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A MGT thermoelastic problem with two relaxation parameters

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Abstract. In this paper, we consider, from both analytical and numerical viewpoints, a thermoelastic problem. The so-called MGT model, with two different relaxation parameters, is used for both the displacements and the thermal displacement, leading to a linear coupled system made by two third-order in time partial differential equations. Then, using the theory of linear semi-groups the existence and uniqueness to this problem is proved. If we restrict ourselves to the one-dimensional case, the exponential decay of the energy is obtained assuming some conditions on the constitutive parameters. Then, using the classical finite element method and the implicit Euler scheme, we introduce a fully discrete approximation of a variational formulation of the thermomechanical problem. A main a priori error estimates result is shown, from which we conclude the linear convergence under suitable additional regularity conditions. Finally, we present some one-dimensional numerical simulations to demonstrate the convergence of the fully discrete approximation, the behavior of the discrete energy decay and the dependence on a coupling parameter.

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1. Introduction

The study of the thermoviscoelastic theories has deserved much interest in the recent years. This is because, in many elastic materials, we can observe viscous mechanical aspects as well as the sensitivity to the thermal effects. We can see that the most general way to propose both aspects can be given by the Kelvin–Voigt dissipation mechanism for the viscosity and the Fourier heat conduction constitutive theory. Unfortunately, both aspects have a relevant drawback. The mechanical (thermal) waves for the theories based on the Kelvin–Voigt (Fourier) theory propagate instantaneously. That is, a mechanical (thermal) deformation imposed in a point in space is felt immediately at any other point in space. This fact violates in a relevant way the so-called causality principle.

This has been the reason why several alternatives theories have been proposed for the heat conduction. However, it is surprising that, although we have a similar effect when we consider the Kelvin–Voigt theory, this fact has not received particular attention.

The most known way to overcome the paradox of the infinite speed of propagation for the thermal waves is the Cattaneo–Maxwell proposition [6], which suggests the introduction of a relaxation parameter. This idea brings to Lord and Shulmann to propose their famous thermoelastic theory [19]. After some time, many other theories were considered. In particular, we can recall the works of Green and Naghdi [12, 13], who presented three new theories that they called type I, II and III, respectively. The difference among them comes from the choice of the independent constitutive variables. Linear version of type I recovers the classical theory of heat conduction, but types II and III became new theories which have deserved much attention over the last twenty-five years. The most general one is the type III theory, which contains the other two theories as limit cases. Unfortunately, type III theory has the same drawback as the classical Fourier law: the thermal waves also propagate instantaneously. For this reason, using a similar idea to the one proposed by Cattaneo and Maxwell, it is natural to introduce a relaxation parameter to bring hyperbolic the equation. Then, we obtain the Moore–Gibson–Thompson equation [21] and so, it is natural to consider the Moore–Gibson–Thompson thermoelasticity.

If we look for the viscous mechanical effect proposed by Kelvin and Voigt, we are in front of an equation (system) which is similar to the type III heat conduction (although with other physical meaning). Then, we also recover the paradox of the infinite speed of propagation of the mechanical waves. It is worth saying that few criticism has deserved this theory, if we compare with the one received by the Fourier or Green–Naghdi theories, but it is also natural to try to overcome the paradox by introducing a new relaxation parameter. This has been done recently in several mechanical situations [1–4, 9–11, 15–17, 20, 22, 23].

In this paper, we want to propose a thermoviscoelastic theory written (only) as partial differential equations in such a way that the thermomechanical waves propagate with finite speed. It is worth noting that this fact was previously proposed in the paper by Conti et al. [8], but, in this case, the authors modified both effects by means of the same parameter. This assumption is consistent, but we can agree that it is also very restrictive. For this reason, in this work we assume that (in the general case) the time relaxation for the thermal and mechanical effects are different. This assumption proposes a new mathematical difficulty since we have to handle with the coupling terms which introduce big mathematical problems. That is, from the mathematical point of view, we are in front of two equations (or systems) of the MGT-type, with different time relaxation parameters, which bring to new *coupling terms*. In this situation, we are going to be able to prove the existence and uniqueness of solutions, as well as the exponential stability (for the one-dimensional case) whenever the difference between the relaxation parameters is not very large in comparison with the remaining parameters proposed in the problem (see condition (4.1)).

The plan of this paper is the following. The model equations and the assumptions required on the constitutive tensors are presented in Sect. 2. The one-dimensional version of this problem is also recalled. Then, in Sect. 3 the existence and uniqueness of solution to the multi-dimensional problem is proved by using the theory of linear semigroups. The exponential decay of the solutions is shown in Sect. 4 when the problem is assumed one-dimensional. Later, by using the classical finite element method and the implicit Euler scheme, a fully discrete approximation is introduced in Sect. 5. A priori error estimates are obtained, and the linear convergence is derived under some additional regularity conditions on the continuous solution. Finally, some one-dimensional numerical simulations are presented in Sect. 6 to demonstrate the numerical convergence, the behavior of the discrete energy decay and the dependence on the coupling parameter.

2. Basic equations

The aim of this section is to propose the basic equations governing the general theory of MGT-thermoviscoelasticity. We are going to consider a bounded region $B \subset \mathbb{R}^d$, $d = 1, 2, 3$, with boundary smooth enough to apply the divergence theorem. In this sense, it is worth recalling that the evolution equations are

$$\begin{aligned} \rho \ddot{u}_i &= t_{ij,j}, \\ \rho T_0 \dot{\eta} &= q_{i,i}. \end{aligned}$$

Here, u_i is the displacement, ρ is the mass density, t_{ij} is the stress tensor, T_0 is the temperature (assumed to be uniform) in the reference configuration but, from now on, we will assume equal to one, η is the entropy and q_i is the heat flux vector.

The constitutive equations are given by

$$t_{ij} = \int_{-\infty}^{t} G_{ijrs}(t-s)\dot{u}_{r,s}(s)\,ds + \beta_{ij}\theta,$$
$$\eta = c\theta - \beta_{ij}u_{i,j},$$
$$\tau_2\dot{q}_i + q_i = K^*_{ij}\alpha_{,j} + K_{ij}\theta_{,j},$$

where

$$G_{ijrs}(s) = C^*_{ijrs} + e^{-s/\tau_1}\Big(\frac{C_{ijrs}}{\tau_1} - C^*_{ijrs}\Big),$$

α is the thermal displacement, $\theta = \dot{\alpha}$ is the temperature, τ_1 and τ_2 are two positive constants, K_{ij} is the thermal conductivity, C^*_{ijrs} is the elasticity tensor and C_{ijkl} is the viscosity, c is the thermal capacity and K^*_{ij} is a tensor which is typical of the works related with the Green and Naghdi thermoelastic theories [12,13].

We impose the following symmetries on the previous tensors:

$$C_{ijrs} = C_{rsij}, \quad C^*_{ijrs} = C^*_{rsij}, \quad K_{ij} = K_{ji}, \quad K^*_{ij} = K^*_{ji}.$$

After substitution of the constitutive equations into the evolution equations, we obtain the following system[1]:

$$\begin{aligned} \rho(\tau_1\dddot{u}_i + \ddot{u}_i) &= \Big(C_{ijrs}\dot{u}_{r,s} + C^*_{ijrs}u_{r,s}\Big)_{,j} + \Big(\beta_{ij}(\theta + \tau_1\dot{\theta})\Big)_{,j}, \\ c(\tau_2\dddot{\alpha} + \ddot{\alpha}) &= \Big(K_{ij}\dot{\alpha}_{,j} + K^*_{ij}\alpha_{,j}\Big)_{,i} + \beta_{ij}(\dot{u}_{i,j} + \tau_2\ddot{u}_{i,j}). \end{aligned} \tag{2.1}$$

We note that this coupling is new in the mathematical studies.

We will consider this system of equations with homogeneous Dirichlet boundary conditions

$$u_i(\boldsymbol{x},t) = \alpha(\boldsymbol{x},t) = 0 \quad \text{for a.e.} \quad \boldsymbol{x} \in \partial B, \tag{2.2}$$

and imposing the initial conditions, for a.e. $\boldsymbol{x} \in B$,

$$\begin{aligned} &u_i(\boldsymbol{x},0) = u_i^0(\boldsymbol{x}), \quad \dot{u}_i(\boldsymbol{x},0) = v_i^0(\boldsymbol{x}), \quad \ddot{u}_i(\boldsymbol{x},0) = a_i^0(\boldsymbol{x}), \\ &\alpha(\boldsymbol{x},0) = \alpha^0(\boldsymbol{x}), \quad \dot{\alpha}(\boldsymbol{x},0) = \theta^0(\boldsymbol{x}), \quad \ddot{\alpha}(\boldsymbol{x},0) = \phi^0(\boldsymbol{x}). \end{aligned} \tag{2.3}$$

Sometimes, it is useful working with the homogeneous one-dimensional case. In this situation, our system of equations becomes:

$$\begin{aligned} \rho(\tau_1\dddot{u} + \ddot{u}) &= C\dot{u}_{xx} + C^*u_{xx} + \beta(\theta_x + \tau_1\dot{\theta}_x), \\ c(\tau_2\dddot{\alpha} + \ddot{\alpha}) &= K\dot{\alpha}_{xx} + K^*\alpha_{xx} + \beta(\dot{u}_x + \tau_2\ddot{u}_x). \end{aligned} \tag{2.4}$$

In order to make the calculations easier, in this case we assume that the boundary conditions are:

$$u(x,t) = \alpha_x(x,t) = 0 \quad \text{for} \quad x \in \{0,\pi\},$$

where now the one-dimensional domain is taken as $B = (0,\pi)$.

We note that the last equation in the system of equations (2.1) can be written as

$$\begin{aligned} c(\tau_1\dddot{\alpha} + \ddot{\alpha}) + \frac{\tau_1}{\tau_2}c\left(1 - \frac{\tau_2}{\tau_1}\right)\ddot{\alpha} &= \frac{\tau_1}{\tau_2}\Big(K_{ij}\dot{\alpha}_{,j} + K^*_{ij}\alpha_{,j}\Big)_{,i} \\ &+\beta_{ij}(\dot{u}_{i,j} + \tau_1\ddot{u}_{i,j}) + \beta_{ij}\frac{\tau_1}{\tau_2}\left(1 - \frac{\tau_2}{\tau_1}\right)\dot{u}_{i,j}. \end{aligned} \tag{2.5}$$

[1] It is worth noting that we could obtain the same system of equations by considering appropriate relaxation functions for the thermoviscoelastic theory proposed by Gurtin [14]. A similar procedure was developed in the appendix of the paper [8], but here we consider more general assumptions.

Therefore, the one-dimensional case becomes:

$$c(\tau_1\dddot{\alpha}+\ddot{\alpha})+\frac{\tau_1}{\tau_2}c\left(1-\frac{\tau_2}{\tau_1}\right)\ddot{\alpha}=\frac{\tau_1}{\tau_2}(K\dot{\alpha}_{xx}+K^*\alpha_{xx})$$
$$+\beta(\dot{u}_x+\tau_1\ddot{u}_x)+\beta\frac{\tau_1}{\tau_2}\left(1-\frac{\tau_2}{\tau_1}\right)\dot{u}_x.$$

We note that we can write the following equality:

$$E(t)+\int_0^t D(s)\,ds=E(0), \tag{2.6}$$

where

$$E(t)=\frac{1}{2}\int_B\Big[\rho(\tau_1\ddot{u}_i+\dot{u}_i)(\tau_1\ddot{u}_i+\dot{u}_i)+c(\tau_1\ddot{\alpha}+\dot{\alpha})^2+\frac{\tau_1}{\tau_2}c\left(1-\frac{\tau_2}{\tau_1}\right)|\dot{\alpha}|^2$$
$$+C^*_{ijrs}(u_{i,j}+\tau_1\dot{u}_{i,j})(u_{r,s}+\tau_1\dot{u}_{r,s})+\frac{\tau_1}{\tau_2}K^*_{ij}(\alpha_{,i}+\tau_1\dot{\alpha}_{,i})(\alpha_{,j}+\tau_1\dot{\alpha}_{,j})$$
$$+\tau_1\overline{C}_{ijrs}\dot{u}_{i,j}\dot{u}_{r,s}+\frac{\tau_1^2}{\tau_2}\overline{K}_{ij}\dot{\alpha}_{,i}\dot{\alpha}_{,j}\Big]\,\mathrm{d}v,$$
$$D(t)=\int_B\Big[\overline{C}_{ijrs}\dot{u}_{i,j}\dot{u}_{r,s}+\frac{\tau_1}{\tau_2}\overline{K}_{ij}\dot{\alpha}_{,i}\dot{\alpha}_{,j}+\frac{\tau_1^2}{\tau_2}c\left(1-\frac{\tau_2}{\tau_1}\right)|\ddot{\alpha}|^2$$
$$+\beta_{ij}\frac{\tau_1}{\tau_2}\left(1-\frac{\tau_2}{\tau_1}\right)\dot{u}_{i,j}(\tau_1\ddot{\alpha}+\dot{\alpha})\Big]\,\mathrm{d}v,$$

with $\overline{C}_{ijrs}=C_{ijrs}-\tau_1C^*_{ijrs}$ and $\overline{K}_{ij}=K_{ij}-\tau_1K^*_{ij}$.

In view of equality (2.6), it will be natural to assume that

(i) $\rho(\boldsymbol{x})\geq\rho_0>0$ and $c(\boldsymbol{x})\geq c_0>0$.
(ii) C^*_{ijrs} and K^*_{ij} are positive definite tensors. That is, there exist two positive constants C and K such that

$$C^*_{ijrs}\xi_{ij}\xi_{rs}\geq C\xi_{ij}\xi_{ij}\quad\text{and}\quad K^*_{ij}\eta_i\eta_j\geq K\eta_i\eta_i$$

for every tensor ξ_{ij} and vector η_i.
(iii) $\overline{C}_{ijrs}$ and $\overline{K}_{ij}$ are positive definite tensors. That is, there exist two positive constants C_1 and K_1 such that

$$\overline{C}_{ijrs}\xi_{ij}\xi_{rs}\geq C_1\xi_{ij}\xi_{ij}\quad\text{and}\quad \overline{K}_{ij}\eta_i\eta_j\geq K_1\eta_i\eta_i$$

for every tensor ξ_{ij} and vector η_i.

We also assume that $\tau_1\geq\tau_2$.

We note that, for the one-dimensional homogeneous case, we can write the above assumptions as follows:

(i*) $\rho>0$, $c>0$, $C^*>0$, $\overline{C}>0$, $K^*>0$, $\overline{K}>0$.

The meaning of condition (i) is obvious. Condition (ii) can be interpreted in the context of the mathematical theory of thermoelastic stability. They are usually imposed. Condition (iii) guarantees that we will have mechanical and thermal dissipation.

3. Existence of solutions

In this section, we propose an existence and uniqueness result for the solutions to the problem defined by system (2.1) with the modified equation (2.5), the boundary conditions (2.2) and the initial conditions (2.3).

We will work on the Hilbert space:

$$\mathcal{H} = \boldsymbol{W}_0^{1,2}(B) \times \boldsymbol{W}_0^{1,2}(B) \times \boldsymbol{L}^2(B) \times W_0^{1,2}(B) \times W_0^{1,2}(B) \times L^2(B).$$

Here, $W_0^{1,2}(B)$ and $L^2(B)$ are the usual Sobolev spaces, and $\boldsymbol{L}^2(B) = [L^2(B)]^d$ and $\boldsymbol{W}_0^{1,2}(B) = [W_0^{1,2}(B)]^d$

In this space $\mathcal{H}$, we will consider the inner product defined as

$$\begin{aligned}\langle U, U^* \rangle = \frac{1}{2} \int_B \Big[& \rho(\tau_1 a_i + v_i)\overline{(\tau_1 a_i^* + v_i^*)} + c(\tau_1 \phi + \theta)\overline{(\tau_1 \phi^* + \theta^*)} \\ & + \frac{\tau_1}{\tau_2} c \left(1 - \frac{\tau_2}{\tau_1}\right) \theta \overline{\theta^*} + C^*_{ijrs}(u_{i,j} + \tau_1 v_{i,j})\overline{(u^*_{r,s} + \tau_1 v^*_{r,s})} \\ & + \frac{\tau_1}{\tau_2} K^*_{ij}(\alpha_{,i} + \tau_1 \theta_{,i})\overline{(\alpha^*_{,j} + \tau_1 \theta^*_{,j})} + \tau_1 \overline{C}_{ijrs} v_{i,j} \overline{v^*_{r,s}} + \frac{\tau_1^2}{\tau_2} \overline{K}_{ij} \theta_{,i} \overline{\theta^*_{,j}} \Big] \, \mathrm{d}v,\end{aligned}$$

where $U = (u_i, v_i, a_i, \alpha, \theta, \phi)$, $U^* = (u_i^*, v_i^*, a_i^*, \alpha^*, \theta^*, \phi^*)$ and the bar over an element of the Hilbert space represents its complex conjugated.

In this situation, we can write our problem defined by system (2.1) with the modified equation (2.5), the boundary conditions (2.2) and the initial conditions (2.3) as the following Cauchy problem:

$$\frac{\mathrm{d}U}{\mathrm{d}t} = \mathcal{A}U, \quad U(0) = (\boldsymbol{u}^0, \boldsymbol{v}^0, \boldsymbol{a}^0, \alpha^0, \theta^0, \phi^0), \tag{3.1}$$

where the matrix operator $\mathcal{A}$ is defined as

$$\mathcal{A} = \begin{pmatrix} 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ \boldsymbol{A}_{31} & \boldsymbol{A}_{32} & \boldsymbol{A}_{33} & 0 & \boldsymbol{A}_{35} & \boldsymbol{A}_{36} \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \\ 0 & A_{62} & A_{63} & A_{64} & A_{65} & A_{66} \end{pmatrix}.$$

In this matrix operator, the elements are given by

$$\begin{aligned} \boldsymbol{A}_{31}\boldsymbol{u} &= \frac{1}{\rho\tau_1}\left(C^*_{ijrs} u_{r,s}\right)_{,j}, \\ \boldsymbol{A}_{32}\boldsymbol{v} &= \frac{1}{\rho\tau_1}\left(C_{ijrs} v_{r,s}\right)_{,j}, \\ \boldsymbol{A}_{33}\boldsymbol{a} &= -\frac{1}{\tau_1} a_i, \\ \boldsymbol{A}_{35}\theta &= \frac{1}{\rho\tau_1}\left(\beta_{ij}\theta\right)_{,j}, \\ \boldsymbol{A}_{36}\phi &= \frac{1}{\rho}\left(\beta_{ij}\phi\right)_{,j}, \\ A_{62}\boldsymbol{v} &= \frac{1}{c\tau_2}\beta_{ij} v_{i,j}, \\ A_{63}\boldsymbol{a} &= \frac{1}{c}\beta_{ij} a_{i,j}, \\ A_{64}\alpha &= \frac{1}{c\tau_2}\left(K^*_{ij}\alpha_{,j}\right)_{,i}, \end{aligned}$$

$$A_{65}\theta = \frac{1}{c\tau_2}\left(K_{ij}\theta_{,j}\right)_{,i},$$
$$A_{66}\phi = -\frac{1}{\tau_2}\phi.$$

The domain of the operator $\mathcal{A}$ is made by the elements $U \in \mathcal{H}$ satisfying $\mathcal{A}U \in \mathcal{H}$. In fact, we note that it is subspace of elements $U \in \mathcal{H}$ such that

$$\boldsymbol{a} \in \boldsymbol{W}_0^{1,2}(B), \quad \phi \in W_0^{1,2}(B),$$
$$\left(C^*_{ijrs}u_{r,s} + C_{ijrs}v_{r,s}\right)_{,j} \in \boldsymbol{L}^2(B),$$
$$\left(K^*_{ij}\alpha_{,j} + K_{ij}\theta_{,j}\right)_{,i} \in L^2(B).$$

It is clear that it is a dense subspace of the Hilbert space $\mathcal{H}$. At the same time, we can obtain the following properties.

Lemma 3.1. *There exists a positive constant L such that*

$$\text{Re}\langle \mathcal{A}U, U\rangle \leq L\|U\|^2 \tag{3.2}$$

for every $U \in Dom(\mathcal{A})$.

Proof. If we recall the energy equality (2.6), it is straightforward to see that the relation (3.2) holds. □

Lemma 3.2. *Zero belongs to the resolvent of operator $\mathcal{A}$.*

Proof. Given $\mathcal{F} = (\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3, f_4, f_5, f_6) \in \mathcal{H}$, we need to prove that the equation

$$-\mathcal{A}U = \mathcal{F}$$

has a solution. We obtain the system:

$$-\boldsymbol{v} = \boldsymbol{f}_1,$$
$$-\boldsymbol{a} = \boldsymbol{f}_2,$$
$$-\boldsymbol{A}_{31}\boldsymbol{u} - \boldsymbol{A}_{32}\boldsymbol{v} - \boldsymbol{A}_{33}\boldsymbol{a} - \boldsymbol{A}_{35}\theta - \boldsymbol{A}_{36}\phi = \boldsymbol{f}_3,$$
$$-\theta = f_4,$$
$$-\phi = f_5,$$
$$-A_{62}\boldsymbol{v} - A_{63}\boldsymbol{a} - A_{64}\alpha - A_{65}\theta - A_{66}\phi = f_6,$$

from which, after easy algebraic manipulations, we conclude that we must solve the simpler system:

$$-\boldsymbol{A}_{31}\boldsymbol{u} = \boldsymbol{f}_3 + \boldsymbol{A}_{32}\boldsymbol{f}_1 + \boldsymbol{A}_{33}\boldsymbol{f}_2 + \boldsymbol{A}_{35}f_4 + \boldsymbol{A}_{36}f_5,$$
$$-A_{64}\alpha = f_6 + A_{62}\boldsymbol{f}_1 + A_{63}\boldsymbol{f}_2 + A_{65}f_4 + A_{66}f_5.$$

It is straightforward to see that the right-hand side of the previous system is in $\boldsymbol{W}^{-1,2}(B) \times W^{-1,2}(B)$.

In view of the assumptions on C^*_{ijkl} and K^*_{ij}, we obtain that this system admits a solution in the space $\boldsymbol{W}^{-1,2}(B) \times W^{-1,2}(B)$. □

From Lemmas 3.1 and 3.2, we can prove that $\mathcal{A}$ generates a quasi-contractive semigroup by using the Lumer–Phillips corollary applied to Hille–Yosida theorem. Therefore, problem (3.1) has a unique solution. This is obtained in the following result.

Theorem 3.3. *The operator $\mathcal{A}$ generates a C^0-semigroup of contractions in the space $\mathcal{H}$. Moreover, for any initial data $U(0)$ in the domain of the operator $\mathcal{A}$, we conclude that there exists at least one solution to Cauchy problem (3.1) with the regularity:*

$$U \in C^1([0,\infty);\mathcal{H}) \cap C([0,\infty);\mathcal{D}(\mathcal{A})).$$

Remark 3.4. It is possible to obtain an existence result under more general assumptions. Even if we could extend this comment to the non-homogeneous case, we now assume that the material is homogeneous. Indeed, we also have the equality:

$$E^*(t) + \int_0^t D^*(s)\, ds = E^*(0),$$

where

$$\begin{aligned} E^*(t) = \frac{1}{2}\int_B \Big[&\rho(\tau_1 \ddot{u}_i + \dot{u}_i)(\tau_1 \ddot{u}_i + \dot{u}_i) \\ &+ C^*_{ijrs}(u_{i,j} + \tau_1 \dot{u}_{i,j})(u_{r,s} + \tau_1 \dot{u}_{r,s}) + \tau_1 \overline{C}_{ijrs}\dot{u}_{i,j}\dot{u}_{r,s} + \frac{\tau_1^2}{\tau_2^2}\Big(c(\tau_2\ddot{\alpha} + \dot{\alpha})^2 \\ &+ K^*_{ij}(\alpha_{,i} + \tau_2\dot{\alpha}_{,i})(\alpha_{,j} + \tau_2\dot{\alpha}_{,j}) + \tau_2 l_{ij}\dot{\alpha}_{,i}\dot{\alpha}_{,j}\Big)\Big]\, \mathrm{d}v, \\ D^*(t) = \int_B \Big[&\overline{C}_{ijrs}\dot{u}_{i,j}\dot{u}_{r,s} + \frac{\tau_1^2}{\tau_2^2} l_{ij}\dot{\alpha}_{,i}\dot{\alpha}_{,j} + \beta_{ij}\Big[\dot{u}_i\theta_{,j} + \tau_1(\ddot{u}_i\theta_{,j} - \dot{u}_{i,j}\dot{\theta}) \\ &+ \frac{\tau_1^2}{\tau_2^2}(\dot{u}_{i,j}\theta + \tau_2(\dot{u}_{i,j}\dot{\theta} - \ddot{u}_i\theta_{,j}))\Big]\, \mathrm{d}v, \end{aligned}$$

and $l_{ij} = K_{ij} - \tau_2 K^*_{ij}$.

In view of this equality, under the assumption proposed previously and that l_{ij} is a positive definite tensor, we can define the inner product in $\mathcal{H}$:

$$\begin{aligned} \langle U, U^*\rangle = \frac{1}{2}\int_B \Big[&\rho(\tau_1 a_i + v_i)\overline{(\tau_1 a_i^* + v_i^*)} + C^*_{ijkl}(u_{i,j} + \tau_1 v_{i,j})\overline{(u^*_{k,l} + \tau_1 v^*_{k,l})} \\ &+ \tau_1 \overline{C}_{ijkl} v_{i,j}\overline{v^*_{k,l}} + \frac{\tau_1^2}{\tau_2^2}\Big(c(\tau_2\phi + \theta)\overline{(\tau_2\phi^* + \theta^*)} + K^*_{ij}(\alpha_{,i} + \tau_2\theta_{,i})\overline{(\alpha^*_{,j} + \tau_2\theta^*_{,j})} \\ &+ \tau_2 l_{ij}\theta_{,i}\overline{\theta^*_{,j}}\Big)\Big]\, \mathrm{d}v. \end{aligned}$$

In this case, we can see that there exists a positive constant L_1 such that

$$\mathrm{Re}\langle \mathcal{A}U, U\rangle \le L_1 \|U\|^2,$$

but we do not need the assumptions required above. Again, the domain is dense and we can also prove that zero belongs to the resolvent of this operator $\mathcal{A}$. Therefore, we can obtain the existence of solutions under the assumptions (i)–(iii), but changing the condition on $\overline{K}_{ij}$ by l_{ij}, which is a weaker condition.

4. One-dimensional case

In this section, we restrict our analysis to the one-dimensional homogeneous case.

If we define the inner product

$$\begin{aligned} \langle U, U^*\rangle = \frac{1}{2}\int_0^\pi \Big[&\rho(\tau_1 a + v)\overline{(\tau_1 a^* + v^*)} + c(\tau_1\phi + \theta)\overline{(\tau_1\phi^* + \theta^*)} \\ &+ \frac{\tau_1}{\tau_2} c\left(1 - \frac{\tau_2}{\tau_1}\right)\theta\overline{\theta^*} + C(u_x + \tau_1 v_x)\overline{(u^*_x + \tau_1 v^*_x)} + \tau_1\overline{C} v_x\overline{v^*_x} + \frac{\tau_1^2}{\tau_2}\overline{K}\theta_x\overline{\theta^*_x} \\ &+ \frac{\tau_1}{\tau_2} K(\alpha_x + \tau_1\theta_x)\overline{(\alpha^*_x + \tau_1\theta^*_x)}\Big]\, \mathrm{d}x, \end{aligned}$$

we have

$$\mathrm{Re}\langle \mathcal{A}U,U\rangle = -\int_0^\pi \Big[\overline{C}v_x\overline{v_x} + \frac{\tau_1}{\tau_2}\overline{K}\theta_x\overline{\theta_x} + \frac{\tau_1^2}{\tau_2}c\left(1-\frac{\tau_2}{\tau_1}\right)\phi\overline{\phi^*} + \frac{1}{2}\beta\frac{\tau_1}{\tau_2}\left(1-\frac{\tau_2}{\tau_1}\right)\left(v_x\overline{(\theta+\tau_1\phi)} + \overline{v_x}(\theta+\tau_1\phi)\right)\Big]\,\mathrm{d}x.$$

Therefore, if we assume that the bilinear form given by the matrix

$$\begin{pmatrix} \overline{C} & \frac{1}{2}\beta\frac{\tau_1}{\tau_2}\left(1-\frac{\tau_2}{\tau_1}\right) & \frac{\tau_1}{2}\beta\frac{\tau_1}{\tau_2}\left(1-\frac{\tau_2}{\tau_1}\right) \\ \frac{1}{2}\beta\frac{\tau_1}{\tau_2}\left(1-\frac{\tau_2}{\tau_1}\right) & \frac{\tau_1}{\tau_2}\overline{K} & 0 \\ \frac{\tau_1}{2}\beta\frac{\tau_1}{\tau_2}\left(1-\frac{\tau_2}{\tau_1}\right) & 0 & \frac{\tau_1^2}{\tau_2}c\left(1-\frac{\tau_2}{\tau_1}\right) \end{pmatrix} \tag{4.1}$$

is positive definite we conclude that $\mathrm{Re}\langle \mathcal{A}U,U\rangle \le 0$ for every $U \in \mathrm{Dom}(\mathcal{A})$.

Remark 4.1. Matrix (4.1) is positive definite if and only if the following conditions hold:

$$\overline{C} > 0, \quad 4\overline{CK}\tau_1\tau_2 - \beta^2(\tau_1-\tau_2)^2 > 0,$$
$$4\overline{CK}c\tau_1\tau_2 - \overline{K}\beta^2\tau_1^2(\tau_1-\tau_2) - c\beta^2(\tau_1-\tau_2)^2 > 0.$$

We see that whenever $\tau_1 - \tau_2$ is small enough, the previous conditions hold.

It is worth noting that, under the assumptions proposed in this section, we can prove that zero belongs to the resolvent of the operator.

Therefore, we have shown the following.

Theorem 4.2. *Under the previous assumptions (i*) and that the matrix (4.1) is positive definite, the solutions decay in an exponential way.*

Proof. Since the semigroup is contractive, we can use the semigroup theory of linear operators as well as the characterization of the exponentially stable semigroups obtained by Pruss (and other authors), which can be recalled in the book of Liu and Zheng [18]. Therefore, in order to prove the theorem it is sufficient to show that the imaginary axis is contained in the resolvent of the operator $\mathcal{A}$ and that the asymptotic condition

$$\overline{\lim_{|\lambda|\to\infty}} \|(i\lambda\mathcal{I} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty \tag{4.2}$$

holds.

We will prove the first condition. In the case that this is not fulfilled, there exist a sequence of real numbers $\lambda_n \to \lambda \neq 0$ and a sequence of unit norm vectors $U_n = (u_n, v_n, a_n, \alpha_n, \theta_n, \phi_n)$ such that

$$\begin{aligned} &i\lambda_n u_n - v_n \to 0 \quad \text{in} \quad W^{1,2}(B),\\ &i\lambda_n v_n - a_n \to 0 \quad \text{in} \quad W^{1,2}(B),\\ &i\lambda_n a_n - A_{31}u_n - A_{32}v_n - A_{33}a_n - A_{35}\theta_n - A_{36}\phi_n \to 0 \quad \text{in} \quad L^2(B),\\ &i\lambda_n \alpha_n - \theta_n \to 0 \quad \text{in} \quad W^{1,2}(B),\\ &i\lambda_n \theta_n - \phi_n \to 0 \quad \text{in} \quad W^{1,2}(B),\\ &i\lambda_n \phi_n - A_{62}v_n - A_{63}a_n - A_{64}\alpha_n - A_{65}\theta_n - A_{66}\phi_n \to 0 \quad \text{in} \quad L^2(B). \end{aligned}$$

In view of the assumption on the dissipation $D(t)$, we have that

$$v_n, \theta_n \to 0 \quad \text{in} \quad W^{1,2}(B), \quad \phi_n \to 0 \quad \text{in} \quad L^2(B),$$

and therefore,

$$\lambda_n u_n, \lambda_n \alpha_n \to 0 \quad \text{in} \quad W^{1,2}(B).$$

If we multiply the above third convergence by v_n, we also obtain that $a_n \to 0$ in $L^2(B)$. This is a contradiction because we assumed that U_n were unit norm vectors.

To prove the asymptotic condition, we can use a similar argument. □

5. A fully discrete scheme: a priori error estimates

In this section, we will introduce a fully discrete approximation of the problem defined by system (2.1), boundary conditions (2.2) and initial conditions (2.3) over a finite time interval $[0, T]$, $T > 0$. First, we need to write this problem in its variational form. Therefore, let us denote $Y = L^2(B)$, $H = [L^2(B)]^d$, $E = H_0^1(B)$ and $V = [H_0^1(B)]^d$. Moreover, for a Hilbert space X, let $(\cdot,\cdot)_X$ and $\|\cdot\|_X$ be the inner product and norm in X, respectively.

Therefore, multiplying the equations of system (2.1) by adequate test functions belonging to the spaces V and E, respectively, and using boundary conditions (2.2), we obtain the following variational formulation of this thermo-mechanical problem, written in terms of the acceleration $\boldsymbol{a}(t) = (a_i(t))$, the thermal acceleration $\phi(t)$, the velocity $\boldsymbol{v}(t) = (v_i(t))$ and the temperature $\theta(t)$.

Find the acceleration $\boldsymbol{a} : [0, T] \to V$ and the thermal acceleration $\phi : [0, T] \to E$ such that $\boldsymbol{a}(0) = \boldsymbol{a}^0$, $\phi(0) = \phi^0$,and for a.e. $t \in (0, T)$ and $\boldsymbol{w} \in V$, $\xi \in E$,

$$\begin{aligned}&\rho(\tau_1\dot{\boldsymbol{a}}(t) + \boldsymbol{a}(t), \boldsymbol{w})_H + \boldsymbol{C}(\boldsymbol{v}(t), \boldsymbol{w}) + \boldsymbol{C}^*(\boldsymbol{u}(t), \boldsymbol{w}) = (\beta_{ij}(\theta(t) + \tau_1\phi(t))_{,j}, w_i)_Y,\\ &c(\tau_2\dot{\phi}(t) + \phi(t), \xi)_Y + \mathcal{K}(\theta(t), \xi) + \mathcal{K}^*(\alpha(t), \xi) = (\beta_{ij}(v_{i,j}(t) + \tau_2 a_{i,j}(t)), \xi)_Y,\end{aligned} \tag{5.1}$$

where the operators $\boldsymbol{C}$, $\boldsymbol{C}^*$, $\mathcal{K}$ and $\mathcal{K}^*$ are given by

$$\begin{aligned}&\boldsymbol{C}(\boldsymbol{w}, \boldsymbol{\xi}) = (C_{ijrs}w_{r,s}, \xi_{i,j})_Y \quad \forall \boldsymbol{w} = (w_i),\ \boldsymbol{\xi} = (\xi_i) \in V,\\ &\boldsymbol{C}^*(\boldsymbol{w}, \boldsymbol{\xi}) = (C^*_{ijrs}w_{r,s}, \xi_{i,j})_Y \quad \forall \boldsymbol{w} = (w_i),\ \boldsymbol{\xi} = (\xi_i) \in V,\\ &\mathcal{K}(w, \xi) = (K_{ij}w_{,j}, \xi_{,i})_Y \quad \forall w,\ \xi \in E,\\ &\mathcal{K}^*(w, \xi) = (K^*_{ij}w_{,j}, \xi_{,i})_Y \quad \forall w,\ \xi \in E,\end{aligned}$$

and the velocity $\boldsymbol{v}(t)$, the temperature $\theta(t)$, the displacements $\boldsymbol{u}(t)$ and the thermal displacements $\alpha(t)$ are recovered from the relations:

$$\begin{aligned}&\boldsymbol{v}(t) = \int_0^t \boldsymbol{a}(s)\,\mathrm{d}s + \boldsymbol{v}^0, \quad \theta(t) = \int_0^t \phi(s)\,\mathrm{d}s + \theta^0,\\ &\boldsymbol{u}(t) = \int_0^t \boldsymbol{v}(s)\,\mathrm{d}s + \boldsymbol{u}^0, \quad \alpha(t) = \int_0^t \theta(s)\,\mathrm{d}s + \alpha^0.\end{aligned} \tag{5.2}$$

Now, we introduce a fully discrete approximation of problem (5.1)–(5.2). We will proceed in two steps. First, we approximate the problem in space. Thus, let us assume that $\overline{B}$ is a polyhedral domain and construct the finite element spaces:

$$\begin{aligned}&E^h = \{\xi^h \in C(\overline{B}) \cap E;\ \xi^h_{Tr} \in P_1(Tr) \quad \forall Tr \in \mathcal{T}^h\},\\ &V^h = \{\boldsymbol{w}^h \in [C(\overline{B})]^d \cap V;\ \boldsymbol{w}^h_{Tr} \in [P_1(Tr)]^d \quad \forall Tr \in \mathcal{T}^h\},\end{aligned} \tag{5.3}$$

where $\mathcal{T}^h$ is a regular finite element triangulation of the domain $\overline{B}$ (in the sense of [7]), and $P_1(Tr)$ represents the space of affine functions in Tr. Moreover, as usual, parameter $h > 0$ denotes the mesh size.

The discrete initial conditions $\boldsymbol{u}^{0h}$, $\boldsymbol{v}^{0h}$, $\boldsymbol{a}^{0h}$, α^{0h}, θ^{0h} and ϕ^{0h} are approximations of the respective initial conditions $\boldsymbol{u}^0$, $\boldsymbol{v}^0$, $\boldsymbol{a}^0$, α^0, θ^0 and ϕ^0 defined as

$$\begin{aligned}&\boldsymbol{u}^{0h} = \mathcal{P}^h_1\boldsymbol{u}^0, \quad \boldsymbol{v}^{0h} = \mathcal{P}^h_1\boldsymbol{v}^0, \quad \boldsymbol{a}^{0h} = \mathcal{P}^h_1\boldsymbol{a}^0,\\ &\alpha^{0h} = \mathcal{P}^h_2\alpha^0, \quad \theta^{0h} = \mathcal{P}^h_2\theta^0, \quad \phi^{0h} = \mathcal{P}^h_2\phi^0.\end{aligned} \tag{5.4}$$

Here, we denote by $\mathcal{P}_1^h$ and $\mathcal{P}_2^h$ the interpolation operators over the finite element spaces V^h and E^h, respectively (see, again, [7]).

Secondly, in order to provide the time discretization, we consider a uniform partition of the time interval $[0,T]$, denoted by $0=t_0<t_1<\cdots<t_N=T$, where $k=T/N$ is the time step size. If f is a continuous function, we denote $f_n=f(t_n)$ and, for the sequence $\{z_n\}_{n=0}^N$, let $\delta z_n=(z_n-z_{n-1})/k$ be its divided differences.

Therefore, using the well-known implicit Euler scheme we can introduce the following fully discrete problem.

Find the discrete acceleration $\boldsymbol{a}^{hk}=\{\boldsymbol{a}_n^{hk}\}_{n=0}^N\subset V^h$ *and the discrete thermal acceleration* $\phi^{hk}=\{\phi_n^{hk}\}_{n=0}^N\subset E^h$ *such that* $\boldsymbol{a}_0^{hk}=\boldsymbol{a}^{0h}$, $\phi_0^{hk}=\phi^{0h}$, *and for all* $n=1,\ldots,N$ *and* $\boldsymbol{w}^h\in V^h$, $\xi^h\in E^h$,

$$\begin{aligned}&\rho(\tau_1\delta\boldsymbol{a}_n^{hk}+\boldsymbol{a}_n^{hk},\boldsymbol{w}^h)_H+\boldsymbol{C}(\boldsymbol{v}_n^{hk},\boldsymbol{w}^h)+\boldsymbol{C}^*(\boldsymbol{u}_n^{hk},\boldsymbol{w}^h)=(\beta_{ij}(\theta_n^{hk}+\tau_1\phi_n^{hk})_{,j},w_i^h)_Y,\\&c(\tau_2\delta\phi_n^{hk}+\phi_n^{hk},\xi^h)_Y+\mathcal{K}(\theta_n^{hk},\xi^h)+\mathcal{K}^*(\alpha_n^{hk},\xi^h)=(\beta_{ij}(v_{in,j}^{hk}+\tau_2a_{in,j}^{hk}),\xi^h)_Y,\end{aligned}\tag{5.5}$$

where the discrete velocity $\boldsymbol{v}_n^{hk}$, *the discrete temperature* θ_n^{hk}, *the discrete displacements* $\boldsymbol{u}_n^{hk}$ *and the discrete thermal displacements* α_n^{hk} *are updated from the relations:*

$$\begin{aligned}&\boldsymbol{v}_n^{hk}=k\sum_{j=1}^n\boldsymbol{a}_j^{hk}+\boldsymbol{v}^{0h},\quad \theta_n^{hk}=k\sum_{j=1}^n\phi_j^{hk}+\theta^{0h},\\&\boldsymbol{u}_n^{hk}=k\sum_{j=1}^n\boldsymbol{v}_j^{hk}+\boldsymbol{u}^{0h},\quad \alpha_n^{hk}=k\sum_{j=1}^n\theta_j^{hk}+\alpha^{0h}.\end{aligned}\tag{5.6}$$

Using assumptions (i)–(iii) and applying Lax–Milgram lemma, it is easy to prove that the above discrete problem has a unique solution.

In the rest of the section, we will obtain some a priori error estimates on the numerical errors $\boldsymbol{a}_n-\boldsymbol{a}_n^{hk}$ and $\phi_n-\phi_n^{hk}$, which we state in the following result.

Theorem 5.1. *Let the assumptions* (i)–(iii) *hold. If we denote by* $(\boldsymbol{u},\boldsymbol{v},\boldsymbol{a},\alpha,\theta,\phi)$ *the solution to problem* (5.1)–(5.2) *and by* $(\boldsymbol{u}^{hk},\boldsymbol{v}^{hk},\boldsymbol{a}^{hk},\alpha^{hk},\theta^{hk},\phi^{hk})$ *the solution to problem* (5.5)–(5.6), *then we have the following a priori error estimates, for all* $\{\boldsymbol{w}_n^h\}_{n=0}^N\subset V^h$, $\{\xi_n^h\}_{n=0}^N\subset E^h$,

$$\begin{aligned}&\max_{0\le n\le N}\Big\{\|\boldsymbol{a}_n-\boldsymbol{a}_n^{hk}\|_H^2+\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_V^2+\|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\|_V^2+\|\phi_n-\phi_n^{hk}\|_Y^2\\&\quad+\|\theta_n-\theta_n^{hk}\|_E^2+\|\alpha_n-\alpha_n^{hk}\|_E^2\Big\}\\&\le Ck\sum_{j=1}^N\Big[\|\dot{\boldsymbol{a}}_j-\delta\boldsymbol{a}_j\|_H^2+\|\dot{\boldsymbol{v}}_j-\delta\boldsymbol{v}_j\|_V^2+\|\boldsymbol{a}_j-\boldsymbol{w}_j^h\|_V^2+\|\dot\phi_j-\delta\phi_j\|_Y^2\\&\quad+\|\dot\theta_j-\delta\theta_j\|_E^2+\|\phi_j-\xi_j^h\|_E^2+\|\dot\alpha_j-\delta\alpha_j\|_E^2+\|\dot{\boldsymbol{u}}_j-\delta\boldsymbol{u}_j\|_V^2+I_j^1+I_j^2\Big]\\&\quad+\frac{C}{k}\sum_{j=1}^{N-1}\Big[\|\boldsymbol{a}_j-\boldsymbol{w}_j^h-(\boldsymbol{a}_{j+1}-\boldsymbol{w}_{j+1}^h)\|_H^2+\|\phi_j-\xi_j^h-(\phi_{j+1}-\xi_{j+1}^h)\|_Y^2\Big]\\&\quad+C\max_{0\le n\le N}\|\boldsymbol{a}_n-\boldsymbol{w}_n^h\|_H^2+C\max_{0\le n\le N}\|\phi_n-\xi_n^h\|_Y^2\\&\quad+C\Big(\|\boldsymbol{a}^0-\boldsymbol{a}^{0h}\|_H^2+\|\boldsymbol{v}^0-\boldsymbol{v}^{0h}\|_V^2+\|\boldsymbol{u}^0-\boldsymbol{u}^{0h}\|_V^2+\|\phi^0-\phi^{0h}\|_Y^2\\&\quad+\|\theta^0-\theta^{0h}\|_E^2+\|\alpha^0-\alpha^{0h}\|_E^2\Big),\end{aligned}$$

where C is a positive constant which does not depend on parameters h and k, and the integration errors I_j^1 and I_j^2 are defined as

$$I_j^1 = \left\| \int_0^{t_j} \theta(s)\,ds - k\sum_{l=1}^{j}\theta_l \right\|_E^2, \quad I_j^2 = \left\| \int_0^{t_j} \boldsymbol{v}(s)\,ds - k\sum_{l=1}^{j}\boldsymbol{v}_l \right\|_V^2. \tag{5.7}$$

Proof. In this proof, in order to simplify the calculations, we will consider that $\tau_1 = \tau_2 = 1$. We note that we can modify the arguments used below to the general case with some minor changes.

First, we will obtain the error estimates on the acceleration term $\boldsymbol{a}_n - \boldsymbol{a}_n^{hk}$. Thus, we subtract variational equation $(5.1)_1$ for a test function $\boldsymbol{w} = \boldsymbol{w}^h \in V^h \subset V$, at time $t = t_n$, and discrete variational equation $(5.5)_1$ to find that

$$\begin{aligned}\rho(\dot{\boldsymbol{a}}_n - \delta\boldsymbol{a}_n^{hk} + \boldsymbol{a}_n - \boldsymbol{a}_n^{hk}, \boldsymbol{w}^h)_H + \boldsymbol{C}(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}, \boldsymbol{w}^h) + \boldsymbol{C}^*(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}, \boldsymbol{w}^h)\\ = (\beta_{ij}(\theta_n - \theta_n^{hk} + \phi_n - \phi_n^{hk})_{,j}, w_i^h)_Y \quad \forall \boldsymbol{w}^h \in V^h.\end{aligned}$$

Therefore, it follows that, for all $\boldsymbol{w}^h \in V^h$,

$$\begin{aligned}&\rho(\dot{\boldsymbol{a}}_n - \delta\boldsymbol{a}_n^{hk} + \boldsymbol{a}_n - \boldsymbol{a}_n^{hk}, \boldsymbol{a}_n - \boldsymbol{a}_n^{hk})_H + \boldsymbol{C}(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}, \boldsymbol{a}_n - \boldsymbol{a}_n^{hk})\\ &\quad + \boldsymbol{C}^*(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}, \boldsymbol{a}_n - \boldsymbol{a}_n^{hk}) - (\beta_{ij}(\theta_n - \theta_n^{hk} + \phi_n - \phi_n^{hk})_{,j}, a_{in} - a_{in}^{hk})_Y\\ &= \rho(\dot{\boldsymbol{a}}_n - \delta\boldsymbol{a}_n^{hk} + \boldsymbol{a}_n - \boldsymbol{a}_n^{hk}, \boldsymbol{a}_n - \boldsymbol{w}^h)_H + \boldsymbol{C}(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}, \boldsymbol{a}_n - \boldsymbol{w}^h)\\ &\quad + \boldsymbol{C}^*(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}, \boldsymbol{a}_n - \boldsymbol{w}^h) - (\beta_{ij}(\theta_n - \theta_n^{hk} + \phi_n - \phi_n^{hk})_{,j}, a_{in} - w_i^h)_Y.\end{aligned}$$

Taking into account that

$$\begin{aligned}&\left(\delta\boldsymbol{a}_n - \delta\boldsymbol{a}_n^{hk}, \boldsymbol{a}_n - \boldsymbol{a}_n^{hk}\right)_H \geq \frac{1}{2k}\left\{\|\boldsymbol{a}_n - \boldsymbol{a}_n^{hk}\|_H^2 - \|\boldsymbol{a}_{n-1} - \boldsymbol{a}_{n-1}^{hk}\|_H^2\right\},\\ &\boldsymbol{C}(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}, \boldsymbol{a}_n - \boldsymbol{a}_n^{hk}) \geq \boldsymbol{C}(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}, \dot{\boldsymbol{v}}_n - \delta\boldsymbol{v}_n)\\ &\quad + \frac{C}{2k}\left\{\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_V^2 - \|\boldsymbol{v}_{n-1} - \boldsymbol{v}_{n-1}^{hk}\|_V^2\right\},\\ &(\beta_{ij}(\phi_n - \phi_n^{hk})_{,j}, a_{in} - w_i^h)_Y = -(\beta_{ij}(\phi_n - \phi_n^{hk}), a_{in,j} - w_{i,j}^h)_Y,\end{aligned}$$

where we have used assumptions (i)–(iii), applying several times Cauchy–Schwarz inequality and Cauchy's inequality $ab \leq \epsilon a^2 + \frac{1}{4\epsilon}b^2$, $a, b, \epsilon \in \mathbb{R}$ with $\epsilon > 0$, we obtain the following error estimates for the acceleration terms:

$$\begin{aligned}&\frac{\rho}{2k}\left\{\|\boldsymbol{a}_n - \boldsymbol{a}_n^{hk}\|_H^2 - \|\boldsymbol{a}_{n-1} - \boldsymbol{a}_{n-1}^{hk}\|_H^2\right\} + \frac{C}{2k}\left\{\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_V^2 - \|\boldsymbol{v}_{n-1} - \boldsymbol{v}_{n-1}^{hk}\|_V^2\right\}\\ &\quad + \boldsymbol{C}^*(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}, \delta\boldsymbol{v}_n - \delta\boldsymbol{v}_n^{hk}) - (\beta_{ij}(\phi_n - \phi_n^{hk})_{,j}, a_{in} - a_{in}^{hk})_Y\\ &\leq C\Big(\|\dot{\boldsymbol{a}}_n - \delta\boldsymbol{a}_n\|_H^2 + \|\boldsymbol{a}_n - \boldsymbol{a}_n^{hk}\|_H^2 + \|\boldsymbol{a}_n - \boldsymbol{w}^h\|_V^2 + \|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_V^2\\ &\quad + \|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V^2 + \|\theta_n - \theta_n^{hk}\|_E^2 + \|\phi_n - \phi_n^{hk}\|_Y^2 + \|\dot{\boldsymbol{v}}_n - \delta\boldsymbol{v}_n\|_V^2\\ &\quad + (\delta\boldsymbol{a}_n - \delta\boldsymbol{a}_n^{hk}, \boldsymbol{a}_n - \boldsymbol{w}^h)_H\Big) \quad \forall \boldsymbol{w}^h \in V^h,\end{aligned} \tag{5.8}$$

where, here and in what follows, C will represent a positive constant which depends on the constitutive tensors and coefficients, but it does not depend on the discretization parameters h and k, and whose value may change even within the same line.

Now, we obtain the error estimates on the thermal acceleration term $\phi_n - \phi_n^{hk}$. Subtracting variational equation $(5.1)_2$, for a test function $\xi = \xi^h \in E^h \subset E$ at time $t = t_n$, and discrete variational equation $(5.5)_2$ it follows that

$$\begin{aligned}c(\dot{\phi}_n - \delta\phi_n^{hk} + \phi_n - \phi_n^{hk}, \xi^h)_Y + \mathcal{K}(\theta_n - \theta_n^{hk}, \xi^h) + \mathcal{K}^*(\alpha_n - \alpha_n^{hk}, \xi^h)\\ -(\beta_{ij}(v_{in,j} - v_{in,j}^{hk} + a_{in,j} - a_{in,j}^{hk}), \xi^h)_Y \quad \forall \xi^h \in E^h.\end{aligned}$$

Therefore, we obtain that, for all $\xi^h \in E^h$,

$$\begin{aligned}
&c(\dot{\phi}_n - \delta\phi_n^{hk} + \phi_n - \phi_n^{hk}, \phi_n - \phi_n^{hk})_Y + \mathcal{K}(\theta_n - \theta_n^{hk}, \phi_n - \phi_n^{hk})\\
&\quad + \mathcal{K}^*(\alpha_n - \alpha_n^{hk}, \phi_n - \phi_n^{hk}) - (\beta_{ij}(v_{in,j} - v_{in,j}^{hk} + a_{in,j} - a_{in,j}^{hk}), \phi_n - \phi_n^{hk})_Y\\
&= c(\dot{\phi}_n - \delta\phi_n^{hk} + \phi_n - \phi_n^{hk}, \phi_n - \xi^h)_Y + \mathcal{K}(\theta_n - \theta_n^{hk}, \phi_n - \xi^h)\\
&\quad + \mathcal{K}^*(\alpha_n - \alpha_n^{hk}, \phi_n - \xi^h) - (\beta_{ij}(v_{in,j} - v_{in,j}^{hk} + a_{in,j} - a_{in,j}^{hk}), \phi_n - \xi^h)_Y.
\end{aligned}$$

Keeping in mind that

$$\begin{aligned}
&\big(\delta\phi_n - \delta\phi_n^{hk}, \phi_n - \phi_n^{hk}\big)_Y \geq \frac{1}{2k}\Big\{\|\phi_n - \phi_n^{hk}\|_Y^2 - \|\phi_{n-1} - \phi_{n-1}^{hk}\|_Y^2\Big\},\\
&\mathcal{K}(\theta_n - \theta_n^{hk}, \phi_n - \phi_n^{hk}) \geq \mathcal{K}(\theta_n - \theta_n^{hk}, \dot{\theta}_n - \delta\theta_n)\\
&\qquad + \frac{C_1}{2k}\Big\{\|\theta_n - \theta_n^{hk}\|_E^2 - \|\theta_{n-1} - \theta_{n-1}^{hk}\|_E^2\Big\},\\
&(\beta_{ij}(a_{in,j} - a_{in,j}^{hk}), \phi_n - \xi^h)_Y = -(\beta_{ij}(a_{in} - a_{in}^{hk}), \phi_{n,j} - \xi^h_{,j})_Y,
\end{aligned}$$

we have, for all $\xi^h \in E^h$,

$$\begin{aligned}
&\frac{c}{2k}\Big\{\|\phi_n - \phi_n^{hk}\|_Y^2 - \|\phi_{n-1} - \phi_{n-1}^{hk}\|_Y^2\Big\} + \frac{C_1}{2k}\Big\{\|\theta_n - \theta_n^{hk}\|_E^2 - \|\theta_{n-1} - \theta_{n-1}^{hk}\|_E^2\Big\}\\
&\quad + \mathcal{K}^*(\alpha_n - \alpha_n^{hk}, \delta\theta_n - \delta\theta_n^{hk}) - (\beta_{ij}(a_{in,j} - a_{in,j}^{hk}), \phi_n - \phi_n^{hk})_Y\\
&\leq C\Big(\|\dot{\phi}_n - \delta\phi_n\|_Y^2 + \|\phi_n - \phi_n^{hk}\|_Y^2 + \|\phi_n - \xi^h\|_E^2 + \|\theta_n - \theta_n^{hk}\|_E^2\\
&\quad + \|\alpha_n - \alpha_n^{hk}\|_E^2 + \|\boldsymbol{a}_n - \boldsymbol{a}_n^{hk}\|_H^2 + \|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_V^2 + \|\dot{\theta}_n - \delta\theta_n\|_E^2\\
&\quad + (\delta\phi_n - \delta\phi_n^{hk}, \phi_n - \xi^h)_Y\Big).
\end{aligned} \tag{5.9}$$

Combining estimates (5.8) and (5.9) and taking into account that

$$-(\beta_{ij}(a_{in,j} - a_{in,j}^{hk}), \phi_n - \phi_n^{hk})_Y = (\beta_{ij}(a_{in} - a_{in}^{hk}), \phi_{n,j} - \phi_{n,j}^{hk})_Y,$$

we find that, for all $\boldsymbol{w}^h \in V^h$, $\xi^h \in E^h$,

$$\begin{aligned}
&\frac{\rho}{2k}\Big\{\|\boldsymbol{a}_n - \boldsymbol{a}_n^{hk}\|_H^2 - \|\boldsymbol{a}_{n-1} - \boldsymbol{a}_{n-1}^{hk}\|_H^2\Big\} + \frac{C}{2k}\Big\{\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_V^2 - \|\boldsymbol{v}_{n-1} - \boldsymbol{v}_{n-1}^{hk}\|_V^2\Big\}\\
&\quad + \frac{c}{2k}\Big\{\|\phi_n - \phi_n^{hk}\|_Y^2 - \|\phi_{n-1} - \phi_{n-1}^{hk}\|_Y^2\Big\} + \frac{C_1}{2k}\Big\{\|\theta_n - \theta_n^{hk}\|_E^2 - \|\theta_{n-1} - \theta_{n-1}^{hk}\|_E^2\Big\}\\
&\quad + \boldsymbol{C}^*(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}, \delta\boldsymbol{v}_n - \delta\boldsymbol{v}_n^{hk}) + \mathcal{K}^*(\alpha_n - \alpha_n^{hk}, \delta\theta_n - \delta\theta_n^{hk})\\
&\leq C\Big(\|\dot{\boldsymbol{a}}_n - \delta\boldsymbol{a}_n\|_H^2 + \|\boldsymbol{a}_n - \boldsymbol{a}_n^{hk}\|_H^2 + \|\boldsymbol{a}_n - \boldsymbol{w}^h\|_V^2 + \|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_V^2\\
&\quad + \|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V^2 + \|\theta_n - \theta_n^{hk}\|_E^2 + \|\phi_n - \phi_n^{hk}\|_Y^2 + \|\dot{\boldsymbol{v}}_n - \delta\boldsymbol{v}_n\|_V^2\\
&\quad + (\delta\boldsymbol{a}_n - \delta\boldsymbol{a}_n^{hk}, \boldsymbol{a}_n - \boldsymbol{w}^h)_H + \|\dot{\phi}_n - \delta\phi_n\|_Y^2 + \|\phi_n - \xi^h\|_E^2 + \|\alpha_n - \alpha_n^{hk}\|_E^2\\
&\quad + \|\dot{\theta}_n - \delta\theta_n\|_E^2 + (\delta\phi_n - \delta\phi_n^{hk}, \phi_n - \xi^h)_Y\Big).
\end{aligned}$$

Multiplying the above estimates by k and summing up to n, we have, for all $\{\boldsymbol{w}_j^h\}_{j=1}^n \subset V^h$, $\{\xi_j^h\}_{j=1}^n \subset E^h$,

$$\begin{aligned}
&\|\boldsymbol{a}_n - \boldsymbol{a}_n^{hk}\|_H^2 + \|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_V^2 + \|\phi_n - \phi_n^{hk}\|_Y^2 + \|\theta_n - \theta_n^{hk}\|_E^2\\
&\quad + k\sum_{j=1}^n \Big[\boldsymbol{C}^*(\boldsymbol{u}_j - \boldsymbol{u}_j^{hk}, \delta\boldsymbol{v}_j - \delta\boldsymbol{v}_j^{hk}) + \mathcal{K}^*(\alpha_j - \alpha_j^{hk}, \delta\theta_j - \delta\theta_j^{hk})\Big]\\
&\leq Ck\sum_{j=1}^n \Big(\|\dot{\boldsymbol{a}}_j - \delta\boldsymbol{a}_j\|_H^2 + \|\boldsymbol{a}_j - \boldsymbol{a}_j^{hk}\|_H^2 + \|\boldsymbol{a}_j - \boldsymbol{w}_j^h\|_V^2 + \|\boldsymbol{v}_j - \boldsymbol{v}_j^{hk}\|_V^2\\
&\quad + \|\boldsymbol{u}_j - \boldsymbol{u}_j^{hk}\|_V^2 + \|\theta_j - \theta_j^{hk}\|_E^2 + \|\phi_j - \phi_j^{hk}\|_Y^2 + \|\dot{\boldsymbol{v}}_j - \delta\boldsymbol{v}_j\|_V^2
\end{aligned}$$

$$+(\delta \boldsymbol{a}_j-\delta \boldsymbol{a}_j^{hk},\boldsymbol{a}_j-\boldsymbol{w}_j^h)_H+\|\dot{\phi}_j-\delta\phi_j\|_Y^2+\|\phi_j-\xi_j^h\|_E^2+\|\alpha_j-\alpha_j^{hk}\|_E^2$$
$$+\|\dot{\theta}_j-\delta\theta_j\|_E^2+(\delta\phi_j-\delta\phi_j^{hk},\phi_j-\xi_j^h)_Y\Big)+C\Big(\|\boldsymbol{a}^0-\boldsymbol{a}^{0h}\|_H^2$$
$$+\|\boldsymbol{v}^0-\boldsymbol{v}^{0h}\|_V^2+\|\phi^0-\phi^{0h}\|_Y^2+\|\theta^0-\theta^{0h}\|_E^2\Big).$$

We note that it is easy to prove that

$$\begin{aligned}
k\sum_{j=1}^n(\delta \boldsymbol{a}_j-\delta \boldsymbol{a}_j^{hk},\boldsymbol{a}_j-\boldsymbol{w}_j^h)_H&=\sum_{j=1}^n(\boldsymbol{a}_j-\boldsymbol{a}_j^{hk}-(\boldsymbol{a}_{j-1}-\boldsymbol{a}_{j-1}^{hk}),\boldsymbol{a}_j-\boldsymbol{w}_j^h)_H\\
&=(\boldsymbol{a}_n-\boldsymbol{a}_n^{hk},\boldsymbol{a}_n-\boldsymbol{w}_n^h)_H+(\boldsymbol{a}^{0h}-\boldsymbol{a}^0,\boldsymbol{a}_1-\boldsymbol{w}_1^h)_H\\
&\quad+\sum_{j=1}^{n-1}(\boldsymbol{a}_j-\boldsymbol{a}_j^{hk},\boldsymbol{a}_j-\boldsymbol{w}_j^h-(\boldsymbol{a}_{j+1}-\boldsymbol{w}_{j+1}^h))_H,\\
k\sum_{j=1}^n(\delta\phi_j-\delta\phi_j^{hk},\phi_j-\xi_j^h)_Y&=\sum_{j=1}^n(\phi_j-\phi_j^{hk}-(\phi_{j-1}-\phi_{j-1}^{hk}),\phi_j-\xi_j^h)_Y\\
&=(\phi_n-\phi_n^{hk},\phi_n-\xi_n^h)_Y+(\phi^{0h}-\phi^0,\phi_1-\xi_1^h)_Y\\
&\quad+\sum_{j=1}^{n-1}(\phi_j-\phi_j^{hk},\phi_j-\xi_j^h-(\phi_{j+1}-\xi_{j+1}^h))_Y,\\
k\sum_{j=1}^n\boldsymbol{C}^*(\boldsymbol{u}_j-\boldsymbol{u}_j^{hk},\delta\boldsymbol{v}_j-\delta\boldsymbol{v}_j^{hk})&\le\boldsymbol{C}^*(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk},\boldsymbol{v}_n-\boldsymbol{v}_n^{hk})+\boldsymbol{C}^*(\boldsymbol{u}^{0h}-\boldsymbol{u}^0,\boldsymbol{v}_1-\boldsymbol{v}_1^{hk})\\
&\quad-k\sum_{j=1}^n\boldsymbol{C}^*(\boldsymbol{v}_j-\boldsymbol{v}_j^{hk},\boldsymbol{v}_j-\boldsymbol{v}_j^{hk})+Ck\sum_{j=1}^n\Big[\|\dot{\boldsymbol{u}}_j-\delta\boldsymbol{u}_j\|_V^2+\|\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}\|_V^2\Big],\\
k\sum_{j=1}^n\mathcal{K}^*(\alpha_j-\alpha_j^{hk},\delta\theta_j-\delta\theta_j^{hk})&\le\mathcal{K}^*(\alpha_n-\alpha_n^{hk},\theta_n-\theta_n^{hk})+\mathcal{K}^*(\alpha^{0h}-\alpha^0,\theta_1-\theta_1^{hk})\\
&\quad-k\sum_{j=1}^n\mathcal{K}^*(\theta_j-\theta_j^{hk},\theta_j-\theta_j^{hk})+Ck\sum_{j=1}^n\Big[\|\dot{\alpha}_j-\delta\alpha_j\|_E^2+\|\theta_j-\theta_j^{hk}\|_E^2\Big],\\
\|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\|_V^2&\le C\Big(\|\boldsymbol{u}^0-\boldsymbol{u}^{0h}\|_V^2+k\sum_{j=1}^n\|\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}\|_V^2+I_j^2\Big),\\
\|\alpha_n-\alpha_n^{hk}\|_E^2&\le C\Big(\|\alpha^0-\alpha^{0h}\|_E^2+k\sum_{j=1}^n\|\theta_j-\theta_j^{hk}\|_E^2+I_j^1\Big),
\end{aligned}$$

where I_j^1 and I_j^2 are the integration errors defined in (5.7).

Thus, using a discrete version of Gronwall's inequality (see [5]) we conclude the desired a priori error estimates. □

We note that we can use the above a priori error estimates to derive the convergence order of the approximations under additional regularity conditions on the continuous solution. Therefore, if we assume that

$$\begin{aligned}
&\boldsymbol{u}\in H^4(0,T;H)\cap W^{2,\infty}(0,T;[H^2(B)]^d)\cap H^3(0,T;V),\\
&\alpha\in H^4(0,T;Y)\cap W^{2,\infty}(0,T;H^2(B))\cap H^3(0,T;E),
\end{aligned}\tag{5.10}$$

we obtain the following.

Corollary 5.2. *Under the additional regularity conditions* (5.10) *and the assumptions of Theorem* 5.1, *we find that the approximations obtained by problem* (5.5)–(5.6) *are linearly convergent; that is, there exists* *a positive constant C, independent of the discretization parameters h and k, such that*

$$\max_{0\le n\le N}\Big\{\|\boldsymbol{a}_n-\boldsymbol{a}_n^{hk}\|_H+\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_V+\|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\|_V+\|\phi_n-\phi_n^{hk}\|_Y$$
$$+\|\theta_n-\theta_n^{hk}\|_E+\|\alpha_n-\alpha_n^{hk}\|_E\Big\}\le C(h+k).$$

6. Numerical results

In this final section, we present the numerical scheme implemented in MATLAB for solving problem (5.5)–(5.6), and we show some numerical examples to demonstrate the accuracy of the approximations, the behavior of the discrete energy decay and the dependence on the coupling coefficient β.

6.1. Numerical scheme for the one-dimensional problem

As a first step, given the solution u_{n-1}^{hk}, v_{n-1}^{hk}, a_{n-1}^{hk}, α_{n-1}^{hk}, θ_{n-1}^{hk} and ϕ_{n-1}^{hk} at time t_{n-1}, variables a_n^{hk} and ϕ_n^{hk} are obtained by solving the discrete linear system, for all w^h, $\xi^h\in V^h$.

$$\rho\left(\frac{\tau_1}{k}a_n^{hk}+\phi_n^{hk},w^h\right)+Ck\left(a_{nx}^{hk},w_x^h\right)+C^*k^2\left(a_{nx}^{hk},w_x^h\right)=\rho\left(\frac{\tau_1}{k}a_{n-1}^{hk},w^h\right)$$
$$-C\left(v_{(n-1)x}^{hk},w_x^h\right)-C^*\left(u_{(n-1)x}^{hk}+kv_{(n-1)x}^{hk},w_x^h\right)+\beta\left(\theta_{nx}^{hk}+\tau_1\phi_{nx}^{hk},w^h\right),$$
$$c\left(\frac{\tau_2}{k}\phi_n^{hk}+\phi_n^{hk},\xi^h\right)+Kk\left(\phi_{nx}^{hk},\xi_x^h\right)+K^*k^2\left(\phi_{nx}^{hk},\xi_x^h\right)=c\left(\frac{\tau_2}{k}\phi_{n-1}^{hk},\xi^h\right)$$
$$-K\left(\theta_{(n-1)x}^{hk},\xi_x^h\right)-K^*\left(\alpha_{(n-1)x}^{hk}+k\theta_{(n-1)x}^{hk},\xi_x^h\right)+\beta\left(v_{nx}^{hk}+\tau_2a_{nx}^{hk},\xi^h\right).$$

This numerical scheme was implemented on a 3.2 GHz PC using MATLAB, and a typical run (using parameters $h=k=0.001$) took about 0.17 s of CPU time

6.2. First example: numerical convergence

As a first simpler example, in order to show the accuracy of the approximations the following problem is considered over the domain $B=(0,1)$.

$$\rho(\tau_1\dddot{u}+\ddot{u})=C\dot{u}_{xx}+C^*u_{xx}+\beta(\theta_x+\tau_1\dot{\theta}_x)+F_1,$$
$$c(\tau_2\dddot{\alpha}+\ddot{\alpha})=K\dot{\alpha}_{xx}+K^*\alpha_{xx}+\beta(\dot{u}_x+\tau_2\ddot{u}_x)+F_2,$$

with the following data:

$$T=1,\quad \rho=1,\quad C=4,\quad C^*=1,\quad \beta=1,\quad c=2,$$
$$K=7,\quad K^*=3,\quad \tau_1=2,\quad \tau_2=1.$$

By using the following initial conditions, for all $x\in B=(0,1)$,

$$u^0(x)=v^0(x)=a^0(x)=\alpha^0(x)=\theta^0(x)=\phi^0(x)=x(x-1),$$

considering homogeneous Dirichlet boundary conditions and the (artificial) supply terms, for all $(x,t)\in(0,1)\times(0,1)$,

$$F_1(x,t)=e^t(3x(x-1)-6x-7),\quad F_2(x,t)=e^t(4x(x-1)-4x-8),$$

the exact solution to the above one-dimensional problem can be easily calculated and it has the form, for $(x,t)\in[0,1]\times[0,1]$:

$$u(x,t)=\alpha(x,t)=e^tx(x-1).$$

TABLE 1. Example 1: Numerical errors for some values of h and k

$h \downarrow k \rightarrow$	0.01	0.005	0.002	0.001	0.0005	0.0002	0.0001
$1/2^3$	0.354205	0.347240	0.343082	0.341698	0.341007	0.340592	0.340454
$1/2^4$	0.184106	0.176962	0.172745	0.171349	0.170654	0.170237	0.170098
$1/2^5$	0.099491	0.092052	0.087753	0.086347	0.085649	0.085232	0.085093
$1/2^6$	0.057764	0.049765	0.045302	0.043873	0.043170	0.042752	0.042613
$1/2^7$	0.037832	0.028905	0.024127	0.022651	0.021936	0.021515	0.021375
$1/2^8$	0.028995	0.018940	0.013633	0.012065	0.011326	0.010897	0.010757
$1/2^9$	0.025472	0.014520	0.008548	0.006818	0.006033	0.005591	0.005449
$1/2^{10}$	0.024203	0.012758	0.006226	0.004275	0.003409	0.002941	0.002795
$1/2^{11}$	0.023793	0.012125	0.005266	0.003114	0.002138	0.001624	0.001471
$1/2^{12}$	0.023677	0.011921	0.004911	0.002634	0.001557	0.000980	0.000812
$1/2^{13}$	0.023647	0.011864	0.004792	0.002457	0.001318	0.000678	0.000490

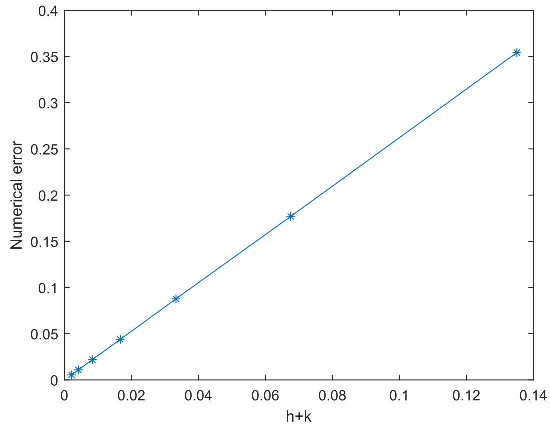

FIG. 1. Example 1: Asymptotic constant error

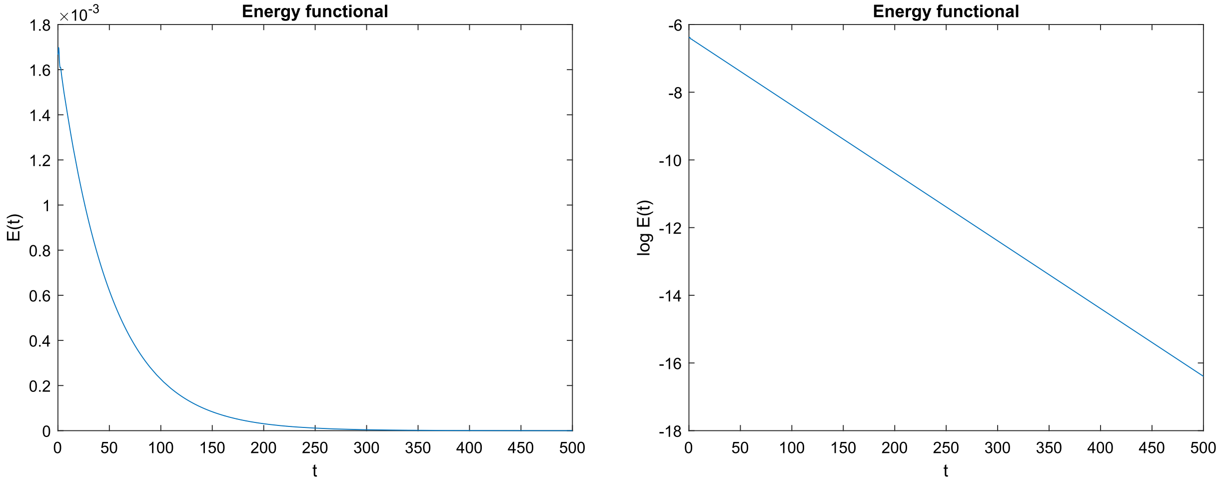

FIG. 2. Example 1: Evolution in time of the discrete energy (natural and semi-log scales)

Therefore, the approximation errors estimated by

$$\max_{0\le n\le N}\Big\{\|a_n-a_n^{hk}\|_Y+\|v_n-v_n^{hk}\|_E+\|u_n-u_n^{hk}\|_E+\|\phi_n-\phi_n^{hk}\|_Y$$
$$+\|\theta_n-\theta_n^{hk}\|_E+\|\alpha_n-\alpha_n^{hk}\|_E\Big\}$$

are presented in Table 1 for several values of the discretization parameters h and k. Moreover, the evolution of the error depending on the parameter $h+k$ is plotted in Fig. 1. We notice that the convergence of the algorithm is clearly observed, and the linear convergence, stated in Corollary 5.2, is achieved.

If we assume that there are not supply terms, and we use the final time $T=500$, the data

$$\rho=1,\quad C=1,\quad C^*=0.01,\quad \beta=1,\quad c=2,\quad K=7,\quad K^*=3,$$
$$\tau_1=1,\quad \tau_2=0.1,$$

and the initial conditions:

$$u^0(x)=x(x-1)\quad \forall x\in(0,1),\quad v^0=a^0=\alpha^0=\theta^0=\phi^0=0,$$

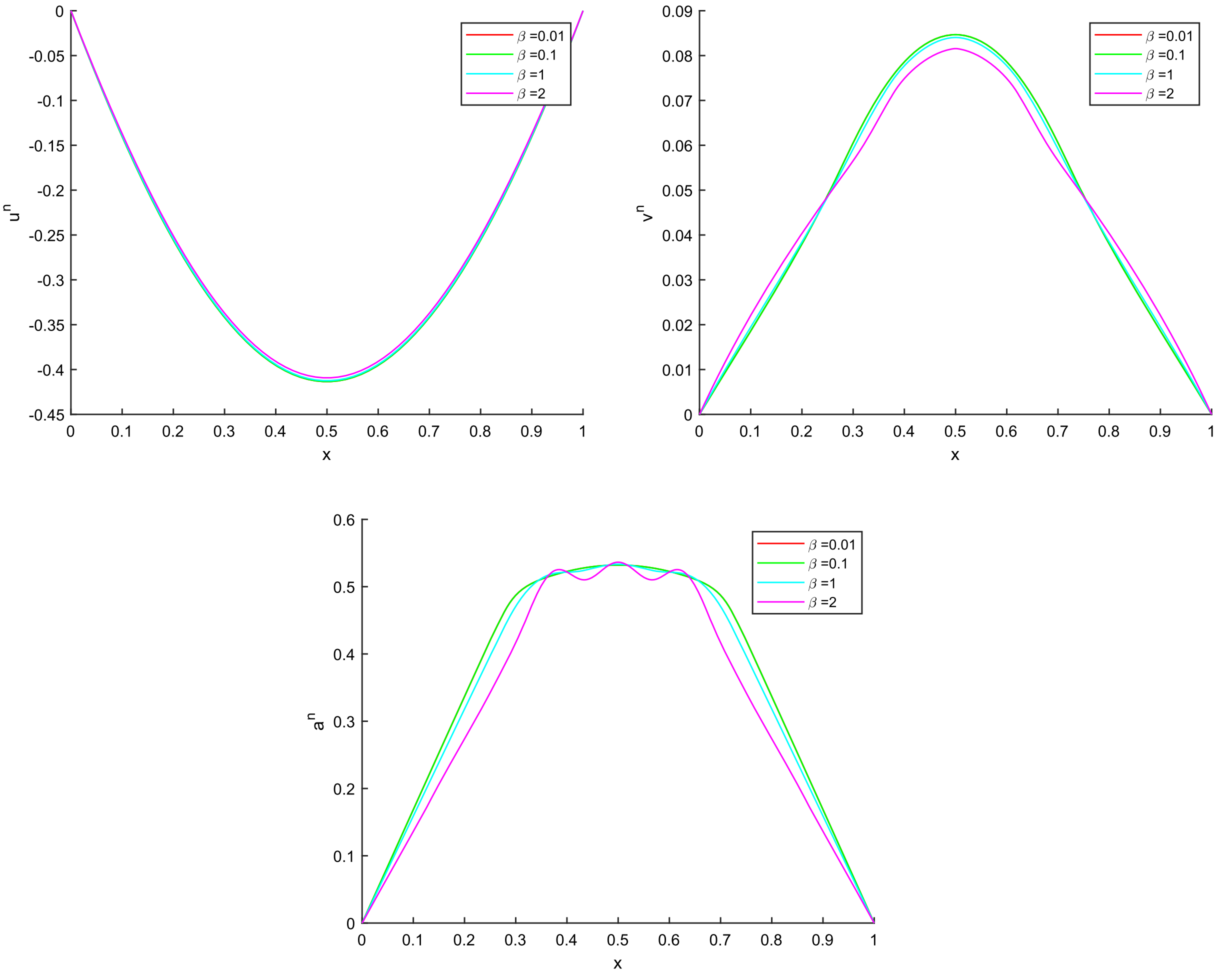

FIG. 3. Example 2: Displacement, velocity and acceleration for different values of β

taking the discretization parameters $h = k = 0.001$, the evolution in time of the discrete energy given by

$$E_n^{hk} = \frac{1}{2}\Big(\rho||\tau_1 a_n^{hk} + v_n^{hk}||_Y^2 + C\tau_1||v_n^{hk}||_E^2 + C^*||u_n^{hk}||_E^2 + c||\tau_2\phi_n^{hk} + \theta_n^{hk}||_Y^2 \\ +K||\tau_1\theta_n^{hk}||_E^2 + K^*||\alpha_n^{hk}||_E^2\Big)$$

is plotted in Fig. 2 (in both natural and semi-log scales). As it is demonstrated in this figure, it converges to zero and an exponential decay seems to be achieved.

6.3. Second example: Dependence of the solution on parameter β

In this last example, we address the dependence of the solution to problem (5.1)–(5.2) on the coupling parameter β.

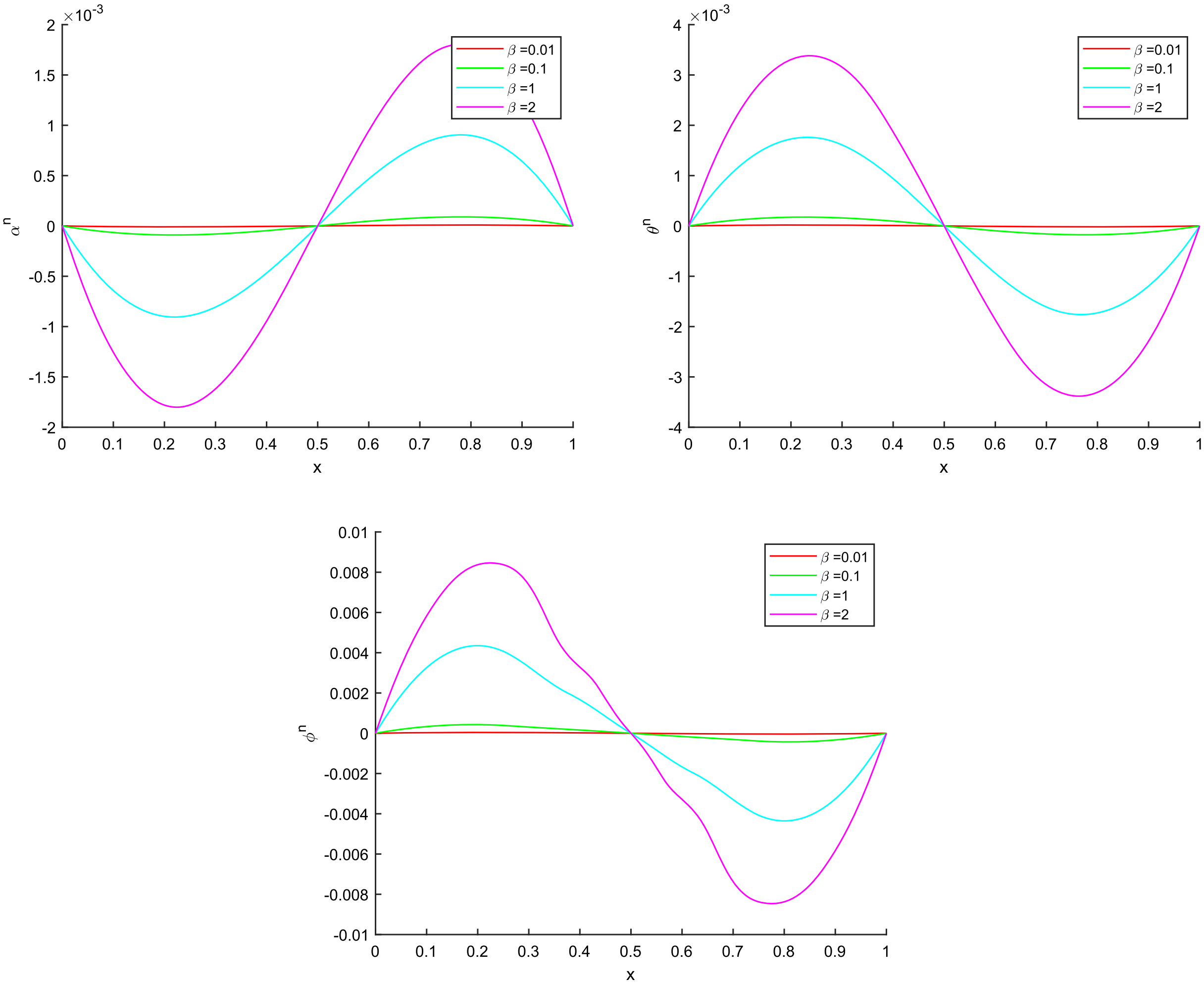

Fig. 4. Example 2: Thermal displacement, temperature and thermal acceleration for different values of β

We assume that there are not supply terms, and we use following data:

$$T=1,\quad \rho=1,\quad C=1,\quad C^*=0.1,\quad c=2,\quad K=7,\quad K^*=3,$$
$$\tau_1=2,\quad \tau_2=0.1,$$

and the initial conditions:

$$u^0(x)=v^0(x)=a^0(x)=x(x-1)\quad \forall x\in(0,1),\quad \alpha^0=\theta^0=\phi^0=0.$$

Then, taking the discretization parameters $h=0.001$ and $k=0.001$, we plot the solution to problem (5.5)–(5.6) in Figs. 3 and 4 for some values of parameter β. As can be seen in Fig. 3, the displacements and velocities are rather similar for all the values but, if we focus on the accelerations, some oscillations appear for the largest value of the parameter.

In Fig. 4, as expected we can see a big dependence on the value of the parameter. The reason is that these thermal displacements are produced by the deformation and so, they increase when the value of the coupling coefficient is greater.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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