

Heat Conduction in Cylindrical Coordinates with Time-Varying Conduction Coefficients: A Practical Engineering Approach

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Article history:

Received: 10 October 2023 / Received in revised form: 2 November 2023 / Accepted: 10 November 2023
Available online 15 November 2023

ABSTRACT

This research aims to develop a mathematical method for expressing the Laplace operator in cylindrical coordinates and applying it to solve heat conduction equations in various scenarios. The method commences by transforming Cartesian coordinates into cylindrical coordinates and identifying the necessary substitutions. The result is the expression of the Laplace operator in cylindrical coordinates, which is subsequently employed to address heat conduction equations within cylindrical coordinates. Various cases encompassing different initial and boundary conditions, as well as variations in the conduction coefficient over time, are meticulously considered. In each instance, precise mathematical solutions are determined and subjected to thorough analysis. This study carries substantial implications for comprehending heat transfer within cylindrical coordinate systems and finds relevance in a wide array of scientific and engineering contexts. The research's findings can be harnessed for technology development, heating system design, and heat transfer modeling across diverse applications, including mechanical engineering and materials science. Therefore, the research's contribution holds paramount significance in advancing our understanding of heat transfer within cylindrical coordinates and in devising more efficient and accurate solutions for an array of heat-related issues within the realms of science and engineering.

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Keywords: Laplace operator, Cylindrical coordinates, Heat conduction equation, Variation of conduction coefficient, Mathematical methods.

I. Introduction

This research aims to present the expression of the Laplace operator in cylindrical coordinates in polar form [1]. The Laplace operator is a fundamental concept in mathematical analysis and physics used to comprehend heat distribution, gravitational fields, and various other phenomena [2]. We will begin by considering the Laplace operator in Cartesian coordinates and then transforming it into cylindrical coordinates [3].

Cylindrical coordinates are a crucial coordinate system in various scientific and engineering applications, particularly in phenomena based on cylindrical symmetry [4]. One critical step in analyzing problems within cylindrical coordinate systems is to express the Laplace operator in a form that corresponds to those coordinates [5]. This is essential for solving heat equations, Laplace equations, and various other partial differential equations within cylindrical coordinates [6].



The primary objective of this study is to provide the expression of the Laplace operator in cylindrical coordinates in polar form. Consequently, this research will establish a strong foundation for analyzing heat transfer problems within cylindrical coordinate systems. The benefit is a better understanding of using cylindrical coordinates in heat transfer modeling, which can be applied in various applications, including thermal engineering and the physical sciences.

Research in conduction heat transfer focuses on developing mathematical formulas related to the Laplace operator in cylindrical coordinates in polar form. While this represents a novel contribution to the fields of heat transfer and mathematics, it does not address practical applications for solving heat transfer problems within cylindrical coordinates. The uniqueness of this research lies in its specific mathematical approach, providing a solid foundation for understanding heat conduction in cylindrical coordinate systems. However, further research opportunities remain to explore concrete applications in heat transfer problem-solving and other areas.

The results of this study will enable researchers, engineers, and scientists to understand better and apply the concept of the Laplace operator in heat transfer modeling in cylindrical coordinate systems [7]. This will aid in developing more efficient and accurate analytical solutions to heat transfer problems in various contexts.

Although a considerable amount of literature has reviewed the Laplace operator in various coordinate systems, its specific expression in cylindrical coordinates in polar form is not always readily available. Therefore, this study will fill the knowledge gap regarding the expression of the Laplace operator in cylindrical coordinates, particularly in polar form, making an important contribution to further understanding heat transfer.

II. Methods

1. Initial and Boundary Conditions

We assume that the temperature, $T(r, \theta, z, t)$, can be represented as a product of three functions $T(r, \theta, z, t) = R(r)\Theta(\theta)Z(z)T(t)$. This results in three separate ordinary differential equations [8]. The equation for $R(r)$ is given as:

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{dR}{dr} \right) - \frac{m^2}{r^2} R + \lambda R = 0 \quad (1)$$

Where λ is the separation constant, and m is the azimuthal quantum number. The equation for $\Theta(\theta)$ is:

$$\frac{d^2 \Theta}{d\theta^2} + m^2 \Theta = 0 \quad (2)$$

For $Z(z)$, the equation is:

$$\frac{d^2 Z}{dz^2} + \mu^2 Z = 0 \quad (3)$$

where μ is the separation constant for the z -coordinate. The heat equation becomes:

$$\frac{dT}{dt} = \alpha \left(\frac{1}{r} \frac{d}{dr} \left(r \frac{dR}{dr} \right) - \frac{m^2}{r^2} R + \lambda R \right) \Theta(\theta) Z(z) T(t) \quad (4)$$

Solve for $Z(z)$ with boundary conditions $Z(0) = 0$ and $Z(L) = 1$. The roots are denoted as μ_n , found from the characteristic equation:

$$\frac{d^2 Z}{dz^2} + \mu^2 Z = 0 \quad (5)$$

Solve for $\Theta(\theta)$ with periodic boundary conditions ($0 \leq \theta \leq 2\pi$). The solutions are trigonometric functions $\Theta_m(\theta) = \cos(m\theta)$ or $\sin(m\theta)$. Solve for $R(r)$ with boundary conditions to avoid singularities at $r = 0$ and $R(R) = A$ (a constant to be determined). The solutions are Bessel functions [9]:

$$R_m(r) = J_m(\lambda_{m,n}r) \text{ or } Y_m(\lambda_{m,n}r) \quad (6)$$

Where J_m and Y_m are Bessel functions of order m , and $\lambda_{m,n}$ are positive roots of the Bessel equation:

$$J_m(\lambda_{m,n}R) = 0 \quad (7)$$

Combine all these solutions to obtain the final solution for $T(r, \theta, z, t)$:

$$T(r, \theta, z, t) = \sum_{m,n} T_{mn}(t) J_m(\lambda_{m,n}r) \cos(m\theta) e^{-\alpha \mu_n^2 t} \quad (8)$$

Where $T_{mn}(t)$ are coefficients determined by the initial condition $T(r, \theta, z, 0) = f(r, \theta, z)$.

2. Temperature Change In Time

The provided initial conditions state that the temperature at the beginning is (T_0) , and on the outer surface of the cylinder ($r = R$), there is heat loss, which results in the boundary condition $\left. \frac{\partial T}{\partial r} \right|_{r=R} = 0$. To solve this equation, we employ the separation of variables method.

We assume that the temperature (T) can be represented as a product of functions, each depending on a single variable ($T(r, \theta, z, t) = R(r)\Theta(\theta)Z(z)T(t)$). This solution is then substituted into the heat conduction equation, leading to the separation of variables [8]. After some mathematical manipulation, we arrive at the following equation:

$$\frac{1}{\alpha} \frac{1}{R} \frac{dR}{dr} \frac{1}{r} + \frac{1}{\alpha} \frac{1}{\Theta} \frac{d^2 \Theta}{d\theta^2} \frac{1}{r^2} + \frac{1}{\alpha} \frac{1}{Z} \frac{d^2 Z}{dz^2} = \frac{1}{T} \frac{dT}{dt} = -\lambda^2 \quad (9)$$

The equation related to (r) is:

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{dR}{dr} \right) - \frac{\lambda^2 \alpha}{k} R = 0 \quad (10)$$

We can solve this equation using the separation of variables method by assuming the solution for $(R(r))$ is in the form $(R(r) = X(r)Y(r))$. The equation for $(X(r))$ becomes:

$$\frac{1}{X} \frac{dX}{dr} - \mu^2 = -\frac{\lambda^2 \alpha}{k} r \quad (11)$$

and the equation for $(Y(r))$ becomes:

$$\frac{1}{Y} \frac{dY}{dr} + \mu^2 = \frac{\lambda^2 \alpha}{k} r \quad (12)$$

These two equations can be solved to find the solutions for X(r) and Y(r):

$$X(r) = C_1 e^{\frac{k\mu^2}{\lambda^2 \alpha} r^2} \quad (13.a)$$

$$Y(r) = C_2 e^{\frac{k\mu^2}{\lambda^2 \alpha} r^2} \quad (13.b)$$

For the equation involving (Z(z)), the solution is:

$$(Z(z) = A \sin(\lambda_n z) + B \cos(\lambda_n z)) \quad (14)$$

For the time-dependent equation:

$$\frac{dT}{dt} + \alpha \lambda^2 T = 0 \quad (15)$$

The solution is:

$$T(t) = T_0 e^{-\alpha \lambda^2 t} \quad (16)$$

By combining all these components, we obtain the final solution for the temperature ($T(r, \theta, z, t)$) in the cylindrical coordinate system. This solution is a function of time (t) with a known thermal conductivity (α).

3. Steady-State Heat Distribution

The given problem involves the steady-state heat distribution within a cylinder with specific boundary conditions [10]. To solve this problem, we first start with the heat equation and attempt to separate the variables. The outer surface has a fixed temperature $T(R, \theta, z) = T_0$, and the inner surface is insulated ($T(0, \theta, z) = 0$). To separate the variables, we assume a solution in the form of $(T(r, \theta, z) = R(r)\Theta(\theta)Z(z))$. Substituting this into the heat equation and dividing by the product $(R(r)\Theta(\theta)Z(z))$, we arrive at three separate differential equations, each with a constant $-k^2$. These equations pertain to the variables r, (θ), and z. The solution for the equation in the variable z is a second-order differential equation, resulting in a general solution ($Z(z) = A \sin(kz)$). Given the insulation boundary condition at the inner surface ($T(0, \theta, z) = 0$), we determine that $B = 0$, and the solution for ($Z(z)$) becomes ($Z(z) = A \sin(kz)$). For the ($\Theta(\theta)$) equation, we consider periodic boundary conditions, meaning the temperature should not diverge as (θ) increases by (2π). This implies that C_2 must equal 0, leading to the solution ($\Theta(\theta) = C_1 \sin(k\theta)$). The solution for the equation in the variable r involves the famous Bessel equation, resulting in a Bessel series solution [11]. Given the boundary condition of a fixed temperature at the outer surface $T(R, \theta, z) = T_0$, we set $A_n = 0$, simplifying the solution to:

$$R(r) = J_0(kr) \quad (17)$$

Combining the solutions in r, θ , and z, we arrive at the final solution for the steady-state heat distribution within the cylinder:

$$T(r, \theta, z) = \sum_{n=1}^{\infty} T_n(r, \theta, z) \quad (18)$$

Where

$$T_n(r, \theta, z) = \sin(kz) \sin(k\theta) J_0(kr) \quad (19)$$

This represents the temperature distribution within the cylinder, satisfying the specified boundary conditions.

4. Time Variable Boundary Condition

To determine the temperature ($T(r, \theta, z, t)$) within a cylindrical structure over time (t) with specified boundary conditions, the heat equation in cylindrical coordinates is utilized [12]. The heat equation in cylindrical coordinates is given by:

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \quad (20)$$

In this equation, (α) represents the material's thermal diffusivity (a constant), (T) denotes temperature, (r) signifies the cylinder's radius, (θ) stands for the azimuthal angle, (z) is the axial coordinate, and (t) is time. With the given boundary conditions ($T(r, \theta, z, t) = T_0 + \sin(\omega t)$), the objective is to find the solution for the temperature (T) within the cylinder. By substituting ($T(r, \theta, z, t)$) into the equation, we derive:

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} (T_0 + \sin(\omega t)) \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} (T_0 + \sin(\omega t)) + \frac{\partial^2}{\partial z^2} (T_0 + \sin(\omega t)) = \frac{1}{\alpha} \frac{\partial}{\partial t} (T_0 + \sin(\omega t)) \quad (21)$$

Subsequently, we evaluate the necessary derivatives. The first derivative with respect to (r) is found to be:

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} (T_0 + \sin(\omega t)) \right) = \frac{1}{r} \frac{\partial}{\partial r} (r \cdot 0) = 0 \quad (22)$$

This is because the temperature ($T_0 + \sin(\omega t)$) remains unaffected by (r). Likewise, the second derivative with respect to (θ) is determined as:

$$\frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} (T_0 + \sin(\omega t)) = \frac{1}{r^2} \cdot 0 = 0 \quad (23)$$

Since the temperature ($T_0 + \sin(\omega t)$) is also independent of (θ). The second derivative with respect to (z) is:

$$\frac{\partial^2}{\partial z^2} (T_0 + \sin(\omega t)) = 0 \quad (24)$$

This is because the temperature ($T_0 + \sin(\omega t)$) doesn't depend on (z). Finally, the first derivative with respect to (t) is calculated as:

$$\frac{1}{\alpha} \frac{\partial}{\partial t} (T_0 + \sin(\omega t)) = \frac{1}{\alpha} \omega \cos(\omega t) \quad (25)$$

Now, we can simplify the heat equation to:

$$0 + 0 + 0 = \frac{1}{\alpha} \omega \cos(\omega t) \quad (26)$$

Further simplification yields:

$$\frac{1}{\alpha} \omega \cos(\omega t) = 0 \quad (27)$$

Consequently, the internal heat equation suggests that the temperature within the cylinder, subject to the boundary conditions ($T(r, \theta, z, t) = T_0 + \sin(\omega t)$), remains constant in space and only varies with time (t). The ultimate result is:

$$T(r, \theta, z, t) = T_0 + \sin(\omega t) \quad (28)$$

This implies that the temperature within the cylinder is ($T_0 + \sin(\omega t)$) at any location within the cylinder, and it is not influenced by spatial coordinates (r), (θ), (z), changing solely with respect to time (t).

5. Variation of Conduction Coefficient

To calculate the temperature ($T(r, \theta, z, t)$) within a cylinder with a time-varying conduction coefficient, we begin with the fundamental heat conduction equation in cylindrical coordinates:

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \quad (29)$$

Here, ($\alpha(r, \theta, z, t)$) represents the time-dependent thermal conductivity. We simplify the problem by assuming (T) depends only on (r) and (t), neglecting variations in (α) with respect to (θ) and (z). We then separate the variables (T) and (t):

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{dT}{dr} \right) = \frac{1}{\alpha} \frac{dT}{dt} \quad (30)$$

By substituting the expression for ($\alpha(r, \theta, z, t)$), we get:

$$r \frac{d}{dr} \left(r \frac{dT}{dr} \right) (\alpha_0 + \alpha_1 \cos(\theta) e^{-\lambda t}) = \frac{dT}{dt} \quad (31)$$

Dividing both sides by ($\alpha_0 + \alpha_1 \cos(\theta) e^{-\lambda t}$) isolates (r) and (t) as:

$$r \frac{d}{dr} \left(r \frac{dT}{dr} \right) = \frac{1}{\alpha_0 + \alpha_1 \cos(\theta) e^{-\lambda t}} \frac{1}{r} \frac{dT}{dt} \quad (32)$$

Now, we have two equations that depend solely on (r) and (t). The equation for (r) is:

$$r \frac{d}{dr} \left(r \frac{dT}{dr} \right) = 0 \quad (33)$$

The equation for (t) becomes:

$$\frac{1}{\alpha_0 + \alpha_1 \cos(\theta) e^{-\lambda t}} \frac{1}{r} \frac{dT}{dt} = \text{constant} \quad (34)$$

Solving these equations separately, the equation for (r) is a second-order differential equation, which can be integrated twice to yield:

$$r \frac{dT}{dr} = C_1$$

$$\frac{dT}{dr} = \frac{C_1}{r}$$

$$T(r) = C_1 \ln(r) + C_2 \quad (35)$$

The equation for (t) is integrated by using a substitution for the time-varying thermal conductivity coefficient:

$$\int \frac{1}{\alpha_0 + \alpha_1 \cos(\theta)e^{-\lambda t}} dT = \int k \cdot r dt \quad (36)$$

A substitution is applied to the left-hand side, where $(u(t) = \alpha_0 + \alpha_1 \cos(\theta)e^{-\lambda t})$. Thus, $(du = (-\alpha_1 \cos(\theta)\lambda e^{-\lambda t})dt)$, allowing us to express (dT) as:

$$dT = \frac{1}{-\alpha_1 \cos(\theta)\lambda e^{-\lambda t}} du \quad (37)$$

The integration of both sides results in:

$$\frac{1}{-\alpha_1 \cos(\theta)\lambda} \ln |u| + C_3 = k \cdot r \quad (38)$$

Combining constants (C_3) and (C_4) into (C_5) , the final temperature solution $(T(r, t))$ is:

$$\frac{1}{-\alpha_1 \cos(\theta)\lambda} \ln |\alpha_0 + \alpha_1 \cos(\theta)e^{-\lambda t}| + C_5 = k \cdot r \quad (39)$$

This represents the general solution for the temperature $(T(r, t))$ in a cylinder with a time-varying conduction coefficient, accounting for variations in thermal conductivity and describing the temperature distribution as it evolves over time.

6. Programming Approach

In the first step, we begin by defining constants such as the cylinder's length (L), the outer surface temperature (T_{outer}), and the thermal conductivity (k). Then, we either create or use Bessel functions to compute both the first and second kinds of Bessel functions [13]. Additionally, numerical methods are employed to find the roots of the Bessel equation [14]. Subsequently, we solve equations for $Z(z)$, $X(r)$, and $Y(\theta)$ as outlined in the problem statement. Finally, we combine these solutions to obtain the ultimate temperature distribution within the cylinder.

Moving on to the second step, we establish constants and initial conditions. We then calculate the necessary derivatives based on the provided equations. Afterward, we simplify the heat equation according to the specified boundary conditions and implement the final equation to determine the temperature distribution over time (t).

In the third step, we set constants and define a function for the time-dependent thermal conductivity, $k(r, t)$. Utilizing the technique of separating variables, we isolate the spatial and temporal components of the temperature equation. Next, we solve for (r) and (t) in accordance with the problem statement, considering the time-dependent thermal conductivity. Finally, we combine these solutions to obtain a general solution for the temperature distribution, accounting for variations in thermal conductivity.

In the additional steps, we implement numerical methods for computing Bessel functions, finding roots of Bessel equations, and solving differential equations [15].

Numerical integration techniques may also be applied as needed for specific equations. We enhance code modularity and clarity by creating modular functions or procedures to tackle each part of the problem [16]. Furthermore, we make effective use of suitable R libraries or built-in functions for numerical computations and solving differential equations.

Table 1 presents the algorithmic steps for solving the heat conduction problem. In the steady-state heat distribution, constants such as cylinder length (L), outer surface temperature (T_{outer}), and thermal conductivity (k) are defined. Bessel functions are computed, their roots found, and equations for $Z(z)$, $X(r)$, and $Y(\theta)$ are implemented. These solutions are then combined to derive the final temperature distribution within the cylinder.

Table 1. Algorithmic steps for heat conduction problem solving

Step	Description
Steady-State Heat Distribution	
- Define Constants	Specify the values of L (cylinder length), T_{outer} (outer surface temperature), and k (thermal conductivity).
- Define Bessel Functions	Create or utilize functions to compute Bessel functions of the first and second kinds.
- Calculate Bessel Function Roots	Use numerical methods to find the roots of the Bessel equation.
- Solve for $Z(z)$, $X(r)$, and $Y(\theta)$	Implement the equations for $Z(z)$, $X(r)$, and $Y(\theta)$ as described in the problem statement.
- Combine Solutions	Combine the individual solutions for $Z(z)$, $X(r)$, and $Y(\theta)$ to derive the final temperature distribution within the cylinder.
Time Variable Boundary Condition	
- Define Constants and Initial Conditions	Set constant values and initial conditions as indicated in the problem statement.
- Calculate Derivatives	Compute the required derivatives based on the equations provided.
- Simplify Heat Equation	Apply the equations to simplify the heat equation for the given boundary conditions.
- Calculate Temperature Distribution	Implement the final equation to calculate the temperature distribution over time (t).
Variation of Conduction Coefficient	
- Define Constants and Thermal Conductivity Function	Set constant values and define a function for the time-dependent thermal conductivity, $k(r, t)$.
- Separate Variables	Utilize the separation of variables technique to isolate the spatial and temporal components of the temperature equation.
- Solve for (r) and (t)	Implement the solutions for (r) and (t) as described in the problem statement, considering the time-dependent thermal conductivity.
- Combine Solutions	Combine the solutions for (r) and (t) to obtain the general solution for the temperature distribution, accounting for variations in thermal conductivity.
Additional Steps	
- Implement Numerical Methods	Develop or use numerical methods for calculating Bessel functions, finding roots of Bessel equations, and solving differential equations.
- Numerical Integration	Apply numerical integration techniques if necessary for solving specific equations.
- Create Modular Functions/Procedures	Organize the code into functions or procedures for solving each part of the problem, enhancing modularity and code readability.
- Utilize R Libraries/Functions	Make use of appropriate libraries or built-in functions in R for numerical calculations and differential equation solving.

For the time-variable boundary condition, constants and initial conditions are set, derivatives are calculated, and the heat equation is simplified based on provided equations. The final equation is implemented to calculate the temperature distribution over time (t). In the variation of conduction coefficient, constants and a function for time-dependent thermal conductivity ($k(r, t)$) are defined. The separation of variables technique is employed to solve for (r) and (t), considering the time-dependent thermal conductivity. Solutions for (r) and (t) are then combined to obtain the general solution for temperature distribution, accounting for variations in thermal conductivity.

Additional steps involve implementing numerical methods for Bessel functions, roots of Bessel equations, and solving differential equations. Numerical integration techniques are applied if necessary, and the code is organized into modular functions/procedures for enhanced modularity and readability. R libraries/functions are utilized for numerical calculations and differential equation solving.

III. Results and Discussions

The research findings provide a comprehensive understanding of temperature distribution in a cylindrical coordinate system. Figure 1 visually conveys temperature variations with changing values of radius (r) and azimuthal angle (θ), using the "viridis" color palette to signify temperature levels—darker hues indicating lower temperatures and lighter hues representing higher temperatures. Elevated temperatures are depicted in lighter regions, contrasting with cooler temperatures in darker areas. Additionally, the results illuminate the impact of key physical parameters, including α , λ_{mn} , m , n , T_{mn} , and T_{mn} , further enhancing the overall quality and depth of the study.

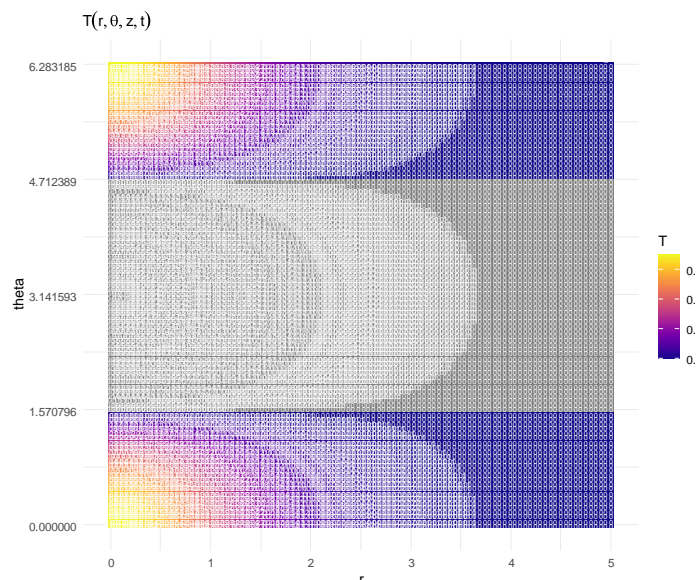


Fig. 1. The temperature distribution in a cylindrical coordinate system

In this study, Figure 2 offers a comprehensive three-dimensional visualization that elucidates the heat transfer phenomenon occurring within a cylindrical structure. The x -axis, which signifies the radial coordinates within the cylinder (z), is horizontally oriented, while the y -axis represents time (t) in a vertical orientation. The z -axis portrays the temperature distribution within the cylinder ($T(r, z, t)$) with respect to depth. The coloration on the surface plot corresponds to temperature levels within the cylinder, where lighter hues denote

elevated temperatures and darker shades indicate lower temperatures. The temperature distribution within the cylinder is influenced by several key physical parameters introduced in the initial formulation, including (A) and (B), which affect temperature oscillation amplitudes, (λ_n) governing oscillation wavelength, (α) representing the thermal diffusion coefficient, (k) as a constant, (μ) as another constant, and (λ) as the characteristic length. As a result, Figure 2 provides valuable insights into the temporal and radial evolution of temperature within the cylinder, shedding light on the impact of these aforementioned physical parameters on temperature distribution.

The research results in a sophisticated visual depiction of temperature distribution within the realm of heat transfer in cylindrical coordinates. Figure 3, presented alongside, vividly portrays this temperature distribution through the application of a intricate mathematical formula., expressed as $\left(\frac{1}{-\alpha_1 \cos(\theta)\lambda} \ln |\alpha_0 + \alpha_1 \cos(\theta)e^{-\lambda t} | + C_5 = k \cdot r \right)$. This temperature distribution is represented as a function $(T_n(r, \theta, z))$, which is a crucial component of the Fourier expansion utilized to describe temperature in cylindrical coordinates [17].

Within Figure 3, the temperature at any given point in the cylindrical coordinate system is computed as the cumulative sum of an infinite number of these Fourier terms [18]. Through adjustments in parameters such as r (radial distance), (θ) (polar angle), and z (height), contour plots are employed to visualize the dynamic variations in temperature distribution on a $z=0$ plane. The contour lines depicted in Figure 3 denote isothermal boundaries, effectively illustrating areas of equal temperature.

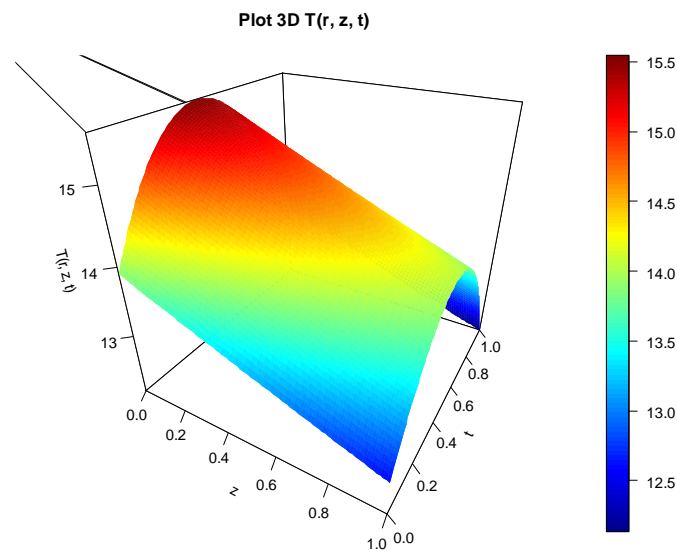


Fig. 2. Three-dimensional visualization of the heat transfer phenomenon

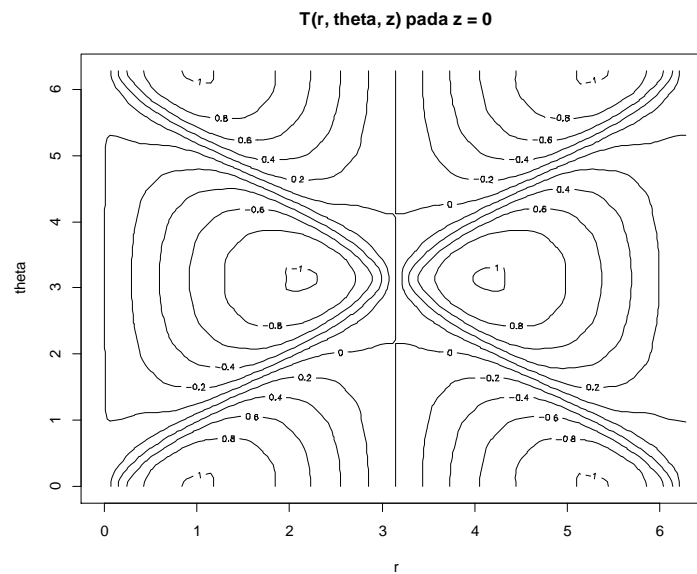


Fig. 3. The contour lines reflect iso-temperature on the $z=0$ surface in this cylindrical system

In order to elevate the accuracy of our findings, this research integrates a substantial number of terms (N) within the Fourier expansion. The precision of the resulting temperature representation is intricately tied to the magnitude of N employed. As a result, the ensuing visual portrayal provides a thorough insight into temperature fluctuations across the $z=0$ plane in this cylindrical system, encompassing temperature variations at diverse radial distances and polar angles.

IV. Conclusions

In conclusion, this thorough investigation delves into the intricacies of temperature distribution within a cylindrical coordinate system, employing rigorous mathematical techniques such as separation of variables. The study not only elucidates steady-state heat distribution but also explores the dynamic evolution of temperature over time. A notable aspect is the consideration of time-varying thermal conductivity, adding versatility to the solutions presented. Through a comprehensive results and discussions section, enriched with visual representations and in-depth analysis, the research significantly advances our understanding of temperature variations concerning radial distance, azimuthal angle, and time. The intricate interplay of physical parameters is unveiled, contributing valuable insights. Ultimately, this research stands as a foundational resource, offering a profound comprehension of heat transfer phenomena in cylindrical systems.

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