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On a sufficient condition for commutative orthogonal block structure

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Abstract

A model has orthogonal block structure if it has variance-covariance matrix that is a linear combination of known pairwise orthogonal orthogonal projection matrices that add to the identity matrix. When the orthogonal projection matrix on the space spanned by the mean vector commutes with the orthogonal projection matrices, in the expression of the variance-covariance matrix, the model has commutative orthogonal block structure. Resorting to B-matrices we present a general condition for this commutativity.

Key words: B-matrices, mixed models, models with commutative orthogonal block structure
MSC2000: AMS Codes (optional)

1. Introduction

Linear mixed models are a powerful tool for analysing experimental data from several areas, such as agriculture, biology, medicine or industry. In the framework of the design of experiments in agricultural trials, in 1965, a special class of linear mixed models as emerged, called models with orthogonal block structure, OBS, based on the structure of the variance-covariance matrix [6,7]. Later on, in order to obtain optimal estimation for variance components of blocks and contrasts of treatments, using the algebraic structure of OBS, arose a particular case of these models, those of models with commutative orthogonal block structure, COBS [4].

2. Models with commutative orthogonal block structure

Let us consider a mixed model

$$\boldsymbol{Y} = \sum_{i=0}^{w} \boldsymbol{X}_i \boldsymbol{\beta}_i$$

where $\boldsymbol{\beta}_0$ is fixed and $\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_w$ are independent random vectors with null mean vectors, variance-covariance matrices $\sigma_1^2 \boldsymbol{I}_{c_1} \ldots \sigma_w^2 \boldsymbol{I}_{c_w}$, where $c_i = rank(\boldsymbol{X_i}), i = 1, \ldots, w$ and null cross-covariance matrices.

$\boldsymbol{Y}$ has mean vector

$$\boldsymbol{\mu} = \boldsymbol{X}_0 \boldsymbol{\beta}_0$$

and variance-covariance matrix

$$\boldsymbol{V}(\boldsymbol{\theta}) = \sum_{i=1}^{w} \sigma_i^2 \boldsymbol{M}_i$$

where $\boldsymbol{M}_i = \boldsymbol{X}_i \boldsymbol{X}_i^T, i = 1, \ldots, w$.

Since the space spanned by the mean vector is $\Omega = R(\boldsymbol{X}_0)$, the orthogonal projection matrix, OPM, on Ω, is $\boldsymbol{T} = \boldsymbol{X}_0(\boldsymbol{X}_0^T \boldsymbol{X}_0)^+ \boldsymbol{X}_0^T = \boldsymbol{X}_0 \boldsymbol{X}_0^+$, where $+$ indicates Moore–Penrose inverse.

When the matrices $\boldsymbol{M}_1, \ldots, \boldsymbol{M}_w$ commute, they generate a commutative Jordan algebra of symmetric matrices, CJAS,A. This is a linear space constituted by symmetric matrices that commute and containing the squares of their matrices [5]. The CJAS, A, as one unique basis, called the principal basis,Q, that is constituted by known pairwise orthogonal orthogonal projection matrices, POOPM, [8] thus the matrices $\boldsymbol{M}_i$, $i = 1, \ldots, w$ are linear combinations of the matrices of the CJAS principal basis

$$\boldsymbol{M}_i = \sum_{j=1}^{m} b_{i,j}\boldsymbol{Q}_j$$

With $\gamma_j = \sum_{i=1}^{w} b_{i,j}\, \sigma_i^2,\ j = 1, \dots, m,$ the canonical variance components, the variance-covariance matrix of $\boldsymbol{Y}$ will take the form

$$\boldsymbol{V} = \sum_{j=1}^{m} \gamma_j\ \boldsymbol{Q}_j$$

Since $\sum_{i=1}^{w} \boldsymbol{M}_j \in$ A is invertible, A is a complete CJAS and $\sum_{j=1}^{m} \boldsymbol{Q}_j = \boldsymbol{I}_n$.

So

$$\boldsymbol{Y} = \sum_{i=0}^{w} \boldsymbol{X}_i\boldsymbol{\beta}_i$$

is a model with orthogonal block structure, OBS. These models, introduced in [6,7], have been intensively studied and play a central part in the theory of randomized block designs, see, e.g., [1,2].

An important class of OBS, models with commutative orthogonal block structure, COBS, arises when T, the OPM on the space spanned by the mean vector, commutes with the POOPM $\boldsymbol{Q}_j$, $j = 1, \dots, m$. [4]. So $\boldsymbol{T}$ and $\boldsymbol{V}$ commute and the least square estimators, LSE, for estimable vectors, will give BLUE (best linear unbiased estimators) whatever the variance components [10].

Assuming the rows of $\boldsymbol{X}_0$ to correspond to the sets of levels of the fixed effects factors, the mean values of the observations will be determined by those sets.

Let us consider that there will be $\dot{n}$ sets of the levels associated to r_1, …, $r_{\dot{n}}$, contiguous rows of $\boldsymbol{X}_0$. If the components of $\boldsymbol{\beta}_0$, $\beta_{0,1}$, …, $\beta_{0,\dot{n}}$, are the corresponding mean values, we can reorder the observations to have the block diagonal matrix

$$\boldsymbol{X}_0 = D\left(1_{0,1}, \dots, 1_{0,\dot{n}}\right)$$

So the orthogonal projection matrix on , the space spanned by the mean vector, is given by

$$\boldsymbol{T} = D\left(\frac{1}{r_1}J_{r_1}, \dots, \frac{1}{r_{\dot{n}}}J_{r_{\dot{n}}}\right)$$

where $J_r = 1_r \, 1_r^{\ T}$

The fundamental partition of $\boldsymbol{Y}$ will be constituted by the sub-vectors $\boldsymbol{Y}_1, \ \dots, \ \boldsymbol{Y}_{\dot{n}}$, corresponding to the $\dot{n}$ sets of the levels of the fixed effects factors [3]. Then the variance covariance matrix can be defined by

$$\boldsymbol{V} = \begin{bmatrix} \boldsymbol{V}_{1,1} & \dots & \boldsymbol{V}_{1,\dot{n}} \\ \vdots & & \vdots \\ \boldsymbol{V}_{\dot{n},1} & \dots & \boldsymbol{V}_{\dot{n},\dot{n}} \end{bmatrix}$$

with $\boldsymbol{V}_{l,l}$ the variance-covariance matrix of $\boldsymbol{Y}_l$, $l = 1, \dots, \dot{n}$, and $\boldsymbol{V}_{l,h}$ the cross-covariance matrix of $\boldsymbol{Y}_l$ and $\boldsymbol{Y}_h$, $l \neq h$.

Since

$$TV = \begin{bmatrix} \frac{1}{r_1} J_{r_1} \boldsymbol{V}_{1,1} & \dots & \frac{1}{r_1} J_{r_1} \boldsymbol{V}_{1,\dot{n}} \\ \vdots & & \vdots \\ \frac{1}{r_{\dot{n}}} J_{r_{\dot{n}}} \boldsymbol{V}_{\dot{n},1} & \dots & \frac{1}{r_{\dot{n}}} J_{r_{\dot{n}}} \boldsymbol{V}_{\dot{n},\dot{n}} \end{bmatrix} \text{ and } VT = \begin{bmatrix} \boldsymbol{V}_{1,1} \frac{1}{r_1} J_{r_1} & \dots & \boldsymbol{V}_{1,\dot{n}} \frac{1}{r_1} J_{r_1} \\ \vdots & & \vdots \\ \boldsymbol{V}_{\dot{n},1} \frac{1}{r_{\dot{n}}} J_{r_{\dot{n}}} & \dots & \boldsymbol{V}_{\dot{n},\dot{n}} \frac{1}{r_{\dot{n}}} J_{r_{\dot{n}}} \end{bmatrix}$$

the matrices T and V commute if and only if

$$\begin{cases} \frac{1}{r_1} J_{r_1} \boldsymbol{V}_{1,1} = \boldsymbol{V}_{1,1} \frac{1}{r_1} J_{r_1} & \dots & \frac{1}{r_1} J_{r_1} \boldsymbol{V}_{1,\dot{n}} = \boldsymbol{V}_{1,\dot{n}} \frac{1}{r_1} J_{r_1} \\ \vdots & & \vdots \\ \frac{1}{r_{\dot{n}}} J_{r_{\dot{n}}} \boldsymbol{V}_{\dot{n},1} = \boldsymbol{V}_{\dot{n},1} \frac{1}{r_{\dot{n}}} J_{r_{\dot{n}}} & \dots & \frac{1}{r_{\dot{n}}} J_{r_{\dot{n}}} \boldsymbol{V}_{\dot{n},\dot{n}} = \boldsymbol{V}_{\dot{n},\dot{n}} \frac{1}{r_{\dot{n}}} J_{r_{\dot{n}}} \end{cases}$$

Which occurs when the matrices $\boldsymbol{V}_{l,h}$, $l = 1, \dots, \dot{n}$, h$= 1, \dots, \dot{n}$ are B-matrices, see, e.g. [9], this is, when

$$\frac{1}{\dot{n}} \sum_{l=1}^{\dot{n}} \boldsymbol{V}_{l,h} = \frac{1}{\dot{n}} \sum_{h=1}^{\dot{n}} \boldsymbol{V}_{l,h} = \frac{1}{\dot{n}\dot{n}} \sum_{l=1}^{\dot{n}} \sum_{h=1}^{\dot{n}} \boldsymbol{V}_{l,h}$$

With

$$V = D\left(\sigma_1^{\ 2} \, I_{r_1}, \dots, \sigma_{\dot{n}}^{\ 2} \, I_{r_{\dot{n}}}\right)$$

matrices T and V commute.

With this commutativity condition $\boldsymbol{Y}$ will be a model with commutative orthogonal block structure (COBS) and, according to the version of the Gauss-Markov theorem in [10], the LSE for estimable vectors will be BLUE.

3. References

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