K}} \neq \emptyset$.

(ii) If either of following conditions is satisfied: (a) *$q = \widetilde{q}$ and $a - \widetilde{a} = |\nabla a - \nabla\widetilde{a}| = 0$ on the boundary $\partial\Omega$ or* (b) *$a = \widetilde{a}$, then the condition $h = \widetilde{h}$ on $\Gamma_0 \times [T_0-\sigma,T]$ implies $(a,q) = (\widetilde{a},\widetilde{q})$.*

In the proof of Theorem A.1, we need the following two lemmas.

Lemma A.1. *The identity $h = \widetilde{h}$ on $\Gamma_2 \times [T_0-\sigma,T_1]$ implies*

$$h_k = \widetilde{h}_k \quad \text{on } \Gamma_2 \times [T_0-\sigma,\infty), \quad \forall k \in \mathbb{N}, \tag{A.1}$$

with $h_k = h_i + h_{b,k}$ which solves problem (1.1) *with $g = g_k$.*

Proof. We prove the assertion by induction. When $k = 1$, by the definition of $\psi_k(t)$, we have $\psi_k = 0$ in $(0,t_3)$ for all $k \geq 2$. Then by Proposition A.1, the condition $h = \widetilde{h}$ on $\Gamma_2 \times [T_0-\sigma,T_1]$ implies $h_1 = \widetilde{h}_1$ on $\Gamma_2 \times [T_0-\sigma,t_3)$, since $[T_0-\sigma,t_3) \subset [T_0-\sigma,T_1]$. By Proposition A.1(i), h_1 and $\widetilde{h}_1$ are $L^2(\partial\Omega)$-valued functions analytic in $t \in (t_2+\varepsilon,\infty)$, and hence $h_1 = \widetilde{h}_1$ for all $t \in [T_0-\sigma,\infty)$. This shows the case for $k = 1$. Now assume that for some $\ell \in \mathbb{N}$, the assertion (A.1) holds for all $k = 1,\ldots,\ell$. Since $\psi_k = 0$ in $(0,t_{2\ell+3})$, for $k \geq \ell+2$, we deduce $\sum_{k=1}^{\ell+1} h_k = h$ in $\Gamma_2 \times (0,t_{2\ell+3})$. Similarly, we have

$$\sum_{k=1}^{\ell+1} h_k = \sum_{k=1}^{\ell+1}\widetilde{h}_k \quad \text{on } \Gamma_2 \times (0,t_{2\ell+3}).$$

From the induction hypothesis, we deduce $h_{\ell+1} = \widetilde{h}_{\ell+1}$ on $\Gamma_2 \times [T_0-\sigma,t_{2\ell+3})$. Use analytic continuation again, we obtain $h_{\ell+1} = \widetilde{h}_{\ell+1}$ on $\Gamma_2 \times [T_0-\sigma,\infty)$. Thus, the assertion (A.1) holds for all $k \in \mathbb{N}$. □

Lemma A.2. *Given a nonempty open subset Γ of $\partial\Omega$, for any fixed $n \in \mathbb{N}^*$, the eigenfunctions $\{\varphi_{n,\ell}\}_{\ell=1}^{m_n}$ corresponding to λ_n are linearly independent on $L^2(\Gamma)$.*

Proof. Suppose that on the contrary: there are $\{c_j\}_{j=1}^{m_n} \subset \mathbb{R}$ such that $\sum_{j=1}^{m_n} c_j\varphi_{n,j} = 0$ on Γ. Let $\varphi = \sum_{j=1}^{m_n} c_j\varphi_{n,j}$. Then φ satisfies $\mathcal{A}\varphi = \lambda_n\varphi$ in Ω, $\partial_\nu\varphi = 0$ on $\partial\Omega$, and $\varphi = 0$ on Γ. Then the regularity on a and q and unique continuation principle [18, Theorem 3.3.1] imply $\varphi \equiv 0$ in Ω. Since $\varphi_{n,j}$ are linearly independent in $L^2(\Omega)$, we obtain $c_j = 0$, $j = 1, \ldots, m_n$, i.e. the desired linear independence. $\square$

Now we can state the proof of Theorem A.1(ii).

Proof of Theorem A.1(ii). By Lemma A.1, we have $h_k = \widetilde{h}_k$ on $\Gamma_2 \times (T_0 - \sigma, \infty)$ for any $k \in \mathbb{N}$. Note that $h_k = h_i + h_{b,k}$ solves problem (1.1) with g replaced by g_k. We have the following representations

$$
\begin{aligned}
h_i(t) &= \rho_0 + \rho_1 t^\alpha + \sum_{n\in\mathbb{K}\cap\mathbb{N}^*} \rho_n E_{\alpha,1}(-\lambda_n t^\alpha),\\
h_{b,k}(t) &= \sum_{n=1}^{\infty}\sum_{j=1}^{m_n} \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n(t-s)^\alpha)\langle g_k(s), \varphi_{n,j}\rangle \,\mathrm{d}s\, \varphi_{n,j}.
\end{aligned}
$$

By the choice of g, the interval $[0,T]$ can be divided into $[0,T_0]$ and $[T_0,T]$. For $t \in (0,T_0)$, $g_k(t) \equiv 0$, Theorem A.1(i) implies that $\{(\rho_\ell, \lambda_\ell)\}_{\ell\in\mathbb{K}} = \{(\widetilde{\rho}_\ell, \widetilde{\lambda}_\ell)\}_{\ell\in\widetilde{\mathbb{K}}}$ and $\alpha = \widetilde{\alpha}$, and hence $h_i(t) = \widetilde{h}_i(t)$ for all $t > 0$. For $t \in [T_0, T]$, this and the condition $h_k(t) = \widetilde{h}_k(t)$ lead to $h_{b,k}(t) = \widetilde{h}_{b,k}(t)$ in $L^2(\Gamma_2)$. Thus,

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\sum_{j=1}^{m_n} \int_{t_1}^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n(t-s)^\alpha)\langle g_k(s), \varphi_{n,j}\rangle_{L^2(\Gamma_1)} \,\mathrm{d}s\, \varphi_{n,j}\\
=&\sum_{n=1}^{\infty}\sum_{j=1}^{\widetilde{m}_n} \int_{t_1}^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\widetilde{\lambda}_n(t-s)^\alpha)\langle g_k(s), \widetilde{\varphi}_{n,j}\rangle_{L^2(\Gamma_1)} \,\mathrm{d}s\, \widetilde{\varphi}_{n,j}, \quad t \in [t_1, \infty).
\end{aligned}
$$

By Proposition A.1(ii), applying Laplace transform on both sides yields

$$
\sum_{n=1}^{\infty}\sum_{j=1}^{m_n} \frac{\langle \widehat{g}_k(z), \varphi_{n,j}\rangle\varphi_{n,j}}{z^\alpha + \lambda_n} = \sum_{n=1}^{\infty}\sum_{j=1}^{\widetilde{m}_n} \frac{\langle \widehat{g}_k(z), \widetilde{\varphi}_{n,j}\rangle\widetilde{\varphi}_{n,j}}{z^\alpha + \widetilde{\lambda}_n}, \quad \forall\, \Re(z) > 0. \tag{A.2}
$$

Next, we repeat the argument of Theorems 3.1 and 3.2 to deduce $\lambda_n = \widetilde{\lambda}_n$, $\forall n \in \mathbb{N}$. To this end, let $U_k \in \mathrm{Dom}(A^{\frac{1}{4}+\varepsilon})$ be the solution of the elliptic equation with a Neumann boundary data $\chi b_k\eta_k$, for all ζ in any compact subset of $\mathbb{C}\setminus\{-\lambda_n, -\widetilde{\lambda}_n\}_{n\in\mathbb{N}}$, we have

$$
\begin{aligned}
&\left\|\sum_{n=1}^{\infty}\sum_{j=1}^{m_n} \frac{\langle \widehat{g}_k(\eta^{\frac{1}{\alpha}}), \varphi_{n,j}\rangle\varphi_{n,j}}{\zeta + \lambda_n}\right\|^2_{\mathrm{Dom}(A^{\frac{1}{4}+\varepsilon})} \le c\sum_{n=1}^{\infty} \lambda_n^{\frac{1}{2}+2\varepsilon} \sum_{j=1}^{m_n} \left|\frac{\langle \chi b_k\eta_k, \varphi_{n,j}\rangle}{\zeta + \lambda_n}\right|^2\\
=&c\sum_{n=1}^{\infty} \lambda_n^{\frac{1}{2}+2\varepsilon} \sum_{j=1}^{m_n} \left|\frac{\lambda_n\,(U_k, \varphi_{n,j})}{\zeta + \lambda_n}\right|^2 \le c\|U_k\|^2_{\mathrm{Dom}(A^{\frac{1}{4}+\varepsilon})} < \infty.
\end{aligned}
$$

Since each term of the series is a $\mathrm{Dom}(A^{\frac{1}{4}+\varepsilon})$-valued function analytic in ζ and the series converges uniformly for ζ in a compact subset set of $\mathbb{C}\setminus\{-\lambda_n, -\widetilde{\lambda}_n\}_{n\in\mathbb{N}}$, by the trace theorem, we deduce that both sides of (A.2) are $L^2(\partial\Omega)$-valued functions analytic in $\zeta \in \mathbb{C}\setminus\{-\lambda_n, -\widetilde{\lambda}_n\}_{n\in\mathbb{N}}$. Assuming $\lambda_j \notin \{\widetilde{\lambda}_n\}_{n\in\mathbb{N}}$, by choosing a small circle centered at $-\lambda_j$ and then using Cauchy integral formula, we obtain

$$
\frac{2\pi\sqrt{-1}}{\lambda_j}\sum_{j=1}^{m_n} \langle \widehat{g}_k, \varphi_{n,j}\rangle\varphi_{n,j}(y) = 0, \quad \forall k \in \mathbb{N}. \tag{A.3}
$$

This and Lemma A.2 (with $\Gamma = \Gamma_2$) imply $\langle \widehat{g}_k, \varphi_{n,j}\rangle = 0$, $\forall k \in \mathbb{N}$, $j = 1, \ldots, m_n$. Since $\widehat{g}_k = \chi b_k \widehat{\psi}\eta_k$, by the density of η_k in $H^{\frac{1}{2}}(\Gamma_1)$, we have $\varphi_{n,j} = 0$ a.e. on Γ_1, $j = 1, \ldots, m_n$. Since $\partial_\nu \varphi_{n,j} = 0$, unique continuation principle [18, Theorem 3.3.1] implies $\varphi_{n,j} \equiv 0$ in Ω, which is a contradiction. Hence, $\lambda_j \in \{\widetilde{\lambda}_n\}_{n\in\mathbb{N}}$ for every $j \in \mathbb{N}$. Likewise, we can prove $\widetilde{\lambda}_j \in \{\lambda_n\}_{n\in\mathbb{N}}$ for every $j \in \mathbb{N}$, and hence $\lambda_n = \widetilde{\lambda}_n$, $\forall n \in \mathbb{N}^*$. It follows directly from (A.2) that

$$\sum_{n=1}^{\infty} \frac{1}{\eta + \lambda_n} \Big(\sum_{j=1}^{m_n} \langle \widehat{g}_k(z), \varphi_{n,j}\rangle \varphi_{n,j}(y) - \sum_{j=1}^{\widetilde{m}_n} \langle \widehat{g}_k(z), \widetilde{\varphi}_{n,j}\rangle \widetilde{\varphi}_{n,j}(y) \Big) = 0, \quad \text{a.e. } y \in \Gamma_2.$$

Using Cauchy integral theorem again, we have

$$\sum_{j=1}^{m_n} \langle \widehat{g}_k(z), \varphi_{n,j}\rangle \varphi_{n,j}(y) = \sum_{j=1}^{\widetilde{m}_n} \langle \widehat{g}_k(z), \widetilde{\varphi}_{n,j}\rangle \widetilde{\varphi}_{n,j}(y), \quad \text{a.e. } y \in \Gamma_2,\ \forall k, n \in \mathbb{N}.$$

By the construction of g_k, it is equivalent to

$$\begin{aligned} & b_k \psi_k(z) \int_{\partial\Omega} \chi(y')\eta_k(y')\Theta_n(y', y)\,\mathrm{d}y' \\ =& b_k \psi_k(z) \int_{\partial\Omega} \chi(y')\eta_k(y')\widetilde{\Theta}_n(y', y)\,\mathrm{d}y', \quad \forall n, k \in \mathbb{N},\ \Re(z) > 0, \end{aligned}$$

with $\Theta_n(y', y) := \sum_{j=1}^{m_n} \varphi_{n,j}(y')\varphi_{n,j}(y)$. Since the set $\{\eta_k\}_{k\in\mathbb{N}}$ is dense in $H^{\frac{1}{2}}(\partial\Omega)$ and $\chi \equiv 1$ on Γ_1', we deduce $\Theta_n(y', y) = \widetilde{\Theta}_n(y', y) \in L^2(\Gamma_1') \times L^2(\Gamma_2)$ for all $n \in \mathbb{N}$. From [5, Theorem 1.1] (see also [30, Lemma 4.1]), we deduce that $m_n = \widetilde{m}_n$ and after an orthogonal transformation

$$\varphi_{n,j}(y) = \widetilde{\varphi}_{n,j}(y), \quad j = 1, \cdots, m_n,\ \forall y \in \partial\Omega,\ n \in \mathbb{N}. \tag{A.4}$$

By [6, Corollary 1.7], the equal Dirichlet boundary spectral data (A.4) imply the desired uniqueness. □

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