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Commutative Medial Near Ring

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Abstract

We deliberate a substructure of a near ring called mediality so as to create a new platform, on which many of the properties like commutativity, regular, idempotent and zero symmetric are applied with. It is shown that every medial near ring is commutative whenever cancellation law holds. Commutative medial near ring is left medial, right medial, reverse, reverse medial, reverse left medial and reverse right medial. It is proved that a non-trivial left medial regular near ring is reduced. Also a structure theorem has been obtained.
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1 Introduction

It is common to refer to algebra as the mathematical language. Algebra and Algorithm both have the same historical antecedent. Near rings are generalized rings: in a ring, we get a near ring if we ignore the commutativity of addition and one distributive law. Gunter Pilz [11] "Near Rings" is a significant compilation of work done in the field of near rings.

This paper mostly focuses on medial near rings. The theory of quasi groups has developed the identity $xyzt = xzyt$. In the year 1987, Silvia Pellegrini Manara [8] defined such near rings as medial near rings. Similar to the Strong IFP near rings, medial near rings feature a subdirect structure. He also attained a condition that is both required and sufficient for a near ring to be regular, medial and subdirectly irreducible. Two years later, G. Birkenmeier and H. Heatherly [6] discussed (left) near rings satisfying the medial identity. On mediality in semigroups several authors such as Attila Nagy [9], Fitore Abdullahu and Abdullah Zejnullahu [1], Muhammad Akram, Naveed Yaqoob and Madad [2] have contributed their work. In 2015, R. Perumal and P. Chinnaraj [10] introduced the concept of medial left bipotent seminear rings and discussed some of its properties. Also, a description of such a seminear ring has also been provided. Later, in the year 2019, A. O. Atagun, H. Kamaci, I. Tastekin and A. Sezgin [3] assumed that N is a near ring and P is an ideal of N and defined P-medial near rings with some properties. Furthermore, numerous examples are used to demonstrate the results.

Now this paper deals with some distinct near ring structures namely left medial, right medial, reverse, reverse medial, reverse left medial, reverse right medial, medial pseudo commutative, medial quasi weak commutative, medial cyclic commutative and medial quasi cyclic weak commutative. These near rings are defined and illustrated with some examples. Using these near rings, several results have been demonstrated.

$\aleph$ represents a right near ring $(\aleph, +, .)$ (That is a right near ring is a non-empty set $\aleph$ combined with two binary operations "+" and "." in such a way that (i) ($\aleph$,+) is a group, (ii) $(\aleph, .)$ is a semigroup, (iii) $(\eta_1 + \eta_2)\eta_3 = \eta_1\eta_3 + \eta_2\eta_3$ for all $\eta_1, \eta_2, \eta_3 \in \aleph$.) [11] in this paper, with at least two elements and '0' represents the identity element of the group $(\aleph, +)$ and we write kl for any two elements k, l of $\aleph$. Obviously $0\eta = 0$ for all $\eta \in \aleph$. If, additionaly, $\eta 0 = 0$ for all $\eta \in \aleph$ then we say that $\aleph$ is zero symmetric [11].

2 Preliminaries

Definition 2.1. *[3],[6],[8],[12] $\aleph$ is referred to as a medial near ring if $klmn = kmln$ for all $k, l, m, n \in \aleph$.*

Definition 2.2. *[13] A near ring $\aleph$ is referred to as a pseudo commutative near ring if $klm = mlk$ for all $k, l, m \in \aleph$.*

Definition 2.3. *[7] A near ring $\aleph$ is referred to as regular if for each $x \in \aleph$, there exists $y \in \aleph$ such that $x = xyx$.*

Definition 2.4. *[5] An element $b \in \aleph$ is referred to be nilpotent if there exists a positive integer k such that $b^k = 0$.*

Definition 2.5. *[5] An element $b \in \aleph$ is referred to be idempotent if $b^2 = b$.*

Definition 2.6. *[4] A near ring $\aleph$ is referred to as a P_k near ring if there exists a positive integer k such that $x^k\aleph = x\aleph x$ for all x in $\aleph$.*

3 Main Results

Definition 3.1. *A near ring $\aleph$ is referred to as Left Medial if $klmn = mkln$ for all $k, l, m, n \in \aleph$.*

Example 3.2. *Consider the near ring $(\aleph, +, .)$ where $(\aleph, +)$ represents the Klein's four group {0,k.l,m} and '.' is defined as follows (scheme 14, p.408 of [11]).*

.	0	k	l	m
0	0	0	0	0
k	0	k	0	m
l	0	0	0	0
m	0	k	0	m

Table 1: Multiplication Table

which is a left medial near ring and medial near ring.

Definition 3.3. *A near ring $\aleph$ is referred to as Right Medial if $klmn = kmnl$ for all $k, l, m, n \in \aleph$.*

Example 3.4. *(i) Example (3.2) demonstrates that $\aleph$ is not a Right Medial near ring (Since $kkkm \neq kkmk$).*
(ii) The near ring $(\aleph, +, .)$ defined on the Klein's four group $\aleph = \{0, k.l, m\}$ and where multiplication is defined as follows (scheme 2, p.408 of [11]).

is a right medial near ring and medial near ring.

.	0	k	l	m
0	0	0	0	0
k	0	0	k	k
l	0	k	l	l
m	0	k	m	m

Table 2: Multiplication Table

Note 3.5. *In any near ring, left mediality and right mediality does not imply mediality.*

Definition 3.6. *A near ring $\aleph$ is referred to as Reverse if $klmn = nmlk$ for all $k, l, m, n \in \aleph$.*

Example 3.7. *(i) Example (3.2) demonstrates that $\aleph$ is not a reverse near ring (Since $kkkm \neq mkkk$).*
(ii) Example (3.4 (ii)) demonstrates that $\aleph$ is not a reverse near ring (Since $lllm \neq mlll$).

Definition 3.8. *A near ring $\aleph$ is referred to as Reverse Medial if $klmn = nlmk$ for all $k, l, m, n \in \aleph$.*

Example 3.9. *(i) Example (3.2) demonstrates that $\aleph$ is not a Reverse Medial near ring (Since $kkkm \neq mkkk$).*
(ii) Example (3.4 (ii)) demonstrates that $\aleph$ is not a reverse medial near ring (Since $mlll \neq lllm$).

Definition 3.10. *A near ring $\aleph$ is referred to as Reverse Left Medial if $klmn = nlkm$ for all $k, l, m, n \in \aleph$.*

Example 3.11. *(i) Example (3.2) demonstrates that $\aleph$ is not a reverse left medial near ring (Since $kkkm \neq mkkk$).*
(ii) Example (3.4 (ii)) demonstrates that $\aleph$ is not a reverse left medial near ring (Since $mlll \neq llml$).

Definition 3.12. *A near ring $\aleph$ is referred to as Reverse Right Medial if $klmn = lnmk$ for all $k, l, m, n \in \aleph$.*

Example 3.13. *(i) The near ring ($\aleph$,+,.) defined on the Klein's four group $\aleph = \{0, k, l, m\}$ where multiplication is defined as follows (scheme 4, p.408, [11]). which is a reverse right medial near ring and medial near ring.*
(ii) Example (3.2) demonstrates that $\aleph$ is not a reverse right medial near ring (Since $kkkm \neq kmkk$).
(iii) Example (3.4 (ii)) demonstrates that $\aleph$ is not a reverse right medial near ring (Since $lmll \neq mlll$).

.	0	k	l	m
0	0	0	0	0
k	0	0	k	k
l	0	k	m	l
m	0	k	l	m

Table 3: Multiplication Table

Definition 3.14. *A near ring $\aleph$ is referred to as Medial Pseudo Commutative if $klmn = knml$ for all $k, l, m, n \in \aleph$.*

Definition 3.15. *A near ring $\aleph$ is referred to as Medial Quasi Weak Commutative if $klmn = mlkn$ for all $k, l, m, n \in \aleph$.*

Definition 3.16. *A near ring $\aleph$ is referred to as Medial Cyclic Commutative if $klmn = lmkn$ for all $k, l, m, n \in \aleph$.*

Definition 3.17. *A near ring $\aleph$ is referred to as Medial Quasi Cyclic Weak Commutative if $klmn = knlm$ for all $k, l, m, n \in \aleph$.*

Proposition 3.18. *Every Medial near ring which satisfies cancellation law is commutative.*

Proof.
Assume $\aleph$ is a medial near ring. Then $klmn = kmln$ for all $k, l, m, n \in \aleph$. By left cancellation law we have $lmn = mln$. Again, by right cancellation law, we have $lm = ml$. Hence $\aleph$ is commutative. □

Proposition 3.19. *Let $\aleph$ be a medial near ring. If $\aleph$ is commutative, then*
(i) $\aleph$ is left medial.
(ii) $\aleph$ is right medial.
(iii) $\aleph$ is reverse.
(iv) $\aleph$ is reverse medial.
(v) $\aleph$ is reverse left medial.
(vi) $\aleph$ is reverse right medial.

Proof. Let $\aleph$ be a medial near ring. Then $klmn = kmln$ for all $k, l, m, n \in \aleph$.
(i) Now $klmn = kmln = (km)ln = (mk)ln = mkln$. That is $klmn = mkln$ for all $k, l, m, n \in \aleph$. Hence $\aleph$ is left medial.
(ii) Now $klmn = kmln = km(ln) = km(nl) = kmnl$. That is $klmn = kmnl$ for all $k, l, m, n \in \aleph$. Hence $\aleph$ is right medial.
(iii) Now $klmn = (kl)(mn) = (lk)(nm) = lknm = lnkm$ (Since $\aleph$ is Medial)

$= (ln)(km) = (nl)(mk) = nlmk = nmlk$ (Since $\aleph$ is Medial) $= nmlk$. That is $klmn = nmlk$ for all $k, l, m, n \in \aleph$. Hence $\aleph$ is reverse.
(iv) Now $klmn = kl(mn) = kl(nm) = klnm = knlm$ (Since $\aleph$ is medial) $= (kn)lm = nklm = nlkm$ (Since $\aleph$ is medial) $= nl(km) = nl(mk) = nlmk$. That is $klmn = nlmk$ for all $k, l, m, n \in \aleph$. Hence $\aleph$ is reverse medial.
(v) Now $klmn = kl(mn) = kl(nm) = klnm = knlm$ (Since $\aleph$ is Medial) $= (kn)lm = (nk)lm = nklm = nlkm$ (Since $\aleph$ is Medial). That is $klmn = nlkm$ for all $k, l, m, n \in \aleph$. Hence $\aleph$ is reverse left medial.
(vi) Now $klmn = (kl)mn = (lk)mn = lkmn = lmkn$ (Since $\aleph$ is Medial) $= lm(kn) = lm(nk) = lmnk = lnmk$ (Since $\aleph$ is Medial). That is $klmn = lnmk$ for all $k, l, m, n \in \aleph$. Hence $\aleph$ is reverse right medial. □

Proposition 3.20. *Let $\aleph$ be a medial near ring. If $\aleph$ is left medial then*
(i) $\aleph$ is medial quasi weak commutative
(ii) $\aleph$ is medial cyclic commutative.

Proof.
(i) Since $\aleph$ is left medial near ring, for all $k, l, m, n \in \aleph$, $klmn = mkln = mlkn$ (Since $\aleph$ is medial). That is $klmn = mlkn$. Hence $\aleph$ is medial quasi weak commutative.
(ii) Since $\aleph$ is a medial near ring, for all $k, l, m, n \in \aleph$, $klmn = kmln = lmkn$ (Since $\aleph$ is left medial). That is $klmn = lmkn$. Hence $\aleph$ is medial cyclic commutative. □

Proposition 3.21. *Every left medial near ring with commutativity is medial.*

Proof. Let $\aleph$ be a left medial near ring. Then for all $k, l, m, n \in \aleph$, $klmn = mkln = (mk)ln = (km)ln = kmln$. That is $klmn = kmln$ for all $k, l, m, n \in \aleph$. Hence $\aleph$ is medial. □

Proposition 3.22. *Let $\aleph$ be a medial quasi cyclic weak commutative near ring. If $\aleph$ is right medial then $\aleph$ is medial.*

Proof. Since $\aleph$ is right medial, then $klmn = kmnl$ for all $k, l, m, n \in \aleph$. It is given that $\aleph$ is medial quasi cyclic weak commutative, $klmn = knlm = kmln$ (Since $\aleph$ is right medial). That is $klmn = kmln$ for all $k, l, m, n \in \aleph$. Hence $\aleph$ is medial. □

Proposition 3.23. *Let $\aleph$ be a medial quasi weak commutative near ring. If $\aleph$ is left medial then $\aleph$ is medial.*

Proof. Since $\aleph$ is medial quasi weak commutative, $klmn = mlkn = kmln$ (Since $\aleph$ is left medial) $k, l, m, n \in \aleph$. That is $klmn = kmln$ for all $k, l, m, n \in \aleph$. Hence $\aleph$ is medial. □

Proposition 3.24. *If $\aleph$ is a medial near ring with medial Quasi weak commutativity then*
(i) $\aleph$ is left medial.
(ii) $\aleph$ is medial cyclic commutative.

Proof. Since $\aleph$ is medial, $klmn = kmln$ for all $k, l, m, n \in \aleph$.
(i) It is given that $\aleph$ is medial quasi weak commutative, $klmn = mlkn = mkln$ (Since $\aleph$ is medial) for all $k, l, m, n \in \aleph$. That is $klmn = mkln$ for all $k, l, m, n \in \aleph$. Hence $\aleph$ is left medial.
(ii) Since $\aleph$ is medial, $klmn = kmln = lmkn$ (Since $\aleph$ is medial quasi weak commutative) for all $k, l, m, n \in \aleph$. That is $klmn = lmkn$ for all $k, l, m, n \in \aleph$. Hence $\aleph$ is medial cyclic commutative. $\square$

Proposition 3.25. *If $\aleph$ is both medial quasi weak commutative and medial cyclic commutative near ring, then every left medial near ring is medial.*

Proof. Since $\aleph$ is a left medial near ring, for all $k, l, m, n \in \aleph$, $klmn = mkln = lkmn$ (Since $\aleph$ is medial quasi weak commutative) $= kmln$ (Since $\aleph$ is medial cyclic commutative). That is $klmn = kmln$ for all $k, l, m, n \in \aleph$. Hence $\aleph$ is medial. $\square$

Proposition 3.26. *If $\aleph$ is both left medial and right medial near ring then $\aleph$ is commutative.*

Proof. Since $\aleph$ is left medial near ring, $klmn = mkln$ for all $k, l, m, n \in \aleph$. Since $\aleph$ is right medial near ring, $klmn = kmnl$ for all $k, l, m, n \in \aleph$. Let $a = cd$ and $b = ef$ for all $a, b, c, d, e, f \in \aleph$. Now $ab = cdef = ecdf = decf = dcfe = fdce = fced = efcd = ba$. That is $ab = ba$ for all $a, b \in \aleph$. Hence $\aleph$ is commutative. $\square$

Proposition 3.27. *Let $\aleph$ be a reverse left medial near ring. If $\aleph$ is a pseudo commutative near ring satisfying the left cancellation law, then $\aleph$ is commutative.*

Proof. Let $\aleph$ be a reverse left medial near ring. then $klmn = nlkm$ for all $k, l, m, n \in \aleph$. Since $\aleph$ is pseudo commutative, $klm = mlk$ for all $k, l, m \in \aleph$. Now $klmn = nlkm = (nlk)m = (kln)m$. That is $klmn = klnm$ for all $k, l, m, n \in \aleph$. Since $\aleph$ satisfies left cancellation law, $mn = nm$. Hence $\aleph$ is commutative. $\square$

Corollary 3.28. *Let $\aleph$ be a pseudo commutative near ring.*
(i) If $\aleph$ is a left medial near ring satisfying the right cancellation law, then $\aleph$ is commutative.
(ii) If $\aleph$ is a right medial near ring satisfying the left cancellation law, then $\aleph$ is commutative.

Proof. Let $\aleph$ be a pseudo commutative near ring. Then $klm = mlk$ for all $k, l, m, \in \aleph$.
(i) Since $\aleph$ is a left medial near ring, for all $k, l, m, n \in \aleph$, $klmn = mkln = (mkl)n = (lkm)n = lkmn$. That is $klmn = lkmn$ for all $k, l, m, n \in \aleph$. Since $\aleph$ satisfies right cancellation law, $kl = lk$. Hence $\aleph$ is commutative.
(ii) Since $\aleph$ is right medial near ring, for all $k, l, m, n \in \aleph$, $klmn = kmnl = k(mnl) = k(lnm) = klnm$. That is $klmn = klnm$ for all $k, l, m, n \in \aleph$. Since $\aleph$ satisfies left cancellation law, $mn = nm$. Hence $\aleph$ is commutative.

Lemma 3.29. *A non-trivial left medial regular near ring is reduced.*

Proof. Let $\aleph$ be a left medial regular near ring. If $\eta \in \aleph$ is nilpotent then $\eta^k = 0$ for some positive integer k. Since $\aleph$ is regular, $\eta = \eta y \eta$ for some $y \in \aleph$. Hence $\eta y = (\eta y \eta)y = \eta\eta yy = \eta^2 y^2 = \eta(\eta y \eta)yy = \eta(\eta y \eta y)y = \eta(\eta\eta yy)y = \eta^3 y^3 ... = \eta^k y^k = 0y^k = 0$. Thus $\eta y = 0$. Hence $\eta = \eta y \eta = 0\eta = 0$. Hence $\aleph$ has no non-zero nilpotent elements. Hence $\aleph$ is reduced.

Note 3.30. *A non-trivial right medial regular near ring is reduced.*

Theorem 3.31. *Let $\aleph$ be both left medial and right medial near ring with regularity. If $\aleph$ is zero symmetric then $\aleph$ satisfies the following condition:*
(i) $ab = 0 \implies ba = b0$ for every $a, b \in \aleph$.
(ii) $a\eta = a\eta a$ for every idempotent $a \in \aleph$ and for every $\eta \in \aleph$.
(iii) $\aleph$ is a P_1 near ring.

Proof. (i) Let $a, b \in \aleph$ such that $ab = 0$. Since $\aleph$ is regular, we have $a = axa$ and $b = byb$ for some $x, y \in \aleph$. Hence $ba = (byb)(axa) = b(ybax)a = b(yaxb)a = b(xyab)a = bxy(ab)a = bxy0 = bx0 = bx00 = b00x = b00 = b0$. Hence $ba = b0$.
(ii) Let $a \in \aleph$ be an idempotent element and let $\eta \in \aleph$. Now $(a\eta - a\eta a)a\eta = a\eta a\eta - a\eta a^2\eta = a\eta a\eta - a\eta a\eta = 0$. That is $(a\eta - a\eta a)a\eta = 0$. Hence by (i) $a\eta(a\eta - a\eta a) = a\eta 0 = a\eta 00 = a00\eta = a0$. That is $a\eta(a\eta - a\eta a) = a0$. Again since $(a\eta - a\eta a)a\eta a = a\eta a\eta a - a\eta a^2\eta a = a\eta a\eta a - a\eta a\eta a = 0$. Hence by (i) we have $a\eta a(a\eta - a\eta a) = a\eta a0 = aa0\eta = aa0 = a^2 0 = a0$. That is $a\eta a(a\eta - a\eta a) = a0$. Hence $(a\eta - a\eta a)^2 = (a\eta - a\eta a)(a\eta - a\eta a) = a\eta(a\eta - a\eta a) - a\eta a(a\eta - a\eta a) = a0 - a0 = 0$. That is $(a\eta - a\eta a)^2 = 0$. By lemma (3.29) and note (3.30) $a\eta - a\eta a = 0$. Hence $a\eta = a\eta a$.
(iii) Let $a \in \aleph$. Since $\aleph$ is regular,$a = axa$ for some $x \in \aleph$. Now $(xa)^2 = (xa)(xa) = x(axa) = xa$. That is xa is an idempotent. Now for any $\eta \in \aleph$, $a\eta = (axa)\eta = a[(xa)\eta] = a(xa)\eta(xa)$ (by (ii)) $= a(xa\eta x)a \in a\aleph a$. That is $a\eta \in a\aleph a$. Hence $a\aleph \subseteq a\aleph a$. Clearly $a\aleph a \subseteq a\aleph$. Thus $a\aleph = a\aleph a$. □

4 Conclusion

Near rings are a key idea in the field of ring theory, and in many fields of mathematics, computer science, and mathematical (theoretical) physics, research is currently engaged. They have extensive applications to the study of geometrical objects and topology and frequently have very direct linkages to other areas of algebra. This paper deals with medial near rings. For a near ring $\aleph$, we study the commutativity on medial near ring and introduced some definitions based on the substructures. Using these definitions we have proved some results to know more about the concepts of mediality in near rings and also structural theorem was further developed. In future, the same medial structures can be platformed in seminear ring and can be applied in cyrptography too. It is hoped that the ideas and findings presented in this work will encourage and improve future research on medial near rings and lead to its advancement in practical application.

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