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Finding Fibonacci : an interdisciplinary liberal arts course based on mathematical patterns

Margaret Stevenson Ribble

To the Graduate Council:

I am submitting herewith a dissertation written by Margaret Stevenson Ribble entitled "Finding Fibonacci : an interdisciplinary liberal arts course based on mathematical patterns." I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Education, with a major in Education.

Donald J. Dessart, Major Professor

We have read this dissertation and recommend its acceptance:

Carl Wagner, Karl Jost, Stephanie Robinson

Accepted for the Council:

Carolyn R. Hodges

Vice Provost and Dean of the Graduate School

(Original signatures are on file with official student records.)

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Stephanie O. Robinson

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Associate Vice Chancellor and
Dean of The Graduate School

FINDING FIBONACCI:
AN INTERDISCIPLINARY LIBERAL ARTS COURSE BASED ON MATHEMATICAL PATTERNS

A Dissertation
Presented for the
Doctor of Education
Degree
The University of Tennessee, Knoxville

Margaret S. Ribble
December 1999

DEDICATION

This dissertation is dedicated to my parents

William Robert Stevenson

and

Margaret Elizabeth Roys Stevenson

whose love and encouragement gave me the confidence to pursue my goals.

ACKNOWLEDGMENTS

My initial interest in the Fibonacci sequence and its connections with areas outside of mathematics came from a course in mathematics history at the University of Tennessee. The professor, William Wade, suggested "The Ubiquitous Fibonacci Sequence" as one topic among others for our assigned paper, and I chose that and became intrigued.

I wish to acknowledge the invaluable help given by colleagues at Maryville College. Among those whose assistance was particularly valuable are Bill Dent, John Nichols, Jerry Pietenpol, and Paul Warne of the Division of Mathematics and Computer Science; Amy Livingstone, Dan Klingensmith, and Chad Berry of the Division of Humanities; Carl Gombert and Sheri Matascik of the Division of Fine Arts; Sherry Kasper, Dean Boldon, and Mary Kay Sullivan of the Division of Social Sciences; Lori Schmied of the Division of Behavioral Sciences; and Roger Myers of Information Systems and Services. In addition to providing resources in their particular disciplines, these persons provided valuable personal encouragement. I especially acknowledge the inspiration of Dean Boldon, formerly Academic Vice President, who first suggested such a course for a Senior Seminar and provided much encouragement in my doctoral studies.

I thank the Appalachian College Association, whose financial support through their program of fellowships for faculty graduate study made possible the achievement of this degree.

Members of the Fibonacci Association who were encouraging and helpful in the study are Herta Freitag, Ron Knott, Calvin Long, Marjorie Johnson, and Piero Filipponi. Special thanks are owed to Marjorie Johnson who kindly gave permission for reproducing her article on the history of the Fibonacci Association journal. Courtney Lix, a high school student from Gatlinburg, Tennessee, was a valuable resource.

The contribution of John Ribble's cover design for the student textbook is clearly worthy of mention. A graphic designer in Madison, Wisconsin, John is a much-loved member of my family. Joyce McCroskey, my "oldest" friend, contributed interest, enthusiasm, and a sculpture of Leonardo of Pisa.

Finally I must mention the members of my doctoral committee. These persons always made me feel that they were on my side. They went above and beyond the required responsibilities to help me produce this study, particularly Carl Wagner of the Mathematics Department. I thank them all for their unfailing encouragement and assistance: Donald Dessart, Stephanie Robinson, Karl Jost, and Carl Wagner.

ABSTRACT

The purpose of this study was to design, teach, and evaluate an undergraduate interdisciplinary mathematics course based on certain patterns, primarily the Fibonacci sequence. Rationale for the course includes the benefits of connected learning and the scarcity of liberal arts courses based on mathematics. The course is intended to emphasize pattern exploration in mathematics as well as in other disciplines. It is hoped that students in the course will find connections between mathematics and history, art, architecture, music, literature, nature, and economics.

Course design includes a syllabus, student textbook, and sample lesson plans. The student textbook explores mathematical connections with the Fibonacci sequence such as the golden ratio, Pascal's triangle, Pythagorean triples, combinatorics, and fractal geometry. Historical background of Leonardo Fibonacci's life and times in the High Middle Ages is used to introduce the course. Applications of Fibonacci numbers in art, architecture, music, literature, nature, and economics are discussed. Students are asked to assess the meaning of these connections in light of their liberal arts experience.

Evaluation of the course, primarily qualitative in nature, gives evidence that the pilot offering of the course enabled students to see relationships between various fields of study in a new way.

TABLE OF CONTENTS

PART I

INTRODUCTION

PART I

INTRODUCTION

The famous "Fibonacci sequence" is a source of great interest for mathematicians, both because of its number theoretic properties and because of its connections with other disciplines such as art and music. A look at its origin in Fibonacci's writing promotes interest in the history of the Middle Ages as well as the history of mathematics. For these reasons, it seems that an interdisciplinary study for undergraduates in a liberal arts setting could be built around this and other mathematical patterns.

Purpose

The purpose of this study is to design an interdisciplinary mathematics course based on the Fibonacci sequence and to determine whether the course improves an individual's ability to make connections between mathematics and other disciplines. The original idea for such a course came as a result of 1996 revisions in the general education curriculum at Maryville College, an undergraduate liberal arts institution located in Maryville, Tennessee. The new curriculum provided for, among other changes, a senior capstone course designed to integrate the various disciplines included in a student's four-year experience. The author believed that mathematical patterns, particularly those found in the Fibonacci sequence, would provide the basis for a useful "Senior Seminar" offering. Guidelines for the senior seminar included the following goals:

1. The creative and critical exercise of the scientific, artistic, and humanistic modes of inquiry, and their integration.
2. Oral communication skills that enable effective comprehension, analysis, and expression.
3. A sense of wonder, curiosity, and a willingness to explore.
4. Global perspective that draws on an understanding of Western and other cultures, including cultures very different from one's own.

The course description and modes of delivery given are as follows: "This course should provide the student with the skills and the opportunity to integrate across at least two modes of inquiry. The course is interdisciplinary in nature. It should follow a thematic approach that examines topics from across two divisions and includes global perspectives. Assignments should include use of primary sources, such as texts, films, and art works. Also, the course should provide the student with opportunities to refine oral communication skills beyond classroom discussion. While some offerings could be developed by teams, the expectation is that individual faculty will model the integration of modes of inquiry." (*Guidelines for Senior Seminar at Maryville College*)

Because of the connections to art, music, literature, nature, and technology found in the Fibonacci sequence, this number pattern seemed a natural basis for such a course. Fibonacci himself is considered by many to be the greatest mathematician of the Middle Ages, and his contributions to mathematics and culture of that time provide global perspective for such a study. The course "Finding Fibonacci," described in Part II of this study, was designed to be proposed as a Senior Seminar in the spring of 2000.

The overall goal of this course is to improve an individual's ability to make connections between mathematics and other disciplines. Four additional goals, related to the Senior Seminar goals, are listed as follows:

(1) Willingness to explore mathematical patterns and to find them in the arts, humanities, natural sciences, and social sciences.

(2) Oral communication skills that enable effective comprehension, analysis, and expression.

(3) The integration of the scientific, artistic, and humanistic modes of inquiry.

(4) Increased interest and fluency in mathematics.

In addition to these stated goals, the author hoped the course would (1) encourage skepticism as well as wonder in searching for patterns in nature and the arts; and (2) alleviate mathematics anxiety if it existed.

Rationale

Historically, mathematics has been central to the liberal arts: the *quadrivium* consisted of arithmetic, geometry, astronomy, and music. Together with the *trivium* (logic, grammar, and rhetoric), these subjects made up the classical liberal arts curriculum. Connections between mathematics and philosophy are well-known, and mathematics obviously provides the basis for many scientific discoveries. Therefore it is persuasive for an interdisciplinary mathematics course to have an important place in the curriculum of a liberal arts college.

One goal of the liberal arts curriculum is connected learning. According to Harlan Cleveland (quoted in Gaff, 1991, p. 52), integration " 'is what is higher about higher education.'" The National Council of Teachers of Mathematics, in its *Curriculum and Evaluation Standards for School Mathematics* (1989), advocates "...a curriculum for all that includes a broad range of content, a variety of contexts, and deliberate connections" (p. 255).

Reuben Hersh (1990) recommends teaching mathematics using an "open, humanistic approach—that concentrates on where mathematics comes from...." He raises the question: "Is mathematics an arcane technical specialty, unrelated to history, philosophy, literature, or art? Is each mathematical subject a self-contained, static, timeless structure, with no meaning or value outside itself?" The obvious answers to these questions underscore the importance of a study relating mathematics to other areas of life.

The importance of connections is described by Newman and Boles (1992): "In this world of overspecialization, much of education deals with discrete bits of information rather than large systems. People, therefore, are not trained to find connections. Without connections, value systems are difficult to develop. In the evolution of civilization, Art and Mathematics are disciplines that have been seen as polarities without connection. Yet, in fact, they are the left and right hand of cultural advance: one is the realm of metaphor, the other, the realm of logic. Our humanness depends upon a place for the fusion of fact and fancy, emotion and reason. Their union allows the human spirit freedom." (p. xiv).

Methodology

Design of the course first involved writing a student textbook. This was primarily historical research, collecting and integrating information about Fibonacci and his sequence as it is found in various fields. In compiling the textbook, the author assimilated information from the sources listed at the end of each section. Full references for these sources are given at the end of the textbook. Lesson plans were created using resources available to mathematics educators.

In order to evaluate the effectiveness of such a course, the author made arrangements with her employer, Maryville College, to pilot it as an experiential offering in the spring of 1999. Enrollees were informed that their responses to certain aspects of the course would be used for evaluation of the course and that confidentiality would be preserved. The textbook, lesson plans, and activities described in Part II of this study were used.

At the end of the course, responses to course evaluation forms, pre- and post-course questions, and other assigned writings were analyzed to determine to what extent the course improved an individual's ability to make connections between mathematics and other disciplines. In addition, the four course goals were assessed through student responses. These results are summarized in Part III.

Certain mathematics problems arose for the author herself to investigate during the course of the writing. This served as a model for the students who were expected to discover and verify number patterns. These "Personal Case Studies" are included in Appendix A. One of these arose from a conjecture by two students in the class.

The author consulted with Dr. Herta Freitag of Roanoke, Virginia, a long-time member of the Fibonacci Association. Her mathematics career as well as her life story became an intriguing aspect of the study. A biographical sketch is therefore included in Appendix C. As a result of this friendship, the author attended the 8^{th} International Conference in Fibonacci Numbers and Their Applications in Rochester, New York, in June 1998. She there became acquainted with several long-time members of the Association such as Marjorie Bicknell-Johnson, Piero Filipponi, and Calvin Long. Interest in the Association and its quarterly journal resulted in inclusion in this study of Ms. Bicknell-Johnson's history of the Fibonacci Journal in Appendix B. Conference attendance provided other enrichment for the study, including acquaintance and many conversations with Ron Knott of the University of Surrey, author of a comprehensive web page on Fibonacci numbers.

Appendix D lists qualitative data obtained from course evaluation forms and pre- and post-course questions. Excerpts from these are included in Part III.

PART II

THE COURSE

SYLLABUS

SYLLABUS

Course: "Fun with Fibonacci" (3 hours experiential credit)

Instructor: Ms. Margie Ribble

Textbook: *Finding Fibonacci,* by M. Ribble ($15)

Description: Students will explore and connect simple mathematical patterns found in the Fibonacci sequence, golden ratio, and Pascal triangle; will discover how these patterns are found in other areas; and will investigate the historical context of these mathematical discoveries. Patterns in art, architecture, music, literature, nature, economics, and technology will be specifically targeted. Students will be expected to prepare an oral presentation on a related topic of interest.

Schedule: Tuesdays, 6-9 p.m., January 5-March 2; Saturday, February 27, 1999

Prerequisite: Statistics 120

Course goals:

(1) Willingness to explore mathematical patterns and to find them in the arts, humanities, natural sciences, and social sciences.

(2) Oral communication skills that enable effective comprehension, analysis, and expression.

(3) The integration of the scientific, artistic, and humanistic modes of inquiry.

(4) Increased interest and fluency in mathematics.

Grading: Each student will prepare a 15-20 minute oral and written presentation on a topic of interest from the course (suggestions for topics will be given in textbook).
Each student will keep a portfolio of assignments and writings.
Each student will be expected to attend class and participate in classroom activities.

Grades will be based on:	40%	presentation
	40%	portfolio
	20%	attendance and participation

STUDENT TEXTBOOK


Finding
FIBONACCI

Finding Fibonacci

an exploration of connections between mathematics, history, art, music, literature, nature, and economics

by Margie Ribble

cover design by John Ribble

Table of Contents

Introduction

This student guide provides readings, assignments, and project suggestions for the course "Fun With Fibonacci." The Historical Background Unit places Fibonacci's work in the context of his time. Students will read and discuss life in the Middle Ages and will summarize mathematical discoveries prior to 1200 C.E. The Mathematics Unit presents concepts in number theory and geometry related to the Fibonacci sequence and the golden ratio. Succeeding units will demonstrate and encourage students to discover mathematical patterns, primarily Fibonacci numbers and the golden ratio, in art, architecture, music, nature, literature, and economics.

Sources given at the end of each section are those used by the author and may also be used by students in further research for topic presentations. Complete references are given in the "Sources" section at the end of the textbook, pages 1-S through 7-S.

Unit 1

Historical Background

1.1 Life in the Middle Ages

Before we look at the contributions Fibonacci made to mathematics, we will examine his life in the context of the times in which he lived. The period from 1050-1300 A.D. (now C.E.—Common Era) in Europe is often called the High Middle Ages. The time was characterized by relative stability following the turmoil of the previous few centuries. The thirteenth century, in particular, was the most prosperous Europe had known since the fall of the Roman Empire. New energy for discovery in political, economic, and scientific areas was evident. Several factors contributed to the growth and cultural revival during the High Middle Ages.

One of the most important developments was the growth of cities and towns. During the previous few centuries, people had clustered around castles for protection from warring Germanic tribes and from rival feudal monarchies. With the increased political stability of the High Middle Ages, more people were able to live in towns and were not as preoccupied with protection and safety. A new middle class of merchants and tradesmen grew up. Cities around the Mediterranean were particularly prominent because of access to shipping trade. Genoa, Pisa, Venice, Milan, and Florence, all in what is now Italy, were independent republics. Paris became the largest city in Europe.

One contributing factor in the growth of these cities was increased trade and commerce throughout Europe. What had been primarily a rural barter economy was becoming more of an urban mercantile system. Italian merchant-bankers were active in trade with most of Europe and north Africa. Spices from the Far East were the most desirable commodities; these were used for seasoning, but were also important for medicines, cosmetics, and food preservation. Such cargoes were transported by ship from the East to Syria or to the Red Sea and then overland to Mediterranean ports where they could easily be taken by ship to Pisa, Venice, and Genoa. This brought increased contact between Europeans and peoples of Arab countries and created new interest in Arab culture and learning.

The Crusades, which took place during the High Middle Ages, also contributed to renewed interest in travel to the Middle East. Idealistically conceived as "Holy Wars" to free Jerusalem from the Muslim infidels, these pilgrimages nevertheless were fueled by mixed motives, including materialistic ones. Ironically, Jerusalem was briefly taken in 1197 by Frederick II, who had been excommunicated by Pope Gregory IX. In any case, the Crusades helped to make Europeans aware of the advanced civilizations of the Arab countries.

In addition to becoming important commercial centers, Italian cities were advanced in cultural and educational areas. The first medical school was established in Salerno in

In addition to becoming important commercial centers, Italian cities were advanced in cultural and educational areas. The first medical school was established in Salerno in southern Italy. Early in the twelfth century the first university was founded in Bologna. Cathedral schools became filled due to renewed interest in learning, and other universities were soon established at Oxford, Padua, Naples, and Cambridge.

The power of European kings during this period depended largely on their individual personalities. The Christian Church, headed by the Pope in Rome and a strong hierarchical structure, was an important influence on people and governments. Monasteries were cloistered centers of learning in which copying and studying Greek and Roman manuscripts was a primary activity. However, during the 12^{th} and 13^{th} centuries monastic orders became more involved in the lives of the people. There was new interest in human aspects of Christianity such as the Nativity and the Virgin Mary. Cathedrals and monasteries were the most important institutions and structures in the towns and cities.

In southern Europe most cathedrals were Romanesque in style, reflecting both Roman and Arab influences. Among characteristics of this style are round arches, massive construction in masonry, and heavy moldings. The cathedral in Pisa, on whose grounds the "Leaning Tower" was constructed, is a good example of Romanesque architecture. However, the High Middle Ages is probably best known today for Gothic architecture which began in France. This style emphasized light and color and was intended to carry the eye upward toward heaven. Pointed arches and elaborate patterns of vaulting characterized Gothic architecture. Stained glass and sculpture told Biblical stories. Many of these buildings are still standing and still magnificent, such as the Cathedral of Notre Dame in Paris. These two architectural styles, Romanesque and Gothic, influenced painting and sculpture as well.

Music of the High Middle Ages that has been preserved was primarily church music, because only clerics were musically literate. Most church music was monophonic or "plainchant," that, is, sung in unison without accompaniment; however it could be varied by being sung antiphonally. Some of the finest music was written by a woman, Hildegard of Bingen (1098-1179), who was a religious visionary, poet, and musician. Without doubt the secular life was enriched with music as well—seasonal peasant celebrations, court ceremonies, and daily rituals—although these songs were not written down.

In literature, Latin was still the predominant language, but the languages of the people began to find their way into some writings. Love lyrics and courtly romance tales promoted the idea of romantic love in such literature as well as in music and art of the time. Marriage remained mostly an arranged institution, however, with husbands often keeping mistresses. The most notable literary work was Dante's *Divine Comedy*, written in the Florentine dialect. The beauty of Dante's verse eventually made this dialect the Italian language.

and re-stated. It was believed that administration of justice was the chief function of government, and the Church, which supported justice as a virtue, upheld this belief. Juries were called to make decisions, and evidence was examined in court.

By the fourteenth century, several disasters brought an end to this period of growth and prosperity. The Black Death, an epidemic of bubonic and pneumonic plague, killed 25 to 45 percent of the population of Europe. The Great Schism of 1379 left Europe with rival popes for many years. The Hundred Years War between France and England caused the decline of France's prominence on the continent.

It is clear why the time in which Fibonacci lived is often called the "The Renaissance of the Twelfth Century." Stability and relative prosperity brought increased interest in culture and learning throughout Europe, but particularly in the cities of the Italian peninsula.

Sources:

Gies & Gies (1969).

Hollister (1972).

LaMonte (1949).

Lucie-Smith (1992).

Microsoft Encarta Encyclopedia (1996).

Strayer (1955).

Van Doren (1991).

1.2 Mathematics History up to the Middle Ages

Our look at the history of mathematics to 1200 will focus on major advances and a few outstanding individuals who made significant contributions to its development. There are, of course, many other persons, known and unknown, to whom progress could be attributed.

Earliest civilizations grew up along the major river valleys in Africa and Asia. The Babylonians, Chinese, and Egyptians are examples of those civilizations which were most advanced in mathematics as well as in other areas. These early peoples used mathematics mostly for counting and measuring. However, evidence has been found as early as 1600 B.C.E. of some quadratic and cubic equations as well as calculations of pi and use of the 3-4-5 triangles to construct right angles.

Greek civilization, beginning in about 500 B.C.E., used Babylonian and Egyptian discoveries in mathematics. One of the early outstanding mathematicians was **Pythagoras**, born about 572 B.C.E. Interestingly, he was a contemporary of Confucius, Buddha, and Lao Tze. He settled in Crotona, in southern Italy, where he founded the famous Pythagorean school. Concerned with the study of philosophy, mathematics, and science, the school was somewhat of a secret society or brotherhood. The basic premise of the school was that whole number is the basis of matter: "All is number" was their motto. This led to what later became the *quadrivium*: arithmetic, geometry, music, and astronomy, the fundamental liberal arts. The Pythagoreans are known for number theory and numerology, and they also worked on the problem of irrational numbers. It is unclear whether Pythagoras actually proved the theorem that bears his name, though proofs dating back 1000 years before his time are known. In fact, there are probably 400 different proofs of the Pythagorean Theorem.

The city of Alexandria was founded in 332 B.C.E. by Alexander the Great, and it was a significant cultural and educational center for the next 700 years. Located in northern Egypt, it was part of Alexander's Macedonian Empire. Following Alexander's death, his empire was divided and Ptolemy took over Egypt and established Alexandria as his capital. He erected the famous University of Alexandria which opened in about 300 B.C.E. He recruited talented men from Athens to staff his university. The university became such a center of mathematical study that almost every mathematician of note during this period of history was associated with Alexandria.

Among Ptolemy's recruits from Athens was the mathematician **Euclid**. Though little is known about his life, one of his writings, the *Elements*, has dominated the teaching of geometry for more than 2000 years. No other work except the Bible has been more widely studied. The *Elements* contains 13 books on the subjects of geometry, number theory, and geometric algebra. More significant than the content is the logical order of his axioms, postulates, and propositions. It is truly a deductive system, showing that each statement follows logically from a previous statement or from preliminary definitions.

The greatest mathematician of antiquity, according to Eves, was **Archimedes** (287-212 B.C.E.). He worked on the problem of calculating pi, showed an early attempt at using integral calculus, and developed geometry. Much of his writing dealt with physics as well as mathematics, and he is known for the quote: "Give me a place to stand on and I will move the earth." You will also remember the story of how he figured out, while taking a bath, that the weight of the water displaced by a body is equal to the weight of the body. He reportedly leapt from the bath and went running naked down the street shouting "Eureka, Eureka!"

Diophantus of Alexandria, whose dates are not exact but probably lived about 250 C.E., is sometimes known as the "Father of Algebra." He used symbols to represent squares, cubes, and unknown numbers. The following riddle is attributed to him:

> *God granted him to be a boy for the sixth part of his life, and adding a twelfth part to this, He clothed his cheeks with down; He lit him the light of wedlock after a seventh part, and five years after his marriage He granted him a son. Alas! late-born wretched child; after attaining the measure of half his father's life, chill Fate took him. After consoling his grief by this science of numbers for four years he ended his life* (Boyer, quoting Cohen and Drabkin, 1958). How old was he when he died?

His major work, the *Arithmetica*, included an assortment of 189 problems and their solutions. The name "Diophantine equations" was later given to a problem with several unknown quantities for which one is interested in integer solutions.

These developments took place in the civilizations around the Mediterranean, from which most of our Western thought can be traced. However, civilization in China flourished even before Greek and Roman times. By 1400 B.C.E. the Chinese had a positional number system with nine symbols. Fewer records of early Chinese mathematics exist today than those of Egyptians and Babylonians, largely because of the difference in climate which affected preservation of written records. Astronomy was of great interest to the Chinese, and inspired much of their mathematical discovery. The oldest arithmetic textbook in existence is the *Nine Chapters on the Mathematical Art*, dating from about 150 B.C.E. Chinese mathematics was more concerned with number and algebra than with geometry, although the oldest known proof of the Pythagorean Theorem is found in a Chinese work, *Arithmetic Classic of the Gnomon and the Circular Paths of Heaven.* During the period known in Europe known as the Dark Ages, Chinese civilization and mathematics flourished, but most of the mathematics was of a practical rather than theoretical nature.

Following the fall of the Greek and Roman Empires, mathematical study in Europe was maintained chiefly in monasteries. Very little new mathematics was accomplished during the next 500 years, with the possible exception of development of the Christian calendar. However, Hindu and Arabian mathematics were thriving during this time, though historical records of their development are scant. Hindu mathematics was undoubtedly influenced by Greek, Babylonian, and Chinese mathematics. Astronomy was the predominant theme; in fact, there was very little of a pure mathematical nature. There was no algebraic symbolism, so problems and solutions were written out in flowery language. **Aryabhata** and **Brahmagupta**, who lived during the fifth through seventh centuries about 100 years apart, contributed greatly to the study of Diophantine equations.

After Mohammed's flight in 622 the Moslem countries became a powerful world force. During the ensuing centuries, many classical Greek works in mathematics and astronomy were translated into Arabic and thus preserved. Among important Arab mathematicians was **Al-Khowarizmi** in the 9th century. It is his name that gave us the word "algorithm," and the word "algebra" came from his treatise *Al-jabr wa'l muqabalah* The number system used in Hindu and Arab countries had obvious advantages over the European Roman numerals.

By about the 10th century, Greek learning began to come into Europe through Christians travelling to Moslem centers of learning. Trade with Arab countries helped with transmission of learning, and important European trade centers of the time were Genoa, Pisa, Venice, Milan, and Florence. The stage is set for the arrival of our hero, **Leonardo of Pisa**, otherwise known as "Fibonacci."

Assignments:

1. Solve Diophantus' riddle.

2. Write a short (one page) summary of one of the following:

   Hypatia
   Euclid's *Elements*
   Apollonius of Perga
   Aristarchus of Samos
   Menelaus of Alexandria
   Boethius
   Chinese mathematics before 1200 A.D.
   Brahmagupta
   Omar Khayyam

Sources:

Boyer (1991).

Burton (1991).

Eves (1964).

Hollister (1974).

Microsoft Encarta Encyclopedia (1996)

Van Doren (1991).

1.3 Leonardo of Pisa

Leonardo of Pisa, also known as Fibonacci ("son of Bonaccio"), is considered the greatest mathematician of the Middle Ages. He lived during the time construction began on the famous "Leaning Tower" of Pisa—approximately 1175-1250 A.D. During his youth he traveled widely around the Mediterranean and was educated in North Africa by Muslim teachers.

There were two classes of mathematicians in the later Middle Ages: those involved in commerce and those in the churches or universities. The mathematics required for trade and commerce was of course more practical in nature, concerned with units of measurement and monetary units. In the universities, mathematicians studied the traditional "liberal arts," consisting of the Trivium (grammar, rhetoric, and dialectic) and the Quadrivium (arithmetic, geometry, music, and astronomy). These had not changed appreciably since the time of Plato and Euclid.

Through his studies and travels, Leonardo learned from the mathematics of the Arabs. He saw that the Hindu-Arabic number system was much more efficient than the Roman numerals used in Europe. This system, as it had evolved at that time, was a place value, positional notation system consisting of ten symbols including a symbol for zero. The value of, say a "7", depended on its "place" in the numeral: 73 means 7 tens and 3 ones, whereas 37 means 3 tens and 7 ones. This seems perfectly logical to us now, but it was new to Europe in the Middle Ages.

In 1202, Leonardo published his *Liber Abaci*, which literally means "A Book About the Abacus" or "A Book About Counting." Unlike other mathematical treatises of the time, it was more concerned with number than with geometry; however in the introduction to the work, Leonardo maintained that arithmetic and geometry are connected and support each other. One purpose of this writing was to show Europeans the advantages of the Hindu-Arabic number system and how efficient it could be in solving number problems. It is widely believed that the second edition of *Liber Abaci*, 1228, was influential in the spread of this numeral system in Europe. The chief objection to the new system was non-standard appearance of some of the digits, and the fact that it was easy to cheat by making a 9 or a 6 look like a 0. Roman numerals were used until about 1550 in many monasteries in Europe, but the invention of printing in the fifteenth century helped standardize the digits and spread the use of Hindu-Arabic numerals.

Most of the problems set forth in *Liber Abaci* are boring word problems concerned with number and money. We will try some of them in class. However, among these was the following problem about breeding rabbits:

> *How many pairs of rabbits will be produced in a year, beginning with a single pair, if in every month each pair bears a new pair which becomes productive from the second month on?*

This is a fairly unrealistic problem in that you have to assume that none of the rabbits die and that incest undoubtedly occurs. But as a mathematics problem it holds great interest. First let's clear up the breeding situation and say that at the end of the first month there is only one pair, the original pair, and at the end of the second month there is still only the original pair. But at the beginning of the third month a pair of babies is born. The parents continue to have pairs of babies at the beginning of each month, and the new pair of babies begins to have pairs of babies at the beginning of the fifth month. The table below summarizes the number of pairs of rabbits at the end of each month:

Months	Adult Pairs	Young Pairs	Total
1	1	0	1
2	1	0	1
3	1	1	2
4	1	2	3
5	2	3	5
6	3	5	8
7	5	8	13
8	8	13	21
9	13	21	34
10	21	34	55
11	34	55	89
12	55	89	144

This problem gave rise to the following sequence of numbers:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, . . .

in which each element in the sequence is the sum of the previous two elements. Although Leonardo probably had no idea of its importance, this sequence provides fascination for mathematicians, artists, musicians, botanists, and others. The formula for the nth Fibonacci number was first written by Albert Girard in 1634 in his work *L'Arithmetique de Simon Stevin de Bruges* (Burton, p. 265):

$$F_1 = F_2 = 1, \qquad F_n = F_{n-1} + F_{n-2} \quad \text{for all } n \geq 3.$$

This formula is therefore "recursive" in that each number can be obtained from previous numbers in the sequence after initial values are established; in fact, the Fibonacci numbers are the first known recursively defined sequence in mathematics. In the 19^{th} century, Edouard Lucas, a number theorist, gave this sequence the name "Fibonacci sequence." Numbers in the sequence are called "Fibonacci numbers."

Leonardo's other significant writing, *Liber Quadratorum* (Book of Squares), investigated diophantine equations of the second degree. One typical problem, which was presented to him by John of Palermo but which had earlier been investigated by Arab writers, required that he find a number for which increasing or decreasing its square by 5 would give a square as the result. No integer solutions can be found; however Leonardo found a rational solution.

In assessing Leonardo's place in history, we read:

"Fibonacci's work indicates a combination of inventive genius and a profound knowledge of earlier writers on mathematics" (Burton, p. 260).

"[*Liber Quadratorum*] marked him as the outstanding mathematician in this field between Diophantus and Fermat. These works [*Liber Abaci, Practica Geometriae,* and *Liber Quadratorum*] were beyond the abilities of most of the contemporary scholars" (Eves, p. 212).

"Leonardo of Pisa was without doubt the most original and most capable mathematician of the medieval Christian world, but much of his work was too advanced to be understood by his contemporaries. [In his writings] there are indeterminate problems reminiscent of Diophantus and determinate problems reminiscent of Euclid, the Arabs, and the Chinese" (Boyer, p. 256).

Exercises:

1. What is the 20^{th} Fibonacci number?

2. Write a computer program (or graphing calculator program) to generate the *n*th Fibonacci number using the recursive formula above.

3. Create a number system using 5 symbols (make them up). Write the numbers 10, 12, 37, and 43 in your new system.

4. Formulate John of Palermo's problem (see above) as a pair of simultaneous Diophantine equations and solve it, if possible.

Project suggestions:

Commerce during the Middle Ages

Unit fractions; how fractions were written in 1200

Lucas sequence

Sources:

Boyer (1991).

Burton (1991).

Eves (1964).

Gies & Gies (1969).

Problems from *Liber Abaci*:

1. A man entered an orchard through seven gates, and there took a certain number of apples, When he left the orchard he gave the first guard half the apples that he had and one apple more. To the second guard he gave half his remaining apples and one apple more. He did the same to each of the remaining five guards, and left the orchard with one apple. How many apples did he gather in the orchard? (Eves, p. 231)

2. Two birds start flying from the tops of two towers 50 feet apart; one tower is 30 feet high and the other 40 feet high. Starting at the same time and flying at the same rate, the birds reach a fountain between the bases of the towers at the same moment. How far is the fountain from each tower? (Burton, p. 262)

3. A merchant doing business in Lucca doubled his money there and then spent 12 denarii 12denarii. On leaving, he went to Florence, where he also doubled his money and spent 12 denarii. Returning home to Pisa, he there doubled his money and again spent 12 denarii, nothing remaining. How much did he have in the beginning? (Burton, p. 262; Gies, p. 101)

4. Three men, each having denarii, found a purse containing 23 denarii. The first man said to the second, "If I take this purse, I will have twice as much as you." The second said to the third, "If I take this purse, I will have three times as much as you." The third man said to the first, "If I take this purse, I will have four times as much as you." How many denarii did each man have? (Burton, p. 262) (In the Gies version, p. 103, the 23 is omitted from the first sentence).

5. A certain lion could eat a sheep in 4 hours, and a leopard could eat one in 5 hours, and a bear in 6 hours; how many hours would it take for them to devour a sheep if it were thrown in among them? (Gies, p. 101)

6. There were two men, of whom the first had 3 small loaves of bread and the other 2; they walked to a spring, where they sat down and ate; and a soldier joined them and shared their meal, each of the three men eating the same amount; and when all the bread was eaten, the soldier departed, leaving 5 bezants to pay for his meal. The first man accepted 3 of the bezants, since he had had 3 loaves; the other took the remaining 2 bezants for his 2 loaves. Was the division fair? (Gies, p. 102)

7. A man whose end was approaching summoned his sons and said: "Divide my money as I shall prescribe." To his eldest son, he said, "You are to have 1 bezant and a seventh of what is left." To his second son he said, "Take 2 bezants and a

seventh of what remains." To the third son, "You are to take 3 bezants and a seventh of what is left." Thus he gave each son 1 bezant more than the previous son and a seventh of what remained, and to the last son all that was left. After following their father's instructions with care, the sons found that they had shared their inheritance equally. How many sons were there, and how large was the estate? (Gies, p. 102)

8. A certain merchant sailed on a certain ship with 13 bales of wool of equal value, a second with 17 bales of the same value. When they arrived in port, the captain asked them for the charge they had agreed upon, but they did not have the cash to pay it. The first merchant said, "Accept 1 of my bales for the price of carrying the 13 bales, and give me back the change." The captain accepted, returning 10 *solidi* for the excess of the value of the bale over the charges for carrying 13 bales. When he collected the fare of the second man, he took one bale from him and returned 3 *solidi*. How much were the bales worth, and what was the shipping charge for each bale? (Gies, p. 102-103)

9. There are four men, of whom the first and the second and third together have 27 *denarii*; the second and the third and the fourth together have 31; the third and the fourth and the first have 34; and the fourth and the first and the second have 37. How much does each have? (Gies, p. 103)

10. Two ants are 100 paces apart, crawling back and forth along the same path. The first goes 1/3 pace forward a day and returns 1/4 pace, the other goes forward 1/5 pace and returns 1/6 pace. How many days before the first ant overtakes the second? (Gies, p. 104)

11. A certain person bought sparrows 3 for a *denarius* and turtledoves 2 for a *denarius* and pigeons for 2 *denarii* apiece; and he bought 30 birds for 30 *denarii*. How many birds of each kind did he buy? (Gies, p. 104)

12. If A gets from B 7 denarii, then A's sum is fivefold B's; if B gets from A 5 denarii, then B's sum is sevenfold A's. How much has each? (Eves, p. 230)

13. A certain king sent 30 men into his orchard to plant trees. If they could set out 1000 trees in 9 days, in how many days would 36 men set out 4400 trees? (Eves, p. 230)

Unit 2

Mathematics

2.1 Preliminary Considerations

In this unit we will review the mathematical concepts required to study the Fibonacci sequence. We will first review some basic notation.

The history of mathematics includes the development of mathematical symbols. We have already mentioned the Hindu-Arabic numerals and how they came to be used in the Western world. Other symbols, such as +, -, =, ( ), x, $\subset$, $\in$, π, were devised beginning in the 16th century. These symbols contribute to what we know as "symbolic algebra" and simplify the work of mathematicians.

We will be dealing with two "families" of numbers in this course. The first is the natural numbers, positive integers, or counting numbers. This set of numbers will be represented by P = {1, 2, 3, . . . }. The series of dots, "ellipsis," means that the numbers go on in that fashion forever.

The second set we will use is the non-negative integers or whole numbers. We will represent them by N = {0, 1, 2, . . . }. Obviously the only difference between these two sets is the inclusion of zero in the set N.

As in algebra you usually used x or y to stand for an unknown number, in this course we will use a lower-case n to represent a general number. This letter n is usually used when one is talking about whole numbers.

You are no doubt familiar with the use of subscripts. A subscript is a number or letter used for identification. For example, F_5 would represent the fifth Fibonacci number, and F_n represents the nth Fibonacci number. Subscripts are often called indices.

2.1.1 Summation Notation

An upper-case Greek letter sigma, Σ, tells you to sum a group of numbers. In statistics we use it in formulas such as that for the sample mean:

$$\frac{\sum x}{n} \quad \text{or} \quad \frac{\sum_{i=1}^{n} x_i}{n}$$

which means we sum (add up) all the x's (individual measurements) and divide the total by the number of measurements.

The summation symbol may be used as follows:

$$\sum_{i=1}^{5} i = 1 + 2 + 3 + 4 + 5 = 15$$

In this case the symbol tells us to add all the positive integers beginning with 1 and ending with 5. Can you think of a general formula for the sum of the first n positive integers?

A teacher once asked his students in their first arithmetic class to add the positive integers 1 to 100, thinking it would keep them busy for awhile and help them practice their sums. One of his young students was Karl Friedrich Gauss (1777-1855), who calculated the sum instantly. Gauss, who was an amazing prodigy, later told (according to Burton, p. 491) that he recognized the sum as pairs of sums equal to 101, such as 1+100, 2+99, etc. He then only had to note that there are 50 such pairs, so that 1 + … + 100 = (101)(50) = 5050.

The problem with this method of summing the first n positive integers is that, if you have an odd number of integers, there is an integer left over in the middle. But here's a way that works whether n is odd or even: Write the numbers twice, forwards and backwards, one underneath the other, and then sum the pairs:

1	2	3	4	5	6	7	8	9	10
10	9	8	7	6	5	4	3	2	1

You can see that there are 10 sums which each equal 11. But this list has each number twice, so we would divide by 10 to get the total of the numbers from 1 to 10. More generally, we would find by this method that the sum of the first n positive integers is

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

This formula can be verbalized in either of two ways: We can say that the sum of the first n positive integers is either (1) the sum of the first plus the last $(n + 1)$ times half the number of terms $\left(\frac{n}{2}\right)$; or (2) the average of the first and the last $\left(\frac{n+1}{2}\right)$ times the number of terms (n). As we will see in Section 2.2.1, these rules hold for summing the terms in any "arithmetic progression."

Here's another example:

$$\sum_{i=1}^{11}(2i-1) = 1+3+5+7+9+11+13+15+17+19+21 = ?$$

Notice that the above sum is a "perfect square." What are the numbers that are summed to get that perfect square? What would be the formula for the sum of the first *n* odd positive integers?

2.1.2 Factorial Notation

If *n* is an element of P, the symbol *n!*, read "*n* factorial" denotes the product of all positive integers between 1 and *n*. For example, 7! = 7 x 6 x 5 x 4 x 3 x 2 x 1 = 5040. A quick way to compute factorials is with the factorial key on your calculator. With graphing calculators, it is generally in the "math" menu.

We have only defined *n!* for $n \in P$, but it is possible to determine 0! as well. You can determine its value on your calculator. We will look at why 0! = 1 in the section on Binomial Coefficients.

2.1.3 Scientific Notation

Factorials can grow very large very fast. Try 14! in your calculator and you will see that the calculator prints the number in scientific notation. This is a method of writing very large or very small numbers which facilitates multiplying, dividing, and comparing them.

Most calculators show 14! as 8.71782912E10. This means $8.71782912 \times 10^{10}$ or 87,178,291,200. The calculator only has room in its display for 9 digits so it rounds off slightly at the end.

Recalling various principles from algebra such as the commutative property and properties of exponents, we know that 2×10^5 times $3 \times 10^7 = 2 \times 3 \times 10^5 \times 10^7 = 6 \times 10^{12}$. That seems to be easier and more accurate than writing out a whole string of zeros.

In scientific notation, there is only one digit to the left of the decimal point. The exponent counts the number of places from where the decimal is placed to the end of the number.

Very small numbers, like .0000025 and .00347 can also be written in scientific notation. Since .0000025 is the same as $\frac{2.5}{1000000}$, it can be written 2.5×10^{-6}. As with very large numbers, there is only one digit to the left of the decimal point; then one counts the number of places the decimal is moved to the right to find the negative exponent.

Exercises:

Write out these sums:

1. $\sum_{j=1}^{10} 2j$

2. $\sum_{i=2}^{6} i^2$

Find the value of each sum:

3. $\sum_{i=1}^{50} i$

4. $\sum_{i=1}^{73} i$

5. What is the sum of the first 1000 positive integers? the first 10,000?

6. Find 10!

7. What is 69!? 70!?

8. Change to scientific notation:

   (a) 3,875,240

   (b) 2,000

   (c) 58,000,000,000
   (d) .000347

   (e) .00000005

9. Write in "expanded form" (as a normal number):

   (a) 2.78423 E 7

   (b) 5.004×10^6

   (c) 2.1×10^{-5}

Sources:

Burton (1991).

2.2 Sequences and Series

A *sequence* of numbers is an ordered list of numbers, which may be finite or infinite. Some examples were seen on our "Search for a pattern" worksheet. An old-fashioned term for *sequence* is *progression*. The indicated sum of terms in a sequence is called a *series.*

The **Fibonacci** sequence is, of course, our primary example:

1, 1, 2, 3, 5, 8, 13, 21, 34, . . .

As we have seen, each term after the first two "initial terms" is the sum of the previous two terms; i.e., the sequence is defined *recursively.*

2.2.1 Arithmetic Sequences

A sequence in which there is a "common difference" between two consecutive terms is known as an arithmetic sequence. Examples are:

1, 5, 9, 13, 17, 21, . . . (the common difference is 4)

7, 20, 33, 46, 59, . . . (the common difference is 13)

In arithmetic sequences, each term after the first is gotten by adding the common difference, denoted by d, to the previous terms. So arithmetic sequences, like the Fibonacci sequence, can be recursively defined. But in the case of arithmetic sequences, it is easy to find a formula for the nth term if we know the first term and the common difference. In the first example above, the 6th term, 21, results from adding the common difference to the first term 5 times. We can derive a formula for the nth term as follows:

$$a_n = a_1 + (n-1)d$$

where d is the common difference. In other words, we multiply the common difference (d) by the number of times we add it to the first term (n – 1), and then add that product to the first term.

We can sum the terms of an arithmetic sequence in a way similar to the way we summed the first n positive integers, which is of course an arithmetic sequence. To sum the first 6 terms of the example used above, we would write the terms, then write them backwards:

1	5	9	13	17	21
21	17	13	9	5	1

Notice that each column adds to 22, which is the sum of the first and last terms. So we have 6 sums, each of which is 22, which means we multiply 6 times 22, then take half of that product (since we have added each number twice), and we have the sum of our six-term sequence, 66. We can now write a formula for the sum of *n* terms of an arithmetic sequence as follows:

$$2S_n = n(a_1 + a_n) \qquad or \qquad S_n = \frac{n(a_1 + a_n)}{2}$$

where S_n is the sum of *n* terms, of which a_1 is the first and a_n is the *n*th.

Without using the formula, find the sum of the first 10 odd positive integers; the sum of the first 10 even positive integers, and the first 10 multiples of 3. Check your answers by using the formula, first finding the 10th term.

2.2.2 Geometric Sequences

A sequence such as the following:

2, 10, 50, 250, 1250, . . .

in which there is a common "multiplier" between consecutive terms is called a geometric sequence. The common multiplier is generally called a "common ratio" and its symbol is *r*. In a geometric sequence, each term after the first is gotten by multiplying the previous term by *r*. So geometric sequences are also recursively defined. As in the case of arithmetic sequences, we can find a formula for the *n*th term of a geometric sequence. Note that the geometric sequence above can be written

$$2,\ 2\cdot 5,\ 2\cdot 5^2,\ 2\cdot 5^3,\ 2\cdot 5^4 \ldots$$

More generally, the *n*th term (g_n) of a geometric sequence with first term g_1 and common ratio *r*, is given by the formula

$$g_n = g_1 r^{n-1}$$

The indicated sum of a geometric sequence is called a geometric series. To find the sum of *n* terms of a geometric sequence, you can use the following procedure:

(1) Write the series: $S_n = g_1 + rg_1 + r^2 g_1 + \cdots + r^{n-1} g_1$

(2) Multiply by r: $rS_n = rg_1 + r^2 g_1 + r^3 g_2 + \cdots + r^n g_1$

Subtracting (2) from (1), we have

$$S_n - rS_n = g_1 + 0 + 0 + 0 + \cdots - r^n g_1 \quad \text{or}$$

$$(1-r)S_n = g_1 - r^n g_1$$

So now $S_n = \dfrac{g_1(1-r^n)}{(1-r)}$ provided $r \neq 1$

The sums of the sequences in the examples used here get larger and larger when more terms are added. It appears that the sum of an infinite number of terms is an infinitely large number. However, let's look at a geometric sequence in which r is between 0 and 1 or between 0 and –1 ($-1 < r < +1$). For example, $r = ½$, $g_1 = 3$. Here are the first five terms of the sequence:

$$g_1 = 3$$

$$g_2 = \left(\frac{1}{2}\right) \cdot 3 = \frac{3}{2}$$

$$g_3 = \left(\frac{1}{2}\right)^2 \cdot 3 = \frac{3}{4}$$

$$g_4 = \left(\frac{1}{2}\right)^3 \cdot 3 = \frac{3}{8}$$

$$g_5 = \left(\frac{1}{2}\right)^4 \cdot 3 = \frac{3}{16}$$

It is clear that the terms get successively smaller and eventually get very close to zero (e.g., the 10^{th} term would be .0029…). What would the 100^{th} term be? Now if we're adding terms, these would be negligible. Is it possible that the sum "approaches" some number as n gets very large, instead of continuing to increase?

Let's substitute our values for r and g_1 in the formula for the sum of the sequence given above, and suppose that n is a very large number:

$$S_n = \frac{g_1(1-r^n)}{1-r} = \frac{3(1-(\frac{1}{2})^n)}{1-\frac{1}{2}} = \frac{3(1-0)}{\frac{1}{2}} = 6 \text{ when } n \text{ is a very large number.}$$

In calculus we would say that the sum of this sequence approaches 6 as its "limit" as n approaches infinity.

2.2.3 Fibonacci, Lucas, and Tribonacci Sequences

As we have seen, the Fibonacci sequence results when the first two terms are defined, and succeeding terms result from summing the previous two terms:

$$F_1 = 1 \qquad F_2 = 1 \qquad F_n = F_{n-1} + F_{n-2}$$

The **Lucas** sequence, 1, 3, 4, 7, 11, 18, 29, . . . , is similar to the Fibonacci sequence in that each term after the initial terms is the sum of the previous two terms. However, the initial terms are 1 and 3 in this case.

Another recursive sequence is called the **Tribonacci** sequence. The initial three terms are 1, 1, and 2, and then each term is the sum of the previous three terms:

1, 1, 2, 4, 7, 13, 24, 44, 81 . . .

2.2.4 Formula for the *n*th Fibonacci number

We have seen a number of examples of sequences which are recursively defined; i.e., arithmetic, geometric, Fibonacci, Lucas, and Tribonacci. In the case of arithmetic and geometric sequences, we have found a formula for the *n*th term (also called a "closed form expression" for the *n*th term). We now find a closed form expression for F_n using the method of generating functions:

Let:

(1) $$F(x) = F_1 + F_2x + F_3x^2 + \cdots + F_nx^{n-1} + F_{n+1}x^n + \cdots$$

Now multiply (1) by x:

(2) $$xF(x) = F_1x + F_2x^2 + \cdots + F_{n-1}x^{n-1} + F_nx^n + \cdots$$

and multiply by x again:

(3) $x^2F(x) = F_1x^2 + F_2x^3 + \cdots + F_{n-2}x^{n-1} + F_{n-1}x^n + F_nx^{n+1} + \cdots$

Now, subtracting (2) and (3) from (1), we have

$$F(x) - xF(x) - x^2F(x) = F_1 + [F_2 - F_1]x + [F_3 - F_2 - F_1]x^2 + \cdots + [F_n - F_{n-1} - F_{n-2}]x^{n-1} + \cdots$$

We know that $F_2 - F_1 = 1 - 1 = 0$, and, in general, $F_n - F_{n-1} - F_{n-2} = 0$ because of our recurrence relation. So,

$(1 - x - x^2)F(x) = 1$, and $F(x) = \dfrac{1}{1 - x - x^2}$.

Using some messy algebra (including partial fractions), it can be shown that

$\dfrac{1}{1 - x - x^2} = \dfrac{A}{1 - \alpha x} + \dfrac{B}{1 - \beta x}$ where $\alpha = \dfrac{1 + \sqrt{5}}{2}$ and $\beta = \dfrac{1 - \sqrt{5}}{2}$

while $A = \dfrac{1}{\sqrt{5}}$ and $B = -\dfrac{1}{\sqrt{5}}$.

For details of this method and derivation, see Wagner, pp. 4-5, 86.

The result is a "closed form expression" for the nth Fibonacci number:

(4) $F_n = \dfrac{1}{\sqrt{5}}\left(\dfrac{1 + \sqrt{5}}{2}\right)^n - \dfrac{1}{\sqrt{5}}\left(\dfrac{1 - \sqrt{5}}{2}\right)^n$

This formula was derived by both Abraham DeMoivre (1667-1754) and Daniel Bernoulli (1700-1782). It is clearly difficult to calculate by hand for even small values of n, but we will store parts of it in our calculators to simplify the process as follows:

Store $\dfrac{1}{\sqrt{5}}$ as X. Then store $\dfrac{(1 + \sqrt{5})}{2}$ as A, and $\dfrac{(1 - \sqrt{5})}{2}$ as B.
(Be sure to put the numerators of A and B in parentheses.)

Now we can find the value of F_n as follows:

(5) $F_n = X * A^n - X * B^n$

Exercises:

1. Find the indicated term:

   (a) 11, 27, 43, 59, 75 (fiftieth term)

   (b) 3, 21, 147, 1029, 7203 (ninth term)

2. Find the sum of the first n terms:

   (a) 25, 44, 63, 82, 101 $(n = 14)$

   (b) 5, 10, 20, 40, 80 $(n = 20)$

3. One penny is put on the first square of a 64-square checkerboard, two pennies on the second square, four pennies on the third square, and so on.

   (a) How much money will be on the 64th square?

   (b) How much money will be on the checkerboard?

4. A creature from Mars lands on Earth. It reproduces itself by dividing into three new creatures each day. How many creatures will populate Earth after 30 days if there is one creature on the first day?

5. Using the formula given above (5), find the 10th Fibonacci number; the 20th; the 40th.

6. Find the sum of the infinite geometric sequence when $g_1 = 12$ and $r = .9$.

Sources:

Conway & Guy (1996).

Sgroi & Sgroi (1993).

Wagner (1996).

Wise, Nation & Crampton (1990).

2.3 Binomial Coefficients

Another symbol with which you are probably already familiar is the symbol for the number of different ways you can choose *k* objects from a set of *n* objects, without regard to order, or in other words, the number of *k*-element subsets of an n-element set. We will call this "*n* choose *k*" and write it $\binom{n}{k}$.

Here is an example:

Suppose a set *S* consists of elements *a, b, c,* and *d*. We would write $S = \{a, b, c, d\}$.

(1) The only subset of *S* with zero elements is the empty set (∅), so $\binom{4}{0} = 1$.

(2) The 1-element sets of *S* are {1}, {2}, {3}, and {4}. So $\binom{4}{1} = 4$.

(3) The 2-element sets of *S* are {1, 2}, {1, 3}, {1, 4}, {2, 3}, {2, 4}, and {3, 4}. So $\binom{4}{2} = 6$.

(4) The 3-element sets of *S* are {1, 2, 3}, {1, 2, 4}, {1, 3, 4}, and {2, 3, 4}. So $\binom{4}{3} = 4$.

(5) And there is only one 4-element set of *S*, the set *S* itself. So $\binom{4}{4} = 1$.

The numerical value of "*n* choose *k*" is as follows: $\dfrac{n!}{k!(n-k)!}$

Your calculator can give you combinations and factorials very simply. Your instructor will demonstrate how this can be done.

What is the logic behind this formula? If we were choosing a president, vice-president, and secretary for a class of 30 people, there would be 30 possibilities for president, then 29 for vice-president (we've already named one as president), then 28 for secretary. For each choice for president, there are 29 choices for vice-president, so clearly you would multiply 30 times 29 to get the number of possible president/vice-president teams. This is called the "multiplication rule." So there are (30)(29)(28) or 24,360 different slates of officers (president, vice-president, and secretary) that are theoretically possible. This is equal to $\dfrac{30!}{27!}$ *or* $\dfrac{30!}{(30-3)!}$. Now, if instead of 3 distinct officers we were to elect a committee of 3 people from a class of 30, there would be fewer possibilities. There are clearly 6 ways the same group of 3 people could be officers as follows:

	President	Vice Pres.	Secretary
(1)	Ann	Ben	Carl
(2)	Ann	Carl	Ben
(3)	Ben	Carl	Ann
(4)	Ben	Ann	Carl
(5)	Carl	Ben	Ann
(6)	Carl	Ann	Ben

There are (from these 3 people) 3 possibilities for president, then 2 for vice-president, then 1 for secretary--which (again using the multiplication rule) gives us 3! or 6. So we would divide the number of slates by 3! to get the number of possible committees. Hence the formula for the number of committees of 3 people from a class of 30 people:
$\binom{30}{3} = \frac{30!}{3!(30-3)!} = 4060$, considerably less than the number of possibilities for slates of officers.

Another example to help clarify this concept is as follows: A pizza parlor offers 10 different toppings: pepperoni, sausage, ham, green olives, black olives, peppers, mushrooms, onions, anchovies, and extra cheese. The order in which they are put on the pizza is not important. How many different kinds of pizza are possible?

The answer can be calculated this way: there is one kind of pizza with no toppings, and one kind with all ten toppings, and combinations of 2, 3, 4, 5, 6, 7, 8, and 9 toppings. The resulting total would look like:

$$\binom{10}{0}+\binom{10}{1}+\binom{10}{2}+\binom{10}{3}+\binom{10}{4}+\binom{10}{5}+\binom{10}{6}+\binom{10}{7}+\binom{10}{8}+\binom{10}{9}+\binom{10}{10}$$

or $\sum_{k=0}^{10}\binom{10}{k}$. The kth term in this sum tells the number of ways you can have a pizza with exactly k ingredients. What is the total number of ways you could have a pizza using some, all, or none of these 10 ingredients?

These expressions, $\binom{n}{k}$, are also called "binomial coefficients." When you raise a binomial (a + b) to a power, the coefficients of the various powers of a and b are of this type. For example:

$$(a+b)^4 = \binom{4}{0}a^4b^0 + \binom{4}{1}a^3b^1 + \binom{4}{2}a^2b^2 + \binom{4}{3}a^1b^3 + \binom{4}{4}a^0b^4$$

Verify this by raising (a + b) to the fourth power!

Note: Why does 0! = 1? Look at the question of how many ways can you choose no things from a group of 10. Obviously there is only one way to chose zero objects (exactly one way to have a pizza with no toppings). By our definition, the answer is $\binom{10}{0}$ or $\frac{10!}{0!(10-0)!}$. The only way that formula can equal 1 is for 0! to equal 1. So we say that 0! = 1 "by definition," because that's the only way the formula is consistent. It may be helpful to remember that $\binom{n}{0} = 1$ and $\binom{n}{n} = 1$

As we have seen, binomial coefficients count the number of *k*-element subsets chosen from an *n*-element set. The term "set" implies that order doesn't matter; that is, the set {1, 2, 3} is the same as {3, 1, 2}.

A classic counting problem is that of determining the number of "lattice paths" from point A to point B on a grid. If one begins at point (0, 0) and moves either north or east to another point (2, 3), the possible paths can be seen as sequences of two steps to the right (east) and three steps north. Possible paths (sequences) are as follows: {EENNN}, {ENENN}, {ENNEN}, {ENNNE}, {NEENN}, {NENEN}, {NENNE}, {NNEEN}, {NNENE}, {NNNEE}. Here we have five "slots" from which we must choose two for the E's. The question becomes, how many ways can we choose two objects from five? The problem could be, alternatively, to determine the number of ways to choose three N's from five slots. Would that give you the same number? In other words, does

$\binom{5}{2} = \binom{5}{3}$? And in general, does $\binom{n}{k} = \binom{n}{n-k}$?

Can you show that this is true?

So the "lattice path" problem becomes a counting problem using binomial coefficients. The number of possible paths from (0, 0) to (r, s), travelling only north and east, is

$\binom{r+s}{r}$ or $\binom{r+s}{s}$.

Now let's look at the pizza question again. Another way to determine the number of possible pizzas with 10 ingredients is to line the ingredients up, take your crust down the line, and say "yes" or "no" to each ingredient. This gives you two choices to each of 10 possibilities, or $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^{10}$. Could this mean that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$?

Exercises:

1. Show that coefficients in the expansion of $(x + y)^3$ are successive binomial coefficients.

2. Calculate the following:

   (a) $\binom{6}{3}$

   (b) $\binom{80}{2}$

   (c) $\binom{35}{12}$

   (d) $\binom{10}{1}$

3. How many different jury panels (12 members each) could be chosen from a pool of 18 jurors?

4. How many different pizzas are possible using some, all, or none of 5 different ingredients?

5. How many possible 5-card poker hands are possible from a deck of 52 cards?

6. How many lattice paths are there from

   (a) (0, 0) to (5, 8)

   (b) (2, 3) to (9, 12)

Sources:

Wagner (1996).

2.4 Patterns in Pascal's Triangle

Earlier we discussed "binomial coefficients." We will now see how these expressions are related to Fibonacci numbers.

If binomial coefficients are arranged in rows, the result is known as "Pascal's triangle" after the French mathematician Blaise Pascal (1623-1662). His original triangle, as published in *Traite du Triangle Arithmetique* (1654), looked something like this:

Version *:

Z	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
2	1	2	3	4	5	6	7	8	9	
3	1	3	6	10	15	21	28	36		
4	1	4	10	20	35	56	84			
5	1	5	15	35	70	126				
6	1	6	21	56	126					
7	1	7	28	84						
8	1	8	36							
9	1	9								
10	1									

In this form, the numbers on the upward-sloping diagonals give the successive coefficients of the binomial expansion of $(a+b)^n$.

The triangle is sometimes written in one of the following forms:

Version **:

n										
0	1									
1	1	1								
2	1	2	1							
3	1	3	3	1						
4	1	4	6	4	1					
5	1	5	10	10	5	1				
6	1	6	15	20	15	6	1			
.	.	.	.	.	.	.	.	.		
.	.	.	.	.	.	.	.	.	.	
.	.	.	.	.	.	.	.	.	.	.

Version ***:

						1					
					1		1				
				1		2		1			
			1		3		3		1		
		1		4		6		4		1	
	1		5		10		10		5		1
1		6		15		20		15		6	

Notice that the last row in that example is $\binom{6}{0}, \binom{6}{1}, \binom{6}{2}, \binom{6}{3}, \binom{6}{4}, \binom{6}{5},$ *and* $\binom{6}{6}.$

Many patterns can be found in this triangle. For example, each entry in the triangle is the sum of the two numbers directly above it. Expressed in terms of binomial coefficients, this could be written:

(1) $$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

If you add the entries on successive upward-sloping diagonals (use version **), you will find Fibonacci numbers. This pattern could be written as follows:

(2) $$\sum_{k=0}^{n} \binom{n-k}{k} = F_n$$

A similar arrangement of binomial coefficients was known by the Chinese around 1000 C.E. The Persian mathematician Omar Khayyam who lived in the 11^{th} or 12^{th} century included such a triangle in his writings. So, although Pascal gets the credit (his name is attached to it!), the arrangement was known by earlier Eastern mathematicians. However, Pascal was the first to make a systematic study of the patterns involved. He listed 19 properties of binomial coefficients that he discovered from the triangle. Among them, according to Burton (1991, p. 416), are:

(II) $$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-2}{r-1} + \binom{n-3}{r-1} + \cdots + \binom{r-1}{r-1}$$

In other words, each number in the triangle is the sum of numbers directly above it (in the ** version of the triangle). For example:

$$\binom{6}{3} = \binom{5}{2} + \binom{4}{2} + \binom{3}{2} + \binom{2}{2} \quad \text{or } 20 = 10 + 6 + 3 + 1$$

(V) $$\binom{n}{r} = \binom{n}{n-r}$$

This property shows the symmetry of the triangle, e.g.: $\binom{6}{2} = \binom{6}{4}$

Pascal applied this triangle to the study of probability. He reportedly discovered it while working on a problem presented to him by a gambler.

Exercises

1. Attached worksheets from Seymour, *Visual patterns in Pascal's triangle.*

2. Pick any number on Pascal's triangle (*** version). Draw a circle which includes your number and the six numbers immediately surrounding it. Add the seven numbers in your circle. Try this with several other numbers on the triangle. Explain how the sums obtained by adding the seven numbers inside the circle are related to one of the numbers outside the circle.

3. What pattern do you discover if you take the numbers in each row as digits of a number (1, 11, 121, 1331, 14641, ...)?

Enrichment exercise:

How many odd numbers are there on the *n*th row of the Pascal triangle?
Is there a pattern to these numbers?

Project ideas:

Life of Blaise Pascal

Further patterns in Pascal's triangle

Sources:

Boyer (1991).

Burton (1991).

Musser& Burger (1994).

Seymour (1986).

2.5 Mathematical Proof

Mathematicians are always trying to prove things. Andrew Wiles of Princeton University spent seven years proving Fermat's Last Theorem, which his predecessors had spent countless years attempting. This theorem says that there are no positive integer solutions to the equation $a^n + b^n = c^n$ for values of *n* greater than 2. We will discuss this more in Section 2.9.

A mathematician typically observes a pattern, forms a conjecture or guess about the pattern, and then sets about proving his or her conjecture. You are no doubt familiar with the "two-column proofs" you learned in high school geometry. These use logic to build a step-by-step argument based on previous knowledge in order to arrive at a conclusion.

There are other methods of proof used in mathematics. We will look at four of them.

2.5.1 Proof by Contradiction

"Eliminate all other factors, and the one which remains must be the truth," said Sherlock Holmes (Bittinger, 1982).

The sentence "Sue is taller than Jan and Sue is not taller than Jan" is always false, not because of its content, but because of its form. In elementary logic we could say *p* is the statement that "Sue is taller than Jan" and $\sim p$ ("not *p*") is the statement "Sue is not taller than Jan." No matter what the content of the two statements, we know they cannot both be true, because one is the negation of the other. A proof by contradiction of a statement *p* is a proof that assumes $\sim p$ and then shows that this implies a statement of the type "$q \wedge \sim q$" which means "*both q and not q are true,*" like "Sue is taller than Jan and Sue is not taller than Jan."

Suppose we wish to prove that $\sqrt{2}$ is an irrational number. We start by assuming that it is rational and then derive a contradiction from that assumption.

A little background: The "Fundamental Theorem of Arithmetic" states that every positive integer has a unique prime factorization except for order. Remember that the prime numbers are positive integers greater than 1 whose only positive divisors are the number itself and 1. (So the number 1 is not a prime number). Another reminder: A rational number is a number that can be written as the quotient of two integers, a and b, where $b \neq 0$. Numbers that are not rational are termed "irrational."

So, if $\sqrt{2}$ is rational, it can be written as the quotient of two integers, a and b, where $b \neq 0$, or $\sqrt{2} = \frac{a}{b}$. We can multiply both sides of that equation by b since $b \neq 0$. Now we have $b\sqrt{2} = a$. Squaring both sides, we see $b^2 \cdot 2 = a^2$. Here is where the contradiction comes in. Both sides of the equation must be equal, but they have a different number of prime divisors so they cannot possibly be equal (by that Fundamental Theorem). We know that a^2 has an even number of factors because whether a has an odd number or an even number of factors, when you square it you'll get an even number (odd plus odd is even, and even plus even is even). And by the same reasoning b^2 has an even number of factors, so when the factor 2 is included, the left-hand side of the equation has an odd number of factors. This therefore contradicts our assumption that $\sqrt{2}$ is rational. That contradiction tells us that it must be irrational.

Proofs by contradiction are widely used in mathematics. In particular, we will see in Section 2.5.3 that the method of proof by mathematical induction follows from the well-ordering principle using a proof by contradiction.

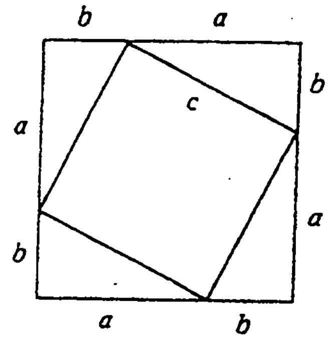
2.5.2 Geometric Proofs

You remember from high school geometry doing "two-column" deductive proofs. In this process, you typically start with a statement that is "given" and proceed step-by-step to show that another statement follows. An example of this is showing that two triangles are congruent (have the same shape and size). The three ways to prove triangle congruence are as follows:

(1) show that all three sides are congruent;
(2) show that two sides and the included angle are congruent;
(3) show that two angles and the included side are congruent.

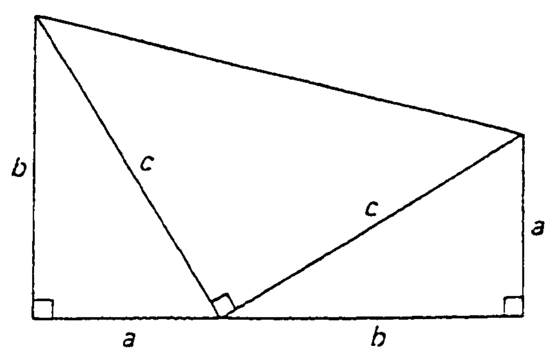
These ways of showing congruence, along with other geometric principles, will be used in proving the Pythagorean Theorem. This well-known theorem states that in a right triangle (one with a 90° angle), the sum of the squares of the two legs (short sides) is equal to the square of the hypotenuse (long side). The theorem was not original with Pythagoras, nor is it clear that he ever proved it, but his name is attached to it because of the Pythagorean school which promoted many mathematical discoveries.

It is known that early Egyptian cultures about 3000 B.C.E. used 3, 4, 5 triangles to construct right angles. A very early (approximately 600 B.C.E.) Chinese proof exists which proves the theorem in a geometric way, using the following diagram, in which a square with sides c is placed inside a larger square with sides $a + b$:



We know that the sum of the areas of the small square and the four triangles equals the area of the large square. It follows from this that $a^2 + b^2 = c^2$, where a and b are the legs of the triangles and c is the hypotenuse. How do you know that the four triangles do indeed have the same area?

A more recent, but similar, proof was published by James A. Garfield in 1876. Garfield was elected President of the United States in 1880, but was shot shortly after his inauguration and held office for only a few weeks. His proof, published in the *New England Journal of Education*, is similar to the early Chinese proof. In the diagram below, we know that the sum of the areas of the three triangles is the same as the area of the trapezoid. From this it can be seen that $a^2 + b^2 = c^2$. (Again, how do we know that the two a's are equal and the two b's are equal? We constructed the trapezoid so that it would contain two congruent triangles. This enables us to determine the angle between the c sides is a right angle.)



2.5.3 Mathematical Induction

According to Burton (1991, p. 416), the principle of mathematical induction is "perhaps the single most useful tool in the mathematician's kit." This method of proof is used extensively in number theory, the branch of mathematics dealing with positive integers. The word "induction" is a bit misleading—proof by induction should not be confused with inductive reasoning, in which a general principle is formed from study of individual cases. Mathematical induction is actually a specific kind of deductive proof. Blaise Pascal was the first to recognize the value of this method and to use it extensively.

The basic process is as follows, where n and k are members of P (positive integers):

(1) Show that the conjecture (or formula) is true for $n = 1$.

(2) Show, for each $k \geq 1$, that if it is true for $n = k$, then it is true for $n = k + 1$.

The principle of mathematical induction then guarantees that the conjecture or formula is true for all positive integers n.

Here is an example:

We observed a pattern when adding 10 consecutive integers beginning with 1: $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 55$. We showed that we could pair numbers in that sequence that added up to 11 ($n + 1$): $1 + 10$, $2 + 9$, and noting that we had 5 ($10 \div 2$) such pairs. This is apparently how Gauss calculated such sums so quickly. Our conjecture for a formula is thus

$$\sum_{1}^{n} i = \frac{n(n+1)}{2}$$

where n is the number of integers in our sequence. We know that this is true for several values, but as mathematicians we must prove it. The proof involves a little algebra, a little arithmetic, and a little clear exposition (clearly stating what we're trying to prove and how we go about proving it):

We wish to show that $\sum_{1}^{n} i = \frac{n(n+1)}{2}$ *

(1) We will show that * is true for $n = 1$:

$$\sum_{1}^{1} i = 1 = \frac{1(1+1)}{2}$$

(2) Let $k \geq 1$ and suppose that * is true for $n = k$, i.e., suppose that $\sum_{i=1}^{k} i = 1 + 2 + \ldots + k = \frac{k(k+1)}{2}$. Let us then show that * is true for $n = k + 1$, i.e., that $\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$:

$$\sum_{i=1}^{k+1} i = 1 + 2 + \ldots + k + (k+1) = \left(\sum_{i=1}^{k} i\right) + (k+1) = \frac{k(k+1)}{2} + (k+1) =$$
$$\frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$$

We have thus shown by mathematical induction that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ for all $n \geq 1$.

How is proof by mathematical induction justified? A basic property of the positive integers is the *Well-Ordering Principle.* This states that any nonempty set of positive integers contains a least integer, with the least integer in the set of all positive integers being 1.

The principle of mathematical induction can now be justified as follows:

Let A be a subset of the positive integers which

(1) contains 1, and

(2) whenever it contains k, it contains k+1.

We wish to show that A = P (the set of all positive integers).

Now let B be all the rest of the positive integers that are not in set A: B = P – A. If B is the empty set, then P = A and the proof is complete. If B is non-empty, it contains (by the well-ordering principle) a least element which we'll call m. We know that $m > 1$ because 1 is in A. So $m - 1 > 0$ (which says that $m - 1$ is a positive integer). But $m - 1$ can't be in B since m is the least integer in B, so it must be in A. But then, by (2), we must have m in A, which contradicts the fact that $m \in$ B.

Here's an example to illustrate this proof. Suppose A = {1, 2, 3}. Then B = {4, 5, ...} Obviously A contains 1, meeting our first criterion above, but if $k = 3$, A does not contain $k + 1$. The only way these two criteria can be met is if A is the set of all positive integers.

A variation on this method of proof by mathematical induction is called Course of Values induction. In this method,

(1) We show that our conjecture is true for $n = 1$;

(2) Show that, for each $k \geq 1$, if the conjecture is true for $n = 1, \ldots, k$, then it is true for $n = k + 1$.

This variation can also be proved using the well-ordering principle as follows:
Let A be a subset of P such that

(1) A contains 1;

(2) For every $k \geq 1$, whenever A contains 1, 2, 3, ... k, it also contains k+1.

We wish to show that A must be the set of all positive integers, P..Again let B = P – A. If B is empty, then A = P and the proof is complete. If B is non-empty, its least element (by the well-ordering principle, there is one!) we'll call m. So the integers 1, 2, 3, ... m-1 are in A. So by (2), we must have $m \in$ A, which contradicts the fact that $m \in$ B.

2.5.3 Combinatorial Proof

A method of proof often used in combinatorics is illustrated in Section 2.10. A combinatorial argument shows that two expressions are equal by showing that they count the same thing.

Exercises:

1. Make a conjecture (guess) about a formula for the sum of the first n Fibonacci numbers. (Generate some values of the sum for n = 1, 2, 3, 4, 5, 6 ..., and then look for a pattern.)

2. Prove your conjecture by mathematical induction. Remember that the definition of Fibonacci numbers says that $F_n = F_{n-1} + F_{n-2}$ (or, equivalently, $F_n + F_{n+1} = F_{n=2}$). Assume that $F_1 = F_2 = 1$.

3. What is the sum of the first n Fibonacci numbers with odd indices? Generate some values, make a conjecture, then prove your conjecture by mathematical induction.

Prove by mathematical induction:

4. $$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

5. $$\sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}$$

6. $2 + 4 + 6 \ldots + 2n = n(n + 1)$

7. For each positive integer n:

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}$$

Sources:

Bittinger (1982).

Burton (1980).

Burton (1991).

Gardner (1979).

Paley & Weichsel (1966).

Sgroi & Sgroi (1993).

Sominskii (1963).

2.6 Patterns in the Fibonacci Sequence

Mathematicians have discovered many intriguing patterns in the Fibonacci sequence. Some examples were seen in the previous exercises. Here are some others:

Every positive integer can be represented as a sum of distinct Fibonacci numbers.

The greatest common divisor of two Fibonacci numbers is also a Fibonacci number.

The difference of the squares of alternate Fibonacci numbers is always a Fibonacci number.

Fibonacci number trick: If 10 consecutive Fibonacci numbers are added together, the sum is always 11 times the 7th number in the list.

Every third Fibonacci number is an even number.

Exercises:

1. Guess a formula for the sum of the squares of the first n Fibonacci numbers.

2. It has been stated that every Fibonacci number with a prime index is a prime number. Can you find a counter-example to disprove this statement? (Remember that a prime number is one that is divisible only by itself and 1.)

3. Explain why the "Fibonacci number trick" works.

4. Find a formula for the sum of the sum of the first n Fibonacci numbers with even indices: $\sum_{i=1}^{n} F_{2i} =$

5. Show that $F_n^{\,2} = F_{n+1}F_{n-1} + (-1)^{n-1}$

6. Represent 50, 75, 100, 125 as sums of distinct Fibonacci numbers.

7. Which Fibonacci numbers are multiples of 3? Multiples of 4? Multiples of 5?

8. Find the sum of the squares of adjacent Fibonacci numbers, that is,
$F_1^2 + F_2^2$, $F_2^2 + F_3^2$, $F_3^2 + F_4^2$, $\cdots$
Extend this sequence to see if there is a pattern.

Sources:

Burton (1980).

Gardner (1979).

Garland (1987).

Knott [on-line].

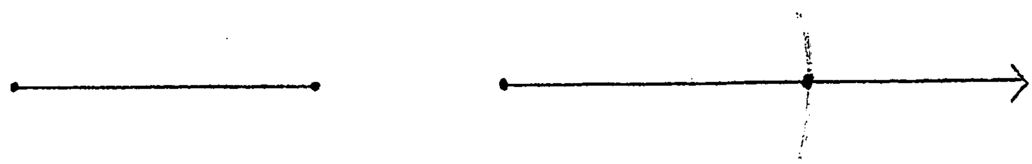
Vorob'ev (1961).

2.7 Geometric Constructions

In order to construct line segments whose lengths are related by the "golden ratio," a relationship closely associated with the Fibonacci sequence, we will review some basic geometric constructions that you probably learned in high school. As far back as the fifth century B.C.E., mathematicians were concerned with constructing certain geometric figures by means of a compass and straight edge. We will review these constructions: copying a line segment, copying an angle, bisecting a line segment (finding its mid-point), bisecting an angle, and constructing a perpendicular to a line. These will be demonstrated in class, but here are instructions:

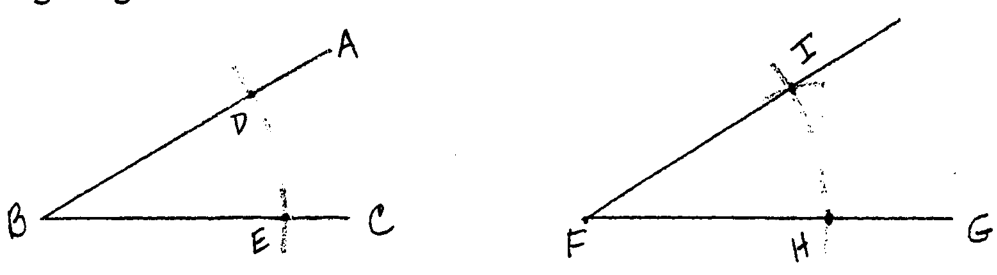
Copying a line segment

Construct a line segment by connecting two points with the straight edge. Then draw another line segment, determining its length by placing the compass on the ends of the first line segment and marking the ends of your new segment.



Copying an angle

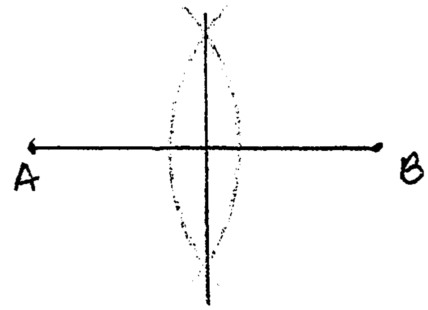
Construct an angle with a straight edge, label it ABC (with B as the vertex). With the point of your compass at B, draw two arcs on the sides of the angle, label the points of intersection with the sides D and E. Draw a new line FG, and draw a large arc (with the same radius you used in the original angle) with the point of your compass at F. Label the point of intersection H. Using your compass to measure, find the distance from D to E on your first angle, and transfer it to the second angle by drawing an arc with the point of your compass at H. Where this arc intersects the previous arc, at I, draw a line with your straight edge to F.



How do you know these angles are equal? Are they corresponding parts of congruent triangles? Show the congruence.

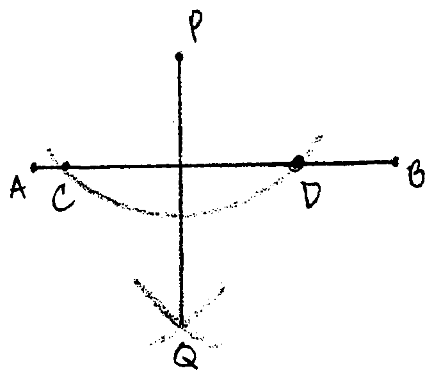
Constructing a perpendicular bisector of a line

Draw a line, label it AB. Use any radius r that is more than half the length of AB to construct a large arc (both above and below the line) with the point of your compass at A. Keeping the radius the same, construct a large arc with the point of your compass at B. Draw a line with your straight edge connect the two points of intersection of the arcs. This is the perpendicular bisector of AB. How do you know?



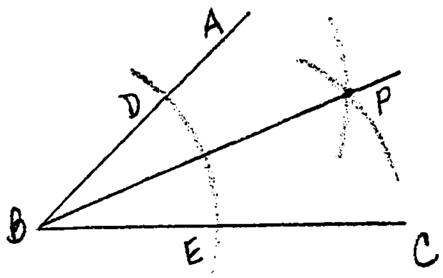
Constructing a perpendicular to a line from a point outside the line

Construct a line AB and a point P outside AB. Draw a large arc that will intersect AB in two places, call them CD. With the point of your compass at C, draw an arc below AB, and draw one the same radius from point D. These arcs will intersect at point Q. Connect the points P and Q with your straight edge. The line PQ is the perpendicular bisector of the line AB. How do you know it is perpendicular? How do you know it bisects AB?



Bisecting an angle

Construct an angle ABC. Use any radius with your compass to construct an arc that will intersect AB and BC at the points D and E. Use your compass (not necessarily the same radius) to construct arcs from D and E that will intersect at point P. Connect the points P and B, which will be the bisector of angle ABC.



In the next section we will see how these constructions will help us construct the "golden ratio."

In addition to these and other basic constructions, the Greeks were concerned with what came to be known as the "Three Construction Problems of Antiquity." Using only a straight edge and compass, is it possible, they asked, to:

1. square a circle; that is, construct a square with exactly the same area as a given circle?
2. duplicate the cube? (find the edge of a cube having a volume twice that of a given cube)
3. trisect an angle? (divide a given angle into three equal angles)

Throughout history these have challenged mathematicians and students alike. In the 19th century it was proved that each of these is impossible. The French mathematician Pierre Wantzel (1814-1848) gave rigorous proofs of the impossibility of duplicating a cube and trisecting an angle.

Those problems will not be assigned for homework.

Practice exercises:

Using only compass and straight edge (ruler):

1. Construct an angle by drawing two intersecting straight lines; copy the angle.

2. Bisect the angle you copied in #1.

3. Draw a line segment; copy it.

4. Draw a line perpendicular to another line.

5. Draw a line segment, then find its midpoint.

6. Construct a square.

Sources:

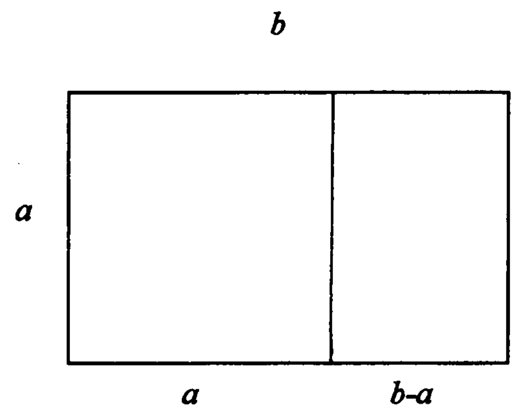
Burton (1991).

Rhoad, Milauskas, & Whipple (1991).

2.8 The Golden Ratio

According to the famous astronomer Johannes Kepler (1571-1630), the two jewels of geometry are the Pythagorean theorem and the golden section. The golden section, also called the golden ratio, the golden mean, the divine ratio, etc., is known as the most pleasing ratio of dimensions of a rectangle or oval. It combines a certain mathematical perfection with widespread aesthetic applications.

A search for perfection in all of life appears to have motivated the ancient Greeks. Among the figures that intrigued them was the "golden" rectangle, one in which a square cut from the rectangle produced a smaller rectangle whose sides were in the same ratio as the original rectangle.



In such a figure, if a is the length of the side of the square, and b is the length of the long side of the large rectangle, then these ratios are equal:

$$\frac{b}{a} = \frac{a}{b-a}$$

This ratio can be calculated multiplying both sides by $a(b - a)$ and using the quadratic formula:

$$b(b-a) = a^2$$

$$b^2 - ab - a^2 = 0$$

Solving for b we find:

$b = a\left(\frac{1 \pm \sqrt{5}}{2}\right)$. Taking the positive root, we see that the ratio of b to a is $\frac{1+\sqrt{5}}{2}$, 1.618..., or the "golden ratio."

The symbol most commonly used for the golden ratio is "phi" or Φ, from the Greek sculptor Phideas. This ratio was believed by the Greeks to be of great aesthetic value, and much of Greek art and architecture displays golden dimensions. Although irrational numbers were not recognized by the Greeks, they considered numbers such as Φ and $\sqrt{2}$ (the measure of a diagonal of a square with sides = 1) "incommensurable" and even possessing certain mystical properties. We will see the ubiquity of these incommensurable numbers in our study of Greek architecture.

Gustav Theodor Fechner (1801-1887) was a noted psychologist and physicist who worked with experimental aesthetics. He collected data from hundreds of people to test their preferences for the golden ratio over other dimensions using rectangles, crosses, lines divided, rectangles within rectangles, ellipses, and figures like a dotted *i*. He found the average choice generally to be close to the golden ratio.

Fechner's experiments, made in 1876, were rather crude. They were repeated by Witmar in 1894, Lalo in 1908, and Thorndike in 1917, with similar results. Results of Fechner's and Lalo's experiments are shown in the table below, along with Fechner's graph. These results show a popular preference for the golden rectangle, or to a shape close to that.

RATIO: WIDTH/LENGTH	BEST RECTANGLE Fechner, %	Lalo, %	WORST RECTANGLE Fechner, %	Lalo, %
1.00	3.0	11.7	27.8	22.5
0.83	0.2	1.0	19.7	16.6
0.80	2.0	1.3	9.4	9.1
0.75	2.5	9.5	2.5	9.1
0.69	7.7	5.6	1.2	2.5
0.67	20.6	11.0	0.4	0.6
0.62	35.0	30.3	0.0	0.0
0.57	20.0	6.3	0.8	0.6
0.50	7.5	8.0	2.5	12.5
0.40	1.5	15.3	35.7	26.6
	100.0	100.0	100.0	100.1

from Huntley, H. (1970), *The divine proportion,* New York, Dover Publications, p. 64

A young student at Gatlinburg's Pi Beta Phi School conducted a more recent exercise to test people's perceptions of what is aesthetically pleasing. In 1998, as part of a Science Fair project, Courtney Lix created a survey of thirty pairs of computer images, one based on *phi* and the other not. She gave the survey to 52 people in her school. Her results showed an 81.1% preference for *phi*. She extended the study to show how road signs and business signs in the Gatlinburg area could be aesthetically improved by having "golden" dimensions.

How does this relate to Fibonacci numbers? Make a chart giving the ratios of adjacent Fibonacci numbers, and you will see the relationship. The ratio gets closer and closer to the exact value of the golden ratio. In calculus, which is the study of rates of change and limits, we would say that:

$\frac{F_{n+1}}{F_n}$ approaches Φ as n approaches infinity (gets larger and larger). The reciprocal of that ratio approaches the value of Φ - 1 or 0.618....

If we refer to our closed form expression for the nth Fibonacci number (Section 2.2.4, (5)), we can find the limit of that ratio as n gets larger and larger. The second term in the formula gets closer and closer to zero for large values of n, since $\frac{1-\sqrt{5}}{2}$ is between 0 and −1, and raising it to a power makes it even closer to zero. Therefore:

$F_n \approx \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n$ and $F_{n-1} \approx \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n-1}$ for large values of n. So, as n approaches infinity (gets larger and larger),

$$\frac{F_n}{F_{n-1}} \approx \frac{\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n}{\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n-1}} = \frac{1+\sqrt{5}}{2}$$

The human body contains many golden proportions. Measure yourself from the top of your head to your navel, and then from your navel to the floor. What is the ratio of those two measurements? Now measure from the top of your head to your neck, and from your neck to your navel. How does that ratio compare? We will look at many other occurrences of the golden ratio in nature, art, and music in subsequent units.

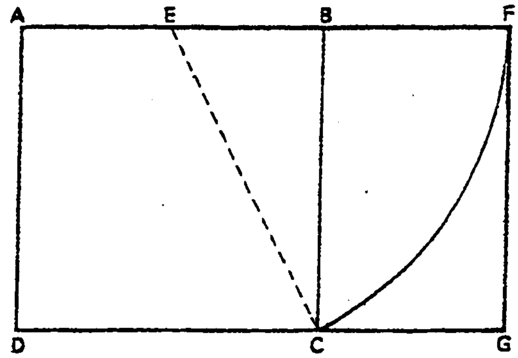
With our newly refreshed skills in geometric construction, we are now ready to construct a golden rectangle:

(1) Construct a square ABCD. Find the midpoint of AB, label it E.

(2) Using your compass, measure the length from E to C, draw an arc that length as indicated to a point on the extension of AB, label it F.

(3) Draw a perpendicular from F to the extension of DC, label the point of intersection G.

The result is a golden rectangle. Prove that AF/FG = Φ. (Hint: Let AB = 2 units of length...) Clearly the smaller rectangle BFGC is similar to rectangle AFGD. The fact that a certain rectangle added to a square formed another rectangle of the same shape was one aspect of the golden ratio that intrigued the Greeks.



Exercises:

1. Make a chart giving ratios of adjacent Fibonacci numbers from $\frac{F_2}{F_1}$ to $\frac{F_{10}}{F_9}$.

2. Calculate the ratio of successive terms in an arithmetic sequence. Does the ratio seem to converge to some number?

3. Calculate the ratio of successive terms in a geometric sequence. What is the result?

4. Calculate the ratio of successive terms in the Lucas sequence. What is the result?

5. Calculate the ratio of successive terms in the Tribonacci sequence? Result?

6. Construct a golden rectangle.

Project ideas:

Pythagoreans

Fechner's experiments (and those of other psychologists) on preference for *phi*

Sources:

Boyer (1991)

Burton (1991)

Garland (1987)

Herz-Fischler (1998)

Huntley (1970)

2.9 Pythagorean Triples

Recall, again, from your high school geometry the well-known "Pythagorean Theorem." This theorem states that in a right triangle, the sum of the squares of the two legs (short sides) is equal to the square of the hypotenuse (long side).

This important geometry theorem led to a problem in arithmetic: For what integer values of *a*, *b*, and *c* does this relationship, $a^2 + b^2 = c^2$, hold true? Resulting sets of integers are called Pythagorean triples; for example, 3, 4, and 5 are Pythagorean triples.

Several formulas have been derived by mathematicians to generate Pythagorean triples. Among mathematicians who worked on this problem were Pythagoras himself, Plato, Euclid, and Diophantus. The most general formula is as follows: If m and n are integers, with $m > n$, and we set $a = 2mn$, $b = m^2 - n^2$, and $c = m^2 + n^2$, then

$$a^2 + b^2 = c^2$$

Fibonacci himself proved (in his treatise *Liber Quadratorum*, 1225) the converse of this: namely that any Pythagorean triple is of this form.

We will now look at a way of generating Pythagorean triples which arises from Fibonacci numbers.

List four consecutive Fibonacci numbers, such as 3, 5, 8, and 13. The first times the fourth will give us *a*, and twice the product of the middle two will give us *b*. It only remains to show that the sum of the squares of these numbers is itself a perfect square.

Here's an interesting historical sidelight to idea of Pythagorean triples:

Since there are so many integer solutions to $a^2 + b^2 = c^2$, early mathematicians began looking for solutions to $a^3 + b^3 = c^3$, and, generally for $a^n + b^n = c^n$ where n is a positive integer greater than 2. None were found, and for centuries people attempted to prove that there were none. Finally, in 1637, Pierre de Fermat wrote a note in the margin of one of his writings that he had a "truly wonderful proof" that there were no integer solutions where $n > 2$, but he said that the margin was too small to contain his proof. This became known as "Fermat's Last Theorem." His marginal note intrigued later mathematicians to try to duplicate his proof. It was not until the 1990's that Andrew Wiles, a mathematician from Princeton, published a proof of this theorem. Because of the sophisticated methods he used, it would not have been possible for Fermat (if he really had a proof, which is doubtful) to have proved it in the same way.

The importance of Fermat's Last Theorem lies not in its usefulness, but in the mathematical tools that were developed over the years in trying to prove it. These tools have been widely used in number theory and algebra, and the proof has generated great interest for over 300 years.

Exercises:

1. Write out a proof of the Pythagorean Theorem. Make sure each step is clearly stated in words.

2. Verify the m, n formulas above by generating several sets of Pythagorean triples from them.

3. Generate Pythagorean triples from the first four Fibonacci numbers.

4. Generate Pythagorean triples from the 6th, 7th, 8th, and 9th Fibonacci numbers.

5. Can you find a general formula for c when the Fibonacci method is used? (Is it a Fibonacci number? If so, which one?)

Sources:

Boulger (1989).

Burton (1991).

2.10 Combinatorial Observations

An interesting problem in combinatorics ("counting") is as follows:

> How many "words" of length n (Wn) can be formed from the alphabet $\{1, 2\}$ in which no two 1's are adjacent?

We'll generate some words by brute force as follows:

n	words	number
1	1, 2	2
2	21, 22, 12	3
3	221, 222, 212, 121, 122	5
4	2221, 2222, 2212, 2121, 2122, 1221, 1222, 1212	8

One way to generate these words systematically is to add "2" to the front of each word in the previous row, and add "12" to each word in the row before that.

5	22221, 22222, 22212, 22121, 22122, 21221, 21222, 21212, 12221, 12222, 12212, 12121, 12122	13

Do you see a pattern? Can you give a recursive formula for Wn? Remember that you have to define initial values, and then show how other values are calculated from previous values.

Once a conjecture has been made, we must prove it by a combinatorial argument. This is another method of proof in mathematics. In a combinatorial proof we must show that both sides of our equation count the same thing; the left-hand side counts the number of words by definition; and the right-hand side counts the number of words in exhaustive, disjoint categories. In our case, those categories are (1) those words beginning with 1 (there are $Wn-2$ of them, formed by adding "12" to the front of each word in the $Wn-2$ row), and (2) those words beginning with 2 (there are $Wn-1$ of them, formed by adding "2" to the front of each word in the previous row).

A "composition" is defined as an ordered sum. Another combinatorics problem asks the question:

> How many compositions of length n are made up of members of the set $\{1, 2\}$?

We can generate compositions, f(n), for n = 1, 2, 3, and 4 by brute force:

f(1) = 1

f(2) = 2 (1+1, 2)

f(3) = 3 (1+1+1, 1+2, 2+1)

f(4) = 5 (1+1+1+1, 1+1+2, 1+2+1, 2+1+1, 2+2)

Again, we are generating the compositions by adding 1 to the front of the compositions from the previous row, and 2 to the front of the compositions on the row before that. We can therefore form a conjecture that, for $n > 2$,

(1) $f(n) = f(n - 1) + f(n - 2)$

The proof of this is again a combinatorial argument; namely, that both sides of (1) count the same thing. The left-hand side counts the number of compositions by definition, and the right-hand side counts the compositions in two exhaustive, disjoint categories: those that begin with 1, and those that begin with 2.

Both of these examples result in Fibonacci-like recursive sequences. Initial values are established by brute force, and subsequent values are generated by a recursive formula.

Here's another conjecture:

$$A(2n) = [f(n)]^2 \text{ where } \{f(n)\} = \{1, 2, 3, 5, 8, 13, \ldots\}$$

Where A(2n) is the number of compositions of the positive even integer 2n made up of 1's, 3's, and 4's.

For example, when n = 1, A(2) = 1 (1+1);
When n = 2, A(4) = 4 (1+1+1+1, 1+3, 3+1, 4)
When n = 3, A(6) = 9 (1+1+1+1+1+1, 1+3+1+1, 3+1+1+1, 1+1+3+1, 1+1+1+3, 3+3, 4+1+1, 1+4+1, 1+1+4)

Using brute force, find A(8). Is this in keeping with our conjecture? How would you go about proving this conjecture?

Exercises:

1. List the "words" in W6 above.

   (a) How many are there?

   (b) How many have all 2's?

   (c) How many have one 1 and five 2's?

   (d) How many have two 1's and four 2's?

   (e) How many have three 1's and three 2's?

   (f) Do any have four 1's and two 2's? If not, why not?

2. Find a recursive formula for the number of words of length n in the alphabet $\{1, 2, \ldots, r\}$ with no two 1's adjacent. Can you prove it with a combinatorial argument?

3. List the compositions of $\{1, 2\}$ for $n = 5, 6, 7$, and 8. Does this bear out our conjecture?

Enrichment exercise:

Prove the conjecture that $A(2n) = [f(n)]^2$ where $\{f(n)\} = \{1, 2, 3, 5, 8, 13, \ldots\}$

Sources:

Meyer (1998).

Wagner (n.d.).

2.11 A Fibonacci Mystery

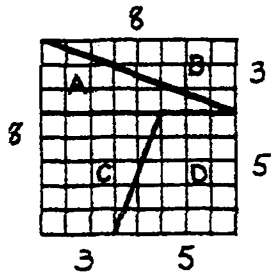
Lewis Carroll (1832-1898), author of *Alice in Wonderland* and *Through the Looking Glass,* was in real life the Oxford mathematics lecturer Charles L. Dodgson. In addition to his popular children's books, he published several mathematical treatises. The best-known of them was *Euclid and His Modern Rivals* (1879).

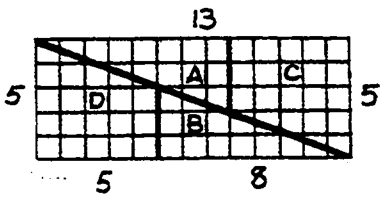
Carroll reportedly liked to puzzle his friends with the following mystery:

Take a square 8 units on a side. Cut it into a 3 x 8 rectangle and a 5 x 8 rectangle, then cut each of those rectangles along the diagonal. Fit the pieces together to form a 5 x 13 rectangle. The area of your original square is 64, but the area of the 5 x 13 rectangle is 65. What happened to that other square inch?

Try the same experiment with a square 13 units on a side. Cut into rectangles 5 x 13 and 8 x 13, then cut each rectangle along its diagonal. Fit the pieces together to form a 21 x 8 rectangle. The area of the original square is 169, but the area of the new rectangle is 168.

The same thing will happen with any Fibonacci square. How can this be? (See next page for solution).





Sources:

Garland (1987).

Lewis Carroll. (http://www.heureka.clara.net/art/carroll.htm)

Solution:

The pieces don't quite fit together along the diagonal—the missing area is exactly to one square unit. You will find that the square of a Fibonacci number always differs from the product of the Fibonacci numbers on either side of it by 1, or:

$$F_n^{\ 2} = F_{n-1}F_{n+1} + (-1)^{n-1}$$

We showed this to be true in Exercise #5, Section 2.6.

2.12 Fractal Geometry

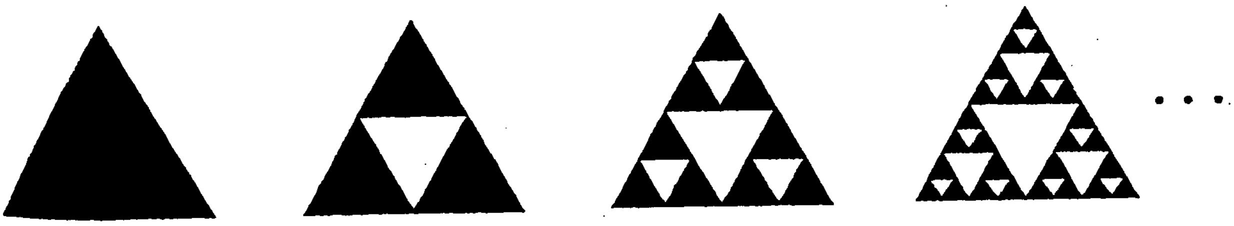
One of the new frontiers in mathematics is the study of fractals. Fractals are shapes that occur in nature and exhibit self-similarity, such as snowflakes, leaves, and coastlines, but they may also be generated by mathematical processes. This new branch of mathematics is the study of non-linear dynamic systems or "the mathematics of chaos."

Benoit Mandelbrot (1924-) investigated a problem posed by Henri Poincare (1854-1912) as follows: *Describe the set of points in the plane which is invariant under inversions (reflections) in a given set of circles.* Mandelbrot reflected a point in several circles and iterated (repeated) this process a large number of times. He called the resulting sets *fractals*. Obviously, high-speed computers have facilitated the iteration process. The Mandelbrot set, discovered in 1980, is the set of all points, c, in the complex plane such that $|z_n| \leq 2$ for all n, where z_n is the nth iterate of 0 under $z_{n+1} = z_n + c$.

The Mandelbrot set

Fibonacci numbers, as we will see, are found in the Mandelbrot set. Before looking at this fractal let's examine a simpler example, the Sierpinski triangle.

To construct a Sierpinski triangle, the midpoints of the three sides of any triangle are connected, forming four triangles the same shape as the original. The center triangle is then removed, and each of the remaining triangles is divided in the same way with the center triangle removed. This process is repeated infinitely as shown in the illustration below.

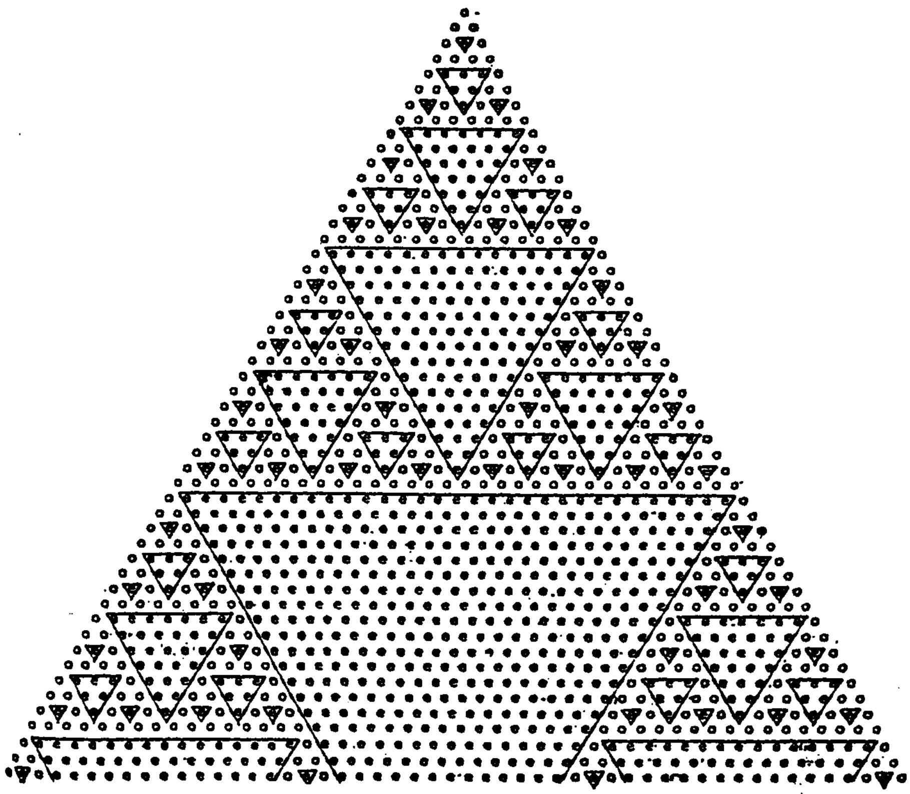


A graphing calculator may be useful in generating fractals, though of course not as quickly as a computer. For example, here is a simple program for the TI-81 calculator to generate a Sierpinski triangle (taken from a handout from NSF College Mathematics Faculty Seminar, Knoxville, TN, May 17-22, 1992):

```
Prgm1: TRIANGLE
:All-Off              (Y-vars)
:ClrDraw              (Draw)
:0→Xscl               (→ = STO, Xscl in VARS, Rng)
:0→Yscl
:1→J
:Lbl 0                (PRGM)
:Rand→{x}(J)          (MATH, PRB)
:Rand→{y}(J)
:IS>(J, 3)            (PRGM)
:Goto 0               (PRGM)
:xSort                (STAT, DATA)
:{x}(1) →Xmin         (VARS, RNG)
:{x}(3) →Xmax
:ySort
:{y}(1) →Ymin
:{y}(3) →Ymax
:Scatter              (STAT, DRAW)
:Rand→A
:Rand→B
:Lbl 1                (PRGM)
:3Rand→N
:1→K
:Lbl 2
:If N < K
:Goto 3
:IS>(K,2)             (PRGM)
:Goto 2
:Lbl 3
:(A+{x}(K))/2→A
:(B+{y}(K))/2→B
:PT-On(A,B)           (DRAW)
:Goto 1
```

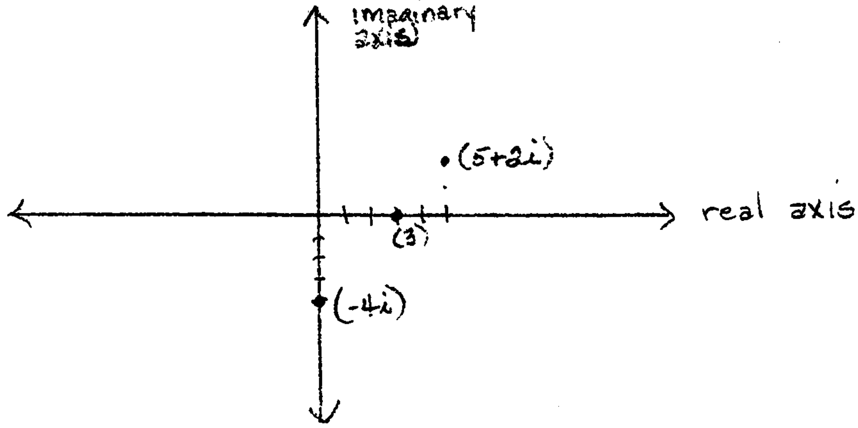
If you wish to construct this graphic one step at a time, you may insert a "Pause" command after "Scatter". Then you must hit the "Enter" button continuously when executing the program. Since this program is an infinite loop, you must stop it by hitting "ON" and then 2 for "Quit".

An interesting connection with the Pascal triangle is shown below. If you locate all the even numbers on the triangle and remove them, you would be left with a Sierpinski triangle.



The Mandelbrot set is a more complex fractal with intricate patterns resulting from iterations of a simple quadratic function. The function, $x^2 + c$, is evaluated where x_o is zero and c is any complex number (real or imaginary). It is then "iterated" many times, and the "orbits" of the function are graphed on the complex plane. The Mandelbrot set consists of those orbits which are not infinite or which don't "march off to infinity." Let's look at what this means. f(x) is a function of x. f(f(x)) is called an "iteration" of the function. In other words, when some value of x is substituted in the original function, the resulting value of the function is then substituted in the original function. Suppose for example that f(x) = 3x. Then f(2) = 3(2) = 6; f(f(2)) = f(6) = 3(6) = 18; f(f(f(2))) = f(f(6) = f(18) = 3(18) = 54. These values, 6, 18, and 54, are successive iterations of the function.

Complex numbers consist of both real and imaginary numbers which may be written in the form $a + bi$ where a and b are real numbers and i is the imaginary unit, $\sqrt{-1}$. We can graph complex numbers on what is called the "complex plane," where the vertical axis represents the imaginary part and the horizontal axis represents the real part. For example, the numbers 3, 5+2i, and –4i are graphed on the following set of axes:



Now let's try some iterations of our Mandelbrot function, $x^2 + c$.

Suppose $x_0 = 0$ (we call that the seed value) and $c = -1$. We will call x_1 the first iteration, etc. Here are the first four iterations of the function:

$$x_1 = 0^2 + (-1) = -1$$
$$x_2 = (-1)^2 + (-1) = 0$$
$$x_3 = 0^2 + (-1) = -1$$
$$x_4 = (-1)^2 + (-1) = 0$$

If we continued this process indefinitely, we can see that the values of x are always 0 and –1. We say that this is the "orbit" of the function, which is the list of numbers in the iteration process. This is known as a 2-cycle orbit.

Let $x_0 = 0$ and $c = 0$. Then

$$x_1 = 0^2 + 0 = 0$$
$$x_2 = 0^2 + 0 = 0$$

and so on. We say that the orbit of this function is a "fixed point."

Now let's let $x_0 = 0$ and $c = +1$. Follow the process again:

$$x_1 = 0^2 + 1 = 1$$
$$x_2 = 1^2 + 1 = 2$$
$$x_3 = 2^2 + 1 = 5$$
$$x_4 = 5^2 + 1 = 25$$

Obviously this orbit does not cycle but "marches off to infinity" with successive iterations.

What happens if $x_0 = 0$ and $c = -1.3$?

$$x_1 = 0^2 + (-1.3) = -1.3$$
$$x_2 = (-1.3)^2 + (-1.3) = .39$$
$$x_3 = (.39)^2 + (-1.3) = -1.1479$$

.

.

.

$$x_{13} = -1.2996...$$
$$x_{14} = .389...$$
$$x_{15} = -1.148...$$
$$x_{16} = .0194...$$

Eventually we can see this function as a period 4 cycle. So this has an orbit of 4.

Calculating these cycles by hand can be tedious, especially when it requires a number of iterations to see what the period is. But a simple computer program can perform successive iterations very quickly. In fact, a graphing calculator can be programmed to generate a large number of iterations of the function when c is a real number. The program below is for the TI-82; commands are similar on other calculators:

```
PROGRAM:ITERATE
:0→X              (stores 0 for X (x0 = 0, the seed value))
:Disp "C="        (displays "C=")
:Input C          (user enters a value for C)
:Lbl 1            (the "goto" statement returns here)
:Disp X²+C        (displays function value)
:Pause            (when running the program, you must "enter" to get past this point)
:X²+C →X          (stores new value to iterate the function)
:Goto 1           (loop)
```

Since this program is an "infinite loop" you must press the ON button, then 2 (quit) to stop running the program.

Let's try the following c values in the program: .3, .2, -1.3, .11, -1.7. If we were to choose c = -1.99, we would see (using our program or a computer) that successive iterations result in chaotic behavior, but the values of the function are always between –2 and +2.

So far c has always been a real number. Let's try $c = i$.

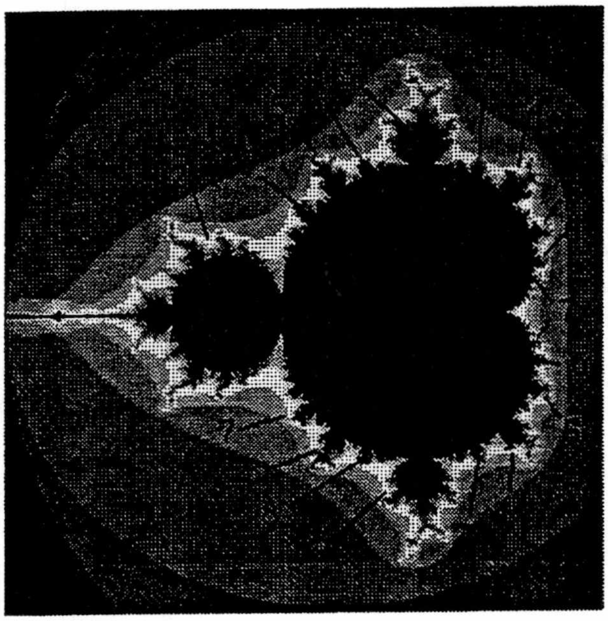
$$x_1 = 0^2 + i = i$$
$$x_2 = i^2 + i = -1 + i$$
$$x_3 = (-1+i)^2 + i = 1 - 2i + i^2 + i = -i$$
$$x_4 = (-i)^2 + i = -1 + i$$
$$x_5 = (-1+i)^2 + i = -i$$

and so on.

This is another example of a period 2 cycle.

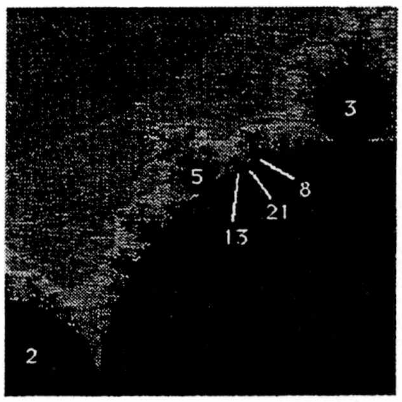
It is clear that, for any complex value of c, orbits either go to infinity or do not. The ones that do not go to infinity are either fixed points or cycle in some way. The Mandelbrot set is defined as the graph of all c values on the complex plane for which the orbit of zero (seed value) does not go to infinity.

It turns out that each "bulb" in the graphed Mandelbrot set represents a different size orbit. The main bulb consists of all values of c whose orbits are fixed points. The largest bulb west of the main bulb represents 2-cycle orbits, and the north and south bulbs represent 3-cycle orbits. The following diagram shows the orbit size for several of the larger bulbs.



From web site: http://math.bu.edu/DYSYS/FRACGEOM2/node3.html

There are many intriguing numerical patterns in the Mandelbrot set. Our main interest, of course, is in Fibonacci numbers, which may be seen as follows: The largest bulb between bulbs 1 and 2 is 3; between 2 and 3 is 5; between 3 and 5 is 8; between 5 and 8 is 13; and on and on.



From web site: http://math.bu.edu/DYSYS/FRACGEOM2/node7.html

It is possible to generate a rough Mandelbrot set with a graphing calculator or a computer. Here is a generic program that can be adapted for computer or graphing calculator (from Devaney (1990), p. 117). The process of running this program takes a long time (remember we are iterating many random numbers) and requires lots of memory.

```
REM  program MANDELBROT1
CLS
FOR I = 1 TO 300
        FOR J = 1 TO 150
                C1 = -2 + 4*I/300
                C2 = 2 – 4*J/300
                X = C1
                Y = C2
                        FOR N = 1 TO 30
                        X1 = X*X – Y*Y + C1
                        Y1 = 2*X*Y + C2
                        R = X1*X1 + Y1+Y1
                        IF R>4 THEN GOTO 1000
                        X = X1
                        Y = Y1
                        NEXT N
                PSET (I, J)
                PSET (I, 300-J)
        1000 NEXT J
NEXT I
END
```

How are these fractals useful in real life? Since the mathematics is so new, practical applications are unclear. But connections between chaos and pattern (order) promise new frontiers in mathematical discovery.

Project Ideas:

Life and work of Benoit Mandelbrot

Julia sets

Sources:

Devaney [On-line].

Devaney (1990).

Devaney (1996)

Musser & Burger (1994).

Unit 3

Art and Architecture

3.1 The Pyramids of Egypt

The primary occurrence of Fibonacci numbers in art and architecture is of course in the golden ratio, which we know is the ratio of two consecutive Fibonacci numbers as the numbers get large. We have noted experiments by Fechner and others who showed that most people find rectangle or oval with "golden" dimensions more aesthetically pleasing than any other.

The pyramids of Egypt are considered to be among the most impressive structures in the world. Constructed more than four thousand years ago, they are listed among the seven wonders of the ancient world. Their size alone is awe-inspiring, given the problems of such massive construction in the ancient world. Mathematicians have analyzed their dimensions and found Phi (the golden ratio), Pi (ratio of circumference to diameter of a circle), the Pythagorean Theorem, and principles of ornament and design.

The largest is found west of Cairo on a plateau known as Giza (there are several alternate spellings such as Gizeh, Djseh, and Jeeseh). Known as the Great Pyramid of Cheops, its base covers 13 acres. This structure, built of more than two million blocks of limestone and granite, contains "more stone than all the cathedrals, churches and chapels built in England since the time of Christ" (Tompkins, p. 1). It is often called the Great Pyramid. Two smaller pyramids on the same plateau are attributed to Cheops' successors, Kephren and Mykerinos.

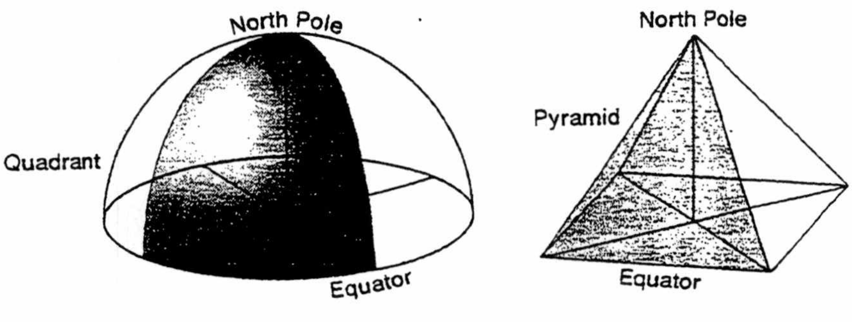
Huntley provides us with a table of dimensions of several pyramids.

Place	Base	Height	Angle
Medum	5,682	3,619	51° 52'
Gizeh (Khufu or Cheops)	9,068	5,776	51° 52'
Gizeh (Khafra or Kephren)	8,475	5,664	53° 10'
Gizeh (Menkaura or Mykerinos)	4,154	2,581	51° 10'
Dahshur (South)	7,459	4,134	43° 5'
			55° 1'
Dahshur (small)	2,064	2,034	44° 34'

We can observe that the ratio of height to base lengths is about the same, very close to 1.6. Which one is closest to the golden ratio? There is additional evidence that the pyramid builders not only knew about the golden ratio but believed it had special significance. Tompkins says, "The pharaonic Egyptians, says Schwaller de Lubicz, considered Φ [Phi] not as a number, but as a symbol of the creative function, or of reproduction in an endless series." (p. 191)

Numerous theories have been advanced about mathematical implications of the measurements of the Great Pyramid. These hypotheses have developed in the last few hundred years due to new archaeological research. Among the first to explore the pyramids was the Arab astronomer Al Mamun in 820 C.E. During the Middle Ages a myths and superstitions developed about the pyramids—ghosts and witches were said to inhabit them. This hindered serious scientific exploration until the beginnings of the Renaissance.

One of the more interesting stories is that of John Taylor, a nineteenth century London newspaper editor and mathematician, who never saw the Great Pyramid but used the measurements and calculations of others to formulate his theories. He discovered that if he divided the perimeter of the Pyramid by twice its height, it gave him a value amazingly close to the value of pi (π). In other words, the height is equal to the radius of a circle whose circumference is the perimeter of the pyramid. Taylor concluded that the Pyramid builders intended to incorporate this irrational number into their building. He then theorized that the perimeter might have represented the circumference of the earth at the equator while the height represented the distance from the earth's center to the pole. He underlined his belief: " 'It was to make a record of the measure of the Earth that it was built....They knew the Earth was a sphere; and by observing the motion of the heavenly bodies over the earth's surface, had ascertained its circumference, and were desirous of leaving behind them a record of the circumference as correct and imperishable as it was possible for them to construct.'" (quoted by Tompkins, p. 72) Is it possible that these ancient Egyptians had such insight? There is evidence in the Rhind Papyrus, dated 1700 B.C.E. and discovered in 1855 C.E., that Egyptians had knowledge of the value of pi (roughly 3.16); but the pyramids were built much earlier.



From Tompkins, P. (1971). *Secrets of the Great Pyramid.* New York: Harper Colophon Books, p. 189.

Taylor concluded that the proportions of the Pyramid were intended to use geometric and astronomical laws and pass on this knowledge to future generations.

Linn maintains that the Egyptian pyramid builders did not use mathematical principles in their construction, but were more concerned with considerations related to their Sun-god. But it is interesting that the proportions, especially that of the Great Pyramid of Gizeh, are so close to the golden section.

Project suggestions:

Early exploration of the pyramids

Tomb of Rameses IX

Tomb of Rameses IV

Sources:

Burton (1991).

Garland (1987).

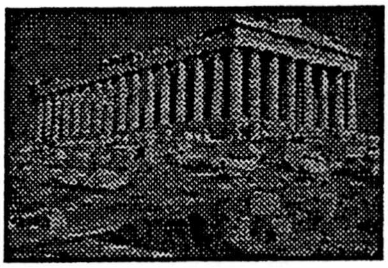
Ghyka (1977).

Huntley (1970)

Linn (1974).

Tompkins (1971).

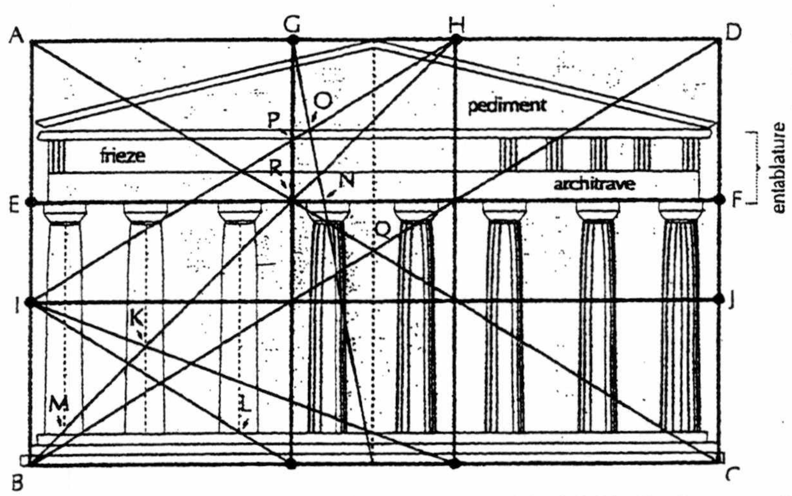
3.2 The Parthenon



The Parthenon, one of the most famous pieces of architecture in the world, stands on the highest part of the acropolis in Athens, Greece. Built during the "Golden Age of Greece" around the fifth century B.C.E., it was intended to serve both as a treasury and as a home for the goddess Athena. The architects Ictinus and Callicrates and the sculptor Phideas are credited with the design.

This temple, built between the years 447 and 432 B.C.E., replaced an earlier one which was destroyed by the Persians in 480 B.C.E. The Parthenon was probably damaged by fire sometime between 150 B.C.E. and 267 C.E. but was repaired and restored. In 600 C.E. it became a Christian church, and in 1687 a small mosque was built in the interior.

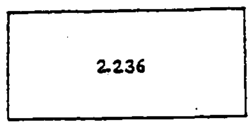
The Parthenon is a Doric temple with a rectangular floor plan, low steps on every side, and a colonnade of Doric columns around the entire structure. There were two interior rooms, the larger of which contained the statue of Athena.



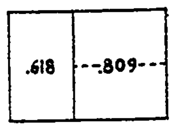
From Newman, R. & Boles, M. (1992). *Universal patterns.* Bradford, MA: Pythagorean Press, p. 69

Clearly the ratio of the greatest height to the greatest width is the golden ratio, making ABCD in the diagram above a golden rectangle, as is AGRE. How many other golden rectangles can you find from the diagram? Squares?

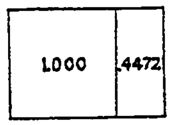
Other ratios frequently seen in the dimensions of the temple, according to Hambridge, are (1) the "root-five" rectangle (ratio: $\sqrt{5}$ to 1 or 2.236 to 1), (2) a combination of a .618 rectangle with half of a 1.618 rectangle (ratio 1.427), (3) a rectangle composed of a square and a root-five rectangle (ratio 1.4472). Examples of each of these are shown below:



(1)



(2)



(3)

Nineteenth century architects and archeologists analyzed the design of Greek monuments to determine whether they were mathematically based or a result of luck and good taste. Zeysing, in about 1850, discussed the presence of the golden ratio in the frontal dimensions of the Parthenon. Other more recent researchers who discuss theories of design are Hambridge, Lund, and Moessel. They are in agreement that Greek symmetry and proportion are based primarily on the golden section.

How did the Greeks know about irrational numbers such as the golden section and "root-five"? Remember that the Golden Age of Greece was an important period in the development of geometry. Irrational lengths can easily be constructed as diagonals of squares or rectangles. For example, the diagonal of a square with sides of length one has a length of root-two. A root-five length is easily constructed as the diagonal of two squares whose sides have length one. The Greeks called these irrational numbers "incommensurable" and believed them to have special "dynamic" meaning compared with integer ratios such as 3:2 which they called "static." Ghyka quotes Plato in his *Theaetetus* as calling such irrational numbers "'commensurable in the square.'"

There is therefore evidence that the use of the golden section was purposeful in the design of the Parthenon. It is likely that the combination of its mathematical and aesthetic aspects made it especially appealing to the Greeks.

Project ideas:

Euclid's geometry

The Parthenon in Nashville, Tennessee

Athena and other prominent Greek gods and goddesses

Sources:

Newman & Boles (1992).

Huntley (1970).

Ghyka (1977).

Hambridge (1924).

Linn (1974).

Parthenon, The [On-line]

3.3 Leonardo da Vinci

Many great painters have used the golden ratio in their works. One of the most notable of these is named Leonardo and happens to come from the town of Vinci, very near Pisa where our mathematician Leonardo came from. The second Leonardo lived in the period of history now known as the Renaissance nearly 300 years after Leonardo of Pisa. He was an exceptionally gifted man in several areas, including mathematics, engineering, music, geology, biology, anatomy, philosophy, architecture, and painting. It is said that his greatest feat was in the diversity of his interests and achievements. Yet the total number of his completed paintings is relatively small, and none of his sculptures was finished. Lucie-Smith claims this is due to "his own reckless impatience, his perfectionism, and also his tendency to abandon painting altogether for long periods, in favor of projects which had little direct connection with art, though Leonardo's thoughts about them were nevertheless recorded in magnificent drawings." (p. 192)

An early biographer, Giorgio Vasari, observes that "'in learning and the study of letters he would have gone far if he had not been so variable and unstable, for he set himself to learn many things and, having once begun them, abandoned them. Thus, he studied arithmetic for a few months and made such progress that he often confounded the master who was teaching him by raising problems and difficulties. He also spent some time on music and quickly resolved to learn to play the lyre. By nature a lofty and refined spirit, he sang divinely while improvising. Despite such a variety of pursuits, he never ceased to draw and work in relief, for these appealed to his fancy more than anything else.'" (quoted in Chastel, p. 7) Today we might identify him as a gifted student with attention deficit disorder!

Leonardo da Vinci, the illegitimate son of a wealthy man and a peasant woman, was born in in 1452 in Vinci in the Tuscany region of the Italian peninsula. He was sent to Florence as a young man to apprentice with a painter Andrea del Verrocchio. He had extraordinary drawing and painting abilities, and he was also interested in architectural plans and elevations. While still a young man, he was the first to suggest that the waters of the Arno River be used for a canal from Pisa to Florence. He designed bridges, aqueducts, cannons, armored vehicles, and even a flying machine (which did not fly but which reflected solid principles of aerodynamics). Most of his scientific writings were not published until after his death.

As a true Renaissance man, Leonardo exemplified the rebirth of interest in science which took place during that period. More than any of his contemporaries, he understood the importance of precise scientific observation. He studied anatomy in order to paint and sculpt the human body more realistically. He dissected bodies of criminals to discover the workings of bones, muscles, and the circulatory system. There are many anatomical drawings showing such detail (see pp. 15, 44, 45, 55, 134, 135 of Chastel).

Leonardo's most famous work is the Mona Lisa, which is a painting of a woman named Lisa del Giaconda. Displayed in the Louvre in Paris, the original painting is surprisingly small—76.8 by 53 cm. The greatness of this painting lies not only in the painter's technique, but in the transience that is conveyed. Lucie-Smith says that "he seems to want to convey the fact that human personality is fluid rather than fixed" (p. 192). Possibly his greatest work was a mural of the Last Supper painted on the wall of a monastery in Milan. Unfortunately it began deteriorating as early as 1500. Numerous restorations have been attempted; the most recent one was completed in the spring of 1999. A reproduction of the Last Supper is in the Louvre in Paris, and a mosaic copy is in the Church of the Minor Brethren in Vienna.

In 1509 a treatise by Luca Pacioli was published, *De Divina Proportione,* illustrated by da Vinci. Martin Gardner called it a "'fascinating compendium of *Phi*'s appearance in various plane and solid figures.'" (quoted in Huntley, p. 25). The golden ratio, or "divine proportion" as Leonardo called it, can be found in much of his art. Interestingly, Pacioli is known as the author of the first printed book on commercial bookkeeping.

Leonardo died in 1519 in France, "'in the arms of the king'" according to Vasari (Chastel, p. 25). Historians give him credit, not only for his contributions to art and science, but also for initiating the "cult of genius" which made it possible for Michelangelo, Raphael, and others to have a say in the subject and design of their works; that is, to become artists rather than artisans.

Many other painters, sculptors, and architects made use of the golden section in their works. A few are listed below as project suggestions.

Project ideas:

Seurat

Durer

Mondrian

Bellows

Le Corbusier

Pacioli

Sources:

Bergamini (1963).

Chastel (1961).

Garland (1987).

Huntley (1970).

Leonardo da Vinci [On-line].

Lucie-Smith (1992).

Microsoft Encarta Encyclopedia (1996).

Unit 4

Music

4.1 Applications of Fibonacci Numbers in Music

" 'The music of nature and the music of man belong to two distinct categories. The transition from the former to the latter passes through the science of mathematics.' " This quote by Eduard Hanslick (1854, quoted by Putz, 1995) reflects the mysterious but widely recognized relationship between music and mathematics.

Fibonacci numbers and the golden section occur frequently in music. For example, in Western music, the most pleasing harmonies are major and minor sixths. A major sixth could consist of C (264 vibrations per second) and A (440 vibrations per second). This ratio reduces to 3/5, a Fibonacci ratio. A minor sixth could consist of E (330) and C (528), which reduces to 5/8, another Fibonacci ratio. One often reads that the pentatonic scale (black keys on the piano) consists of 5 notes, the diatonic scale consists of 8 notes, and the chromatic scale consists of 13 notes; all of those numbers are of course Fibonacci numbers. However, the diatonic scale actually has only 7 different pitches (first and last are the same pitch, differing only by an octave), and likewise the chromatic scale has only 12 different pitches.

The mathematician and musician Joseph Schillinger originated a system of musical composition in which successive notes in a melody are successive Fibonacci units (such as 1, 2, 3, 5, 8, and 13) above or below each other. However, when larger numbers are used in this way the result becomes too extreme and is not as pleasing to hear.

The major connection between Fibonacci numbers and music is in the golden ratio in numbers of measures. This ratio seems to result in an aesthetic sense of balance and perfection of which composers probably were not conscious. Rogers says that the basis of most first movements in sonata form of Mozart and Beethoven piano sonatas and string quartets and of Beethoven and Brahms symphonies is a golden section. He discovered too that compositions by Schubert, Mendelssohn, Chopin, Schumann, Tchaikovsky, Dvorak, Delius, Scriabin, Debussy and Schoenberg show some features of the golden section. These composers likely made use of the balance of the golden section intuitively, though no one knows for sure. The Hungarian composer Bela Bartok used both Fibonacci numbers and the golden section, as we will see in more detail in 4.3. Recent research in musicology indicates that his use of these elements was purposeful rather than intuitive.

In the video we will see of Rostropovich playing and talking about Bach's music, the golden section in phrasing will be evident. He discusses the "intake of breath" as the rise of the phrase, and the exhaling as the relaxing of the phrase, and shows how the lengths of these are in approximately the golden ratio.

Project ideas:

The Schillinger system of musical composition

Sources:

Breakfast with the Arts video
Garland (1987).
Putz (1995).
Rogers (1977).

4.2 Music of Mozart

Wolfgang Amadeus Mozart is considered to be one of the greatest and most prolific composers who ever lived. He was clearly a child prodigy, playing the piano at age three, performing in public and composing at five, and performing all over Europe with his family at seven. In his short 35-year life span, he wrote over 600 compositions. He seemed to write music effortlessly, and he displayed an unusual aural sensibility. According to Elias (p. 60), "Musical inventions flowed from him as dreams emanate from a sleeping person." He adds, "What we feel to be the perfection of many of his works is due equally to his rich imagination, his comprehensive knowledge of the musical material, and the spontaneity of his musical conscience" (pp. 60-61).

The extraordinary balance and perfection of Mozart's music has led to theories about its effects on intelligence and creativity. Don Campbell, author of *The Mozart effect,* claims that exposure to sound, music, and other forms of vibration, beginning before birth, can have lifelong effects on health, learning, and behavior. Advocates of this theory claim that certain sounds promote neural activity in the brain creating dendrites which help us think.

Rauscher, Shaw, and Ky researched the effects of listening to music on "higher brain functions." In their experiment, three groups of college students were given an I.Q. test. Before the test, one group listened to 10 minutes of Mozart (Sonata for Two Pianos in D major, K488), a second group listened to 10 minutes of a relaxation tape, and the third group sat in silence for 10 minutes. The Mozart group scored significantly higher than either of the other groups ($p = .002$ and $p = .0008$). These researchers were careful to state that no causal relationship had been determined and that their project was limited to one composer. They theorized that "music lacking complexity or which is repetitive may interfere with, rather than enhance, abstract reasoning."

What could it be about Mozart's music that it might enhance reasoning or creativity? Pam Gildrie, a music teacher in Maryville, TN, says, " 'The music of the classical period, Mozart in particular, seems to have the most rewarding effect [on learning]. The music of this period is highly organized, rhythmically interesting without being distracting and has a fine balance of melody, harmony and texture'" (*McCarter-Hall,* p. 12A).

Is the balance found in such music mathematical? John Putz of Michigan's Alma College examined movements of all Mozart piano sonatas that were in the sonata-allegro form. Such movements have two distinct parts: (A) exposition, in which the musical theme is introduced, and (B) development/recapitulation, in which the theme is developed and revisited. He counted the measures in each part for each movement, and the results are summarized below:

Kochel	A	B	A+B
279, I	38	62	100
279, II	28	46	74
279, III	56	102	158
280, I	56	88	144
280, II	24	36	60
280, III	77	113	190
281, I	40	69	109
281, II	46	60	106
282, I	15	18	33
282, III	39	63	102
283, I	53	67	120
283, II	14	23	37
283, III	102	171	273
284, I	51	76	127
309, I	58	97	155
311, I	39	73	112
310, I	49	84	133
330, I	58	92	150
330, III	68	103	171
332, I	93	136	229
332, III	90	155	245
333, I	63	102	165
333, II	31	50	81
457, I	74	93	167
533, I	102	137	239
533, II	46	76	122
545, I	28	45	73
547A, I	78	118	196
570, I	79	130	209

Let's look at the ratio of A to B and then the ratio of B to A+B. One way to do this would be to construct a scatter plot of B against A + B to see the correlation. Another way is to calculate the ratio in each movement and find the average. We will do these as a class activity. It becomes clear that this ratio is very close to the golden ratio, though there is some variation.

Did Mozart consciously use mathematics in his music? It is known that he was enthusiastic about arithmetic. Alfred Einstein, one of Mozart's biographers, stated: "The pleasure of playing with figures remained with Mozart all his life long. Thus he once took up the problem, very popular at the time, of composing minuets 'mechanically,' by putting two-measure melodic fragments together in any order. And we possess a page of musical sketches on which he had begun to figure out the sum which the chess player would have received from the King in the famous Oriental story" (p. 25).

But was his use of the golden section deliberate or instinctive? We will never know, of course. Putz says, "Perhaps the golden section does, indeed, represent the most pleasing proportion, and perhaps Mozart, through his consummate sense of form, gravitated to it as the perfect balance between extremes. It is a romantic thought" (p. 281).

Sources:

Einstein (1945).

Elias (1993).

McCarter-Hall (1998, October 29).

Putz (1995)

Rauscher, Shaw, & Ky (1993, October 14).

Rogers (1977).

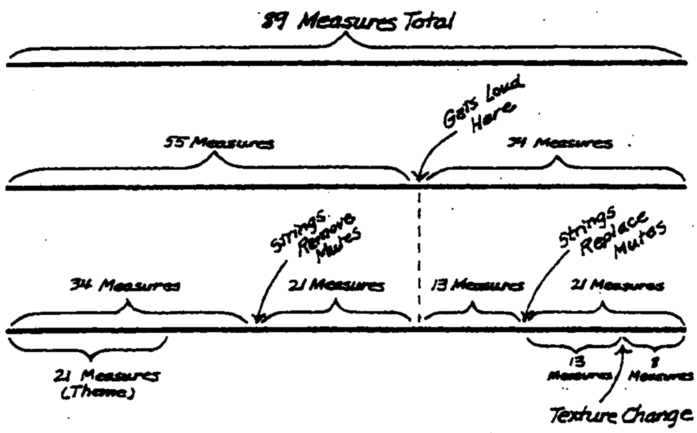
4.3 Music of Bartok

Bela Bartok (1881-1945) was one of the most outstanding composers of the 20th century. His music combined the best of Hungarian folk music and Western classical music. According to Lendvai (p. 97), "Bartok achieved something that no one had before his time, the symbolic handshake between East and West: a synthesis of the music of Orient and Occident."

Bartok was born in the Hungarian district of Torontal. His parents were both schoolteachers, but his father died when Bela was eight years old. A child prodigy, he performed in public at age 10. He studied piano and composition at the Budapest Conservatory where he was steeped in the German influence which predominates Western classical music. He began studying Hungarian folk tunes, but along with Kodaly, rebelled against the conventional view of such music. After the First World War, his fame as a composer and pianist spread throughout Europe and America. In 1940 he went to America to perform piano recitals with his wife, Ditta Pasztory Bartok. He received an honorary Doctor of Music from Columbia University. His well-known *Concerto for Orchestra* was composed while he was in the U.S., though his health was failing and he was in financial difficulties. He died at the age of 64 in New York.

According to Evans, the "chief characteristic of Bartok's music throughout his career has been its intense dynamism and rhythmic strength. As expressed in his music, his is no gentle spirit."

Bartok frequently incorporated Fibonacci numbers in his compositions. For example, the first movement of his *Music for Strings, Percussion, and Celeste* consists of 89 measures. The point at which the piece gets loud is just after 55 measures, with 34 measures remaining. Other divisions suggest sections of 34, 21, 13, and 21 measures, as seen in the diagram below:



From Garland, T. (1987). *Fascinating Fibonacci: Mystery and magic in numbers. Palo Alto, CA: Dale Seymour Publications*, p. 37.

Movement III of this piece also reflects these numbers, with the first theme making up 21 measures, the second theme 13 measures, the "roaring of the wind" section comprising 34 measures, and recapitulation of the two themes, 13 and 8 measures respectively.

Bartok's use of chords and intervals also reflects Fibonacci numbers. He frequently uses major seconds, minor thirds, perfect fourths, minor sixths, and augmented octaves. These intervals consist of the following numbers of half-tones: 2, 3, 5, 8, and 13. Lendvai's analysis discusses this in much more detail, the understanding of which requires fairly extensive knowledge of music theory.

The golden section is seen frequently in Bartok's compositions. In the first movement of his *Sonata for Two Pianos and Percussion*, there are 443 measures, with the recapitulation beginning at bar 274. (Calculate the ratio!) The first movement of *Contrasts* has 93 measures, with the recapitulation beginning in the middle of bar 57. Other examples of the golden section in his compositions are in *Divertimento* (first movement), *Free Variations, Broken Chords,* and *From the Diary of a Fly.* Lendvai states, "It can be observed that GS [golden section] always coincides with the *most* significant turning point of the form" (p. 20).

It is unclear whether Bartok purposely made use of Fibonacci numbers and the golden section in his music. However, because of his interest in plant growth and because of his frequent use of these numerical patterns, one could hypothesize that it was purposeful. He once wrote, "We follow nature in composition," (Lendvai, p. 29) and his favorite plant was the sunflower. Lendvai says that he was extremely happy whenever he found pine cones on his desk. He said "folk music is a phenomenon of nature. Its formations developed as spontaneously as other living natural organisms; the flowers, animals, etc." (quoted in Lendvai, p. 29).

Bush, in his introduction to Lendvai's analysis of Bartok's music, says that Bartok "evolved for himself a method of integrating all the elements of music; the scales, the chordal structures with the melodic motifs appropriate to them, together with the proportions of length as between movements in a whole work...according to one single basic principle, that of the Golden Section." (p. vii). He maintains that Bartok refrained "as far as is known, from expounding to anyone during his lifetime, either in writing or by word of mouth," (p. vii) the theoretical principles which he worked out. Recent musicology scholarship, however, reveals more convincing evidence that Bartok's use of Fibonacci and *phi* were purposeful.

Sources:

Evans (1975).

Lendvai (1971).

Unit 5

Literature

5.1 Poetry

Poetry and music have much in common, including beauty that seems to defy definition. However, since early times people have attempted to measure and define beauty in music and poetry. Pythagoras, who lived in the 6^{th} century B.C.E., related everything to mathematics, including music and poetry. His quote "The beautiful in sound must depend upon a succession of notes related to each other and a prime by the simplest possible ratios" exemplifies this principle. Early poetry was read aloud because only the elite could read; therefore it could be considered a type of music. A Princeton scholar, Dr. George Duckworth, analyzed Vergil's *Aeneid* and reportedly discovered use of the Fibonacci sequence to create golden proportions.

Throughout the ages, mathematicians and poets have attempted to analyze poetry in a mathematical way. Edgar Allan Poe in his essays "The Rationale of Verse" and "The Philosophy of Composition" discusses his method of analysis. He points out the importance of both harmony and melody in poetry (Poe, p. 50).

In a broader way, philosophers, mathematicians, and others have attempted to quantify the measurement of beauty. Descartes, Euler, and Sylvester were three mathematicians who expressed views on this subject. Philosophers such as Burke, Kant, and Spencer discussed the problem, and Frans Hemsterhuis said "The beautiful is that which gives the greatest number of ideas in the shortest space of time." (Linn, p. 66).

George David Birkhoff, an American mathematician, (1884-1944), devised a system for measuring beauty in his book *Aesthetic Measure*, but was careful to point out the dangers of such a system. He analyzed music, art, and poetry with his numerical system. His criteria for poetry included the following:

rhyme;
repetition of vowel sounds
alliteration (words with the same beginning letter);
musical vowel sounds ("a" as in art; "u" as in tuneful, "o" as in ode);
ease of speaking

He created a mathematical formula for M, musical quality in poetry, as follows:

$$M = \frac{O}{C} = \frac{aa + 2r + 2m - 2ae - 2ce}{C}$$

where O is harmony, symmetry and order: made up of *aa*—alliteration and assonance, *r*—the element of rhyme, *m*—the number of musical vowels (as he defines them), *ae*—alliterative and assonantal excess, *ce*—the element of consonantal excess; and C is the complexity of any part of a poem ("...the total number of elementary sounds therein, increased by the word-junctures involving two adjacent consonantal sounds of the same line, which do not admit of liaison.") (p. 177).

Examples of his analysis follow:

In Xanadu, did Kubla Khan
A stately pleasure-dome decree:
Where Alf, the sacred river, ran
        Through caverns measureless to man
        Down to a sunless sea.

From Coleridge's *Kubla Khan* (M = .83)

Tell me not, in mournful numbers,
  Life is but an empty dream!—
For the soul is dead that slumbers,
  And things are not what they seem.

From Longfellow's *A Psalm of Life* (M = .73)

Come into the garden, Maud,
        For the black bat, Night, has flown,
Come into the garden, Maud,
        I am here by the gate alone;
And the woodbine spices are wafted abroad,
        And the musk of the roses blown.

From Tennyson's *Maud* (M = .77)

In discussing measurement of aesthetic value, Birkhoff maintains that "...it is the fundamental problem of aesthetics to determine, within each class of aesthetic objects, those specific attributes upon which the aesthetic value depends." (p. 3). He proposes similar systems for the measurement of beauty in art and music.

Assignment:

Write a one-page essay on the following questions:

Do you believe there is an objective way to measure beauty? Does a mathematical formula for aesthetic value take away some of the mystique of poetry, music, or art for you? How do you explain the nearly universal appeal of some works of art?

Project ideas:

Poe's method of analysis of poetry

Duckworth's analysis of Vergil's *Aeneid*

George David Birkhoff

Sources:

Birkhoff (1933).

Garland (1987).

Linn (1974).

Poe (1968).

5.2 Limericks

A limerick is a short humorous verse made up of 5 lines and a total of 13 beats grouped in 3's and 2's. Do those numbers sound familiar? Many limericks also have 8 syllables in the first and second lines, 5 in the third and fourth lines, and 8 in the last line, resulting in a total of 34 syllables.

Limericks are often described as "indecorous." We will limit our examples to those which are humorous but only mildly indecorous at worst. However, the academic community seems to have decided, according to Baring-Gould, that "...smutty stories and ribald verse are socially significant" (p. 12), and of course limericks are our major literary connection with Fibonacci numbers.

Clifton Fadiman, in an essay in *Any Number Can Play,* wrote of the limerick's perfection: " 'It has progression, development, variety, speed, climax, and high mnemonic value.'" (quoted by Baring-Gould, p. 16).

Here are some examples that relate to topics in our study.

The thoughts of the rabbit on sex
Are seldom, if ever, complex;
    For a rabbit in need
    Is a rabbit indeed,
And does just as a person expects. (Baring-Gould, p. 71)

The golden ratio symbol, *phi*, you'll recall, came from the Greek sculptor Phideas. He is remembered in the following:

There once was a sculptor named Phidias
Whose manners in art were invidious:
    He carved Aphrodite
    Without any nightie,
Which startled the ultrafastidious. (Baring-Gould, p. 12)

'Tis a favorite project of mine
A new value of *pi* to assign,
    I would fix it at 3
    For its simpler, you see,
Than 3 point 1 4 1 5 9.
(Harvey Carter, Historian, from Baring-Gould, p. 13)

There was a young lady named Bright
Whose speed was far faster than light;
She went out one day,
In a relative way,
And returned the previous night.
(A. H. Reginald Buller, Botanist, from Baring-Gould, p. 13)

Many limericks have inspired sequels. One of the best-known limerick sequences is the following:

There was an old man of Nantucket
Who kept all his cash in a bucket;
But his daughter, named Nan,
Ran away with a man,
And as for the bucket, Nantucket.

Pa followed the pair to Pawtucket
(The man and the girl with the bucket)
And he said to the man,
"You're welcome to Nan,"
But as for the bucket, Pawtucket.

Then the pair followed Pa to Manhasset,
Where he still held the cash as an asset;
And Nan and the man
Stole the money and ran,
And as for the bucket, Manhasset.

Two well-known masters of the limerick were Edward Lear (1812-1888) and Ogden Nash (1902-1971). But statesmen, scientists, mathematicians, and others from all walks of life have contributed to the limerick literature. Numerous web sites invite contributions from amateur versifiers.

Assignment:

Write two limericks related to this course or to mathematics.

Project suggestions:

Edward Lear

Sources:

Asimov, I. (1992).

Baring-Gould, W. (1968).

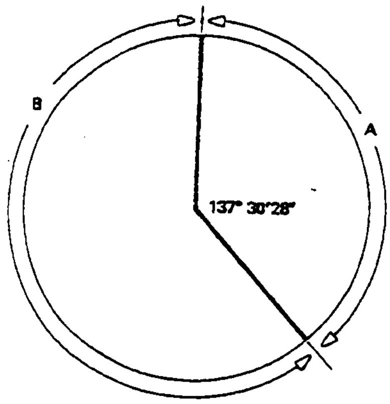
Unit 6

Nature

6.1 Plant Growth

The term *phyllotaxis* (or *phyllotaxy*) refers to the patterns in which leaves grow from a stem. Fibonacci numbers have been found in these patterns. According to Bell, "The study of phyllotaxis has led to an extensive terminology and also to a preoccupation with the Fibonacci series." (p. 218)

The phyllotaxis of plants is often described in terms of fractions or ratios: ½, 1/3, 2/5, etc. In ½ phyllotaxis, there are two leaves per node, opposite each other, with 180° between them; 2/5 means there is an angle of 144° between the leaves which spiral around the stem (2/5 of 360° is 144°). These fractions found in plant growth are almost always ½, 1/3, 2/5, 3/8, 5/13, 8/21, ... We can see these as Fibonacci ratios where the numerators and denominators are successive Fibonacci numbers. The angles between the leaves in each of these fractions are 180°, 120°, 144°, 135°, 138.46°, 137.14°, ... The ratio seems to approach the value 137.52...° which divides the area of a circle into the golden section:



From Bell, A. (1991). *Plant form: An illustrated guide to flowering plant morphology.* Oxford: Oxford University Press, p. 221

It seems that these growth patterns enable the maximum amount of sunlight to reach each leaf. If the leaves (and branches) were spaced up the stem at intervals of exactly 137° 30' 28", then no leaf or branch would be directly above another and therefore not shaded by another.

Similar patterns can be found in fruits of many plants. Pine cones, for example, exhibit a spiral growth pattern. According to Garland (p. 9), Brousseau shows that there is a 99 percent likelihood that the numbers of spirals on any pine cone will be Fibonacci numbers. Pineapples, artichokes, cauliflower, and sunflower heads are other plants exemplifying such patterns in their spirals.

The numbers of petals on most flowers are Fibonacci numbers. Lilies and irises have three petals; buttercups and larkspurs, five; cosmos and delphiniums, eight; field daisies, 34, and so on. It is rare to discover a flower with four or six petals; Conway and Guy (p. 123) state that those flowers with six petals are often organized as two generations of three petals each. We have seen earlier that pentagons and pentagrams exhibit both Fibonacci numbers and the golden ratio. Garland says that more flowers bloom in pentagons than any other shape.

We must remember, however, that not all plants exhibit Fibonacci numbers. Knott (p. 14 of his web page link, "Fibonacci numbers and nature,") quotes H. S. M. Coxeter: " 'It should be frankly admitted that in some plants the numbers do not belong to the sequence of f's[Fibonacci numbers] but to the sequence of g's[Lucas numbers] or even to the still more anomalous sequences

$$3, 1, 4, 5, 9, \ldots \text{ or } 5, 2, 7, 9, 16, \ldots$$

Thus we must face the fact that phyllotaxis is really not a universal **law** but only a fascinatingly prevalent **tendency**.'"

Sources:

Bell (1991).

Conway & Guy (1996).

Garland (1987)

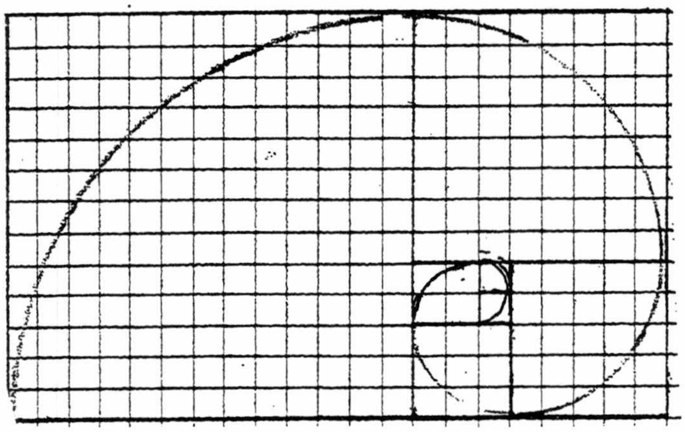
Knott [on-line]

6.2 Logarithmic Spirals

One of the most beautiful of all nature's creations is the chambered nautilus. A cross-section of this shell is shown below.

This is approximately what mathematicians call a logarithmic or an equiangular spiral. We can construct such a spiral, which exhibits Fibonacci numbers, as follows:

Begin with a 1-unit square. Add another 1-unit square to it. Using the long side of the resulting rectangle, construct a 2-unit square attached to your two 1-unit squares. Using the 3-unit side of the resulting rectangle, construct a 3-unit square. Continue this process until you run out of paper! Now draw connecting arcs, using the corner of the square as the center of a circle and the side of the square as the radius, in each square beginning in the original square as follows:



You will see that the sides of the squares are consecutive Fibonacci numbers. Golden rectangles abound in the finished construction. The continuous arc you have drawn is an approximation of a logarithmic or equiangular spiral. Huntley derives the polar equation for this curve (for details, see Huntley, pp. 172-173):

$$r = ae^{\theta \cot \alpha}$$

In addition to the chambered nautilus shell, examples of this spiral in nature are thought to be horns of antelopes, wild goats and sheep, elephant tusks, and spider webs—but of course these are more difficult to measure. Ghyka also lists *Haliotis Splendens* or Abalone Shell of California as an example similar to the chambered nautilus.

Many seashells grow in a spiral shape as the animal living in the "house" grows and adds rooms to her house. Each new room becomes progressively larger, as the outer surface grows more than the inner surface. According to Stevens (p. 89), "forms curl so that the faster growing or longer surface lies outside and the slower growing or shorter surface lies inside, there being more room outside than inside." He maintains that if the rates of growth of two surfaces are unequal, the material curls so that the slower growing surface is inside the faster growing surface. This produces a spiral shape, which Stevens considers one of the basic patterns of nature.

T. A. Cook further maintains that the spiral is fundamental to the structure of plants, shells, and the human body, and a key to understanding organic nature. He says that the spiral or helix may lie at the core of life's first principle—that of growth. However, he points out subtle differences between nature and mathematical perfection: "...nothing which is alive is ever simply mathematical.... The nautilus is alive and, therefore, it cannot be exactly expressed by any simple mathematical conception." (p. ix)

Sources:

Cook (1979).

Garland (1987).

Ghyka (1977).

Hoffer (1975, October).

Huntley (1970).

Stevens (1974).

6.3 The Human Body

Some believe that the Egyptian and Greek ideal of the golden proportion was based on proportions of the human body. It is easy to find these proportions in Greek classical sculpture. Although proportions of an individual person may vary somewhat from the "golden" ideal, here are some examples to calculate:

(1) Measure your height, compare it with the distance from your waist to the floor.

(2) Now compare the distance from your waist to the floor with the distance from the top of your head to your waist.

(3) Measure your head from its top to your chin, compare it with the width of your face.

(4) Measure from your waist to your knee, compare it to the distance from your knee to the floor.

(5) Measure from your neck to your waist, compare it to the distance from your neck to the top of your head.

(6) Bend your index finger (pointing finger) as far as you can, and observe your own golden rectangle! Measure to see how close it comes.

The occurrence of spiral-like formations in the human body is noted in detail by Cook. He points out fingerprints, umbilical cord, muscular fibers of the heart, the cochlea, and others as examples of types of spirals. He also discusses right- and left-handedness as possible results of reversing of certain spirals. Was Leonardo da Vinci left-handed? Cook believes so, based on evidence that he often reversed letters and wrote from right to left. We know that many identical twins have reverse-handedness.

Sources:

Cook (1979).

Garland (1987).

Newman & Boles (1992).

6.4 Reproduction of Rabbits and Bees

Recall that Fibonacci's original problem was about rabbit reproduction. We observed that rabbits really do not reproduce exactly that way. However, there is an example in nature of exactly that pattern of reproduction—the male bee.

Bees reproduce in this way: the female has both a mother and a father (comes from a fertilized egg), but the male, called a "drone," has only a mother since he comes from an unfertilized egg. Ron Knott (p. 4 of "Fibonacci numbers and nature", a link on his web page) provides us with an explanation of the drone bee's family tree:

1. He has 1 parent, a female.
2. He has 2 grandparents, since his mother had two parents, a male and a female.
3. He has 3 great-grandparents: his grandmother had two parents but his grandfather had only one.
4. How many great-great-grandparents did he have?
5. How many great-great-great grandparents did he have?

Now calculate these numbers for a female bee. Fill in the chart below.

	Male bee	Female bee
Parents		
Grandparents		
Great-grandparents		
Great-great-grandparents		
Great-great-great-grandparents		

What about the family tree of humans? What sort of sequence results when you trace your family back five or six generations?

Knott also mentions a variation on Fibonacci's problem written by the English puzzlist, Henry E. Dudeney (1857-1930) as follows:

> *If a cow produces its first she-calf at age two years and after that produces another single she-calf every year, how many she-calves are there after 12 years, assuming none die?*

Does this puzzle result in a Fibonacci sequence? Do you think it is more realistic than the rabbit problem?

Assignment:

Trace your family tree back to your great-great-great grandparents. Draw a family tree with names if possible. How many parents, grandparents, great-grandparents, etc., would you have if they were all alive?

Sources:

Garland (1987).

Knott [on-line]

6.5 Astronomy

The Swedish astronomer Carl-Gustav Danver discovered logarithmic spirals in the outward twirling of galaxies in space. Cook's work contains a photograph of spiral nebula on page 2.

Fibonacci numbers have been found in a formula used to predict the distances of the moons of Jupiter, Saturn, and Uranus from their respective planets. B. A. Read used statistics to infer patterns on these distances. He concluded that (p. 428) "The Fibonacci Series is shown to predict the distances of the moons of Jupiter, Saturn and Uranus from their respective primary. The planets are shown to have a trend which follows the Fibonacci Series with individual offsets attributed to planetary densities." He further hypothesized that (p. 437) "a particular moon's position is dependent upon the positions of the previous two moons closer to the primary." As we know, each Fibonacci number is dependent upon the previous two numbers.

Sources:

Cook (1979).

Garland (1987).

Hoffer (1975).

Read (1970).

Unit 7

Management and Economics

7.1 Patterns in the Stock Market and Commodities Trading

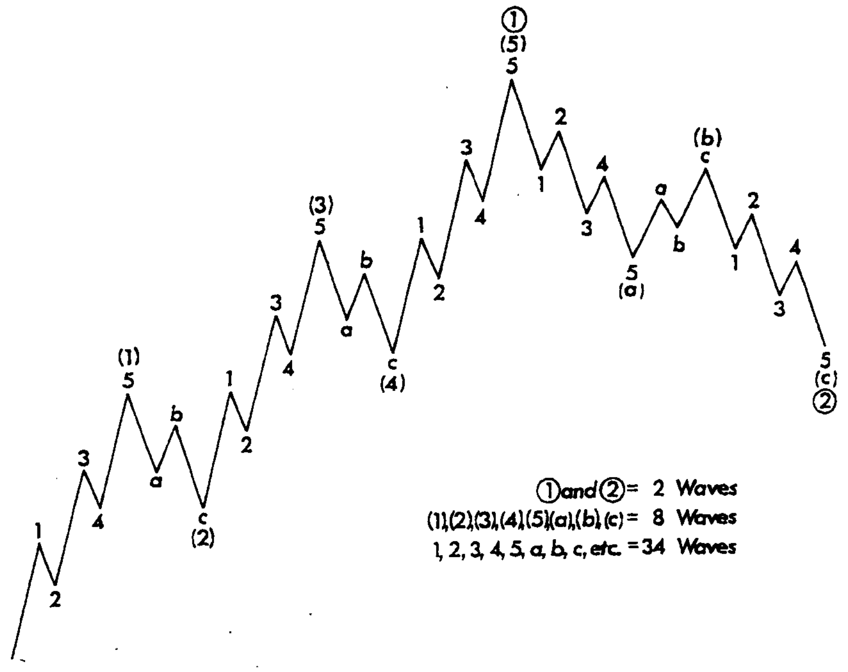
The stock market is often a reflection of a country's economic growth. Many factors are thought to influence its ups and downs, such as employment, interest rates, world events, and general prosperity. Naturally, investors would like to understand patterns of these ups and downs, and various theories have been promoted. One interesting example is known as the Super Bowl Indicator: stocks do poorly in any year that an original member of the AFL wins the Super Bowl, and do well after a victory by any other NFL team. This theory did not hold up in 1998, when the Denver Broncos (original AFL member) won the game. Another theory is called the January effect, which maintains that stocks do well in any year in which they do well in January.

Ralph Nelson Elliott (1871-1948) was an accountant and business consultant who, in the 1930's, analyzed stock prices and formulated a theory about fluctuations in the stock market. His theory relates these fluctuations to Fibonacci numbers. His book, *The Wave Principle*, was published in 1938. In it he writes:

> "'No truth meets more general acceptance than that the universe is ruled by law. Without law, it is self-evident there would be chaos, and where chaos is, nothing is.... Very extensive research in connection with...human activities indicates that practically all developments which result from our social-economic processes follow a law that causes them to repeat themselves in similar and constantly recurring serials of waves or impulses of definite number and pattern.... The stock market illustrates the wave impulse common to social-economic activity.... It has its law, just as is true of other things throughout the universe.'" (quoted in Prechter, *The R. N. Elliott Story*, p. 7).

In 1939 Elliott first published his contention that the Dow Jones Index moves in rhythms. He compared such rhythms to the tides—low tide follows high tide, reaction follows action. According to Fischer, Elliott maintained that " 'All human activities have three distinctive features, pattern, time and ratio, all of which observe the Fibonacci summation series'" (p. 12).

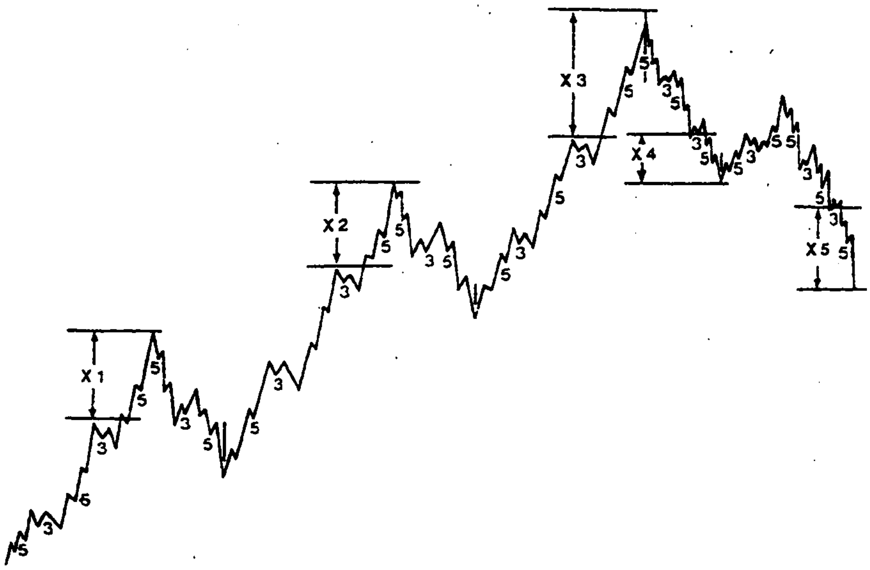
The "Elliott Wave Principle," as his theory is called, says in general that mankind's progress (including the stock market) does not occur in a straight line, does not occur randomly, and does not occur cyclically. The pattern of progress is more a "three steps forward, two steps backward" one, a form that he thought nature prefers. The pattern Elliot describes for the stock market consists of impulse waves and corrective waves. An impulse wave is composed of five subwaves and moves in the same direction as the trend of the next larger size. A corrective wave is composed of three subwaves and moves against the trend of the next larger size. This is illustrated below:



from Frost & Prechter (1984). *Elliott Wave Principle.* Gainesville, GA: New Classics Library (p. 21)

The theory gets much more complicated, of course. But many feel that it can be used to predict lows and highs of the stock market; that it does not provide certainty about future trends, but provides "an objective means of assessing the relative probabilities of possible future paths for the market." (Capsule summary..., p. 3)

Fischer believes that the Fibonacci ratio is one of the most important mathematical presentations of natural phenomena ever discovered. He applies this ratio to equity and commodity price swings as well as to the stock market. An example of his recommendations, if one follows the Elliott principles, is that one should not invest in an uptrend at the end of wave 3 (p. 19). He reasons that this theory cannot resolve the dilemma of whether (1) a correction is part of a long-term trend, or (2) a correction is the beginning of a new trend in the opposite direction. The following chart shows areas of uncertainty out of the Elliott concept:



from Fischer, R (1993). *Fibonacci applications and strategies for traders.* New York: John Wiley & Sons, Inc.

The Capsule summary web site generalizes Elliott's theories as follows:

"On a philosophical level, the Wave Principle suggests that the nature of mankind has within it the seeds of social change. As an example simply stated, prosperity ultimately breeds reactionism, while adversity eventually breeds a desire to achieve and succeed. The social mood is always in flux at all degrees of trend, moving toward one of two polar opposites in every conceivable area, from a preference for heroic symbols to a preference for anti-heroes, from joy and love of life to cynicism, from a desire to build and produce to a desire to destroy. Most important to individuals, portfolio managers and investment corporations is that the Wave Principle indicates in advance the relative magnitude of the next period of social progress or regress." (p. 4)

Sources:

Capsule summary of the Wave Principle

Currier (1999, January 25).

Fischer (1993).

Frost & Prechter (1984).

Prechter (1994).

7.2 Univariate Optimization – Fibonacci Search

In the field of management science, there are several mathematical methods of finding the maximum or minimum value of a function over a closed bounded interval. Four of these are given below:

(1) Uniform search method $n \geq \dfrac{b_1 - a_1}{l/2}$

(2) Dichotomous search method $(1/2)^{n/2} \geq \dfrac{1}{b_1 - a_1}$

(3) Golden section method: $(0.618)^{n-1} \geq \dfrac{1}{b_1 - a_1}$

(4) Fibonacci search method: $F_n \geq \dfrac{b_1 - a_1}{l}$

In each case, the interval in which one is interested is $[a_1, b_1]$, the length of the final interval of uncertainty is l, and the number of iterations required is n. The most efficient of these algorithms is the Fibonacci search method. Though calculus is often used to find maxima and minima, it can only be used for continuous functions. Examples of functions which are unimodal (exactly one local minimum or maximum), but whose minima cannot be found using calculus, are given below.

(a) (b) (c)

from Berman, G. & Fryer, K. (1972). *Introduction to combinatorics.* New York: Academic Press, p. 251.

Here is an example of how the *Fibonacci algorithm for the minimum* works:

Suppose we want to find the minimum value of the function $y = |x+3|$ in the interval (-4, 1). The width of the interval is 5, so let's take n = 10 so that $F_{10} = 55$. We divide the interval into 55 sections, each of which had width 1/11, and label the points of division 0, 1, 2, ... 54, 55 as shown below:

Now let's mark off 21 units and 34 units of 1/11 to the right of -4 so that

$$x_1 = -4 + F_8(1/11) = -4 + 21(1/11) = -23/11$$

$$x_2 = -4 + F_9(1/11) = -4 + 34(1/11) = -10/11$$

$$f(x_1) = |-23/11 + 3| = 10/11$$

$$f(x_2) = |-10/11 + 3| = 23/11$$

and obviously $f(x_1) < f(x_2)$. This means the minimum is in the interval (-4, x_2) and we can forget (x_2, 1).

Now we let $x_3 = -4 + F_7(1/11) = -4 + 13/11 = -31/11$

and we see that $f(x_3) < f(x_1)$ so we can discard the interval $(x_1,\ x_2)$.

Continuing this process with F_6, F_5, etc., we find smaller and smaller intervals that contain the minimum. Ultimately, we find that the minimum is located approximately at $x = x_6 = -3$ with a possible margin of error of 1/11, and the minimum value of the function is approximately 0.

We happen to know, of course, that the minimum value of this function is exactly zero, so the method works.

Sources:

Bazaraa, M., Sherall, H., & Shetty, C. (1993).

Berman, G. & Fryer, K. (1972).

Unit 8

Philosophical Considerations

8.1 Why Is the Golden Ratio Appealing?

We have observed many examples of the almost universal aesthetic appeal of the golden ratio--from road signs to ancient architecture to measures of music. Is this purely cultural? Perhaps it comes from ancient Greek ideals--after all, much of Western civilization can be traced to the Greeks. Or maybe it is an innate, inborn human preference. A number of theories have been advanced to explain the appeal. Rogers (1977) outlines some of them.

(1) The principle of "dynamic symmetry" as discussed by Zeising, Church, Coleman, and Hambidge. Hambidge compares dynamic symmetry, that is, the golden section as used by the Greeks, and static symmetry, the square as used by Romans. The idea is that the golden section is the only ratio that is capable of reproducing the same proportion in two different ways from only two different lengths. This theory maintains that "incommensurable" or irrational dimensions are more naturally appealing than commensurable or integral dimensions.

(2) The Gestalt principle of closure, or resolution of tension. This theory says that the work of the brain is dependent on patterns that strive for balance. New brain research may shed light on this theory.

(3) Perimetric hypothesis. The perimeter of our field of vision, according to ophthalmologists, is approximately a golden rectangle. Does this mean that the most beautiful rectangle for a one-eyed person would be a square?

(4) Learned factor of social setting. Although the golden section was the "common aesthetic ground for the largest number of people" (Rogers, p. 113), in some experiments Japanese people preferred squares.

(5) Geometric analogies of the three basic civilizations. This theory states that the linear model was preferred by Chinese civilizations, the circular model in India, and the logarithmic model in Western Europe.

(6) Environmental influences. Western artists have been more attracted to mathematical models than Oriental artists. This probably derives from classical models of ancient Greece.

(7) Archetype theory. Our brains have "well-worn memory grooves" or genetically inherited schema.

(8) Avoidance of extremes. The major portion should dominate the minor but not too strongly. The best ratio is somewhere between equality and doubling.

(9) Maternal heartbeat. The average ratio of long to short beats of the human heart has been found to be .62. Infants are influenced by this before and after birth and find it comforting and appealing.

(10) Bird song. Golden proportions have been found in the music of birds, which humans have always heard.

(11) The "golden section hypothesis" as outlined by Benjafield and Adams-Webber. This theory states that positive and negative aspects of life (sometimes known as Yang and Yin) usually fall into golden proportions, with the positive (Yang) occupying .62 of the whole and the negative (Yin) occupying .38. This theory was tested by several experiments; for example, by having subjects choose positive and negative adjectives from pairs of words. This may occur because the smaller proportion will stand out more—when subjects are asked to arrange colors to make one stand out against the others, that color is used about 38% of the time. Benjafield and Adams-Webber conclude (p. 14) that "The golden section hypothesis suggests that, while we construe most events positively, we attempt to create a harmony between positive and negative events such that the latter make a maximal contribution to the whole."

Which of these theories do you think are most valid? Why do you think the golden section is appealing in art and music?

Project ideas:

The Benjafield-Adams-Webber study

Dynamic symmetry

Sources:

Benjafield & Adams-Webber (1976).

Hambidge (1923).

Rogers (1977).

8.2 What Factors Contribute to the Making of a Genius?

In the course of our investigation of Fibonacci numbers and the golden section, we have looked at a few individuals who were unquestionably considered geniuses: Mozart, Bartok, Leonardo da Vinci, and, of course, Fibonacci himself. Many other names will spring to mind in the fields of mathematics, the arts, sciences, and humanities. Does "genius" result from inborn intellectual talents, or does the environment play a part? Although this topic does not directly relate to our topic, it is an interesting question to contemplate.

The question of genius has intrigued people for centuries. Defining the term is difficult, and establishing criteria for a given definition is tricky at best. We will use the following definition: A genius is an eminently gifted person who creates and contributes something new and useful in the arts, sciences, or humanities and is widely recognized for this contribution.

Several studies have been published on this topic, and each used different criteria for identifying the people they considered geniuses. Goertzel and Goertzel chose their subjects based on people who lived into the 20th century and who had at least two books about them in the biography section of the Montclair, New Jersey, public library. Berry (Radford, 1990) studied Novel prize winners. Eisenstadt chose individuals whose entries in the *Encyclopedia Britannica* or the *Encyclopedia Americana* occupied at least one-half page.

Although selection criteria vary widely, factors that contribute to the phenomenon of genius emerge from such case studies. Researchers today generally support the importance of inborn talent and genetic influences, particularly in the arts, though few would believe these are the only pathways to genius. Social and cultural factors play a part, as do religious/ethnic background, gender, and birth order. Childhood experiences and other environmental influences are also believed to be major contributors.

For centuries it was believed that genius was due to divine inspiration. Ideas were breathed into people by divine force or supernatural powers. This provided a way of explaining unusual gifts, and as well it helped explain the connection between genius and madness. Frances Galton, 1822-1911, was the first major researcher to study genius; he concluded that genetics played the most important part. Galton was primarily an anthropologist who came from an intellectual, inbred family that included Charles Darwin. He showed that eminent people have many eminent relatives, thereby supporting his view that intellectual activity depends on biological processes or inborn gifts. He maintained that even the capacity to work hard was an inherited trait. Although genetics is not now considered to be the whole story, his research influenced later scholars to consider the effects of early experiences and personality characteristics.

The Terman studies in the early 20th century were an important step in the study of gifted individuals. In one study, 1300 gifted children were tracked for 60 years. It was found that IQ does not necessarily predict transcendent achievement (Ochse). The second Terman study analyzed 300 famous historical figures, concluding that their success was largely due to perseverance.

Freud was influential in the development of explanations of genius, though, like many of his views, his explanation is not considered to tell the whole story. He believed that the sublimation of sexual and aggressive energy into creative work resulted in artistic works. He admitted, however, that "psychoanalysis 'can do nothing toward elucidating the nature of the artistic gift, nor can it explain the means by which the artist works'" (quoted by Ochse, p. 15). Later a more humanistic view of creativity was espoused by such people as Adler, Rank, and Maslow. They believed, generally, that self-actualization was the motive for creativity, and that people have a positive drive to improve the self. These explanations largely ignored the intellectual aspects involved in creative thinking.

Inborn or inherited talent is surely a prerequisite for genius. This may be particularly true in the visual arts and music. Winner (p. 273) maintains that "drawing precocity has an innate, biological component" and that no amount of practice will produce a visual artist without such talent. The same is surely true of musical talent. Some researchers claim, according to Andrew Solomon in his study of the pianist and composer Evgeny Kissin, that musical predisposition occurs in children who are hypersensitive to sound, who are "driven to order the noise around them, so that it becomes less disturbing" (p. 119). Many such gifted children use music as a way to communicate, and they become more proficient with that form of communication than with language. Solomon recounts that Kissin could sing a Bach fugue his sister was practicing at 11 months, and he began improvising and composing original music at age three. It is well known that some people are born with perfect pitch and therefore able to sing a certain pitch on command or to tell readily when an instrument or an orchestra is out of tune. This clearly is an inborn talent, though of course without musical training one would not be aware of this gift.

Ochse recounts that clusters of geniuses have appeared throughout history during "golden ages." American anthropologist Alfred Kroeber researched fluctuations in creativity and concluded that these clusters do not occur strictly by chance. It appears from his study that "creativity in a society waxes and wanes as a cultural pattern . . . becomes saturated and its possibilities become exhausted" (Ochse, p. 50). Perhaps scientific contributions are more dependent on cultural and environmental influences than artistic contributions. Scientific discovery often depends on previous developments, which helps explain why such advances as the invention of the calculus was made by two individuals working independently, Newton and Leibnitz, at about the same time. Scientists stand on the shoulders of their predecessors in a unique way.

Though it is generally acknowledged that musical talent is largely innate, there is some disagreement. DeNora and Mehan argue that Beethoven's success as a composer, for example, depended on an "organization predisposition for musical celebrities" (p. 166) in Vienna during the 1790's. They point out that recognized geniuses have been predominantly male and musical geniuses predominantly Germanic, which reinforces the strength of cultural influences.

Creative writers acknowledge the contributions of others to their work. Ochse quotes Goethe as saying, " 'The greatest genius will never amount to anything if he wants to limit himself to his own resources. . . . It's simply unconscious conceit not to admit frankly that one is a plagiarist'" (pp. 53-54). Obviously cultural influences have a bearing on the development of talent in any field.

Montuori and Purser argue against the isolationist or "lone genius" myth, maintaining that a more contextual view of creativity is needed. Collaboration and independence seem to them to be necessary in an ecological view of genius. They quote Rogers who defined the creative process as: " 'the emergence in action of a novel relational product, growing out of the uniqueness of the individual on the one hand, and the materials, events, people, or circumstances of his life on the other'" (p. 82). This view clearly points out the importance of cultural factors.

According to Ochse, there has been a disproportionate number of high achievers among Jews, and Catholics have been under-represented in recognized highly creative people. Statistics help illustrate this: eighty percent of American Jews go to college, compared to forty percent of gentiles; twenty-seven percent of American Nobel prize winners have been Jews, but only three percent of the population is Jewish. It is difficult to explain this phenomenon. Heredity is obviously a factor, and it is believed that culture plays a more significant role than religion.

One important "genius cluster" in history was the mathematical center at Gottingen in Germany during the early 20th century. Tragically, in the spring of 1933 the Nazi regime dismissed all "racially undesirable" professors (Jews), and many of them, including Albert Einstein, Emmy Noether, Hermann Weyl, and Paul Bernays, emigrated to the United States. When David Hilbert, who had remained at Gottingen, was asked by a Nazi official how mathematics was progressing now that the university was freed of Jewish influence, he was reported to have replied, " 'Mathematics at Gottingen? There is really none any more'" (Burton, 1991, p. 632). These events helped shape the Institute for Advanced Study at Princeton, largely due to the presence of Einstein, Weyl, and Kurt Goedel.

Gender issues in occurrence of genius are difficult to analyze because of the male-dominated culture up to the latter part of the 20th century. Any listing of eminent creative people up to the present time will contain more males than females. However, the percentage of females attaining literary eminence is higher than in other fields. One gender-related issue mentioned in the literature concerns parents of eminent persons: the Goertzel study found many dominating mothers, but few dominating fathers (often even failure-prone fathers), among their sample.

Creative individuals tend to be first-born or only children, according to studies reported by Ochse. This was borne out in general in the Goertzel study; however in their sample, politicians seemed more likely to be middle children. Schachter (reported by Ochse) found that there was a "marked over-representation of first-borns in college and graduate students in families of all sizes." Other studies support the conclusion that first-born children are more likely to be more intelligent than others, possibly due to a higher level of achievement motivation.

One of the most intriguing factors in the making of genius is the effect of childhood experiences. A common thread in the study of eminent individuals is the occurrence of misfortunes of childhood, such as a broken home, illness, physical handicap, rejecting or dominating parents, or bereavement. The Goertzel study reported that three-fourths of their sample endured such troubled youths. Interestingly, another common thread is the drive for achievement by one or both parents and the value placed on education. They report that in almost all of the homes in their study "there is a love for learning in one or both parents, often accompanied by a physical exuberance and a persistent drive toward goals" (p. 272). Similarly, Ochse reports that studies of early life-experiences of eminent people include a disproportionate number of those who were orphaned, experienced isolation and loneliness, were treated cruelly, or suffered ugliness, deformity, or disease. However, there was considerable intellectual stimulation in the homes of such children, and they learned to value achievement.

Therivel maintains that the "challenged personality" is a pre-condition for sustained creativity. He defines "challenged" as including both the presence of misfortunes such as those listed above and the supports offered by friendly help from parents or mentors, medical care, cultural advantages, education, and free time to pursue interests. He thus distinguishes "challenged" from "crushed" and "pathological" persons due to the presence of such supports. One possible explanation for the occurrence of giftedness in so many persons with fairly miserable childhoods is the effect of periods of solitude in which such children may dream, think, write, and perhaps escape reality through reading, music, or art. Winston Churchill's biographer William Manchester stated that great men are frequently products of boyhood loneliness. Although this is often a result of circumstances in his or her life, it can be due partially to the child's own inclinations and nature. Einstein, for example, said, " 'I . . . have never belonged to my country, my friends, or even my immediate family with my whole heart; in the face of all these ties, I have never lost a sense of distance and a need for solitude—feelings which increase with the years.'" (quoted in Ochse, pp. 78-79).

Marvin Eisenstadt (1989) further believes that the specific misfortune he calls orphanhood (loss of one or both parents) often motivates a child to excel. He studied 573 subjects whose articles were given a half page or more in the 1963 *Encyclopedia Britannica* or the 1964 *Encyclopedia Americana* and found the incidence of parental loss in this sample much greater than in the general population. In a psychoanalytic approach to this phenomenon, he believes that the trauma of such loss may impose pressure on the psyche which leads to the creativity necessary to resolve issues of frustration, identity, and feelings of emptiness. He also points out that the loss of a parent often significantly changes the make-up and circumstances of the family.

We have seen that a true genius emerges when a number of factors combine to produce a truly creative person who contributes something of value to the culture. Although this may be a rare occurrence, the appreciation and understanding of that product may be nearly universal. There has been only one Mozart, but millions of people's lives are enriched by what he created. Fibonacci numbers and/or golden proportions may be employed consciously or unconsciously by the creator, but the masses of people who listen to or view the results do not need to be aware of the reasons for the exceptional balance and beauty to appreciate the result.

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8.3 What Does All This Mean?

The reader (student) is to write this section. The ubiquity of Fibonacci numbers and the golden ratio in the arts, sciences, and humanities is only one example of the connections between mathematics and other fields. Write a page or two on your conclusions from this study of such connections.

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SAMPLE LESSON PLANS

Course Goals

The primary goal of the course is to enable students to find connections between mathematics and other disciplines.

Other goals, referred to in the sample lesson plans, are:

(1) Willingness to explore mathematical patterns and to find them in the arts, humanities, natural sciences, and social sciences.

(2) Oral communication skills that enable effective comprehension, analysis, and expression.

(3) The integration of the scientific, artistic, and humanistic modes of inquiry.

(4) Increased interest and fluency in mathematics.

Lesson Plan #1

Unit 1 – Historical Background

Lesson – Introduction to the course

Goal(s) Addressed: #1

Objective of Lesson: The student will observe and extend numerical patterns and other patterns by completing worksheets. They will discuss how patterns can be used to predict.

Prerequisite Topics: None

Previous Assignment: None

Outline of Lesson:

(1) Instructor will introduce the course and discuss expectations listed in syllabus.

(2) Students will complete attached worksheets, working in pairs.

(3) Instructor will lead discussion based on responses to sequences. Students will observe that some of the worksheet examples could have more than one pattern. The usefulness of patterns in predicting behavior will be discussed.

Assignment: Read and be prepared to discuss "Life in the Middle Ages" in the textbook.

Materials: Copies of worksheets:
(1) Pattern activity from *Games,* April 1999
(2) "Search for a pattern..."

Assessment: Informal: Teacher observation of participation in discussion and activities

Search for a pattern . . .

For each of the following sequences, give the next element. State in your own words what you think the patterning rule is.

1. 80, 40, 20, 10, ______

2. James, Jill, Joan, John, ______

3. 1, 8, 27, 64, 125, ______

4. Styx, Beatles, Who, Kansas, ______

5. Alvin, Barbara, Carla, Dennis, ______

6. 6, 30, 150, 750, ______

7. 0, 2, 24, 252, ______

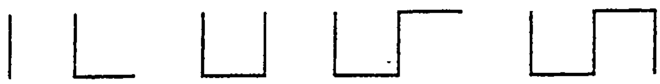
8. 1, 1, 2, 4, 7, 13, 24, ______

9. 1/2, 2/3, 3/4, 4/5, ______

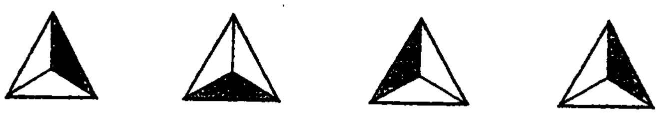
10. O, T, T, F, F, S, S, ______

Identify and draw the indicated term:

11. thirteenth: 

12. tenth: 

13. fifteenth: 

14. sixth: 

15. twentieth: 

Lesson Plan #2

Unit 1 – Historical Background

Lesson – Life in the Middle Ages

Goal(s) Addressed: #3

Objective of Lesson: The student will explore the culture of the Middle Ages

Prerequisite Topics: None

Previous Assignment: Read and be prepared to discuss "Life in the Middle Ages" in the textbook.

Outline of Lesson:

Guest lecturer or
Discussion of 12th and 13th centuries in Western Europe with: slides of monasteries, Gothic cathedrals, Pisa's Leaning Tower; tapes of music from period.

Materials: Slides, projector, tapes, tape player

Assessment: Informal, based on class discussion

Lesson Plan #3

Unit 1 – Historical Background

Lesson – Mathematics History up to the Middle Ages

Goal(s) Addressed: #2, #3, #4

Objective of Lesson: The student will see contributions of individuals and cultures to the development of mathematics before 1200 C.E.

Prerequisite Topics: Life in the Middle Ages

Previous Assignment: Read "Mathematics History up to the Middle Ages" in text; Exercises 1 and 2 (prepare oral report for #2)

Outline of Lesson:

(1) Class discussion of reading

(2) Oral reports from Exercise #2

(3) Solve Diophantus' Riddle, discuss approaches to problem

Assignment: Read "Leonardo of Pisa" in text.

Materials: None

Assessment: Formal: Evaluation of oral report using rubric
Informal: Teacher observation of participation in discussion and problem solving exercise

Lesson Plan #4

Unit 1 – Historical Background

Lesson – Leonardo of Pisa

Goal(s) Addressed: #1, #3, #4

Objective of Lesson: The student will recognize Leonardo's contributions to the mathematics of the Middle Ages.

Prerequisite Topics: Life in the Middle Ages, Mathematics History Up to the Middle Ages

Previous Assignment: Read "Leonardo of Pisa" in text

Outline of Lesson:

(1) Class discussion of Leonardo

(2) Lecture on characteristics of various number systems

Assignment: Exercises 1, 2 or 3, and 4 from "Leonardo of Pisa" in text

Materials: Transparencies of number systems, slide or photograph of statue of Leonardo.

Assessment: Informal: Teacher observation of participation in discussion.

Lesson Plan #5

Unit 1 – Historical Background

Lesson – Liber Abaci

Goal(s) Addressed: #1, #2, #4

Objective of Lesson: The student will solve number problems from *Liber Abaci* and recognize the significance of these problems.

Prerequisite Topics: Life in the Middle Ages, Mathematics History Up to the Middle Ages, Leonardo of Pisa

Previous Assignment: Exercises 1, 2 or 3, and 4 from "Leonardo of Pisa" (1.3) in text

Outline of Lesson:

(1) Teacher introduction of problems and their significance: number rather than geometry; demonstrating efficiency of Hindu-Arabic number system; practical problems.

(2) Students will work in groups or pairs to solve certain problems from *Liber Abaci*, will share their solutions and methods with class.

Assignment: Finish assigned problems from *Liber Abaci*.

Materials: Lists of *Liber Abaci* problems from text

Assessment: Formal: Evaluation of exercises 1, 2 or 3, and 4, Section 1.3
Informal: Teacher observation of problem-solving exercise.

Lesson Plan #6

Unit 2 – Mathematics

Lesson – Preliminary Considerations

Goal(s) Addressed: #1, #4

Objective of Lesson: The student will review (or learn) summation, factorial, and scientific notation.

Prerequisite Topics: None

Previous Assignment: None

Outline of Lesson:

(1) Instructor will demonstrate notation used for summation, factorial, and scientific notation.

(2) Students will practice using notation by working some exercises in section 2.1.

Assignment: Exercises in 2.1

Materials: None

Assessment: Informal: Teacher observation of student practice
Formal: Grading of assigned exercises in 2.1

Lesson Plan #7

Unit 2 – Mathematics

Lesson – Sequences and Series

Goal(s) Addressed: #1, #4

Objective of Lesson: The student will explore arithmetic, geometric, Fibonacci, Lucas, and Tribonacci sequences. They will learn to store the formula for the *n*th Fibonacci number in a graphing calculator.

Prerequisite Topics: Preliminary Considerations (Section 2.1), Leonardo of Pisa (Section 1.3)

Previous Assignment: Exercises in 2.1

Outline of Lesson:

(1) Instructor will review sequences from 2.1, will discuss the concept of recursive sequences, will demonstrate Fibonacci, Lucas, and Tribonacci sequences. The formula for the *n*th Fibonacci will be explained.

(2) Students will, under direction of instructor, store parts of the formula for the *n*th Fibonacci number in their graphing calculators. They will then find the 10^{th} and 20^{th} Fibonacci number in two ways (recursively, and by the formula).

Assignment: Exercises in 2.2

Materials: Graphing calculators

Assessment: Formal: Grading exercises in 2.2
Informal: Teacher observation of student activities

Lesson Plan #8

Unit 2 – Mathematics

Lesson – Binomial Coefficients

Goal(s) Addressed: #1, #4

Objective of Lesson: The student will learn the meaning of and how to calculate binomial coefficients.

Prerequisite Topics: None

Previous Assignment: Exercises in 2.2

Outline of Lesson:

(1) Students will, by brute force, calculate how many 0-, 1-, 2-, and 3-element sets are possible from a 3-element set.

(2) Instructor will demonstrate how these numbers are the coefficients when (a + b) is raised to the 3rd power. Notation will be explained.

(3) Students will practice calculating binomial coefficients.

Assignment: Exercises in 2.3

Materials: Graphing calculators

Assessment: Formal: Grading exercises in 2.3
Informal: Teacher observation of student practice

Lesson Plan #9

Unit 2 – Mathematics

Lesson – Pascal's Triangle

Goal(s) Addressed: #1, #2, #4

Objective of Lesson: The student will learn the history of Pascal's triangle and what it represents. Patterns in Pascal's triangle will be explored.

Prerequisite Topics: 2.3

Previous Assignment: Exercises in 2.3

Outline of Lesson:

(1) History of Pascal's triangle will be explored, including earlier versions. Instructor will demonstrate that entries on the triangle are binomial coefficients, and will lead class to discover that the sum of elements on the *n*th row is 2^n, that each element is the sum of two in the previous row, and that the sum of numbers along diagonals are Fibonacci numbers.

(2) Students will work in groups to explore patterns in the triangle using worksheets from Seymour.

Assignment: Exercises in 2.4

Materials: Worksheets from Dale Seymour, *Visual Patterns in Pascal's Triangle.*

Assessment: Formal: Grading exercises in 2.4
Informal: Teacher observation of students in group assignment

Lesson Plan #10

Unit 2 – Mathematics

Lesson – Mathematical Proof

Goal(s) Addressed: #1, #2, #4

Objective of Lesson: The student will observe methods of mathematical proof and will learn to do simple proofs by mathematical induction.

Prerequisite Topics:

Previous Assignment: Exercises in 2.4

Outline of Lesson:

(1) Instructor will demonstrate geometric proofs of the Pythagorean Theorem, discuss their history.

(2) Instructor will demonstrate proofs by mathematical induction. Students will practice simple such proofs such as the sum of the first *n* positive integers.

Assignment: Exercises in 2.5

Materials:

Assessment: Formal: Grading exercises in 2.5

Lesson Plan #11

Unit 2 – Mathematics

Lesson – Patterns in the Fibonacci Sequence

Goal(s) Addressed: #1, #4

Objective of Lesson: The student will observe patterns in the Fibonacci sequence, using brute force and "looking for a pattern" to form a conjecture and, in some cases, mathematical induction to prove a conjecture.

Prerequisite Topics: Leonardo of Pisa (1.3) and Mathematical Proof (2.5)

Previous Assignment: Exercises in 2.5

Outline of Lesson:

(1) Instructor will demonstrate the brute force method to be used in forming conjectures. Students will practice using examples in the text.

Assignment: Exercises in 2.6

Materials:

Assessment: Formal: Grading exercises in 2.6

Lesson Plan #12

Unit 2 – Mathematics

Lesson – Geometric Constructions

Goal(s) Addressed: #1, #4

Objective of Lesson: The student will review simple mathematical constructions using compass and straight edge in preparation for constructing a golden rectangle.

Prerequisite Topics:

Previous Assignment: Exercises in 2.6

Outline of Lesson:

(1) Instructor will demonstrate, and students will practice, copying a line segment, copying an angle, constructing a perpendicular bisector, and bisecting an angle.

Assignment: Exercises in 2.7

Materials: Compasses and rulers for each student

Assessment: Formal: Grading exercises in 2.7
Informal: Teacher observation of student practice

Lesson Plan #13

Unit 2 – Mathematics

Lesson – Golden Ratio

Goal(s) Addressed: #1, #2, #4

Objective of Lesson: The student will discover the relationship between Fibonacci numbers and the golden ratio; will calculate the golden ratio; and will construct a golden rectanble.

Prerequisite Topics: Leonardo of Pisa (1.3), Geometric Constructions (2.7)

Previous Assignment: Exercises in 2.7

Outline of Lesson:

(1) Using worksheets, students will calculate the ratio of adjacent Fibonacci numbers up to the 10^{th}.

(2) Using the ratio a/b = b/(a+b), students will calculate the golden ratio numerically for certain values of a.

(3) Experiments by psychologists (Fechner, etc.) will be discussed to determine if preference for the golden ratio is intuitive.

Assignment: Exercises in 2.8, survey 10 people to determine their favorite rectangle

Materials: Worksheet to calculate ratios, Sheet of rectangles of various shapes

Assessment: Formal: Grading exercises in 2.8
Informal: Teacher observation of student completion of worksheets and discussion

Calculate the ratios of successive Fibonacci numbers. Remember that $F_1 = F_2 = 1$, each succeeding F is the sum of the previous two.

$F_2 / F_1 = 1/1 =$ ________

$F_3 / F_2 = 2/1 =$ ________

$F_4 / F_3 =$ ____________

$F_5 / F_4 =$ ____________

$F_6 / F_5 =$ ____________

$F_7 / F_6 =$ ____________

$F_8 / F_7 =$ ____________

$F_9 / F_{10} =$ ____________

$F_{10} / F_9 =$ ____________

Lesson Plan #14

Unit 2 – Mathematics

Lesson – Pythagorean Triples

Goal(s) Addressed: #1, #4

Objective of Lesson: The student will discover a method of generating Pythagorean triples from four consecutive Fibonacci numbers.

Prerequisite Topics: Leonardo of Pisa (1.3)

Previous Assignment: Exercises in 2.8

Outline of Lesson:

(1) Instructor will explain the term "Pythagorean Triples" and demonstrate methods of generating them.

(2) A sequence of four consecutive Fibonacci numbers will be used to generate Pythagorean triples.

(3) Students will practice using these methods.

Assignment: Exercises in 2.9

Materials:

Assessment: Formal: Grading exercises in 2.9

Lesson Plan #15

Unit 3 – Art and Architecture

Lesson – The Parthenon

Goal(s) Addressed: #1, #2, #3

Objective of Lesson: The student will discover the golden ratio and other "incommensurable" measurements in the dimensions of the Greek Parthenon.

Prerequisite Topics: The Golden Ratio (2.8)

Previous Assignment: Read "The Parthenon" (3.2)

Outline of Lesson:

(1) A visiting lecturer from the art department will discuss historical background of the Parthenon and will point out various dimensions and patterns in the structure.

(2) Students will discuss the use of "incommensurable" or irrational numbers in the Golden Age of Greece.

Assignment: Find a source (book, journal, or web site) on the Parthenon and summarize it.

Materials: Slides, video, or photographs of the Parthenon.

Assessment: Formal: Evaluation of summary of Parthenon source
Informal: Teacher observation of discussion

Lesson Plan #16

Unit 4 – Music

Lesson – Music of Mozart

Goal(s) Addressed: #1, #3

Objective of Lesson: The student will discover the golden ratio in the music of Wolfgang Amadeus Mozart.

Prerequisite Topics: The Golden Ratio (2.8)

Previous Assignment: Read "Music of Mozart" (4.2)

Outline of Lesson:

(1) Students will use the data provided to see, in Mozart's piano sonatas, how close the ratio of "exposition" to "development and recapitulation" sections comes to the golden ratio.

(2) Students will discuss research on whether listening to Mozart's music enhances reasoning and creativity, and whether there is a connection between that and the prevalence of the golden ratio.

(3) Students will listen to another Mozart composition, with orchestral parts provided, to determine whether the golden ratio is present.

Assignment: Find experimental research on "The Mozart Effect" and be prepared to report to class.

Materials: Recording and orchestral parts of a Mozart composition (movement in sonata/allegro form).

Assessment: Formal: Evaluation of report on research.
Informal: Teacher observation of discussion.

Lesson Plan #17

Unit 5 – Literature

Lesson – Limericks

Goal(s) Addressed: #1, #2, #3

Objective of Lesson: The student will find Fibonacci numbers in the syllables and beats in a limerick and will compose two limericks related to subject matter in the course.

Prerequisite Topics:

Previous Assignment: Read "Limericks" (5.2)

Outline of Lesson:

(1) The instructor will demonstrate how to count syllables and beats in each line of selected limericks and will briefly discuss the history of the limerick.

(2) Students will compose two limericks each related to the subject matter of this course and will read them aloud to the class.

Assignment: Find and report on a web site related to limericks.

Materials:

Assessment: Formal: Students and instructor will evaluate limericks written by students; Instructor will evaluate web site results.

Lesson Plan #18

Unit 6 – Nature

Lesson – Logarithmic Spirals

Goal(s) Addressed: #1, #3

Objective of Lesson: The student will become aware of spirals in nature and will construct a logarithmic spiral using graph paper and squares with Fibonacci dimensions.

Prerequisite Topics: "The Golden Ratio" (2.8)

Previous Assignment: Read "Logarithmic Spirals" (6.2)

Outline of Lesson:

(1) Instructor will show examples of spirals in nature, such as the chambered nautilus shell, antelope horns, spider webs, and elephant tusks.

(2) The class will discuss why Stevens considers the spiral "one of the basic patterns of nature."

(3) Students will construct a logarithmic spiral as indicated in the textbook directions.

Assignment:

Materials: Chambered nautilus shell, pictures of other spirals in nature, graph paper

Assessment: Informal: Teacher observation of participation in discussion.

PART III

RESULTS

PART III

RESULTS

The course "Fun with Fibonacci" was offered to Maryville College students as an experiential elective course in the spring of 1999. The purpose was to determine whether the course that is described in this study accomplishes its goals. Fifteen students, whose majors included biology, chemistry, mathematics, English, history, business management, psychology, and computer science, were enrolled in the course. One student, an English major, dropped before mid-term due to other demands on his time; one additional student, an art major, audited. The format was a 10-week evening schedule with classes held Tuesdays from 6 to 9 p.m. and one Saturday class from 9 a.m. to 1 p.m., January 6 through March 6. As an experiential course, grades given were Satisfactory and Unsatisfactory rather than letter grades. Three hours credit were awarded to students who passed the course, which could be counted toward their experiential education requirement.

The overall goal of the course was to improve an individual's ability to make connections between mathematics and other disciplines. Other goals, as outlined in the syllabus, were:

(1) Willingness to explore mathematical patterns and to find them in the arts, humanities, natural sciences, and social sciences;

(2) Oral communication skills that enable effective comprehension, analysis, and expression;

(3) The integration of the scientific, artistic, and humanistic modes of inquiry;

(4) Increased interest and fluency in mathematics.

In addition to these formal goals, the instructor hoped that students would be curious, even skeptical, about course material. She further hoped that students who were uneasy or anxious about learning mathematics would develop confidence in that area.

Teaching the Class

Because of the three-hour format of the class, I chose not to teach the mathematics topics consecutively, but to combine one or one-and-one-half hours of mathematics with other topics related to the Fibonacci sequence each evening. Guest speakers provided interest and depth in various areas. Courtney Lix, a high school student from Gatlinburg, brought her winning science fair project on the golden ratio in art and nature. She was able to demonstrate how signs and billboards in Gatlinburg, a gateway city to a scenic area, could be made more aesthetically appealing merely by making their shapes golden rectangles or ovals. Carl Gombert of the Maryville College art department presented a program on the Parthenon and Leonardo da Vinci. Amy Livingstone, a medieval historian, discussed the 11^{th} and 12^{th} centuries in Europe, the time when Fibonacci lived. She showed slides of medieval sites such as Gothic and Romanesque cathedrals. Mary Kay Sullivan of the MC management department talked to the class about the stock market and possible patterns in its fluctuations.

In addition to hearing and interacting with these guest speakers, the class viewed three videotapes that enriched certain subject areas: "Donald Duck in Mathmagicland" discusses the golden ratio; a "Breakfast with the Arts" segment shows Mstislav

Rostropovich pointing out the golden section in the phrasing of Bach's music; and an A&E video on the Pyramids of Egypt explores these phenomenal structures. Actual lecture time was kept to a minimum; these and other student activities helped break up the three-hour block of time.

Although the course was pass/fail, certain requirements had to be met to pass the course. Homework assignments (mostly mathematics exercises in the textbook) were graded and totaled 150 possible points. Worksheets, essays, limericks, and other assignments, in addition to homework, made up a student's portfolio which comprised 40% of his/her grade. Attendance and participation made up 20% of a student's grade, and 40% of the grade resulted from a major oral presentation on the last day of class. Topics chosen for these are listed in Appendix D.

Quantitative Results

The only quantitative data collected were the results of a Likert scale included as part of Maryville College's course evaluation form. The form, administered near the end of the course, asked students to respond to the following ten statements. Responses ranged from 5 (strongly agree) to 1 (strongly disagree).

1. The basic objectives and purposes of the course were stated clearly.
2. Actual course content was consistent with the syllabus and stated objectives of the course.
3. The prerequisites, if any, provided adequate background for the course. (Note: The only prerequisite for the course was Introductory Statistics.

The purpose of that was to ensure that students had adequate algebra background and could interpret basic statistics.)

4. Organization of course content contributed to my ability to learn.
5. The textbook(s) (or other required readings and materials) contributed to my learning.
6. Assignments contributed to my understanding of the course content.
7. The course challenged me to think seriously about and become involved with this subject.
8. My responsibilities were clearly defined.
9. The methods of evaluation were clearly stated.
10. The methods of evaluation measured my performance accurately.

Results for this course, as well as average results for all Maryville College courses for the school year 1998-99, are given in the following table:

	"Fun with Fibonacci" Average	Average of all MC courses
#1	4.833	4.4
#2	4.750	4.4
#3	4.400	4.0
#4	4.583	4.1
#5	4.750	4.0
#6	4.833	4.1
#7	4.583	4.1
#8	4.9167	4.3

#9	4.9167	4.3
#10	4.833	4.1
Average	4.74	4.2

There was a significant difference between the Fibonacci course and the average for all Maryville Courses ($p < .05$) in every category except #3, "The prerequisites, if any, provided adequate background for the course." The only prerequisite was Introductory Statistics to ensure adequate algebra background; however statistics were directly used in only one lesson. Two students answered "3 – Neutral" for this item, and one left it blank. Items #6 "Assignments contributed to my understanding of the course content," #8 "My responsibilities were clearly defined," #9 "The methods of evaluation were clearly stated," and #10 "The methods of evaluation measured my performance accurately," resulted in $p \approx .0000$. Quantitative results thus show that the course was received positively. It should be pointed out, however, that this course was elective and experiential, so that more positive results would be expected than for required courses.

Qualitative Results

Student comments, both on the evaluation form and on other writing assignments, give a more complete picture of the effectiveness of the course in connecting mathematics with other areas of the curriculum. On the standard evaluation form, students were asked to comment on strengths of the course and to make suggestions for improvement. These responses are listed in full in Appendix D.

Other qualitative data was gained from the pre- and post-course questions:

(1) What is mathematics?

(2) How does mathematics connect with the world at large?

These responses are also listed in full in Appendix D.

The most useful data came from an assignment at the end of the course in section 8.3: "The reader (student) is to write this section. The ubiquity of Fibonacci numbers and the golden ratio in the arts, sciences, and humanities is only one example of the connections between mathematics and other fields. Write a page or two on your conclusions from this study of such connections."

Other data came from teacher notes following each class session and from homework assignments and classroom activities throughout the course.

Discussion

Evidence that the overall goal of the course was achieved came from several sources. Students verbalized their increased ability to make connections between mathematics and other disciplines in the final course essays excerpted below:

"It has often been thought that the world is full of intricate connections, but their complexity may leave them unnoticed. It was fun to actually discover some of the links between the arts, humanities, and natural sciences, and find their connections to mathematics…."

"Many of the disciplines offered at Maryville College seem distinct in themselves, but this course has created a link between many of them."

"Although in some cases we may be stretching it, it does look like the Fibonacci sequence and the golden ratio have much to do with everything.... It is certain that this is one of the reasons why mathematics is easily related from one subject to the next."

"Like the circle, this pattern [the Fibonacci sequence] is prolific across all cultural and subject boundaries."

Results from the pre- and post-course questions, "What is mathematics?" and "How does mathematics connect with the world at large?" were somewhat disappointing. I felt that students were trying to give a profound answer to these rather deep questions both at the beginning and at the end of the course. They had a limited time to answer them during class, so less thought and reflection are seen in their responses than in the final essays. However, there were some examples of increased awareness of connections due to the course in responses to the question "How does mathematics connect with the world at large?"

Student #14, initial response: "It [mathematics] is directly linked with global economy, government, trade (import/export), etc. Everything." This appeared to reflect her Business and Organizational Management background. However, her final response was as follows: "Mathematics provides balance and harmony. It helps us have greater understanding of nature, architecture, music, art, and practically everything around us." Clearly the course had an effect on her belief about how mathematics connects with the world at large.

Student #10, a mathematics major, wrote initially: "Mathematics is in everything! It is the universal language. In any task, subject, or thought mathematics may be involved in a very large to a very small way! (But it IS involved.)" Final response: "It is in art, music, science, etc. It is everywhere. It is everything."

Student #8, a biology major, wrote: "Mathematics is used by everyone every day be it simply making change while shopping or as complex as using a computer simulation." At the end of the course he responded: "Through searching for patterns in occurrences both in natural sciences and art, one can connect mathematics with just about anything."

Student #6, another Business and Organizational Management major, answered the original question: "Completely." Final response: "Mathematics is the connector for the world at large, for it is also the language of art, music, and other creative endeavors. It orders nature, organizes the universe. It is beautiful."

On the Maryville College course evaluation form, one student commented: "This is a great course. It's really useful to learn how math connects to the world around us." Another wrote: "It was an interesting course that tied together a lot of different fields of study." A third student said: "It is a good course in that it brings all aspects of the world into one field."

Course goals were achieved as follows:

Goal 1. Willingness to explore mathematical patterns and find them in the arts, humanities, natural sciences, and social sciences.

Students explored patterns in Lesson 1, discovered patterns in the Fibonacci sequence in Section 2.6 and in Pascal's triangle in Section 2.4 They found the Fibonacci numbers and the golden ratio in music, art, architecture, poetry, botany, astronomy, and economics. Classroom activities and assignments indicate this goal was achieved.

One of the more rewarding events of the course was when two students derived a new (to me) formula for the sum of the squares of the first n Fibonacci numbers. This conjecture resulted from observing patterns in this sum for various values of n. I proved that their conjecture is equivalent to the usual formula for this sum in Appendix A, Case Study 2.

On the Maryville College course evaluation form one student commented: "This is a very interesting course. I knew nothing about patterns before."

Goal 2. Oral communication skills that enable effective comprehension, analysis, and expression.

Each class session included some group discussion and/or activities. Cooperative learning took place in finding patterns in the *Games* magazine activity and in the Pascal triangle. In addition, students made two individual oral presentations. The first was a brief report to the class on a mathematician who lived before the time of Fibonacci; the second was a more extensive presentation on a topic of their choice, with suggestions in the textbook. Topics and descriptions of these are listed in Appendix D. For the most part these presentations were interesting and informative. Students were encouraged to choose a topic unrelated to their major field, and most did. They were required to include audio-visual aids or audience participation. All demonstrated effective comprehension, analysis, and expression.

Examples of those which used various modes of expression were: a discussion of fractal geometry and the Mandelbrot set by a senior chemistry major, including illustrations and excerpts from the Devaney video; and a performance and discussion of the music of Mozart by a senior mathematics major.

Goal 3. The integration of the scientific, artistic, and humanistic modes of inquiry.

The course included mathematical problem solving, observation of nature (pine cones, pineapples, etc.), listening to music while following a score, writing verse, and reading history. Clearly a variety of modes of inquiry was integrated into the course.

Goal 4. Increased interest and fluency in mathematics.

As evidence by written assignments, students were able to extend and interpret patterns, prove simple conjectures using mathematical induction, calculate binomial coefficients, generate Pythagorean triples, and perform geometric constructions. Those with limited mathematical background found these activities non-threatening and learnable. Further evidence that this goal was achieved was given by these comments on the course evaluation form: "Lessening fear of math for those who detest the subject" and "I knew nothing about patterns before."

Comments in the instructor's notes showed that one particular student blossomed in the area of mathematical interest and confidence:

1-5-99	#6 seemed subdued. She's a non-traditional student.
1-26-99	#6 and #4 want help with the math problems. Made appointments with both of them.
1-29-99	#6 came by for help with math problems, calculator, etc. Catches on fast! She said she never thought about these patterns before.... She was thrilled with first homework grade (20/20) – surprised.

Other goals.

As discussed in the Rationale in Part I, I believe in the ubiquity of the Fibonacci sequence and the golden ratio, but maintain a healthy skepticism about some aspects of the topic. I hoped to communicate this belief *and* skepticism to students. Such skepticism resulted from several explorations of patterns in nature. For example, as a group we had difficulty counting rows on pine cones and were not able to draw conclusions about the presence of the Fibonacci sequence there. We had better luck with pineapples; one student determined that the artichoke had layers of eight leaves. We did find the golden ratio in measurements of the human body by averaging the heights and navel-to-floor distances of each person, but were not as successful in measuring faces. Because I had always heard that the first movement Mozart's Symphony #40 was "the most perfect movement in music," and because it is in sonata-allegro form as are most movements of the piano sonatas, I borrowed orchestral parts and obtained a recording so that we could try to find the golden ratio between the two sections, exposition and development/ recapitulation. We did not find the golden ratio—were not even close—but the experience of listening to the music and following a score was new to many of the students and was a valuable exercise in itself. We concluded that Mozart's intuition was not always "golden."

When Carl Gombert spoke on the golden ratio in art and architecture, one student asked if he used this ratio in his art. He replied that he purposely tries *not* to, just because it is so prevalent, but that it does occur since he paints human faces. Mary Kay Sullivan reviewed the theories of Fibonacci numbers in fluctuations of the stock market, but stated that she believed these were not very useful in predicting market activity.

Evidence of such healthy skepticism showed up on one student's final paper: "Although in some cases *we may be stretching it*, it does look like the Fibonacci sequence and the golden ratio have much to do with everything."

A second hope of the instructor was that any students who were uneasy or anxious about mathematics would gain confidence in that area. Clearly Student #6 demonstrated success in this area. One student's course evaluation form comment confirms that, for that person at least, "Lessening fear of math for those who detest the subject" took place as a result of the course.

Conclusions

It appears from the previous discussion that both the overall purpose of the course and the individual course goals were achieved. As stated in the rationale for the course in Part I, one of the goals of the liberal arts curriculum is connected learning. This course, therefore, could be a useful offering in any undergraduate liberal arts setting.

The Academic Life Council of Maryville College gave its approval for this course to be offered as a Senior Seminar in the spring of 2000.

One student's final essay exemplifies the success of the course:

> Our study of the Fibonacci sequence and the golden ratio has certainly been intriguing. It has often been thought that the world is full of intricate connections, but their complexity may leave them unnoticed. It was fun to actually discover some of the links between the arts, humanities, and natural sciences, and find their connections to mathematics (namely the Fibonacci numbers and the golden ratio). *Finding Fibonacci* has demonstrated the presence of not only the Fibonacci numbers, but the Fibonacci sequence as well, in nature, music, literature, art, and architecture, as well as many divisions of math. In addition, while examining these connections, we often found evidences of the golden ratio.

I was amazed to see how the Fibonacci sequence and the golden ratio showed up time and again in fields other than mathematics. I think that the guest speakers really helped to emphasize the importance of the sequence in their respective topics. Many of the disciplines offered at Maryville College seem distinct in themselves, but this course has created a link between many of them. The result is the realization that possibly everything has some special bond, and the knowledge that is available is endless.

I think that this was a wonderful and challenging study. I was encouraged to take a closer look into many of the subjects that I have studied for my major. For many people, math can be confusing or unrealistic, so I was excited to see that the math behind the Fibonacci sequence was very applied in this course, and the students that did not have strong math backgrounds were able to visualize and understand the applications.

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APPENDICES

APPENDIX A

PERSONAL CASE STUDIES IN DISCOVERY AND VERIFICATION OF NUMBER PATTERNS

Case Study #1

In 2.10 of the student textbook, combinatorial proof is introduced and a counting problem is investigated. Two similar counting problems follow:

Let $g(n)$ denote the number of compositions of the positive integer n with all parts in the set $\{1, 2\}$, and with no 1's adjacent, and let $h(n)$ denote the number of such compositions with no 2's adjacent. Find recursive formulae for g and h. What would finding closed form expressions for $g(n)$ and $h(n)$ involve?

We can generate initial values of $g(n)$ by brute force as follows:

g(n)	compositions
g(1) = 1	(1)
g(2) = 1	(2)
g(3) = 2	(1+2, 2+1)

For all $n > 3$:

* $\quad g(n) = g(n - 2) + g(n - 3)$

Each side of # counts the same thing: the left-hand side counts the compositions by definition; the right-hand side counts the compositions in two exhaustive, disjoint subclasses: (a) the compositions with first part equal to 2, and (b) the compositions with first part equal to 1.

We know that (a) contains $n - 2$ compositions because they begin with 2 and have a sum equal to n. Similarly, (b) contains $n - 3$ compositions because they must begin with 1 + 2 in order to avoid adjacent 1's and they must have a sum equal to n.

Generating initial values of $h(n)$ by brute force, we find:

h(n)	compositions
h(1) = 1	(1)
h(2) = 2	(1+1, 2)
h(3) = 3	(1+2, 2+1, 1+1+1)

For all $n > 3$:

** $\quad h(n) = h(n - 1) + h(n - 3)$

Each side of ** counts the same thing. The left-hand side counts the compositions by definition; the right-hand side counts the compositions in two exhaustive, disjoint subclasses, i.e., (c) the compositions with first part equal to 1, and (d) the compositions with first part equal to 2.

We know that (c) contains n - 1 compositions because they begin with 1 and have a sum equal to n. Similarly, (d) contains n - 3 compositions because they must begin with 2 + 1 in order to avoid adjacent 2's and they must have a sum equal to n.

Finding closed-form expressions for $g(n)$ and $h(n)$ would involve the method of characteristic polynomials. Using the recursive formula for $g(n)$ and letting $n = 3$, we find

$$g(3) = g(1) + g(0), \text{ so } g(0) = 1$$

$$g(n) - g(n - 2) - g(n - 3) = 0$$

The characteristic polynomial, p(x), is

$$p(x) = x^3 - x - 1$$

By the method of graphing to find approximate solutions, this polynomial has two imaginary solutions and one real solution approximately equal to 1.3.

Similarly, the characteristic polynomial for $h(n)$, which we'll call $q(x)$ is given by

$$q(x) = x^3 - x^2 - 1$$

which has two imaginary solutions and one real solution approximately equal to 1.45.

Case Study #2

During the class investigation of patterns in the Fibonacci sequence (2.6), two students (#5 and #15) came up with this result when assigned to find a formula for the sum of the squares of the first n Fibonacci numbers:

$$\sum_{i=1}^{n} f_i^{\,2} = f_{n+1}^{\,2} - f_n^{\,2} - (-1)^n$$

The usual result given for this sum is:

$$\sum_{i=1}^{n} f_i^{\,2} = f_n \cdot f_{n+1}$$

I will show, by mathematical induction, that the two results are equal.

Proof:

I wish to show that

$$f_{n+1}^{\,2} - f_n^{\,2} - (-1)^n = f_n \cdot f_{n+1}$$

This conjecture is true for $n = 1$:

$$f_2^{\,2} - f_1^{\,2} - (-1)^1 = 1^2 - 1^2 - (-1) = 1 - 1 + 1 = 1 = f_1 \cdot f_2$$

If the conjecture is true for $n = k$, I will show that it is true for $n = k + 1$.

$$f_{k+1}^{\,2} - f_k^{\,2} - (-1)^k = f_k \cdot f_{k+1} \qquad \text{(assumed to be true)}$$

Substituting: $f_k = f_{k+2} - f_{k+1}$ by the recursive nature of Fibonacci numbers, we have

$$f_{k+1}^{\,2} - f_k^{\,2} - (-1)^k = (f_{k+2} - f_{k+1})f_{k+1}$$

$$= f_{k+2}f_{k+1} - f_{k+1}^{\,2}$$

Adding $f_{k+1}^{\,2}$ to both sides, we have:

$$2f_{k+1}^{\ 2} - f_k^{\ 2} - (-1)^k = f_{k+2}f_{k+1}$$

Substituting again for f_k (as above):

$$2f_{k+1}^{\ 2} - (f_{k+2} - f_{k+1})^2 - (-1)^k = f_{k+2}f_{k+1}$$

$$2f_{k+1}^{\ 2} - \left[f_{k+2}^{\ 2} - 2f_{k+2}f_{k+1} + f_{k+1}^{\ 2}\right] - (-1)^k = f_{k+2}f_{k+1}$$

$$2f_{k+1}^{\ 2} - f_{k+2}^{\ 2} + 2f_{k+2}f_{k+1} - f_{k+1}^{\ 2} - (-1)^k = f_{k+2}f_{k+1}$$

$$f_{k+1}^{\ 2} - f_{k+2}^{\ 2} - (-1)^k = -f_{k+2}f_{k+1}$$

Multiplying both sides by (-1) we have the desired result:

$$f_{k+2}^{\ 2} - f_{k+1}^{\ 2} - (-1)^{k+1} = f_{k+2}f_{k+1}$$

We have thus shown by mathematical induction that the sum of the squares of the first n Fibonacci numbers can be given by the students' formula:

$$\sum_{i=1}^{n} f_i^{\ 2} = f_{n+1}^{\ 2} - f_n^{\ 2} - (-1)^n$$

Case Study #3

An investigation of Pythagorean triples is suggested in 2.9 of the student textbook. In William Boulger's article, "Pythagoras Meets Fibonacci" (*Mathematics Teacher,* April, 1989), a connection between Fibonacci numbers and Pythagorean triples is demonstrated. If any four consecutive Fibonacci numbers are taken, the product of the first and fourth give side A of a right triangle; twice the product of the second and third gives side B of a right triangle; and the sum of these squared gives the square of the hypotenuse of such a triangle, such hypotenuse also being a Fibonacci number. In other words:

$$* \qquad (F_n \cdot F_{n+3})^2 + (2F_{n+1}\,F_{n+2})^2 = (F_{2n+3})^2$$

Proof:

Let the *n*th Fibonacci number be *a*, and the (*n+1*)th be *b*. Then a sequence of four Fibonacci numbers, beginning with the *n*th, would be:

a
b
$a+b$
$a+2b$

Following the pattern above to form the first two terms of the Pythagorean Theorem,

$$[a(a+2b)]^2 + [2(b)(a+b)]^2 =$$
$$(a^2+2ab)^2 + (2ab+2b^2)^2 = a^4 + 4a^3b + 4a^2b^2 + 4a^2b^2 + 8ab^3 + 4b^4 =$$
$$a^4 + 4a^3b + 8a^2b^2 + 8ab^3 + 4b^4 = (a^2+2ab+2b^2)^2$$

This can now be written $[(a+b)^2 + b^2]^2$ which is the square of the sum of the squares of the two middle numbers in the original sequence and is equal to the left side of *. If we substitute this for the left side of * and remove the squares on both sides, our conjecture becomes:

$$F_{n+2}{}^2 + F_{n+1}{}^2 = F_{2n+3}.$$

A well-known result (see note below for proof) is $F_{m+n} = F_{m-1}F_n + F_mF_{n+1}$

If we let $m = n + 3$, then

$$F_{2n+3} = F_{n+2}F_n + F_{n+3}F_{n+1}$$

Using our recursive properties of Fibonacci numbers, this can be written:

$$F_{n+2}(F_{n+2} - F_{n+1}) + (F_{n+1} + F_{n+2})F_{n+1} = F_{n+2}{}^2 - F_{n+2}F_{n+1} + F_{n+1}{}^2 + F_{n+2}F_{n+1}$$

$$= F_{n+2}{}^2 + F_{n+1}{}^2$$

Note:

To prove that $F_{m+n} = F_{m-1}F_n + F_mF_{n+1}$, we will use course-of-values induction as follows:

This conjecture is true for $n = 1$ and $n = 2$:

$$F_{m+1} = F_{m-1}F_1 + F_mF_2 = F_{m-1} + F_m \quad (F_1 = F_2 = 1,\ \ F_k = F_{k-1} + F_{k-2} \quad \forall \quad k > 2)$$

$$F_{m+2} = F_{m-1}F_2 + F_mF_3 = F_{m-1} + 2F_m = (F_{m-1} + F_m) + F_m = F_{m+1} + F_m$$

If this conjecture is true for $n = k$ and $n = k + 1$, then we will show that it is true for $n = k + 2$.

Assume $F_{m+k} = F_{m-1}F_k + F_mF_{k+1}$ and $F_{m+k+1} = F_{m-1}F_{k+1} + F_mF_{k+2}$

Now $F_{m+k+2} = F_{m+k} + F_{m+k+1} = F_{m-1}F_k + F_mF_{k+1} + F_{m-1}F_{k+1} + F_mF_{k+2}$

$$= F_{m-1}(F_k + F_{k+1}) + F_m(F_{k+1} + F_{k+2}) = F_{m-1}(F_{k+2}) + F_m(F_{k+3})$$

Thus our conjecture is true for all values of $n \in$ N by the principle of mathematical induction.

Case Study #4

The method of generating functions to determine a closed-form expression for the *n*th Fibonacci number is discussed in 2.2.4 of the student textbook. The project here is to find the ordinary generating function for the Fibonacci numbers with negative index:

$$\sum_{n=1}^{\infty} F_{-n}x^n \quad \text{where} \quad F_0 = F_1 = 1 \quad \text{and} \quad F_n = F_{n-1} + F_{n-2} \quad \forall n \in Z$$

Generating a few values of F_n:

n	-6	-5	-4	-3	-2	-1	0	1	2	3	4
F_n	5	-3	2	-1	1	0	1	1	2	3	5

Let *F(x)* be the ordinary generating function for the Fibonacci numbers with negative index.

$$F(x) = x \cdot F_{-1} + x^2 \cdot F_{-2} + x^3 \cdot F_{-3} + x^4 \cdot F_{-4} + x^5 \cdot F_{-5} + \ldots$$

Multiplying both sides by x and then by x^2, we have:

$$xF(x) = \quad x^2 \cdot F_{-1} + x^3 \cdot F_{-2} + x^4 \cdot F_{-3} + x^5 \cdot F_{-4} + \ldots$$

$$x^2F(x) = \quad x^3 \cdot F_{-1} + x^4 \cdot F_{-2} + x^5 \cdot F_{-3} + \ldots$$

Combining these three equations by adding the first two and subtracting the third, we have:

$$\begin{aligned} F(x) + xF(x) - x^2F(x) &= [F_{-1}]x + [F_{-2} + F_{-1}]x^2 + [F_{-3} + F_{-2} - F_{-1}]x^3 + \\ &\quad [F_{-4} + F_{-3} - F_{-2}]x^4 \ldots + [F_{-n} + F_{-n+1} - F_{-n+2}]x^n + \ldots \\ &= (0)x + (1+0)x^2 + (-1+1-0)x^3 + (2-1-1)x^4 + \ldots \end{aligned}$$

Clearly the coefficients of each term after the first two will be zero because $F_n + F_{n+1} - F_{n+2} = 0$. Therefore:

$$F(x) + xF(x) - x^2F(x) = x^2$$

$$(1 + x - x^2)F(x) = x^2 \text{ and } \qquad F(x) = \frac{x^2}{1 + x - x^2}$$

This expression thus represents the ordinary generating function for the Fibonacci numbers with negative index.

Case Study #5

Patterns in Pascal's triangle are discussed in 2.4 of the student textbook. It is observed that the sums of entries on consecutive diagonals of the triangle are consecutive Fibonacci numbers. Here is a proof of that conjecture; in other words, for all $n \in N$, $\sum_{k=0}^{n}\binom{n-k}{k} = F_n$, the nth Fibonacci number.

---

* $\quad n \in N, \ \sum_{k=0}^{n}\binom{n-k}{k} = F_n$

Proof: We'll use the following variation on ordinary induction:
We'll show * is true for $n = 0$ and $n = 1$. Then we'll show that if * is true for $n - 1$ and n (for all $n \geq 1$), it is true for $n + 1$.

(1) We'll show that * is true for $n = 0$: $\binom{0}{0} = 1 = F_0$

(2) And show that * is true for $n = 1$: $\binom{1}{0} + \binom{0}{1} = 1 + 0 = 1 = F_1$

(3) We'll assume that * is true for $n = 2, 3, \ldots n - 1, n$, and then show that * is true for $n + 1$:

(a) $n - 1$: $\quad \sum_{k=0}^{n-1}\binom{n-1-k}{k} = F_{n-1}$

(b) n: $\quad \sum_{k=0}^{n}\binom{n-k}{k} = F_n$

Now we'll add those two together, term by term. The right-hand side will sum to the $n + 1$ Fibonacci number by definition. We'll show that the left-hand side sums to the following:

$$\sum_{k=0}^{n+1}\binom{n+1-k}{k}$$

(a) $\binom{n-1}{0}+\binom{n-2}{1}+\binom{n-3}{2}+\ldots+\binom{1}{n-2}+\binom{0}{n-1}$

(b) $\binom{n}{0}+\binom{n-1}{1}+\binom{n-2}{2}+\binom{n-3}{3}+\ldots+\binom{1}{n-1}+\binom{0}{n}$

Adding together and grouping, we have:

$$\binom{n}{0}+\left[\binom{n-1}{0}+\binom{n-1}{1}\right]+\left[\binom{n-2}{1}+\binom{n-2}{2}\right]+\left[\binom{n-3}{2}+\binom{n-3}{3}\right]+\ldots+\left[\binom{0}{n-1}+\binom{0}{n}\right]$$

By Theorem 3.4 (Wagner, p. 21), we know that $\binom{n}{k}=\binom{n-1}{k-1}+\binom{n-1}{k}$

Therefore we can simplify our left-hand sum as follows:

$$\binom{n}{0}+\binom{n}{1}+\binom{n-1}{2}+\binom{n-2}{3}+\ldots+\binom{1}{n}$$

Since $\binom{n}{0}=\binom{n+1}{0}=1$, and $\binom{0}{n+1}=0$ for $n>1$, this expression is the same as

$$\binom{n+1}{0}+\binom{n}{1}+\binom{n-1}{2}+\binom{n-2}{3}+\ldots+\binom{1}{n}+\binom{0}{n+1} = \sum_{k=0}^{n}\binom{n+1-k}{k}$$

We have thus proved by complete induction that $\sum_{k=0}^{n}\binom{n-k}{k}=F_n$.

APPENDIX B

THE FIBONACCI ASSOCIATION

A Short History of the Fibonacci Quarterly

By Marjorie Bicknell-Johnson
Published in *The Fibonacci Quarterly,* February 1987

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This volume marks the 25th year of publication of *The Fibonacci Quarterly,* prompting memories of just how it all started. As a long-time observer and participant, I was asked to write a short history of the early organization.

In the beginning, the Fibonacci Association grew out of the bond of friendship formed by those sharing an interest in the Fibonacci numbers. Professor Verner E. Hoggatt, Jr., San Jose State College, had become interested in the Fibonacci sequence in the late 1950s. Vern's colleague Dmitri Thoro introduced him to Brother Alfred Brousseau, St. Mary's College, in the early 1960s. Vern and Brother Alfred began a long friendship and met frequently to discuss Fibonacci numbers and often sang songs, accompanied by Brother Alfred's accordion. (I recall a ballad written by Brother Alfred, "Do What Comes Fibernaturally!", to the tune of "The Blue-Tail Fly.")

As time went on, their intense interest in the Fibonacci sequence began to take a more organized direction. Brother Alfred, for example, compiled a bibliography of more than 700 Fibonacci references, ranging from recreational to serious research, to disseminate to interested initiates. Both took any and every opportunity to lecture on the sequence, so much so that Vern soon became fondly known as "Professor Fibonacci."

By December of 1962, the group also included Professor Paul Byrd, I. Dale Ruggles, Stanley L. Basin, and Terrance A. Brennan. It was this group of men who founded the Fibonacci Association to provide an opportunity for those who shared an interest in the Fibonacci numbers to exchange ideas.

So much interest in the Fibonacci numbers was apparent to the "founding fathers" that they decided to publish *The Fibonacci Quarterly*, despite limited support and all the other problems that beset a new venture. Vern and Brother Alfred wanted a journal to provide rapid dissemination of the ever expanding research on the Fibonacci numbers and to invite teachers and students to share their enthusiasm for mathematics.

With a very small amount of money from subscriptions and donations, and a large amount of volunteer labor from students, friends, and family, the first issue of *The Fibonacci Quarterly* was published in February 1963, with Editor Verner E. Hoggatt, Jr., and Managing Editor Brother U. Alfred.

Due to shoestring economics, the first issue was typed by Brother Alfred; after that, several professional technical typists came and went. Keeping a good typist almost caused Vern to have a nervous breakdown, until he met someone who needed him to complete a golf foursome and discovered a technical typist in the course of getting acquainted!

The first printer was a photocopy shop with a small press, but this proved inadequate and costly. Then Brother Alfred approached William Descalso, who had done printing for St. Mary's College since 1948, to take on the printing of the *Quarterly*. Descalso had a large web press which could print 16 pages at one time. (This explains why we had 80 , 96, or 112 pages, but never 89.) These signatures and the cover were put into a folding machine, and the journal was assembled, stapled, and trimmed in one operation. Mr. Descalso took special interest in the *Quarterly* for many years, and I suspect that he helped us to continue by making personal sacrifices. Also, he used to deliver the *Quarterly* to Brother Alfred for mailing, then bring the reprints to Vern's home in a big truck for stapling and mailing.

At first, subscriptions came in slowly (59 on January 31, 1963), but with some advertising and favorable notices in various magazines, especially *Scientific American* (June 1963, p. 152), the tempo increased. As a result, by September 1963 there were 659 subscribers, and 915 subscribers by the end of the first year of publication. From this point on, it was a matter of maintaining this momentum. While researching this article, I found a handwritten page entitled "back-sliders" among Vern's notes; he had personally called every person who failed to renew his or her subscription for the second year!

The Fibonacci Quarterly slowly began to draw attention. While at the first meeting in December 1962, Professor Paul Byrd had wondered how we would obtain enough material for such a specialized journal. Ironically, the problem, over the years, turned out to be a superabundance of material. Vern answered all of the many inquiries addressed to the *Quarterly* personally, in longhand. Brother Alfred wrote and published the booklet, *Fibonacci Discovery*, as an aid to beginners and as another source of income for the Association. Many articles were written especially to interest beginners in the study of Fibonacci numbers. (Subsequently, these early articles were collected together and published as *A Primer for the Fibonacci Numbers.*) *The Fibonacci Quarterly* was mentioned in Martin Gardner's column in *Scientific American* in March 1969, and Brother Alfred and Vern were interviewed in an article in *Time*, April 4, 1969, pages 48 and 50. Vern was asked to write a series of articles for *Math Log*, published by Mu Alpha Theta, and his book, *Fibonacci and Lucas Numbers*, was published by Houghton

Mifflin in 1969. (I know that he had to write two complete drafts of this book because I typed both versions!) With a little fame, Vern was given a small grant by San Jose State College, and a semester-long sabbatical leave.

In those early days, the Editor carried everyone's address, telephone number, and research paper in his head. Although carrying a full teaching load, Vern still answered all correspondence personally, often writing more than 50 letters a week. He carried on such a prolific correspondence on Fibonacci matters that he frequently slept for only four hours a night. While I lived only across town, I would receive two or three letters each week because Vern wanted to put his thoughts on paper. Then he would call me for feedback, often before I had received the letters! Vern put his family to work stapling reprints and mailing them to the authors, and gave his graduate students proofreading, typing, and other tasks. I once spent many hours proofreading the first 571 Fibonacci numbers (F_{571} has 119 digits) in an attempt to make the project perfect; however, the printer's helper dropped the tray of lead characters, transposing 50 digits of F_{521} and F_{522}! Nevertheless, that article, which appeared in the October 1962 issue of *Recreational Mathematics Magazine*, was a good source of publicity for the soon-to-appear *Fibonacci Quarterly*. I also remember that he had such a concern for struggling foreign authors that he asked me to do a bit of ghost-writing because he didn't have the heart to reject their papers.

As Managing Editor, Brother Alfred kept track of all subscription and book orders and the mailing list. He mailed everything from St. Mary's College and soon had an entire basement devoted to storing Fibonacci magazines and books. When the fifty pound boxes of magazines arrived from the printer, he had to carry them to the basement and then haul them back upstairs to mail them. Because of the large volume of manuscripts, whenever the Association could raise extra money, they published an extra issue, so there were five or six issues a year at times after 1966. Storage space kept filling up; when the back issues and books were transferred to Santa Clara University in 1975, there were 257 boxes. (A Fermat number!)

Brother Alfred wrote a number of elementary articles to interest and stimulate beginners, teachers, and students, and compiled several books of tables which are still available from the Fibonacci Association. He could generate new pages for the books at such a prodigious rate that I found it difficult to keep up with the proofreading. He gave lectures at nearly every meeting of mathematics teachers in California for years. And, of course, all of this was in addition to his teaching load.

Brother Alfred seemed always to have a new Fibonacci-related problem or a new approach to present. He was interested in phyllotaxis and collected more than 6000 pinecones, including cones from the twenty native pine trees of California, because the Fibonacci sequence occurred in the spirals of the cones. Vern once sent him a "Lucas" sunflower that exhibited Lucas numbers instead of the expected Fibonacci sequence; Vern had grown the sunflower himself especially to count its spirals.

In January 1968, the Board of the Fibonacci Association was formed to set policy and to provide continuity for The Fibonacci Association and its publications. The members of the original Board of the Fibonacci Association were: Brother Alfred Brousseau, Verner E. Hoggatt, Jr., G. L. Alexanderson, George Ledin, I. Dale Ruggles, and myself. For many years, a research conference was held annually, and a special conference for high school teachers and their students was held at the University of San Francisco for five consecutive years.

Brother Alfred continued as Managing Editor for 13 years, until his retirement in 1975, and Vern Hoggatt served as Editor for 18 years, until his death on August 11, 1980. It is hard to imagine *The Fibonacci Quarterly* having been published for so long if it had not been for the propitious meeting and enduring friendship of two such talented men and their interest in an obscure mathematical sequence, 1, 1, 2, 3, 5, 8, ...

The 1987 volume marks the twenty-fifth year of publication of *The Fibonacci Quarterly*, which has evolved into a research journal with international subscribers. (There are over 200 foreign subscribers, mostly from West Germany, Canada, Japan, Australia, The United Kingdom, Greece, and Italy, but representing 36 other countries as well.)

Long live Fibonacci!

Conferences of the Fibonacci Association

A week-long conference is held by the Fibonacci Association every two years. These alternate between U.S. and European sites. The conferences to date were held as follows:

First	August 1984	University of Patras, Greece
Second	August 1986	San Jose State University, CA
Third	July 1988	Pisa, Italy
Fourth	July 1990	Wake Forest University, NC
Fifth	July 1992	St. Andrews, Scotland
Sixth	July 1994	Washington State University, WA
Seventh	July 1996	Technische Universitat, Graz, Austria
Eighth	June 1998	Rochester Institute of Technology, NY

Four persons have attended all eight conferences. They are Herta T. Freitag, Roanoke, VA; A. F. Horadam, Armidale, Australia; A. G. Shannon, Sydney, Australia; and Lawrence Somer, Washington, D.C. More than 50 people have attended each conference. The ninth is planned for Luxembourg in the year 2000.

APPENDIX C

HERTA TAUSSIG FREITAG

Herta Taussig Freitag

The November 1996 issue of *The Fibonacci Quarterly* was dedicated to Herta Taussig Freitag. The dedication page reads as follows:

> This issue of *The Fibonacci Quarterly* is dedicated to Herta Taussig Freitag as she enters her 89th year, in recognition of her years of outstanding service and achievement in the mathematics community through excellence in teaching, problem solving, lecturing, and research.
>
> During Dr. Freitag's years at Hollins College, she earned many honors, among them the prestigious Algernon Sidney Sullivan Award. She was the first faculty member to receive the Hollins Medal, and the first recipient of the Virginia College Mathematics Teacher of the Year Award. She was the first woman to become President of the Virginia, Maryland, and District of Columbia Section of the Mathematical Association of America, after having served as Vice-President and Secretary.
>
> Although she officially retired in 1971, Dr. Freitag continues her professional activities—research, publishing, and lecturing—throughout the region and abroad. Of her many accomplishments, she is perhaps most proud of her perfect attendance at the seven International Conferences of the Fibonacci Association. Herta has presented at least one paper at each conference and considers participants as not merely mathematical colleagues, but virtual family members. The problem section is the first page Herta turns to in *The Fibonacci Quarterly*, and here is a story she often tells: When two non-mathematicians meet on the street and one says, "I've got problems," the other answers, "I'm so sorry for you." When two mathematicians meet and one says, "I've got problems," the other says, "Oh, goody!"
>
> We would like to take the opportunity here to thank Herta in this small way for her innumerable contributions to the mathematics community.

Herta Taussig was born in Vienna in December 1908, eleven months after her brother Walter. She tells that her mother's physician advised her to have an abortion since the two children would be less than a year apart; her mother happily refused. She finished high school in Vienna in 1927. When she later applied for admission to Columbia University's graduate program, they equated her high school diploma with two years of college in America. She studied two languages, one for eight years and another for six, mathematics through calculus, physics, chemistry, biology, psychology, philosophy and logic, history and geography, and art appreciation.

At the University of Vienna, which was founded in the 12th century, Herta found that students had complete freedom. There was no required class attendance and no student advisors. Comprehensive examinations were given for graduation—eight hours on two successive days. Herta reports that there was "terrific anti-semitism" among students and professors, and that cheating was rampant on the comprehensive exams. The feeling seemed to be that cheating was not wrong unless you were caught. Herta received her degree in 1934, the first in her entering group to finish.

In 1938 the Nazis took over Austria, an event for which Herta has vivid memories. She recalls her father standing motionless on the day of the take-over, and saying in a toneless voice as she arrived home, "Have you not yeard? Hitler has overrun Austria—there is no Austria any more!" Since her father was a newspaper editor and one of her great-grandparents was not "pure Arian," the family feared for their safety and began to try to leave the country. According to American immigration laws, a person entering the U.S. was required to have an American citizen as guarantor and a document from this person stating that the émigré would not be a financial burden. The Taussigs were unable to make such arrangements; meanwhile the Nazis took over their apartment in Vienna.

England had a shortage of domestic workers at the time, and wished to help people in Nazi countries. Even though Herta had her first graduate degree, she applied and was hired to be a housemaid in England. Her hope was that she would later be able to bring her parents to England. It was on her 30th birthday that she started this job, working

12 to 14 hours a day (with a half day off each week) and bearing the brunt of terrific class distinction. She had studied English in school, but was rusty after 12 years and was unable to exchange ideas, feelings, and thoughts with anyone. Over time, however, she became acquainted with others in the community who were obviously impressed with her intelligence and abilities, and she was soon invited to become a governess to an 8-year-old boy. Her employers (two spinsters) were furious and called her ungrateful, so she waited a year before taking the position. In the meantime her parents had arrived in England and were living in the same village. She eventually was asked to teach at a London school which had been evacuated to the country due to the German bombing of London. Sadly, her father died while they were in England, but had lived long enough to know the war was coming to an end.

Finally, in 1944, visas were obtained and Herta and her mother were able to sail to the United States on a freighter. They were reunited with her brother in New York. She recalls the naturalization ceremony in 1949 and the words of Judge Crane who officiated: "Don't forget the ideals you brought with you from the old country. Remember your thoughts, your values, your aspirations. These are some of the things that make our country great. At the same time I ask you to look around you from day to day to observe and understand our ideals and ways and, as we fervently hope, eventually to get to love us." Herta remembers, "When we walked out there, each of us pressing those citizenship papers to our breasts, there were tears on all eyes. And those tears tell the story better than words can ever tell it…, the story of what it feels like, finally to be able to say: 'I am an American.'" More detail about Herta's long journey to America can be found in *One-*

Way Ticket: The True Story of Herta Taussig Freitag by Mary Ann Johnson, available from the Hollins College Bookstore.

Herta's first job in America (1944-48) was teaching at Greer School, a private school in New York state, whose students were mostly poor children from broken homes. She met her husband, Arthur H. Freitag, at Greer (they were married in 1950). Though the school no longer exists, reunions of students and teachers are still held. She finished her master's degree in 1948 and started right away on the Ph.D. at Columbia, completing it in 1953.

Herta began teaching at Hollins College, Roanoke, Virginia, in 1948. She reports that it was a one-woman department at the time, so she was the head "and the foot." Her husband taught high school mathematics in Roanoke. While looking for visiting lecturers, she found the name of Dr. Lida Barrett in the AMS directory and invited her to speak to the Hollins students. Dr. Barrett was on the faculty of the University of Tennessee at the time, and her husband was chair of the mathematics department. She and Herta became friends and she invited Herta to teach at U-T during a sabbatical semester in the mid-60's.

Herta retired from Hollins in 1971. A widow since 1980, she makes her home at Friendship Manor retirement village in Roanoke. Until recent health problems prevented it, she illustrated the Friendship Manor newsletter and swam daily, in addition to maintaining her interest in number theory and the Fibonacci Association. In 1997 she received the Humanitarian Award from the National Conference of Christians and Jews. The nomination for this award reads, in part, "As a refugee from the Nazi persecution of the Jews in Austria, Herta has an understanding of the horrors of prejudice. Rather than let such injustice embitter her, she has devoted her life to demonstrating that there is a better way to live. What would have been a life-shattering experience for many set her on a course of personal and professional achievement directed toward helping everyone, regardless of race, sex, color, ethnic background, religious persuasion or social class reach their maximum potential. And she does it in such a way as to make one feel that she is traveling with you, rather than leading the way."

Incidentally Herta's brother Walter, at the time of this writing, is still active as an associate conductor at the Metropolitan Opera in New York. An aunt, Tina Blau (1845-1916) was a noted Austrian painter whose works can be seen at the Osterr. Galerie, Vienna.

My life has been greatly enriched by the friendship of Herta Freitag. I visited with her three times in Roanoke, as well as at the Fibonacci conference in Rochester, and we maintained a lively correspondence until a few months ago. Joyce McCroskey and I attended a talk she gave entitled "One-Way Ticket," an account of her journey from Austria to America, to the students at City School in Roanoke. She was most interested in my dissertation project and provided resources and encouragement.

APPENDIX D

QUALITATIVE DATA

Responses to:
"Based on the statements above, please comment on strengths of this course."
(Maryville College standard course evaluation form)

"Keep the text—it was invaluable to learning process, interesting and relevant. The course is challenging and varied."

"The course was very interesting. The information was something I would have never known if I had not taken the course."

"I liked how we broke up the 3-hour class."

"It was an interesting course that tied together a lot of different fields of study."

"Lessening fear of math for those who detest the subject."

"This is a very interesting course. I knew nothing about patterns before."

"This was a great class. I would recommend it to everyone. It is not your typical math class."

"Great class. Maybe it's the best class I've taken. It's definitely top 3."

"It is a good course in that it brings all aspects of the world into one field. It is applied mathematics which is a wonderful field and should be offered as a major."

"One of the major strengths of the class is how the topic itself and the assignments provoked thought and learning. The text was well laid out and went well with class discussion. Special speakers were fantastic!"

"This is a great course. It's really useful to learn how math connects to the world around us."

Responses to:
"Based on the statements above, please provide suggestions for improvement."
(Maryville College standard course evaluation form)

"None."

"I cannot think of any improvements."

"The only thing I would improve on is a better numbering system for the book."

"Nope."

"Go more indepth with specific examples of Fibonacci in music, art, etc."

"The course would be more enjoyable if it were not a 3-hour class."

"Somehow renumber the chapter pages—it's kind of confusing. Maybe just keep the chapter number off of the heading and number consecutively. I think the class would be better as a J-term. A week is so long in between classes."

"More class time to learn more about the subject."

Responses to pre- and post-course question:
"What is mathematics?"

Student	Pre-Test	Post Test
#1 (Business/Computer Science major)	The study of M.!	
#2 (Mathematics)	A means of using numbers to explain (or understand) life in its purest form, whether simple or complex.	
#3 (Mathematics)	M. is a method of describing phenomena in the world around us. The description can be universally understood through numbers and graphs, etc. It is also a collection of systems such as the real # system, Euclidean geometry, etc., built on axioms.	A universal way of describing natural phenomena with numbers and variables.
#4 (English)	I think of M. as the composition of numbers, how they relate to each other, and overall how they relate to the world around us.	M. is something we face everyday in every aspect of life. From adding numbers to looking at signs and measurements.
#5 (Management)	The key	More than a study of numbers, it includes how numbers relate to each other; it is numerical patterns; the language of

		science and technology.
#6 (Management)		
#7 (Biology)	M. is a system by which one can attempt to explain or predict the outcome of realistic or theoretical phenomena. It is a pure science in that it is often the foundation on which other sciences are derived.	M. is a totally objective science which seeks to describe the workings of nature in terms of numbers and equations.
#8 (English)	The use of numbers representing quantities (size, age, mass, value, etc.) of things in order to better understand those things and their qualities.	
#9 (Undeclared)	The study of numbers and the sequence of numbers. Also, the study of words to demonstrate just like numbers. How numbers can be complex or simple to make something even more complex or more simple.	M. is the study of numbers, how numbers are used, words, and how they connect with numbers. M. can be used in many ways.
#10 (Mathematics)	Study of modeling the world! All mathematics has an application whether known or unknown. There is no such thing as new mathematics, there is only math that has not been discovered. At this point if all of mathematics were the Atlantic Ocean, all that we know is what we can see while standing on the shoreline at Myrtle Beach, but it is still all out there we just have to build a boat to get to it!	M. is everything. It is everywhere. And it has application to everything. M. is cool!
#11 (Psychology)	A representation of value, dimension, and change.	A way to abstract and organize the world--reality
#12 (History)	M. is the process of adding, subtracting, multiplying, or	M. is the study of numbers and how they apply to us in

	dividing numbers to solve problems that need to be solved that way.	the world.
#13 (Chemistry)	M. is the study of numbers and patterns and their relationship with and to real world examples and problems.	M. is a system of numbers and systems that can be used to describe the world around us. It is also known as the "universal language."
#14 (Management)	M. to me is in every aspect of life. In school too many times children really dread the thoughts of mathematically applications because of the way it is sometimes presented. I always cringe at the thoughts of math because I feel it is one of my weaker points.	M. is a system of formulas and patterns that are connected to everything around us. It is a system consisting of numeric algorithms.
#15 (Mathematics)	M. is the basic fundamental science including algebra, calculus, physics, sequences, linear algebra, analysis, statistics, etc. M. is also the application of these ideas to solving problems and understanding diverse patterns of our society.	The study of numbers and patterns. It seems to be the principle [sic] science of all the sciences because it is used in every science and in business, and the liberal arts.

Responses to pre- and post-course question:
"How does mathematics connect with the world at large?"

Student	Pre-	Post-
#1 (Computer Science/Business)	It is the "universal language." Everything in the world is related to math in some way!	
#2 (Mathematics)	M. is everywhere! It acts as a basis to the world. It would take pages to give a complete answer to this question.	
#3	--aids scientists to	--universal communication

(Mathematics)	understand physical occurrences (Ex. statistical analysis) --develops people's thinking and problem-solving skills --method of communication between cultures of like or differing languages	--modeling real-life occurrence in order to make predications and explanations
#4 (English)	M. has a great impact on the world. Without this system the world would not and could not advance. In many ways mathematics is our legs. It keeps us standing and stable.	It's everywhere you look.
#5 (Management)		
#6 (Management)	Completely	M. <u>is</u> the connector for the world at large, for it is also the language of art, music, and other creative endeavors. It orders nature, organizes the universe. It is beautiful.
#7 (English)	Too many ways to list. From the growth rate of trees in relation to species, water amount, and sunlight amount to the fluctuations of economies	
#8 (Biology)	M. is used by everyone everyday be it simply making change while shopping or as complex as using a computer simulation.	Through searching for patterns in occurrences both in natural sciences and art, one can connect mathematics with just about anything.
#9 (Undeclared)	Everything in today's world has to do with numbers. For example, money is dealing with numbers and all over the world there is some form of money. The world has a time scale which deals with numbers.	Almost all things in the world at large deal with numbers. Our money system, our weighing system, and many other things in the world. It probably safe to assume mathematics is everywhere

	So, the most simple things that we take for granted have to do with numbers. Numbers are everywhere whether or not we choose to see them or acknowledge them is our own fault.	we look.
#10 (Mathematics)	Mathematics is in everything! It is the universal language. In any task, subject, or thought mathematics may be involved in a very large to a very small way! (But it IS involved)	It is in art, music, science, etc. It is everywhere. It is everything.
#11 (Psychology)	Conflict abatement--"The numbers will/will not support a belief."	Mathematics allow humans to explain the world without the supernatural. Reality becomes science rather than thought.
#12 (History)	M. is seen in many things in the world. We are always around it.	It is everywhere.
#13 (Chemistry)	M. helps describe different world phenomena numerically. It is a tool to help unlock "secrets" of life and the workings of the physical universe.	Mathematics describes phenomena in the world using equations and numbers. Certain patterns can be discovered and described with these numbers.
#14 (Management)	It is directly linked with global economy, government, trade (import/export), etc. Everything.	M. provides balance and harmony. It helps us have greater understanding of nature, architecture, music, art, and practically everything around us.
#15 (Mathematics)	Mathematical symbols can be understood the world around. Math seems to be a global language to helping humans solve and understand their problems and questions. It seems to me that nearly everything has a trace back to M.	It helps to explain the laws that govern our world (i.e. gravity, electricity, rates of change, etc.). It is a universal language that brings together people and ideas from all around the globe.

Topics of Student Presentations

Student #1	Computer program to generate Fibonacci numbers and to calculate and graph ratios of successive Fibonacci numbers
Student #2	Survey and demonstration of web sites showing applications of Fibonacci numbers
Student #3	Music of Mozart, including performance of horn concerto
Student #4	(dropped the course)
Student #5	Board game using Fibonacci trivia learned during course (with #15)
Student #6	Hildegarde of Bingen's life and music
Student #7	Fictitious drugs
Student #8	Edward Lear's life and limericks
Student #9	The goddess Athena and statue by Phideas
Student #10	College basketball players
Student #11	Golden section in Maryville College's Anderson Hall
Student #12	Golden ratio in sports fields
Student #13	Fractals and the Mandelbrot set
Student #14	Luca Paciola
Student #15	Board game (with #5)

VITA

Margaret Stevenson Ribble was born in Ithaca, New York on January 28, 1940. She graduated from Twinsburg (Ohio) High School in 1957 and Maryville (Tennessee) College in 1961. She received secondary education endorsement in mathematics through Maryville College in 1986, and the Master of Mathematics degree from The University of Tennessee in 1991. In the spring of 1996 she began work on the doctorate in education (Ed.D.), specializing in mathematics education, at The University of Tennessee and received this degree in December 1999.

Since 1989 she has been mathematics instructor at Maryville College.