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On Wijsman Strongly $\mathfrak{I}_2$ - Lacunary Convergence of Double Sequences in Neutrosophic Metric Spaces

J. Johnsy *
M. Jeyaraman †

Abstract

The concept of Wijsman $\mathfrak{I}_2$ - Statistical Convergence ($\mathfrak{W}\mathfrak{I}_2\mathfrak{S}\mathfrak{t}\mathfrak{C}$), Wijsman $\mathfrak{I}_2$ - Lacunary Statistical Convergence ($\mathfrak{W}\mathfrak{I}_2\mathfrak{L}\mathfrak{S}\mathfrak{t}\mathfrak{C}$), Wijsman Strongly $\mathfrak{I}_2$ - Lacunary Convergence ($\mathfrak{W}\mathfrak{S}\mathfrak{I}_2\mathfrak{L}\mathfrak{C}$) and Wijsman Strongly $\mathfrak{I}_2$ - Cesaro Convergence ($\mathfrak{W}\mathfrak{S}\mathfrak{I}_2\mathfrak{C}\mathfrak{e}\mathfrak{C}$) of double sequences in the Neutrosophic Metric Spaces ($\mathfrak{N}\mathfrak{M}\mathfrak{S}$) are examined in this paper. Additionally, we introduce the concepts of Wijsman Strongly $\mathfrak{I}_2^*$ - Lacunary Convergence ($\mathfrak{W}\mathfrak{S}\mathfrak{I}_2^*\mathfrak{L}\mathfrak{C}$), Wijsman Strongly $\mathfrak{I}_2$ - Lacunary Cauchy ($\mathfrak{W}\mathfrak{S}\mathfrak{I}_2\mathfrak{L}\mathfrak{C}\mathfrak{a}$), and Wijsman Strongly $\mathfrak{I}_2^*$ - Lacunary Cauchy ($\mathfrak{W}\mathfrak{S}\mathfrak{I}_2^*\mathfrak{L}\mathfrak{C}\mathfrak{a}$) sequence in $\mathfrak{N}\mathfrak{M}\mathfrak{S}$ and establish impressive results.

Keywords: Fixed point; Neutrosophic Metric Spaces;
Wijsman strongly $\mathfrak{I}_2$ - lacunary convergent and lacunary Cauchy.
2020 AMS subject classifications: 54H25, 47H10 [1]

*Research Scholar, P.G. and Research Department of Mathematics, Raja Doraisingam Govt. Arts College, Sivagangai. Affiliated to Alagappa University, Karaikudi, Tamilnadu, India; e-mail johnsy.math@gmail.com, ORCID:https://orcid.org/0009-0007-7790-0321

†P.G. and Research Department of Mathematics, Raja Doraisingam Govt. Arts College, Sivagangai. Affiliated to Alagappa University, Karaikudi, Tamilnadu, India; e-mail jeya.math@gmail.com, ORCID: https://orcid.org/0000-0002-0364-1845

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1 Introduction

Fuzzy sets were initially described by Zadeh [20]. Various article's publishing has far-reaching consequences throughout scientific disciplines. The concept has real-world relevance, yet it doesn't offer satisfactory answers for various issues. These difficulties inspire creative investigations. Atanassov [1] looked the study of intuitionistic fuzzy sets and found that they work well in this kind of scenario. The idea of intuitionistic fuzzy metric space has been presented by Park [14]. Jeyaraman et. al and Sowndararajan et. al proposed the Neutrosophic Metric Spaces concept and outlined several fixed-point solutions [8,9,10,16,17,18]. Das et al. [4] investigated I and I* convergence sequences, while Ulusu and Nuray [19] presented Wijsman Lacunary Statistical Convergence of sequences. Numerous authors had a significant role in ideal and Wijsman ideal convergence sequence[7,13]. Mursaleen et. al. [12] were described the seperability concept. Fridy and Orhan [6] developed the idea of lacunary Statistical convergence via Lacunary sequence. Major article's publishing had a significant impact across all disciplines of science. There are several lacunary statistical convergence sequence [2,3,5,11,15] had a significant impact across all disciplines of mathematics and science.

We have indicated through this entire work $\mathfrak{I}_2$ - to be the admissible ideal in $\mathbb{N} \times \mathbb{N}, \omega_2 = \{(j_u, k_s)\}$ to be a double lacunary sequence, $(\Omega, \psi, \varrho, \varphi, *, \Diamond, \otimes)$ to be the $\mathfrak{NMS}$ and $\{F_{wq}\}$ to be nonempty closed subsets of Ω.

In the present paper, we define the concept of $\mathfrak{WI}_2\mathfrak{S}t\mathfrak{C}$, $\mathfrak{WI}_2\mathfrak{LS}t\mathfrak{C}$, $\mathfrak{WSI}_2\mathfrak{LC}$ and $\mathfrak{WSI}_2\mathfrak{CeC}$ of double sequences in the $\mathfrak{NMS}$ are examined. Also, we give the notions of $\mathfrak{WSI}_2^*\mathfrak{LC}$, $\mathfrak{WSI}_2\mathfrak{LC}a$, and $\mathfrak{WSI}_2^*\mathfrak{LC}a$ set sequence in $\mathfrak{NMS}$ and establish results. Also $\mathfrak{I}_2$ and $\mathfrak{I}_2^*$ -convergence of double sequences in the setting of $\mathfrak{NMS}$ and established some relationship between these types of convergence.

2 Preliminaries

Definition 2.1. *A sequence $\Upsilon_{w\lambda}$ of nonempty closed subsets of Ω is known as $\mathfrak{WI}_2\mathfrak{StC}$ to Υ or $\mathfrak{S}(\mathfrak{I}_{\mathfrak{W}_2}^{\psi,\varrho,\varphi})$ - convergent to Υ with regard to $\mathfrak{NM}(\psi, \varrho, \varphi)$, if for every $\varepsilon \in (0,1), \tau > 0$, for each $\xi \in \Omega$ and for every $\varpi > 0$,*

$$\left\{ \frac{1}{st} \left| \begin{array}{c} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| \leq 1 - \varepsilon \\ or\, |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| \geq \varepsilon \\ and\, |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| \geq \varepsilon \end{array} \right| \geq \tau \right\} \in \mathfrak{I}_2$$

We demonstrate this symbolically by
$\Upsilon_{w\lambda}\mathfrak{S}\left(\mathfrak{I}_{\mathfrak{W}_2}^{(\psi,\varrho,\varphi)}\right)\Upsilon$ or $\Upsilon_{w\lambda} \to \Upsilon\left(\mathfrak{S}\left(\mathfrak{I}_{\mathfrak{W}_2}^{(\psi,\varrho,\varphi)}\right)\right)$.

The set of all $\mathfrak{WI}_2\mathfrak{StC}$ sequences in $\mathfrak{NMS}$ is indicated by $\mathfrak{S}\left(\mathfrak{I}_{\mathfrak{W}_2}^{(\psi,\varrho,\varphi)}\right)$.

Example 2.1. *Let $\Omega = \mathfrak{R}^2$ and double sequence $\{\Upsilon_{w\lambda}\}$ be determined as follows:*
$$\Upsilon_{w\lambda} = \begin{cases} (a,b) \in \mathfrak{R}^2 : (a+w)^2 + (b+\lambda)^2 = 1, & \text{if } w \text{ and } \lambda \text{are square integers},\\ \{(1,1)\}, & \text{otherwise.} \end{cases}$$
If $\mathfrak{I}_2 = \mathfrak{I}_2^{\delta}\mathfrak{I}_2^{\delta}$ is the class of $K \subset \mathbb{N}\times\mathbb{N}$ (with density of ζ equal to 0), then the sequence $\{\Upsilon_{w\lambda}\}$ is $\mathfrak{W}\mathfrak{I}_2\mathfrak{S}\mathfrak{t}\mathfrak{C}$ to $\Upsilon = \{(1,1)\}$ with regard to $\mathfrak{N}\mathfrak{M}(\psi, \varrho, \varphi)$.

Definition 2.2. *A sequence $\{\Upsilon_{w\lambda}\}$ is $\mathfrak{W}\mathfrak{S}\mathfrak{I}_2\mathfrak{C}\mathfrak{e}\mathfrak{S}$ to Υ or $\mathfrak{C}_1\left[\mathfrak{I}_{\mathfrak{W}_2}^{(\psi,\varrho,\varphi)}\right]$- summable to Υ with regard to $\mathfrak{N}\mathfrak{M}(\psi, \varrho, \varphi)$, if for every $\varepsilon \in (0,1)$, for each $\xi \in \Omega$ and for all $\varpi > 0$,*
$$\left\{\begin{array}{c} \frac{1}{st}\sum\limits_{w,\lambda=1,1}^{s,t} |\psi(\xi,\Upsilon_{w\lambda},\varpi) - \psi(\xi,\Upsilon,\varpi)| \le 1-\varepsilon \\ \text{or } \frac{1}{st}\sum\limits_{w,\lambda=1,1}^{s,t} |\varrho(\xi,\Upsilon_{w\lambda},\varpi) - \varrho(\xi,\Upsilon,\varpi)| \ge \varepsilon \\ \text{and } \frac{1}{st}\sum\limits_{w,\lambda=1,1}^{s,t} |\varphi(\xi,\Upsilon_{w\lambda},\varpi) - \varphi(\xi,\Upsilon,\varpi)| \ge \varepsilon \end{array}\right\} \in \mathfrak{I}_2.$$
We write $\Upsilon_{w\lambda} \to \mathfrak{C}_1\left[\mathfrak{I}_{\mathfrak{W}_2}^{(\psi,\varrho,\varphi)}\right]\Upsilon$ or $\Upsilon_{w\lambda} \to \Upsilon\left(\mathfrak{C}_1\left[\mathfrak{I}_{\mathfrak{W}_2}^{(\psi,\varrho,\varphi)}\right]\right)$.

Example 2.2. *Let $\Omega = \mathfrak{R}^2$ and double sequence $\{\Upsilon_{w\lambda}\}$ be determined as follows:*
$$\Upsilon_{w\lambda} = \begin{cases} (a,b) \in \mathbb{R}^2 : (a+1)^2 + b^2 = \frac{1}{w\lambda}; & \text{if } w \text{ and } \lambda \text{ are square integers},\\ \{(0,1)\}; & \text{otherwise.} \end{cases}$$
If $\mathfrak{I}_2 = \mathfrak{I}_2^{f}\left(\mathfrak{I}_2^{f}\right)$ is the class of finite subsets of $\mathbb{N}\times\mathbb{N}$, then the sequence $\{\Upsilon_{w\lambda}\}$ is $\mathfrak{W}\mathfrak{S}\mathfrak{I}_2\mathfrak{C}\mathfrak{e}\mathfrak{S}$ to $\Upsilon = \{(0,1)\}$ with regard to $\mathfrak{N}\mathfrak{M}(\psi, \varrho, \varphi)$.

Definition 2.3. *The sequence $\{\Upsilon_{w\lambda}\}$ is known as $\mathfrak{W}\mathfrak{I}_2\mathfrak{L}\mathfrak{S}\mathfrak{t}\mathfrak{C}$ to Υ or $\mathfrak{S}_{\omega_2}\left(\mathfrak{I}_{\mathfrak{W}_2}^{(\psi,\varrho,\varphi)}\right)$- convergent to Υ with regard to (ψ, ϱ, φ), if for every $\varepsilon \in (0,1), \tau > 0$, for each $\xi \in \Omega$ and for all $\varpi > 0$,*
$$\left\{\frac{1}{\mathfrak{h}_{us}}\left|\begin{array}{c} |\psi(\xi,\Upsilon_{w\lambda},\varpi) - \psi(\xi,\Upsilon,\varpi)| \le 1-\varepsilon \\ \text{or } |\varrho(\xi,\Upsilon_{w\lambda},\varpi) - \varrho(\xi,\Upsilon,\varpi)| \ge \varepsilon \\ \text{and } |\varphi(\xi,\Upsilon_{w\lambda},\varpi) - \varphi(\xi,\Upsilon,\varpi)| \ge \varepsilon \end{array}\right| \ge \tau\right\} \in \mathfrak{I}_2.$$
We write $\Upsilon_{w\lambda} \to \mathfrak{S}_{\omega_2}\left[\mathfrak{I}_{\mathfrak{W}_2}^{(\psi,\varrho,\varphi)}\right]\Upsilon$ or $\Upsilon_{w\lambda} \to \Upsilon\left(\mathfrak{S}_{\omega_2}\left[\mathfrak{I}_{\mathfrak{W}_2}^{(\psi,\varrho,\varphi)}\right]\right)$.

Example 2.3. *Let $\Omega = \mathfrak{R}^2$ and double sequence $\{\Upsilon_{w\lambda}\}$ be determined as follows:*
$$\Upsilon_{w\lambda} = \begin{cases} (a,b) \in \mathfrak{R}^2 : (a-w)^2 + (b+\lambda)^2 = 1, & \text{if } (w,\lambda) \in \mathfrak{I}_{us};\\ \{(-1,1)\}, & \text{otherwise.} \end{cases}$$
If we take $\mathfrak{I}_2 = \mathfrak{I}_2^{\delta}$, then the sequence $\{\Upsilon_{w\lambda}\}$ is $\mathfrak{W}\mathfrak{I}_2\mathfrak{L}\mathfrak{S}\mathfrak{t}\mathfrak{C}$ to $\Upsilon = \{(-1,1)\}$ with regard to $\mathfrak{N}\mathfrak{M}(\psi, \varrho, \varphi)$.

Definition 2.4. *A sequence $\{\Upsilon_{w\lambda}\}$ is Wijsman Strong $\mathfrak{I}_2$-Lacunary Summable ($\mathfrak{W}\mathfrak{S}\mathfrak{I}_2\mathfrak{L}\mathfrak{S}$) to Υ or $\mathfrak{N}_{\omega_2}\left[\mathfrak{I}_{\mathfrak{W}_2}^{(\psi,\varrho,\varphi)}\right]$- summable to Υ with regard to $\mathfrak{N}\mathfrak{M}(\psi, \varrho, \varphi)$,* *if for every* $\varepsilon \in (0,1)$*, for all* $\varpi > 0$ *and for each* $\xi \in \Omega$*.*

$$\left\{ \begin{array}{c} \frac{1}{\mathfrak{h}_{us}} \sum\limits_{(w,\lambda)\in\mathfrak{I}_{us}} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| \leq 1-\varepsilon \\ \textit{or}\ \frac{1}{\mathfrak{h}_{us}} \sum\limits_{(w,\lambda)\in\mathfrak{I}_{us}} |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| \geq \varepsilon \\ \textit{and}\ \frac{1}{\mathfrak{h}_{us}} \sum\limits_{(w,\lambda)\in\mathfrak{I}_{us}} |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| \geq \varepsilon \end{array} \right\} \in \mathfrak{I}_2.$$

We write $\Upsilon_{w\lambda} \overset{\mathfrak{N}_{\omega_2}\left[\mathfrak{I}_{\mathfrak{W}_2}^{(\psi,\varrho,\varphi)}\right]}{\longrightarrow} \Upsilon$ *or* $\Upsilon_{w\lambda} \to \Upsilon\left(\mathfrak{N}_{\omega_2}\left[\mathfrak{I}_{\mathfrak{W}_2}^{(\psi,\varrho,\varphi)}\right]\right)$.

Example 2.4. *Let* $\Omega = \mathfrak{R}^2$ *and double sequence* $\{\Upsilon_{w\lambda}\}$ *be determined as follows:*

$$\Upsilon_{w\lambda} = \left\{ \begin{array}{ll} (a,b) \in \mathfrak{R}^2 : a^2 + (b-1)^2 = \frac{1}{w\lambda}; & \textit{if}\ (w,\lambda) \in \mathfrak{I}_{us};\ w, \\ \{(1,0)\}; & \textit{otherwise.} \end{array} \right.$$

If $\mathfrak{I}_2 = \mathfrak{I}_2^f$*, then the sequence* $\{\Upsilon_{w\lambda}\}$ *is* $\mathfrak{W}\mathfrak{S}\mathfrak{I}_2\mathfrak{L}\mathfrak{S}$ *to* $\Upsilon = \{(1,0)\}$ *with regard to* $\mathfrak{N}\mathfrak{M}(\psi, \varrho, \varphi)$*.*

3 Main Results

Theorem 3.1. *Let* $\omega_2 = \{(j_u, k_s)\}$ *be a Double Lacunary Sequence* $(\mathfrak{D}\mathfrak{L}\mathfrak{S})$*. Then* $\Upsilon_{w\lambda} \to \Upsilon\left(\mathfrak{N}_{\omega_2}\left[\mathfrak{I}_{\mathfrak{W}_2}^{(\psi,\varrho,\varphi)}\right]\right) \Rightarrow \Upsilon_{w\lambda} \to \Upsilon\left(\mathfrak{S}_{\omega_2}\left[\mathfrak{I}_{\mathfrak{W}_2}^{(\psi,\varrho,\varphi)}\right]\right)$.

Proof. Let $\varepsilon \in (0,1)$ and $\Upsilon_{w\lambda} \to \Upsilon\left(\mathfrak{N}_{\omega_2}\left[\mathfrak{I}_{\mathfrak{W}_2}^{(\psi,\varrho,\varphi)}\right]\right)$. At that time, for every $\xi \in \Omega$, we get

$$\sum_{(w,\lambda)\in\mathfrak{I}_{us}} \left\{ \begin{array}{c} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| \text{ or } |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| \\ \text{and } |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| \end{array} \right\}$$

$$\geq \sum_{\substack{(w,\lambda)\in\mathfrak{I}_{us}:|\psi(\xi,\Upsilon_{w\lambda},\varpi)-\psi(\xi,\Upsilon,\varpi)|\leq 1-\varepsilon \\ \text{or } |\varrho(\xi,\Upsilon_{w\lambda},\varpi)-\varrho(\xi,\Upsilon,\varpi)|\geq\varepsilon \\ \text{and } |\varphi(\xi,\Upsilon_{w\lambda},\varpi)-\varphi(\xi,\Upsilon,\varpi)|\geq\varepsilon}} \left\{ \begin{array}{c} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| \\ \text{or } |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| \\ \text{and } |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| \end{array} \right\},$$

$$\geq \varepsilon \left| \begin{array}{c} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| \leq 1-\varepsilon \text{ or } |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| \geq \varepsilon \\ \text{and } |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| \geq \varepsilon \end{array} \right|$$

and so

$$\frac{1}{\varepsilon\mathfrak{h}_{us}} \sum_{(w,\lambda)\in\mathfrak{I}_{us}} \left\{ \begin{array}{c} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| \text{ or } |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| \\ \text{and } |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| \end{array} \right\}$$

$$\geq \frac{1}{\mathfrak{h}_{us}} \left| \begin{array}{c} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| \leq 1-\varepsilon \\ \text{or } |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| \geq \varepsilon \text{ and } |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| \geq \varepsilon \end{array} \right|$$

Then, for any $\tau > 0$, for each $\xi \in \Omega$,

$$\left\{ \left\{ \frac{1}{\mathfrak{h}_{us}} \left| \begin{array}{c} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| \leq 1 - \varepsilon \\ \text{or } |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon, p)| \geq \varepsilon \\ \text{and } |\varphi(\xi, \Upsilon_{w\lambda}, p) - \varphi(\xi, \Upsilon, p)| \geq \varepsilon \end{array} \right| \right\} \geq \tau \right\}$$

$$\subseteq \left\{ \begin{array}{c} \frac{1}{\mathfrak{h}_{us}} \sum_{(w,\lambda) \in \mathfrak{I}_{us}} |\psi(\xi, \Upsilon_{w\lambda}, p) - \psi(\xi, \Upsilon, p)| \leq 1 - \varepsilon.\tau \\ \text{or } \frac{1}{\mathfrak{h}_{us}} \sum_{(w,\lambda) \in \mathfrak{I}_{us}} |\varrho(\xi, \Upsilon_{w\lambda}, p) - \varrho(\xi, \Upsilon, p)| \geq \varepsilon.\tau \\ \text{and } \frac{1}{\mathfrak{h}_{us}} \sum_{(w,\lambda) \in \mathfrak{I}_{us}} |\varphi(\xi, \Upsilon_{w\lambda}, p) - \varphi(\xi, \Upsilon, p)| \geq \varepsilon.\tau \end{array} \right\}. \qquad \square$$

Theorem 3.2. *Let $\omega_2 = \{(j_u, k_s)\}$ be a $\mathfrak{DLS}$. Then, $\{\Upsilon_{w\lambda}\}$ is bounded $(\{\Upsilon_{w\lambda}\} \in L^2_\infty(\Omega))$ and $\Upsilon_{w\lambda} \to \Upsilon \left(\mathfrak{S}_{\omega_2}\left[\mathfrak{I}^{(\psi,\varrho,\varphi)}_{\mathfrak{W}_2}\right]\right) \Rightarrow \Upsilon_{w\lambda} \to \Upsilon \left(\mathfrak{N}_{\omega_2}\left[\mathfrak{I}^{(\psi,\varrho,\varphi)}_{\mathfrak{W}_2}\right]\right)$. The set of all bounded double sequences of sets in $\mathfrak{NMS}$ is indicated by $\mathfrak{L}^2_\infty(\Omega)$.*

Proof. Assume that $\Upsilon_{w\lambda} \to \Upsilon \left(\mathfrak{S}_{\omega_2}\left[\mathfrak{I}^{(\psi,\varrho,\varphi)}_{\mathfrak{W}_2}\right]\right)$ and $\{\Upsilon_{w\lambda}\} \in \mathfrak{L}^2_\infty(\Omega)$. To be noted at this point, there is an $\mathfrak{K} > 0$ such that $|\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| \geq 1 - \mathfrak{K}$ or $|\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| \leq \mathfrak{K}$ and $|\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| \leq \mathfrak{K}$ for every $\xi \in \Omega$ and $w, \lambda \in \mathbb{N}$. Given $\varepsilon \in (0, 1)$, we obtain

$$\frac{1}{\mathfrak{h}_{us}} \sum_{(w,\lambda) \in \mathfrak{I}_{us}} \left\{ \begin{array}{c} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| \text{ or} |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| \\ \text{and } |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| \end{array} \right\}$$

$$= \frac{1}{\mathfrak{h}_{us}} \sum_{\substack{(w,\lambda) \in \mathfrak{I}_{us} : |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| \leq 1 - \varepsilon/2 \\ |\varrho(\xi, \Upsilon, p) - \varrho(\xi, \Upsilon, \varpi)| \geq \varepsilon/2 \\ |\varphi(\xi, \Upsilon, \varpi) - \varphi(\xi, \Upsilon, \varpi)| \geq \varepsilon/2}} \left\{ \begin{array}{c} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| \\ \text{or } |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| \\ \text{and } |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| \end{array} \right\}$$

$$+ \frac{1}{\mathfrak{h}_{us}} \sum_{\substack{(w,\lambda) \in \mathfrak{I}_{us} : |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| > 1 - \varepsilon/2 \\ |\varrho(\xi, \Upsilon, \varpi) - \varrho(\xi, \Upsilon, \varpi)| < \varepsilon/2 \\ |\varphi(\xi, \Upsilon, \varpi) - \varphi(\xi, \Upsilon, \varpi)| < \varepsilon/2}} \left\{ \begin{array}{c} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| \\ \text{or } |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| \\ \text{and } |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| \end{array} \right\}$$

$$\leq \frac{\mathfrak{K}}{\mathfrak{h}_{us}} \left| \begin{array}{c} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| \leq 1 - \frac{\varepsilon}{2} \\ \text{or } |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| \geq \frac{\varepsilon}{2} \\ \text{and } |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| \geq \frac{\varepsilon}{2} \end{array} \right| + \frac{\varepsilon}{2}.$$

As a consequence, for each $\xi \in \Omega$, we get

$$\left\{ \begin{array}{c} \frac{1}{\mathfrak{h}_{us}} \sum_{(w,\lambda)\in\mathfrak{I}_{us}} |\psi(\xi,\Upsilon_{w\lambda},\varpi)-\psi(\xi,\Upsilon,\varpi)| \leq 1-\varepsilon \\ \text{or } \frac{1}{\mathfrak{h}_{us}} \sum_{(w,\lambda)\in\mathfrak{I}_{us}} |\varrho(\xi,\Upsilon_{w\lambda},\varpi)-\varrho(\xi,\Upsilon,\varpi)| \geq \varepsilon \\ \text{and } \frac{1}{\mathfrak{h}_{us}} \sum_{(w,\lambda)\in\mathfrak{I}_{us}} |\varphi(\xi,\Upsilon_{w\lambda},\varpi)-\varphi(\xi,\Upsilon,\varpi)| \geq \varepsilon \end{array} \right\}$$

$$\subseteq \left\{ \frac{1}{\mathfrak{h}_{us}} \left| \begin{array}{c} |\psi(\xi,\Upsilon_{w\lambda},\varpi)-\psi(\xi,\Upsilon,\varpi)| \leq 1-\frac{\varepsilon}{2} \\ \text{or } |\varrho(\xi,\Upsilon_{w\lambda},\varpi)-\varrho(\xi,\Upsilon,\varpi)| \geq \frac{\varepsilon}{2} \\ \text{and } |\varphi(\xi,\Upsilon_{w\lambda},\varpi)-\varphi(\xi,\Upsilon,\varpi)| \geq \frac{\varepsilon}{2} \end{array} \right| \geq \frac{\varepsilon}{2\mathfrak{K}} \right\} \in \mathfrak{I}_2.$$

□

Corollary 3.1. *We have the following result:*
$\left\{\mathfrak{S}_{\omega_2}\left[\mathfrak{I}_{\mathfrak{W}_2}^{(\psi,\varrho,\varphi)}\right]\right\} \cap \mathfrak{L}^2_\infty(\Omega) = \left\{\mathfrak{N}_{\omega_2}\left[\mathfrak{I}_{\mathfrak{W}_2}^{(\psi,\varrho,\varphi)}\right]\right\} \cap \mathfrak{L}^2_\infty(\Omega).$

Theorem 3.3. *If* $\liminf_u \lambda_u > 1$ *and* $\liminf_s \lambda_s > 1$, *then* $\Upsilon_{w\lambda} \to \Upsilon\left(\mathfrak{S}\left(\mathfrak{I}_{\mathfrak{W}_2}^{(\psi,\varrho,\varphi)}\right)\right)$ *implies* $\Upsilon_{w\lambda} \to \Upsilon\left(\mathfrak{S}_{\omega_2}\left(\mathfrak{I}_{\mathfrak{W}_2}^{(\psi,\varrho,\varphi)}\right)\right)$.

Proof. Assume that $\liminf_u \lambda_u > 1$ and $\liminf_s \lambda_s > 1$.
Then, there are $\eta > 0, \vartheta > 0$ such that $\lambda_u \geq 1+\eta$ and $\lambda_s \geq 1+\vartheta$.
For sufficiently large u,s which gives that $\frac{\mathfrak{h}_{us}}{j_u k_s} \geq \frac{\eta\vartheta}{(1+\eta)(1+\vartheta)}$.
Assume that $\Upsilon_{w\lambda} \to \Upsilon\left(\mathfrak{S}\left(\mathfrak{I}_{\mathfrak{W}_2}^{(\psi,\varrho,\varphi)}\right)\right)$.
For each $\varepsilon \in (0,1)$, for all $\varpi > 0$, and for each $\xi \in \Omega$, we have

$$\frac{1}{j_u k_s} \left| \begin{array}{c} |\psi(\xi,\Upsilon_{w\lambda},\varpi)-\psi(\xi,\Upsilon,\varpi)| \leq 1-\varepsilon \\ \text{or } |\varrho(\xi,\Upsilon_{w\lambda},\varpi)-\varrho(\xi,\Upsilon,\varpi)| \geq \varepsilon \\ \text{and } |\varphi(\xi,\Upsilon_{w\lambda},\varpi)-\varphi(\xi,\Upsilon,\varpi)| \geq \varepsilon \end{array} \right|$$
$$\geq \frac{1}{j_u k_s} \left| \begin{array}{c} |\psi(\xi,\Upsilon_{w\lambda},\varpi)-\psi(\xi,\Upsilon,\varpi)| \leq 1-\varepsilon \\ \text{or } |\varrho(\xi,\Upsilon_{w\lambda},\varpi)-\varrho(\xi,\Upsilon,\varpi)| \geq \varepsilon \\ \text{and } |\varphi(\xi,\Upsilon_{w\lambda},\varpi)-\varphi(\xi,\Upsilon,\varpi)| \geq \varepsilon \end{array} \right|$$
$$= \frac{\mathfrak{h}_{us}}{j_u k_s} \frac{1}{\mathfrak{h}_{us}} \left| \begin{array}{c} |\psi(\xi,\Upsilon_{w\lambda},\varpi)-\psi(\xi,\Upsilon,\varpi)| \leq 1-\varepsilon \\ \text{or } |\varrho(\xi,\Upsilon_{w\lambda},\varpi)-\varrho(\xi,\Upsilon,\varpi)| \geq \varepsilon \\ \text{and } |\varphi(\xi,\Upsilon_{w\lambda},\varpi)-\varphi(\xi,\Upsilon,\varpi)| \geq \varepsilon \end{array} \right|$$
$$\geq \frac{\eta\vartheta}{(1+\eta)(1+\vartheta)} \frac{1}{\mathfrak{h}_{us}} \left| \begin{array}{c} |\psi(\xi,\Upsilon_{w\lambda},\varpi)-\psi(\xi,\Upsilon,\varpi)| \leq 1-\varepsilon \\ \text{or } |\varrho(\xi,\Upsilon_{w\lambda},\varpi)-\varrho(\xi,\Upsilon,\varpi)| \geq \varepsilon \\ \text{and } |\varphi(\xi,\Upsilon_{w\lambda},\varpi)-\varphi(\xi,\Upsilon,\varpi)| \geq \varepsilon \end{array} \right|.$$

Thus, for any $\tau > 0$,

$$\left\{ \frac{1}{\mathfrak{h}_{us}} \left| \begin{array}{c} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| \leq 1 - \varepsilon \\ \text{or } |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| \geq \varepsilon \\ \text{and } |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| \geq \varepsilon \end{array} \right| \geq \tau \right\}$$
$$\left\{ \frac{1}{j_u k_s} \left| \begin{array}{c} |\psi(\xi, V_w\lambda, \varpi) - \psi(\xi, \Upsilon, \varpi)| \leq 1 - \varepsilon \\ \text{or } |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| \geq \varepsilon \\ \text{and } |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| \geq \varepsilon \end{array} \right| \geq \frac{\eta\vartheta\tau}{(1+\eta)(1+\vartheta)} \right\}$$

Consequently, by our notion, the set on the right side belongs to $\mathfrak{I}_2$, and obviously the set on the left side belongs to $\mathfrak{I}_2$.
As a result, we obtain $\Upsilon_{w\lambda} \to \Upsilon \left(\mathfrak{S}_{\omega_2} \left[\mathfrak{I}_{\mathfrak{W}_2}^{(\psi,\varrho,\varphi)} \right] \right)$. □

Theorem 3.4. *If* $\limsup_u \lambda_u < \infty$ *and* $\limsup_s \lambda_s < \infty$*, then* $\Upsilon_{w\lambda} \to \Upsilon \left(\mathfrak{S}_{\omega_2} \left[\mathfrak{I}_{\mathfrak{W}_2}^{(\psi,\varrho,\varphi)} \right] \right)$ *implies* $\mathfrak{u}_{w\lambda} \to \Upsilon \left(\mathfrak{S} \left[\mathfrak{I}_{\mathfrak{W}_2}^{(\psi,\varrho,\varphi)} \right] \right)$.

Proof. Presume that $\limsup_u \lambda_u < \infty$ and $\limsup_s \lambda_s < \infty$. Then, there are $\mathfrak{P}, \mathfrak{R} > 0$ such that $\lambda_u < \chi$ and $\lambda_s < \mathfrak{R}$ for all u and s.
Assume that $\Upsilon_{w\lambda} \to \Upsilon \left(\mathfrak{S}_{\omega_2} \left[\mathfrak{I}_{\mathfrak{W}_2}^{(\psi,\varrho,\varphi)} \right] \right)$ and let
$$\mathfrak{K}_{us} = \left| \begin{array}{c} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| \leq 1 - \varepsilon \\ \text{or } |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| \geq \varepsilon \\ \text{and } |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| \geq \varepsilon \end{array} \right|.$$
Since $\Upsilon_{w\lambda} \to \Upsilon \left(\mathfrak{S}_{\omega_2} \left[\mathfrak{I}_{\mathfrak{W}_2}^{(\psi,\varrho,\varphi)} \right] \right)$, it holds for each $\varepsilon \in (0,1), \tau > 0$, for each $\xi \in \Omega$ and for all $\varpi > 0$,

$$\left\{ \frac{1}{\mathfrak{h}_{us}} \left| \begin{array}{c} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| \leq 1 - \varepsilon \\ \text{or } |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| \geq \varepsilon \\ \text{and } |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| \geq \varepsilon \end{array} \right| \right\} \geq \tau = \left\{ \frac{\mathfrak{K}_{us}}{\mathfrak{h}_{us}} \geq \tau \right\} \in \mathfrak{I}_2.$$

So, we can select positive integers $u_0, s_0 \in \mathbb{N}$ such that $\frac{\mathfrak{K}_{us}}{\mathfrak{h}_{us}} < \tau$ for all $u \geq u_0, s \geq s_0$.
Now, take $\mathfrak{D} = \max\{\mathfrak{K}_{us} : 1 \leq u \leq u_0, 1 \leq s \leq s_0\}$, and let m and n be integers providing $j_{u-1} < m \leq j_u$ and $k_{s-1} < n \leq k_s$.

Then, for every $\varepsilon > 0$ and each $\xi \in \Omega$, we get

$$\frac{1}{mn}\left|\begin{array}{c}|\psi(\xi,\Upsilon_{w\lambda},\varpi)-\psi(\xi,\Upsilon,\varpi)|\leq 1-\varepsilon\\ \text{or }|\varrho(\xi,\Upsilon_{w\lambda},\varpi)-(\xi,\Upsilon,\varpi)|\geq\varepsilon\\ \text{and }|\varphi(\xi,\Upsilon_{w\lambda},\varpi)-\varphi(\xi,\Upsilon,\varpi)|\geq\varepsilon\end{array}\right|$$

$$\leq \frac{1}{j_{u-1}k_{s-1}}\left|\begin{array}{c}|\psi(\xi,\Upsilon_{w\lambda},\varpi)-\psi(\xi,\Upsilon,\varpi)|\leq 1-\varepsilon\\ \text{or }|\varrho(\xi,\Upsilon_{w\lambda},\varpi)-(\xi,\Upsilon,\varpi)|\geq\varepsilon\\ \text{and }|\varphi(\xi,\Upsilon_{w\lambda},\varpi)-\varphi(\xi,\Upsilon,\varpi)|\geq\varepsilon\end{array}\right|$$

$$= \frac{1}{j_{u-1}k_{s-1}}\left\{\mathfrak{K}_{11}+\mathfrak{K}_{12}+\mathfrak{K}_{21}+\mathfrak{K}_{22}+\cdots+\mathfrak{K}_{u_0s_0}+\cdots+\mathfrak{K}_{us}\right\}\leq\frac{1}{j_{u-1}k_{s-1}}$$

$$\leq \frac{u_0s_0}{j_{u-1}k_{s-1}}\left(\max_{\substack{1\leq w\leq u_0\\1\leq\lambda\leq s_0}}\{\mathfrak{K}_{w\lambda}\}\right)+\frac{1}{j_{u-1}k_{s-1}}\left\{\begin{array}{c}\mathfrak{h}_{u_0(s_0+1)}\frac{\mathfrak{K}_{u_0(s_0+1)}}{\mathfrak{h}_{u_0(s_0+1)}}+\mathfrak{h}_{(u_0+1)s_0}\frac{\mathfrak{K}_{(u_0+1)s_0}}{\mathfrak{h}_{(u_0+1)s_0}}\\ +\mathfrak{h}_{(u_0+1)(s_0+1)}\frac{\mathfrak{K}_{(u_0+1)(s_0+1)}}{\mathfrak{h}_{(u_0+1)(s_0+1)}}+\dots\mathfrak{h}_{us}\frac{\mathfrak{K}_{us}}{\mathfrak{h}_{us}}\end{array}\right\}$$

$$\leq \frac{u_0s_0\mathfrak{D}}{j_{u-1}k_{s-1}}+\frac{1}{j_{u-1}k_{s-1}}\left(\max_{\substack{u>u_0\\s>s_0}}\frac{\mathfrak{H}_{us}}{\mathfrak{h}_{us}}\right)\left(u,s\sum_{\substack{u>u_0\\s>s_0}}\mathfrak{h}_{w\lambda}\right)$$

$$\leq \frac{u_0s_0\mathfrak{D}}{j_{u-1}k_{s-1}}+\tau\frac{(j_u-j_{u_0})(k_s-k_{s_0})}{j_{u-1}k_{s-1}}\leq\frac{u_0s_0\mathfrak{D}}{j_{u-1}k_{s-1}}+\tau\lambda_u\lambda_s\leq\frac{u_0s_0\mathfrak{D}}{j_{u-1}k_{s-1}}+\tau\mathfrak{P}\mathfrak{R}.$$

Since $j_{u-1}k_{s-1}\to\infty$ as $m,n\to\infty$, it concludes that for each $\xi\in\Omega$,

$$\frac{1}{mn}\left|\begin{array}{c}|\psi(\xi,\Upsilon_{w\lambda},\varpi)-\psi(\xi,\Upsilon,\varpi)|\leq 1-\varepsilon\\ \text{or }|\varrho(\xi,\Upsilon_{w\lambda},\varpi)-\varrho(\xi,\Upsilon,varpi)|\geq\varepsilon\\ \text{and }|\varphi(\xi,\Upsilon_{w\lambda},\varpi)-\varphi(\xi,\Upsilon,\varpi)|\geq\varepsilon\end{array}\right|\to 0$$

and as a result, for any $\tau_1>0$, the set

$$\left\{\frac{1}{mn}\left|\begin{array}{c}|\psi(\xi,\Upsilon_{w\lambda},\varpi)-\psi(\xi,\Upsilon,\varpi)|\leq 1-\varepsilon\\ \text{or }|\varrho(\xi,\Upsilon_{w\lambda},\varpi)-(\xi,\Upsilon,\varpi)|\geq\varepsilon\\ \text{and }|\varphi(\xi,\Upsilon_{w\lambda},\varpi)-\varphi(\xi,\Upsilon,\varpi)|\geq\varepsilon\end{array}\right|\geq\tau_1\right\}\in\mathfrak{I}_2.$$

It gives that $\Upsilon_{w\lambda}\to\Upsilon\left(\mathfrak{S}\left[\mathfrak{I}^{(\psi,\varrho,\varphi)}_{\mathfrak{W}_2}\right]\right)$. □

Theorem 3.5. *Let* ω_2 *be a* $\mathfrak{DLS}$. *If* $1<\liminf_u\lambda_u<\limsup_u u\lambda<\infty$ *and* $1<\liminf_s\lambda_s<\limsup_s s\lambda<\infty$, *then* $\Upsilon_{w\lambda}\to\Upsilon\left(\Upsilon_{\omega_2}\left[\mathfrak{I}^{(\psi,\varrho,\varphi)}_{\mathfrak{W}_2}\right]\right)$ *if* $\Upsilon_{w\lambda}\to\Upsilon\left(\mathfrak{S}\left[\mathfrak{I}^{(\psi,\varrho,\varphi)}_{\mathfrak{W}_2}\right]\right)$.

Proof. It is clearly understood from Theorem (3.3) and theorem (3.4). □

Theorem 3.6. *Let* $\mathfrak{I}_2$ *be a Strongly Admissible Ideal* $(\mathfrak{SAI})$ *providing feature* $(\mathfrak{AP}_2),\omega_2\in\Upsilon(\mathfrak{I}_2)$. *If* $\{\Upsilon_{w\lambda}\}\in\mathfrak{S}\left[\mathfrak{I}^{(\psi,\varrho,\varphi)}_{\mathfrak{W}_2}\right]\cap\mathfrak{S}_{\omega_2}\left[\mathfrak{I}^{(\psi,\varrho,\varphi)}_{\mathfrak{W}_2}\right]$, *then* $\mathfrak{S}\left[\mathfrak{I}^{(\psi,\varrho,\varphi)}_{\mathfrak{W}_2}\right]-\lim_{w,\lambda\to\infty}\Upsilon_{w\lambda}=\mathfrak{S}_{\omega_2}\left[\mathfrak{I}^{(\psi,\varrho,\varphi)}_{\mathfrak{W}_2}\right]-\lim_{w,\lambda\to\infty}\Upsilon_{w\lambda}$

Proof. Assume that $\mathfrak{S}\left[\mathfrak{I}_{\mathfrak{W}_2}^{(\psi,\varrho,\varphi)}\right] - \lim_{w,\lambda\to\infty} \Upsilon_{w\lambda} = \mathfrak{U}$ and

$\mathfrak{S}_{\omega_2}\left[\mathfrak{I}_{\mathfrak{W}_2}^{(\psi,\varrho,\varphi)}\right] - \lim_{w,\lambda\to\infty} \Upsilon_{w\lambda} = \mathfrak{Z}$ and $\mathfrak{Y} \neq \mathfrak{Z}$.

Let $0 < \varepsilon < \frac{1}{2}|\psi(\xi,\mathfrak{Y},\varpi) - \psi(\xi,\mathfrak{Z},\varpi)|, 0 < \varepsilon < \frac{1}{2}|\varrho(\xi,\mathfrak{Y},\varpi) - \varrho(\xi,\mathfrak{Z},\varpi)|$ and $0 < \varepsilon < \frac{1}{2}|\varphi(\xi,\mathfrak{Y},\varpi) - \varphi(\xi,\mathfrak{Z},\varpi)|$, for every $\xi \in \Omega$.

Since $\mathfrak{I}_2$ provides the feature $(\mathfrak{AP}_2)$, then there is $\mathfrak{Q} \in \Upsilon(\mathfrak{I}_2)$ such that for every $\xi \in \Omega$ and for $(m,n) \in \mathfrak{Q}$.

Let $\lim_{m,n\to\infty} \frac{1}{mn}\left|\begin{array}{c} |\psi(\xi,\Upsilon,\varpi) - \psi(\xi,\mathfrak{Y},\varpi)| \leq 1-\varepsilon \\ \text{or } |\varrho(\xi,\Upsilon_{w\lambda},\varpi) - \varrho(\xi,\mathfrak{Y},\varpi)| \geq \varepsilon \\ \text{and } |\varphi(\xi,\Upsilon_{w\lambda},\varpi) - \varphi(\xi,\mathfrak{Y},\varpi)| \geq \varepsilon \end{array}\right| = 0$

$\mathfrak{D} = \left\{\begin{array}{c} |\psi(\xi,\Upsilon_{w\lambda},\varpi) - \psi(\xi,\mathfrak{Y},\varpi)| \leq 1-\varepsilon \\ \text{or} |\varrho(\xi,\Upsilon_{w\lambda},\varpi) - \varrho(\xi,\mathfrak{Y},\varpi)| \geq \varepsilon \\ \text{and } |\varphi(\xi,\Upsilon_{w\lambda},\varpi) - \varphi(\xi,\mathfrak{Z},\varpi)| \geq \varepsilon \end{array}\right\}$

$\mathfrak{S} = \left\{\begin{array}{c} |\psi(\xi,\Upsilon_{w\lambda},\varpi) - \psi(\xi,\mathfrak{Z},\varpi)| \leq 1-\varepsilon \\ \text{or } |\varrho(\xi,\Upsilon_{w\lambda},\varpi) - \varrho(\xi,\mathfrak{Z},\varpi)| \geq \varepsilon \\ \text{and } |\varphi(\xi,\Upsilon_{w\lambda},\varpi) - \varphi(\xi,\mathfrak{Z},\varpi)| \geq \varepsilon \end{array}\right\}$

Then, $mn = |\mathfrak{D} \cup \mathfrak{S}| \leq |\mathfrak{D}| + |\mathfrak{S}|$. This gives that $1 \leq \left(\frac{|\mathfrak{D}|}{mn}\right) + \left(\frac{|\mathfrak{S}|}{mn}\right)$.

Since $\left(\frac{|\mathfrak{S}|}{mn}\right) \leq 1$ and $\lim_{m,n\to\infty)} \frac{|\mathfrak{D}|}{mn} = 0$, we have to get $\lim_{m,n\to\infty} \frac{|\mathfrak{S}|}{mn} = 1$.

Let $\mathfrak{Q}^* = \mathfrak{Q} \cap \omega_2 \in \Upsilon(\mathfrak{I}_2)$.

Then, for every $\xi \in \Omega$ and $(w_k,\lambda_j) \in \mathfrak{Q}^*$, the $w_k\lambda_j^{\text{th}}$ term of the statistical limit expression

$\frac{1}{mn}\left|\begin{array}{c} |\psi(\xi,\Upsilon_{w\lambda},\varpi) - \psi(\xi,\mathfrak{Z},\varpi)| \leq 1-\varepsilon \\ \text{or } |\varrho(\xi,\Upsilon_{w\lambda},\varpi) - \varrho(\xi,\mathfrak{Z},\varpi)| \geq \varepsilon \\ \text{and} |\varphi(\xi,\Upsilon_{w\lambda},\varpi) - \varphi(\xi,\mathfrak{Z},\varpi)| \geq \varepsilon \end{array}\right|$, is

$\frac{1}{w_k\lambda_j}\left|\begin{array}{c} |\psi(\xi,\Upsilon_{w\lambda},\varpi) - \psi(\xi,\mathfrak{Z},\varpi)| \leq 1-\varepsilon \\ \text{or} |\varrho(\xi,\Upsilon_{w\lambda},\varpi) - (\xi,\mathfrak{Z},\varpi)| \\ \text{and} |\varphi(\xi,\Upsilon_{w\lambda},\varpi) - \varphi(\xi,\mathfrak{Z},\varpi)| \geq \varepsilon \end{array}\right|$

$= \frac{1}{\bigcup_{u,s=1,1}^{k,j} \mathfrak{h}_{us}} \bigcup_{u,s=1,1}^{k,j} \mathfrak{V}_{us}\mathfrak{h}_{us}$, where

$$\mathfrak{V}_{us} = \frac{1}{\mathfrak{h}_{us}}\left|\begin{array}{c} |\psi(\xi,\Upsilon_{w\lambda},\varpi) - \psi(\xi,\Upsilon,\varpi)| \leq 1-\varepsilon \\ \text{or} |\varrho(\xi,\Upsilon_{w\lambda},\varpi) - \varrho(\xi,\Upsilon,\varpi)| \geq \varepsilon \\ \text{and} |\varphi(\xi,\Upsilon_{w\lambda},\varpi) - \varphi(\xi,\Upsilon,\varpi)| \geq \varepsilon \end{array}\right| \xrightarrow{\mathfrak{I}_2} 0 \qquad (1)$$

because $\left(\mathfrak{S}_{\omega_2}\left[\mathfrak{I}_{\mathfrak{W}_2}^{(\psi,\varrho,\varphi)}\right]\right) - \lim_{w,\lambda} \Upsilon_{w\lambda} = \mathfrak{Z}$.

Since ω_2 is a lacunary sequence, (1) is a regular weighted mean transform of $\mathfrak{V}_{us}$'s and as a result, it is $\mathfrak{I}_2$-convergent to 0 as $k,j\to\infty$ and also it has a subsequence which is convergent to 0 since $\mathfrak{I}_2$ provides the feature $(\mathfrak{AP}_2)$.

Anyway, because this is a sequence of

$$\left\{\frac{1}{mn}\left|\begin{array}{c}|\psi(\xi,\Upsilon_{w\lambda},\varpi)-\psi(\xi,\mathfrak{Z},\varpi)|\leq 1-\varepsilon\\ \text{or}|\varrho(\xi,\Upsilon,\varpi)-(\xi,\mathfrak{Z},\varpi)|\geq\varepsilon\\ \text{and}|\varphi(\xi,\Upsilon_{w\lambda},\varpi)-\varphi(\xi,\mathfrak{Z},\varpi)|\geq\varepsilon\end{array}\right|\right\}_{(m,n)\in\mathfrak{M}'}$$

We conclude that

$$\left\{\frac{1}{mn}\left|\begin{array}{c}|\psi(\xi,\Upsilon_{w\lambda},\varpi)-\psi(\xi,\mathfrak{Z},\varpi)|\leq 1-\varepsilon\\ \text{or}|\varrho(\xi,\Upsilon,\varpi)-(\xi,\mathfrak{Z},\varpi)|\geq\varepsilon\\ \text{and}|\varphi(\xi,\Upsilon_{w\lambda},\varpi)-\varphi(\xi,\mathfrak{Z},\varpi)|\geq\varepsilon\end{array}\right|\right\}_{(m,n)\in\mathfrak{M}'}$$

which is not convergent to 1. The contradiction here shows that we cannot have $\mathfrak{Y}\neq\mathfrak{Z}$. □

Theorem 3.7. *If* $\liminf_u \lambda_u > 1$ *and* $\liminf_s \lambda_s > 1$ *then* $\left(\mathfrak{C}_1\left[\mathfrak{I}_{\mathfrak{W}_2}^{(\psi,\varrho,\varphi)}\right]\right)\subseteq\left(\mathfrak{N}_{\omega_2}\left[\mathfrak{I}_{\mathfrak{W}_2}^{(\psi,\varrho,\varphi)}\right]\right)$.

Proof. Let $\liminf_u \lambda_u > 1$ and $\liminf_s \lambda_s > 1$.
Then, there are $\eta,\vartheta>0$ such that $\lambda_u\geq 1+\eta$ and $\lambda_s\geq 1+\vartheta$, for all u and s which gives that $\frac{j_uk_s}{\mathfrak{h}_{us}}\leq\frac{(1+\eta)(1+\vartheta)}{\eta\vartheta}$ and $\frac{j_{u-1}k_{s-1}}{\mathfrak{h}_{us}}\leq\frac{1}{\eta\vartheta}$.
Assume that $\Upsilon_{w\lambda}\to\Upsilon\left(\mathfrak{C}_1\left[\mathfrak{I}_{\mathfrak{W}_2}^{(\psi,\varrho,\varphi)}\right]\right)$. For each $\xi\in\Omega$, we get

$$\begin{aligned}
&\frac{1}{\mathfrak{h}_{us}}\sum_{(w,\lambda)\in\mathfrak{I}_{us}}|\psi\left(\xi,\Upsilon_{w\lambda},\varpi\right)-\psi\left(\xi,\Upsilon,\varpi\right)|-1\\
&\quad=\frac{1}{\mathfrak{h}_{us}}\sum_{w,\lambda=1,1}^{j_u,k_s}|\psi(\xi,\Upsilon_{w\lambda},\varpi)-\psi(\xi,\Upsilon,\varpi)|\\
&\quad\quad-\frac{1}{\mathfrak{h}_{us}}\sum_{w,\lambda=1,1}^{j_{u-1},k_{s-1}}|\psi(\xi,\Upsilon,\varpi)-\psi(\xi,\Upsilon,\varpi)|-1\\
&\quad=\frac{j_uk_s}{\mathfrak{h}_{us}}\left[\frac{1}{j_uk_s}\sum_{w,\lambda=1,1}^{j_u,k_s}|\psi(\xi,\Upsilon,\varpi)-\psi(\xi,\Upsilon,\varpi)|-1\right]\\
&\quad\quad-\frac{j_{u-1}k_{s-1}}{\mathfrak{h}_{us}}\left[\frac{1}{j_{u-1}k_{s-1}}\sum_{w,\lambda=1,1}^{j_{u-1},k_{s-1}}|\psi(\xi,\Upsilon_{w\lambda},\varpi)-\psi(\xi,\Upsilon,\varpi)|-1\right].
\end{aligned}$$

Since $\Upsilon_{w\lambda}\to\Upsilon\left(\mathfrak{C}_1\left[\mathfrak{I}_{\mathfrak{W}_2}^{(\psi,\varrho,\varphi)}\right]\right)$, then for each

$\frac{1}{j_uk_s}\sum_{w,\lambda=1,1}^{j_u,k_s}|\psi(\xi,\Upsilon_{w\lambda},\varpi)-\psi(\xi,\Upsilon,\varpi)|-1)\overset{\mathfrak{I}_2}{\to}0$ and

$\frac{1}{j_{u-1}k_{s-1}}\sum_{w,\lambda=1,1}^{j_{u-1},k_{s-1}}|\psi(\xi,\Upsilon_{w\lambda},\varpi)-\psi(\xi,\Upsilon,\varpi)|-1)\overset{\mathfrak{I}_2}{\to}0.$

As a result, when the above equality is checked, for every $\xi \in \Omega$, we have
$\frac{1}{\mathfrak{h}_{us}} \sum_{(w,\lambda)\in\mathfrak{I}_{us}} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| - 1 \xrightarrow{\mathfrak{I}_2} 0.$
Similarly, we obtain
$\frac{1}{\mathfrak{h}_{us}} \sum_{(w,\lambda)\in\mathfrak{I}_{us}} |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| \xrightarrow{\mathfrak{I}_2} 0,$
$\frac{1}{\mathfrak{h}_{us}} \sum_{(w,\lambda)\in\mathfrak{I}_{us}} |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| \xrightarrow{\mathfrak{I}_2} 0.$
That is, $\Upsilon_{w\lambda} \to \Upsilon \left(\mathfrak{N}_{\omega_2}\left[\mathfrak{I}_{\mathfrak{W}_2}^{(\psi,\varrho,\varphi)}\right]\right)$.
As a result, we obtain $\left(\mathfrak{C}_1\left[\mathfrak{I}_{\mathfrak{W}_2}^{(\psi,\varrho,\varphi)}\right]\right) \subseteq \left(\mathfrak{N}_{\omega_2}\left[\mathfrak{I}_{\mathfrak{W}_2}^{(\psi,\varrho,\varphi)}\right]\right)$. □

Theorem 3.8. *If* $\liminf_u \lambda_u = 1$ *and* $\liminf_s \lambda_s = 1$ *then* $\left(\mathfrak{N}_{\omega_2}\left[\mathfrak{I}_{\mathfrak{W}_2}^{(\psi,\varrho,\varphi)}\right]\right) \subseteq \left(\mathfrak{C}_1\left[\mathfrak{I}_{\mathfrak{W}_2}^{(\psi,\varrho,\varphi)}\right]\right)$.

Proof. Take $\liminf_u \lambda_u = 1$ and $\liminf_s \lambda_s = 1$, and $\Upsilon_w\lambda \in \mathfrak{N}_{\omega_2}\left[\mathfrak{I}_{\mathfrak{W}_2}^{(\psi,\varrho,\varphi)}\right]$.
Then for every $\varpi > 0$, we acquire

$$\left.\begin{aligned}
\mathfrak{h}_{us} &= \frac{1}{\mathfrak{h}_{us}} \sum_{(w,\lambda)\in\mathfrak{I}_{us}} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| \xrightarrow{\mathfrak{I}_2} 1,\\
\mathfrak{h}'_{us} &= \frac{1}{\mathfrak{h}_{us}} \sum_{(w,\lambda)\in\mathfrak{I}_{us}} |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| \xrightarrow{\mathfrak{I}_2} 0,\\
\mathfrak{h}''_{us} &= \frac{1}{\mathfrak{h}_{us}} \sum_{(w,\lambda)\in\mathfrak{I}_{us}} |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| \xrightarrow{\mathfrak{I}_2} 0
\end{aligned}\right\} \text{ as } u, s \to \infty.$$

Then for $\varepsilon > 0$, there are $u_0, s_0 \in \mathfrak{N}$ such that $\mathfrak{h}_{us} < 1 + \varepsilon$ for all $u > u_0, s > s_0$.
Also, we can find $\zeta > 0$ such that $\mathfrak{h}_{us} < \zeta$, $\mathfrak{h}'_{us} < \zeta$ and $\mathfrak{h}''_{us} < \zeta, u, s = 1, 2, \ldots$.
Let m and n be an integer with $j_{u-1} < m \leq j_u$ and $k_{s-1} \leq n \leq k_s$. Then,

$$\begin{aligned}
&\frac{1}{mn} \sum_{w,\lambda=1,1}^{m,n} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)|\\
&\leq \frac{1}{j_{u-1}k_{s-1}} \sum_{w,\lambda=1,1}^{j_{u-1},k_{s-1}} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)|\\
&= \frac{1}{j_{u-1}k_{s-1}} \left[\begin{array}{c} \sum_{(w,\lambda)\in\mathfrak{I}_{11}} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| + \cdots + \\ \sum_{(w,\lambda)\in\mathfrak{I}_{us}} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| \end{array}\right]\\
&= \sup_{\substack{1\leq u\leq u_0\\1\leq s\leq s_0}} \mathfrak{h}_{us} \frac{j_{u_0}k_{s_0}}{j_{u-1}k_{s-1}} + \frac{h_{(u_0+1)(s_0+1)}}{j_{u-1}k_{s-1}} \mathfrak{K}_{(u_0+1)(s_0+1)} + \cdots + \frac{\mathfrak{h}_{us}}{j_{u-1}k_{s-1}} \mathfrak{h}_{us}
\end{aligned}$$

$$< \zeta \frac{j_{u_0} k_{s_0}}{j_{u-1} k_{s-1}} + (1+\varepsilon) \frac{j_{u_0} k_{s_0}}{j_{u-1} k_{s-1}}.$$

Since $j_{u-1}k_{s-1} \to \infty$ as $m, n \to \infty$, it follows that

$\frac{1}{mn} \sum_{w,\lambda=1,1}^{m,n} |\psi(\xi, \Upsilon_{w\lambda}, p) - \psi(\xi, \Upsilon, p)| \xrightarrow{\mathfrak{I}_2} 1.$

Similarly, we can show that

$\sum_{w,\lambda=1,1}^{m,n} |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| \xrightarrow{\mathfrak{I}_2} 0$

and $\sum_{w,\lambda=1,1}^{m,n} |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| \xrightarrow{\mathfrak{I}_2} 0.$

Hence $\{\Upsilon_w \lambda\} \in \left(\mathfrak{C}_1 \left[\mathfrak{I}_{\mathfrak{W}_2}^{(\psi,\varrho,\varphi)}\right]\right).$ □

Theorem 3.9. *If* $\{\Upsilon_w \lambda\} \in \mathfrak{N}_{\omega_2} \left[I_{\mathfrak{W}_2}^{(\psi,\varrho,\varphi)}\right] \cap \mathfrak{C}_1 \left[\mathfrak{I}_{\mathfrak{W}_2}^{(\psi,\varrho,\varphi)}\right]$, *then* $\mathfrak{N}_{\omega_2} \left[\mathfrak{I}_{\mathfrak{W}_2}^{(\psi,\varrho,\varphi)}\right] - \lim \Upsilon_{w\lambda} = \mathfrak{C}_1 \left[\mathfrak{I}_{\mathfrak{W}_2}^{(\psi,\varrho,\varphi)}\right] - \lim \Upsilon_{w\lambda}$.

Proof. Let $\Upsilon_{w\lambda} \to \Upsilon_1 \left(\mathfrak{N}_{\omega_2} \left[\mathfrak{I}_{\mathfrak{W}_2}^{(\psi,\varrho,\varphi)}\right]\right)$ and $\Upsilon_{w\lambda} \to \Upsilon_2 \left(\mathfrak{C}_1 \left[\mathfrak{I}_{\mathfrak{W}_2}^{(\psi,\varrho,\varphi)}\right]\right)$.
Assume $r \in \mathbb{N}$ and $\varepsilon > 0$ in such a way that $r > \frac{2}{\varepsilon}$. Then, for any $p > 0$, there are $u_0, s_0 \in \mathbb{N}$ such that

$$\left.\begin{array}{l} \frac{1}{\mathfrak{h}_{us}} \sum_{(w,\lambda)\in \mathfrak{I}_{us}} \left|\psi\left(\xi, \Upsilon_{w\lambda}, \frac{\varpi}{2}\right) - \psi\left(\xi, \Upsilon_1, \frac{\varpi}{2}\right)\right| > 1 - \frac{1}{r}, \\ \frac{1}{\mathfrak{h}_{us}} \sum_{(w,\lambda)\in \mathfrak{I}_{us}} \left|\varrho\left(\xi, \Upsilon_{w\lambda}, \frac{\varpi}{2}\right) - \varrho\left(\xi, \Upsilon_1, \frac{\varpi}{2}\right)\right| < \frac{1}{r} \\ \text{and } \frac{1}{\mathfrak{h}_{us}} \sum_{(w,\lambda)\in \mathfrak{I}_{us}} \left|\varphi\left(\xi, \Upsilon_{w\lambda}, \frac{\varpi}{2}\right) - \varphi\left(\xi, \Upsilon_1, \frac{\varpi}{2}\right)\right| < \frac{1}{r}, \end{array}\right\}$$

for all$u > u_0, s > s_0$.
Also, there are $m_0, n_0 \in \mathbb{N}$ such that

$$\left.\begin{array}{l} \frac{1}{mn} \sum_{w,\lambda=1,1}^{m,n} \left|\psi\left(\xi, \Upsilon_{w\lambda}, \frac{\varpi}{2}\right) - \psi\left(\xi, \Upsilon_1, \frac{\varpi}{2}\right)\right| > 1 - \frac{1}{r}, \\ \frac{1}{mn} \sum_{w,\lambda=1,1}^{m,n} \left|\varrho\left(\xi, \Upsilon_{w\lambda}, \frac{\varpi}{2}\right) - \varrho\left(\xi, \Upsilon_1, \frac{\varpi}{2}\right)\right| < \frac{1}{r} \\ \text{and } \frac{1}{mn} \sum_{w,\lambda=1,1}^{m,n} \left|\varphi\left(\xi, \Upsilon_{w\lambda}, \frac{\varpi}{2}\right) - \varphi\left(\xi, \Upsilon_1, \frac{\varpi}{2}\right)\right| < \frac{1}{r}, \end{array}\right\}$$

for all $m > m_0, n > n_0$.
Take $r_1 = \max\{u_0, m_0\}$ and $r_2 = \max\{s_0, n_0\}$. Then we take $k, t \in \mathbb{N}$ such

that

$$\begin{aligned}
&\left|\psi\left(\xi,\Upsilon_k t,\frac{\varpi}{2}\right)-\psi\left(\xi,\Upsilon_1,\frac{\varpi}{2}\right)\right| \\
&\quad\geq \frac{1}{\mathfrak{h}_{us}}\sum_{(w,\lambda)\in\mathfrak{I}_{us}}\left|\psi\left(\xi,\Upsilon_{w\lambda},\frac{p}{2}\right)-\psi\left(\xi,\Upsilon_1,\frac{\varpi}{2}\right)\right|>1-\frac{1}{r} \\
&\left|\psi\left(\xi,\Upsilon_k t,\frac{\varpi}{2}\right)-\psi\left(\xi,\Upsilon_2,\frac{\varpi}{2}\right)\right| \\
&\quad\geq \frac{1}{mn}\sum_{w,\lambda=1,1}^{m,n}\left|\psi\left(\xi,\Upsilon_{w\lambda},\frac{\varpi}{2}\right)-\psi\left(\xi,\Upsilon_2,\frac{\varpi}{2}\right)\right|>1-\frac{1}{r} \\
&\left|\varrho\left(\xi,\Upsilon_k t,\frac{\varpi}{2}\right)-\varrho\left(\xi,\Upsilon_1,\frac{\varpi}{2}\right)\right| \\
&\quad\leq \frac{1}{\mathfrak{h}_{us}}\sum_{(w,\lambda)\in\mathfrak{I}_{us}}\left|\varrho\left(\xi,\Upsilon_{w\lambda},\frac{\varpi}{2}\right)-\varrho\left(\xi,\Upsilon_1,\frac{\varpi}{2}\right)\right|<\frac{1}{r} \\
&\left|\varrho\left(\xi,\Upsilon_k t,\frac{\varpi}{2}\right)-\varrho\left(\xi,\Upsilon_2,\frac{\varpi}{2}\right)\right| \\
&\quad\leq \frac{1}{mn}\sum_{w,\lambda=1,1}^{m,n}\left|\varrho\left(\xi,\Upsilon_{w\lambda},\frac{\varpi}{2}\right)-\varrho\left(\xi,\Upsilon_2,\frac{\varpi}{2}\right)\right|<\frac{1}{r} \text{ and} \\
&\left|\varphi\left(\xi,\Upsilon_k t,\frac{\varpi}{2}\right)-\varphi\left(\xi,\Upsilon_1,\frac{\varpi}{2}\right)\right| \\
&\quad\leq \frac{1}{\mathfrak{h}_{us}}\sum_{(w,\lambda)\in\mathfrak{I}_{us}}\left|\varphi\left(\xi,\Upsilon_{w\lambda},\frac{\varpi}{2}\right)-\varphi\left(\xi,\Upsilon_1,\frac{\varpi}{2}\right)\right|<\frac{1}{r} \\
&\left|\varphi\left(\xi,\Upsilon_k t,\frac{\varpi}{2}\right)-\varphi\left(\xi,\Upsilon_2,\frac{\varpi}{2}\right)\right| \\
&\quad\leq \frac{1}{mn}\sum_{w,\lambda=1,1}^{m,n}\left|\varphi\left(\xi,\Upsilon_{w\lambda},\frac{\varpi}{2}\right)-\varphi\left(\xi,\Upsilon_2,\frac{\varpi}{2}\right)\right|<\frac{1}{r}.
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
&|\psi(\xi,\Upsilon_1,\varpi)-\psi(\xi,\Upsilon_2,\varpi)| \\
&\quad\geq \left|\psi\left(\xi,\Upsilon_k t,\frac{\varpi}{2}\right)-\psi\left(\xi,\Upsilon_1,\frac{\varpi}{2}\right)\right|+\left|\psi\left(\xi,\Upsilon_k t,\frac{\varpi}{2}\right)-\psi\left(\xi,\Upsilon_2,\frac{\varpi}{2}\right)\right| \\
&\quad> \left(1-\frac{1}{r}\right)+\left(1-\frac{1}{r}\right)>1-\varepsilon, \\
&|\varrho(\xi,\Upsilon_1,\varpi)-\varrho(\xi,\Upsilon_2,\varpi)| \\
&\quad\leq \left|\varrho\left(\xi,\Upsilon_k t,\frac{\varpi}{2}\right)-\varrho\left(\xi,\Upsilon_1,\frac{\varpi}{2}\right)\right|+\left|\varrho\left(\xi,\Upsilon_k t,\frac{\varpi}{2}\right)-\varrho\left(\xi,\Upsilon_2,\frac{\varpi}{2}\right)\right| \\
&\quad< \left(\frac{1}{r}\right)+\left(\frac{1}{r}\right)<\varepsilon\text{and}
\end{aligned}$$

$$\begin{aligned}&|\varphi(\xi,\Upsilon_1,\varpi)-\varphi(\xi,\Upsilon_2,\varpi)|\\&\quad\le\left|\varphi\left(\xi,\Upsilon_k t,\frac{\varpi}{2}\right)-\varphi\left(\xi,\Upsilon_1,\frac{\varpi}{2}\right)\right|+\left|\varphi\left(\xi,\Upsilon_k t,\frac{\varpi}{2}\right)-\varphi\left(\xi,\Upsilon_2,\frac{\varpi}{2}\right)\right|\\&\quad<\left(\frac{1}{r}\right)+\left(\frac{1}{r}\right)<\varepsilon.\end{aligned}$$

Since $\varepsilon>0$ is arbitrary, we get
$|\psi(\xi,\Upsilon_1,\varpi)-\psi(\xi,\Upsilon_2,\varpi)|=1, |\varrho(\xi,\Upsilon_1,\varpi)-\varrho(\xi,\Upsilon_2,\varpi)|=0$ and
$|\varphi(\xi,\Upsilon_1,\varpi)-\varphi(\xi,\Upsilon_2,\varpi)|=0$, for all $\varpi>0$, which yields $\Upsilon_1=\Upsilon_2$.
As we go through the definitions and theorems that follow,
let us consider $(\Omega,\psi,\varrho,\varphi,*,\diamond,\otimes)$ to be a separable $\mathfrak{NMG}$ and $\mathfrak{I}_2$ to be $\mathfrak{SAI}$.
□

Definition 3.1. *The sequence $\{\Upsilon_{w\lambda}\}$ is $\mathfrak{WSI}_2\mathfrak{LC}a$ if for each $\varepsilon\in(0,1)$ for each $\xi\in\Omega$ and for all $\varpi>0$, there are $s=s(\varepsilon,\xi), t=t(\varepsilon,\xi)\in\mathbb{N}$ such that*

$$\mathfrak{Y}(\varepsilon,\xi,\varpi)=\left\{\begin{array}{c}\frac{1}{\mathfrak{h}_{us}}\sum\limits_{(w,\lambda)\in\mathfrak{I}_{us}}|\psi(\xi,\Upsilon_{w\lambda},\varpi)-\psi(\xi,\Upsilon_{st},\varpi)|\le 1-\varepsilon\\ \textit{or}\ \frac{1}{\mathfrak{h}_{us}}\sum\limits_{(w,\lambda)\in\mathfrak{I}_{us}}|\varrho(\xi,\Upsilon_{w\lambda},\varpi)-\varrho(\xi,\Upsilon_{st},\varpi)|\ge\varepsilon\\ \textit{and}\ \frac{1}{\mathfrak{h}_{us}}\sum\limits_{(w,\lambda)\in\mathfrak{I}_{us}}|\varphi(\xi,\Upsilon_{w\lambda},\varpi)-\varphi(\xi,\Upsilon_{st},\varpi)|\ge\varepsilon\end{array}\right\}\in\mathfrak{I}_2$$

Theorem 3.10. *Every $\mathfrak{WSI}_2\mathfrak{LC}$ sequence of closed sets $\{\Upsilon_{w\lambda}\}$ is $\mathfrak{WSI}_2\mathfrak{LC}a$ with regard to $\mathfrak{NM}(\psi,\varrho,\varphi)$.*

Proof. Let $\Upsilon_{w\lambda}\to\mathfrak{N}_{\omega_2}\left[\mathfrak{I}_{\mathfrak{W}_2}^{(\psi,\varrho,\varphi)}\right]\Upsilon$. At that case, for each $\varepsilon\in(0,1)$, for every $\xi\in\Omega$ and for all $\varpi>0$,

$$\mathfrak{Y}(\varepsilon,\xi,\varpi)=\left\{\begin{array}{c}\frac{1}{\mathfrak{h}_{us}}\sum\limits_{(w,\lambda)\in\mathfrak{I}_{us}}|\psi(\xi,\Upsilon_{w\lambda},\varpi)-\psi(\xi,\Upsilon,\varpi)|\le 1-\varepsilon\\ \text{or}\ \frac{1}{\mathfrak{h}_{us}}\sum\limits_{(w,\lambda)\in\mathfrak{I}_{us}}|\varrho(\xi,\Upsilon_{w\lambda},\varpi)-\varrho(\xi,\Upsilon,\varpi)|\ge\varepsilon\\ \text{and}\ \frac{1}{\mathfrak{h}_{us}}\sum\limits_{(w,\lambda)\in\mathfrak{I}_{us}}|\varphi(\xi,\Upsilon_{w\lambda},\varpi)-\varphi(\xi,\Upsilon,\varpi)|\ge\varepsilon\end{array}\right\}\in\mathfrak{I}_2.$$

Since $\mathfrak{I}_2$ is $\mathfrak{SAI}$, the set

$$\mathfrak{Y}^c(\varepsilon,\xi,\varpi)=\left\{\begin{array}{c}\frac{1}{\mathfrak{h}_{us}}\sum\limits_{(w,\lambda)\in\mathfrak{I}_{us}}|\psi(\xi,\Upsilon_{w\lambda},\varpi)-\psi(\xi,\Upsilon,\varpi)|>1-\varepsilon\\ \text{and}\ \frac{1}{\mathfrak{h}_{us}}\sum\limits_{(w,\lambda)\in\mathfrak{I}_{us}}|\varrho(\xi,\Upsilon_{w\lambda},\varpi)-\varrho(\xi,\Upsilon,\varpi)|<\varepsilon\\ \text{and}\ \frac{1}{\mathfrak{h}_{us}}\sum\limits_{(w,\lambda)\in\mathfrak{I}_{us}}|\varphi(\xi,\Upsilon_{w\lambda},\varpi)-\varphi(\xi,\Upsilon,\varpi)|<\varepsilon\end{array}\right\}\in\mathfrak{I}_2$$

is nonempty and belongs to $\Upsilon(\mathfrak{I}_2)$. So, we select positive integers u and s such that $(u,s)\neq\mathfrak{Y}(\varepsilon,\xi,\varpi)$ and we get

$$\left.\begin{array}{l} \frac{1}{\mathfrak{h}_{us}} \sum\limits_{(w_0,\lambda_0)\in\mathfrak{I}_{us}} |\psi(\xi,\Upsilon_{w_0\lambda_0},\varpi)-\psi(\xi,\Upsilon,\varpi)| > 1-\varepsilon, \\ \text{and } \frac{1}{\mathfrak{h}_{us}} \sum\limits_{(w_0,\lambda_0)\in\mathfrak{I}_{us}} |\varrho(\xi,\Upsilon_{w_0\lambda_0},\varpi)-\varrho(\xi,\Upsilon,\varpi)| < \varepsilon, \\ \text{and } \frac{1}{\mathfrak{h}_{us}} \sum\limits_{(w_0,\lambda_0)\in\mathfrak{I}_{us}} |\varphi(\xi,\Upsilon_{w_0\lambda_0},\varpi)-\varphi(\xi,\Upsilon,\varpi)| < \varepsilon. \end{array}\right\}$$

Now, presume that

$$\mathfrak{Z}(\varepsilon,\xi,\varpi) = \left\{\begin{array}{l} \frac{1}{\mathfrak{h}_{us}} \sum\limits_{(w,\lambda),(w_0,\lambda_0)\in\mathfrak{I}_{us}} |\psi(\xi,\Upsilon_{w\lambda},\varpi)-\psi(\xi,\Upsilon_{w_0\lambda_0},\varpi)| \leq 1-2\varepsilon \\ \text{or } \frac{1}{\mathfrak{h}_{us}} \sum\limits_{(w,\lambda),(w_0,\lambda_0)\in\mathfrak{I}_{us}} |\varrho(\xi,\Upsilon_{w\lambda},\varpi)-\varrho(\xi,\Upsilon_{w_0\lambda_0},\varpi)| \geq 2\varepsilon \\ \text{and} \frac{1}{\mathfrak{h}_{us}} \sum\limits_{(w,\lambda),(w_0,\lambda_0)\in\mathfrak{I}_{us}} |\varphi(\xi,\Upsilon_{w\lambda},\varpi)-\varphi(\xi,\Upsilon_{w_0\lambda_0},\varpi)| \geq 2\varepsilon. \end{array}\right\}$$

Consider the inequality

$$\begin{aligned} &\frac{1}{\mathfrak{h}_{us}} \sum_{(w,\lambda),(w_0,\lambda_0)\in\mathfrak{I}_{us}} |\psi(\xi,\Upsilon_{w\lambda},\varpi)-\psi(\xi,\Upsilon_{w_0\lambda_0},\varpi)| \\ &\quad \leq \frac{1}{\mathfrak{h}_{us}} \sum_{(w,\lambda)\in\mathfrak{I}_{us}} |\psi(\xi,\Upsilon_{w\lambda},\varpi)-\psi(\xi,\Upsilon,\varpi)| \\ &\qquad + \frac{1}{\mathfrak{h}_{us}} \sum_{(w_0,\lambda_0)\in\mathfrak{I}_{us}} |\psi(\xi,\Upsilon_{w\lambda},\varpi)-\psi(\xi,\Upsilon_{w_0\lambda_0},\varpi)|, \\ &\frac{1}{\mathfrak{h}_{us}} \sum_{(w,\lambda),(w_0,\lambda_0)\in\mathfrak{I}_{us}} |\varrho(\xi,\Upsilon_{w\lambda},\varpi)-\varrho(\xi,\Upsilon_{w_0\lambda_0},\varpi)| \\ &\quad \geq \frac{1}{\mathfrak{h}_{us}} \sum_{(w,\lambda)\in\mathfrak{I}_{us}} |\varrho(\xi,\Upsilon_{w\lambda},\varpi)-\varrho(\xi,\Upsilon,\varpi)| \\ &\qquad + \frac{1}{\mathfrak{h}_{us}} \sum_{(w_0,\lambda_0)\in\mathfrak{I}_{us}} |\varrho(\xi,\Upsilon_{w_0\lambda_0},\varpi)-\varrho(\xi,\Upsilon,\varpi)|, \\ &\frac{1}{\mathfrak{h}_{us}} \sum_{(w,\lambda),(w_0,\lambda_0)\in\mathfrak{I}_{us}} |\varphi(\xi,\Upsilon_{w\lambda},\varpi)-\varphi(\xi,\Upsilon_{w_0\lambda_0},\varpi)| \\ &\quad \geq \frac{1}{\mathfrak{h}_{us}} \sum_{(w,\lambda)\in\mathfrak{I}_{us}} |\varphi(\xi,\Upsilon_{w\lambda},\varpi)-\varphi(\xi,\Upsilon,\varpi)| \\ &\qquad + \frac{1}{\mathfrak{h}_{us}} \sum_{(w_0,\lambda_0)\in\mathfrak{I}_{us}} |\varphi(\xi,\Upsilon_{w_0\lambda_0},\varpi)-\varphi(\xi,\Upsilon,\varpi)| \end{aligned}$$

Notice this if $(u,s) \in \mathfrak{Z}(\varepsilon,\xi,\varpi)$, therefore,

$$\frac{1}{\mathfrak{h}_{us}} \sum_{(w,\lambda)\in\mathfrak{I}_{us}} |\psi(\xi,\Upsilon_{w\lambda},\varpi)-\psi(\xi,\Upsilon,\varpi)| + \frac{1}{\mathfrak{h}_{us}} \sum_{(w_0,\lambda_0)\in\mathfrak{I}_{us}} |\psi(\xi,\Upsilon_{w\lambda},\varpi)-\psi(\xi,\Upsilon_{w_0\lambda_0},\varpi)| \leq 1-2\varepsilon,$$

$$\frac{1}{\mathfrak{h}_{us}} \sum_{(w,\lambda)\in\mathfrak{I}_{us}} |\varrho(\xi,\Upsilon_{w\lambda},\varpi)-\varrho(\xi,\Upsilon,\varpi)| + \frac{1}{\mathfrak{h}_{us}} \sum_{(w_0,\lambda_0)\in\mathfrak{I}_{us}} |\varrho(\xi,\Upsilon_{w\lambda},\varpi)-\varrho(\xi,\Upsilon_{w_0\lambda_0},\varpi)| \geq 2\varepsilon,$$

$$\frac{1}{\mathfrak{h}_{us}} \sum_{(w,\lambda)\in\mathfrak{I}_{us}} |\varphi(\xi,\Upsilon_{w\lambda},\varpi)-\varphi(\xi,\Upsilon,\varpi)| + \frac{1}{\mathfrak{h}_{us}} \sum_{(w_0,\lambda_0)\in\mathfrak{I}_{us}} |\varphi(\xi,\Upsilon_{w\lambda},\varpi)-\varphi(\xi,\Upsilon_{w_0\lambda_0},\varpi)| \geq 2\varepsilon.$$

From another point of view, since $(u,s) \neq \mathfrak{Y}(\varepsilon,\xi,\varpi)$, we get

$$\left.\begin{array}{l} \frac{1}{\mathfrak{h}_{us}} \sum_{(w_0,\lambda_0)\in\mathfrak{I}_{us}} |\psi(\xi,\Upsilon_{w\lambda},\varpi)-\psi(\xi,\Upsilon,\varpi)| > 1-\varepsilon, \\ \text{or } \frac{1}{\mathfrak{h}_{us}} \sum_{(w_0,\lambda_0)\in\mathfrak{I}_{us}} |\varrho(\xi,\Upsilon_{w_0\lambda_0},\varpi)-\varrho(\xi,\Upsilon,\varpi)| < \varepsilon \\ \text{and } \frac{1}{\mathfrak{h}_{us}} \sum_{(w_0,\lambda_0)\in\mathfrak{I}_{us}} |\varphi(\xi,\Upsilon_{w_0\lambda_0},\varpi)-\varphi(\xi,\Upsilon,\varpi)| < \varepsilon \end{array}\right\}$$

$$\left.\begin{array}{l} \frac{1}{\mathfrak{h}_{us}} \sum_{(w,\lambda)\in\mathfrak{I}_{us}} |\psi(\xi,\Upsilon_{w\lambda},\varpi)-\psi(\xi,\Upsilon,\varpi)| \leq 1-\varepsilon \\ \text{or} \frac{1}{\mathfrak{h}_{us}} \sum_{(w,\lambda)\in\mathfrak{I}_{us}} |\varrho(\xi,\Upsilon_{w\lambda},\varpi)-\varrho(\xi,\Upsilon,\varpi)| \geq \varepsilon \\ \text{and} \frac{1}{\mathfrak{h}_{us}} \sum_{(w,\lambda)\in\mathfrak{I}_{us}} |\varrho(\xi,\Upsilon_{w\lambda},\varpi)-\varrho(\xi,\Upsilon,\varpi)| \geq \varepsilon. \end{array}\right\}$$

Hence, $(u,s) \in \mathfrak{Y}(\varepsilon,\xi,\varpi)$. This gives that $\mathfrak{Z}(\varepsilon,\xi,\varpi) \subset \mathfrak{Y}(\varepsilon,\xi,\varpi) \in \mathfrak{I}_2$, so the sequence is Wijsman strongly $\mathfrak{I}_2$-lacunary sequence. □

Definition 3.2. *The sequence* $\{\Upsilon_{w\lambda}\}$ *is* $\mathfrak{WSI}_2\mathfrak{LC}to\Upsilon$ *iff there is a set* $\mathfrak{Q} = \{(w,\lambda) \in \mathbb{N}\times\mathbb{N}\}$ *such that* $\mathfrak{M}' = \{(w,\lambda) \in \mathfrak{I}_{us}\} \in \Upsilon(\mathfrak{I}_2)$ *for each* $\xi \in \Omega$*,*

$$\lim_{u,s\to\infty} \frac{1}{\mathfrak{h}_{us}} \sum_{(w,\lambda)\in\mathfrak{I}_{us}} |\psi(\xi,\Upsilon_{w\lambda},\varpi)-\psi(\xi,\Upsilon,\varpi)| = 1,$$

$$\lim_{u,s\to\infty)} \frac{1}{\mathfrak{h}_{us}} \sum_{(w,\lambda)\in\mathfrak{I}_{us}} |\varrho(\xi,\Upsilon_{w\lambda},\varpi)-\varrho(\xi,\Upsilon,\varpi)| = 0,$$

$$\lim_{u,s\to\infty} \frac{1}{\mathfrak{h}_{us}} \sum_{(w,\lambda)\in\mathfrak{I}_{us}} |\varrho(\xi,\Upsilon_{w\lambda},\varpi)-\varrho(\xi,\Upsilon,\varpi)|=0.$$

In this case, we write $\Upsilon_{w\lambda}\to\Upsilon\left(\mathfrak{N}_{\omega_2}\left[\mathfrak{I}_{\mathfrak{W}_2}^{(\psi,\varrho,\varphi)}\right]\right)$.

Theorem 3.11. *If the sequence* $\{\Upsilon_{w\lambda}\}$ *is* $\mathfrak{WSI}_2^*\mathfrak{LC}t$ *to* Υ*, then* $\{\Upsilon_{w\lambda}\}$ *is* $\mathfrak{WSI}_2\mathfrak{LC}$ *to* Υ.

Proof. Assume that $\Upsilon_{w\lambda}\to\Upsilon\left(\mathfrak{N}_{\omega_2}\left[\mathfrak{I}_{\mathfrak{W}_2}^{(\psi,\varrho,\varphi)}\right]\right)$.
Then, there is a set $\mathfrak{Q}=\{(w,\lambda)\in\mathbb{N}\times\mathbb{N}\}$ such that $\mathfrak{M}'=\{(w,\lambda)\in\mathfrak{I}_{us}\}\in\Upsilon(\mathfrak{I}_2)$, for each $\xi\in\Omega$,

$$\left.\begin{array}{l}\frac{1}{\mathfrak{h}_{us}}\sum_{(w,\lambda)\in\mathfrak{I}_{us}}|\psi(\xi,\Upsilon_{w\lambda},\varpi)-\psi(\xi,\Upsilon,\varpi)|>1-\varepsilon,\\ \frac{1}{\mathfrak{h}_{us}}\sum_{(w,\lambda)\in\mathfrak{I}_{us}}|\varrho(\xi,\Upsilon_{w\lambda},\varpi)-\varrho(\xi,\Upsilon,\varpi)|<\varepsilon,\\ \frac{1}{\mathfrak{h}_{us}}\sum_{(w,\lambda)\in\mathfrak{I}_{us}}|\varphi(\xi,\Upsilon_{w\lambda},\varpi)-\varphi(\xi,\Upsilon,\varpi)|<\varepsilon,\end{array}\right\}$$

for every $\varepsilon>0$ and for all $w,\lambda\geq k_0=k_0(\varepsilon,\xi)\in\mathbb{N}$.
Hereby for each $\varepsilon>0$ and $\xi\in\Omega$, we get

$$\chi(\varepsilon,\xi,p)=\left\{\begin{array}{c}\frac{1}{\mathfrak{h}_{us}}\sum_{(w,\lambda)\in\mathfrak{I}_{us}}|\psi(\xi,\Upsilon_{w\lambda},\varpi)-\psi(\xi,\Upsilon,\varpi)|\leq 1-\varepsilon\\ \text{or } \frac{1}{\mathfrak{h}_{us}}\sum_{(w,\lambda)\in\mathfrak{I}_{us}}|\varrho(\xi,\Upsilon_{w\lambda},\varpi)-\varrho(\xi,\Upsilon,\varpi)|\geq\varepsilon\\ \text{and } \frac{1}{\mathfrak{h}_{us}}\sum_{(w,\lambda)\in\mathfrak{I}_{us}}|\varphi(\xi,\Upsilon_{w\lambda},\varpi)-\varphi(\xi,\Upsilon,\varpi)|\geq\varepsilon\end{array}\right\}$$

$\subset\mathfrak{K}\cup(\mathfrak{M}'\cap((\{1,2,\ldots,(k_0-1)\}\times\mathbb{N})\cup(\mathfrak{N}\times\{1,2,\ldots,(k_0-1)\}))).$
For $\mathbb{N}\times\mathbb{N}\backslash\mathfrak{M}'=\mathfrak{K}\in\mathfrak{I}_2$. Since $\mathfrak{I}_2$ is an $\mathfrak{AI}$, we obtain
$\mathfrak{K}\cup(\mathfrak{M}'\cap((\{1,2,\ldots,(k_0-1)\}\times\mathbb{N})\cup(\mathbb{N}\times\{1,2,\ldots,(k_0-1)\})))\in\mathfrak{I}_2$
and so $\chi(\varepsilon,\xi,\varpi)\in\mathfrak{I}_2$. Hence $\{\Upsilon_{w\lambda}\}\to\Upsilon\left(\mathfrak{N}_{\omega_2}\left[\mathfrak{I}_{2_{\mathfrak{W}_2}}^{(\psi,\varrho,\varphi)}\right]\right)$. □

Theorem 3.12. *Let* $\mathfrak{I}_2$ *be a* $\mathfrak{SAI}$ *involving feature* $(\mathfrak{AP}_2)$. *Then* $\{\Upsilon_{w\lambda}\}\to\Upsilon\left(\mathfrak{N}_{\omega_2}\left[\mathfrak{I}_{2_{\mathfrak{W}_2}}^{(\psi,\varrho,\varphi)}\right]\right)$ *implies* $\{\Upsilon_{w\lambda}\}\to\Upsilon\left(\mathfrak{N}_{\omega_2}\left[\mathfrak{I}_{2_{\mathfrak{W}_2}}^{*(\psi,\varrho,\varphi)}\right]\right)$.

Definition 3.3. *The sequence* $\{\Upsilon_{w\lambda}\}$ *is known as* $\mathfrak{WSI}_2^*\mathfrak{LC}a$ *sequence if for each* $\varepsilon\in(0,1)$ *for all* $\xi\in\Omega$ *and for all* $\varpi>0$*, there is a set* $\mathfrak{Q}=\{(w,\lambda)\in\mathbb{N}\times\mathbb{N}\}$ *such that* $\mathfrak{M}'=\{(w,\lambda)\in\mathfrak{I}_{us}\}\in\Upsilon(\mathfrak{I}_2)$ *and* $\mathbb{N}=\mathbb{N}(\epsilon,\xi)\in\mathbb{N}$ *such that*

$$\left.\begin{array}{l}\frac{1}{\mathfrak{h}_{us}}\sum_{(w,\lambda),(s,t)\in\mathfrak{I}_{us}}|\psi(\xi,\Upsilon_{w\lambda},\varpi)-\psi(\xi,\Upsilon_{st},\varpi)|>1-\varepsilon\\ \frac{1}{\mathfrak{h}_{us}}\sum_{(w,\lambda),(u,s)\in\mathfrak{I}_{us}}|\varrho(\xi,\Upsilon_{w\lambda},\varpi)-\varrho(\xi,\Upsilon_{st},\varpi)|<\varepsilon\\ \textit{and } \frac{1}{\mathfrak{h}_{us}}\sum_{(w,\lambda),(u,s)\in\mathfrak{I}_{us}}|\varphi(\xi,\Upsilon_{w\lambda},\varpi)-\varphi(\xi,\Upsilon_{st},\varpi)|<\varepsilon,\end{array}\right\}$$

for every $w, \lambda, s, t \geq \mathbb{N}$.

Theorem 3.13. *Every* $\mathfrak{WSI}_2^*\mathfrak{LC}a$ *sequence of closed sets is* $\mathfrak{WSI}_2\mathfrak{LC}a$ *in* $\mathfrak{NMS}(\psi, \varrho, \varphi)$.

Proof. If the hypothesis is provided, then for each $\varepsilon \in (0,1)$, for each $\xi \in \Omega$, and for all $\varpi > 0$, there is a set $\mathfrak{Q} = \{(w,\lambda) \in \mathbb{N} \times \mathbb{N}\}$ such that $\mathfrak{M}' = \{(w,\lambda) \in \mathfrak{I}_{us}\} \in \Upsilon(\mathfrak{I}_2)$ and $\mathbb{N} = \mathbb{N}(\epsilon, \xi) \in \mathbb{N}$ such that

$$\left.\begin{array}{l} \frac{1}{\mathfrak{h}_{us}} \sum\limits_{(w,\lambda),(s,t)\in\mathfrak{I}_{us}} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon_{st}, \varpi)| > 1 - \varepsilon \\ \frac{1}{\mathfrak{h}_{us}} \sum\limits_{(w,\lambda),(u,s)\in\mathfrak{I}_{us}} |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon_{st}, \varpi)| < \varepsilon \text{ and} \\ \frac{1}{\mathfrak{h}_{us}} \sum\limits_{(w,\lambda),(u,s)\in\mathfrak{I}_{us}} |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon_{st}, \varpi)| < \varepsilon, \end{array}\right\} \text{ for each } w, \lambda, s, t \geq$$

$\mathbb{N}$.

Let $\mathfrak{K} = \mathbb{N} \times \mathbb{N}\mathfrak{M}'$. It is clear that $\mathfrak{K} \in \mathfrak{I}_2$ and

$$\chi(\varepsilon, \xi, \varpi) = \left\{\left\{\begin{array}{l} \frac{1}{\mathfrak{h}_{us}} \sum\limits_{(w,\lambda),(s,t)\in\mathfrak{I}_{us}} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon_{st}, \varpi)| > 1 - \varepsilon \\ \text{or } \frac{1}{\mathfrak{h}_{us}} \sum\limits_{(w,\lambda),(u,s)\in\mathfrak{I}_{us}} |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon_{st}, \varpi)| < \varepsilon \\ \text{and } \frac{1}{\mathfrak{h}_{us}} \sum\limits_{(w,\lambda),(u,s)\in\mathfrak{I}_{us}} |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon_{st}, \varpi)| < \varepsilon \end{array}\right\}\right\}$$

$\subset \mathfrak{K} \cup (\mathfrak{M}' \cap ((\{1, 2, \ldots, (\mathbb{N}-1)\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1, 2, \ldots, (\mathbb{N}-1)\})))$.

As $\mathfrak{I}_2$ be a $\mathfrak{SAI}$, then

$\mathfrak{K} \cup (\mathfrak{M}' \cap ((\{1, 2, \ldots, (\mathfrak{N}-1)\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1, 2, \ldots, (\mathfrak{N}-1)\}))) \in \mathfrak{I}_2$.

Therefore, we obtain $\chi(\varepsilon, \xi, \varpi) \in \mathfrak{I}_2$; that is $\{\Upsilon_{w\lambda}\}$ is $\mathfrak{WSI}_2\mathfrak{LC}a$ with regard to (ψ, ϱ, φ). ☐

Theorem 3.14. *Let* $\mathfrak{I}_2$ *be an* $\mathfrak{AI}$ *involving property* $(\mathfrak{AP}_2)$. *Then, the concept of* $\mathfrak{WSI}_2\mathfrak{LC}a$ *of sets coincides with* $\mathfrak{WSI}_2^*\mathfrak{LC}as$ *of sets.*

Proof. If a set sequence is $\mathfrak{WSI}_2^*\mathfrak{LC}a$, then it is $\mathfrak{WSI}_2\mathfrak{LC}a$ sequence according to theorem (3.13), where $\mathfrak{I}_2$ need not have the feature $(\mathfrak{AP}_2)$.

Now it is adequate to demonstrate that a sequence $\{\Upsilon_{w\lambda}\}$ in Ω is $\mathfrak{WSI}_2^*\mathfrak{LC}a$ sequence under assumption that it is a $\mathfrak{WSI}_2\mathfrak{LC}a$. Let $\{\Upsilon_{w\lambda}\}$ be a $\mathfrak{WSI}_2\mathfrak{LC}a$ sequence. In this case, for each $\varepsilon \in (0,1)$, for all $\xi \in \Omega$, there is a number $s = s(\epsilon, \xi), t = t(\epsilon, \xi) \in \mathbb{N}$ such that

$$\mathfrak{Y}(\varepsilon, \xi, \varpi) = \left\{\begin{array}{l} \frac{1}{\mathfrak{h}_{us}} \sum\limits_{(w,\lambda)\in\mathfrak{I}_{us}} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon_{st}, \varpi)| \leq 1 - \varepsilon \\ \text{or } \frac{1}{\mathfrak{h}_{us}} \sum\limits_{(w,\lambda)\in\mathfrak{I}_{us}} |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon_{st}, \varpi)| \geq \varepsilon \\ \text{and } \frac{1}{\mathfrak{h}_{us}} \sum\limits_{(w,\lambda)\in\mathfrak{I}_{us}} |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon_{st}, \varpi)| \geq \varepsilon \end{array}\right\} \in \mathfrak{I}_2$$

Let

$$\chi_j(\varepsilon,\xi,\varpi)=\left\{\begin{array}{c}\frac{1}{\mathfrak{h}_{us}}\sum\limits_{(w,\lambda),(s,t)\in\mathfrak{I}_{us}}|\psi(\xi,\Upsilon_{w\lambda},\varpi)-\psi(\xi,\Upsilon_{st},\varpi)|>1-\frac{1}{j},\\ \text{or }\frac{1}{\mathfrak{h}_{us}}\sum\limits_{(w,\lambda),(u,s)\in\mathfrak{I}_{us}}|\varrho(\xi,\Upsilon_{w\lambda},\varpi)-\varrho(\xi,\Upsilon_{st},\varpi)|<\frac{1}{j}\\ \text{and }\frac{1}{\mathfrak{h}_{us}}\sum\limits_{(w,\lambda),(u,s)\in\mathfrak{I}_{us}}|\varphi(\xi,\Upsilon_{w\lambda},\varpi)-\varphi(\xi,\Upsilon_{st},\varpi)|<\frac{1}{j}\end{array}\right\}$$

where $s(j)=s\left(\frac{1}{j}\right)$ and $t(j)=t\left(\frac{1}{j}\right)$, $j=1,2,\ldots$.

Clearly, for $j=1,2,\ldots,\chi_j(\varepsilon,\xi,\varpi)\in\Upsilon(\mathfrak{I}_2)$. Since $\mathfrak{I}_2$ has the property $(\mathfrak{AP}_2)$, there is $\chi\subset\mathbb{N}\times\mathbb{N}$ so that $\chi\in\Upsilon(\mathfrak{I}_2)$ and $\chi\backslash\chi_j$ is finite for all j.

Now, we demonstrate that

$$\lim_{w,\lambda,s,t\to\infty}\frac{1}{\mathfrak{h}_{us}}\sum_{(w,\lambda)\in\mathfrak{I}_{us},(s,t)\in\mathfrak{I}_{us}}|\psi(\xi,\Upsilon_{w\lambda},\varpi)-\psi(\xi,\Upsilon_{st},\varpi)|=1,$$

$$\lim_{w,\lambda,s,t\to\infty}\frac{1}{\mathfrak{h}_{us}}\sum_{(w,\lambda)\in\mathfrak{I}_{us},(s,t)\in\mathfrak{I}_{us}}|\varrho(\xi,\Upsilon_{w\lambda},\varpi)-\varrho(\xi,\Upsilon_{st},\varpi)|=0,$$

$$\lim_{w,\lambda,s,t\to\infty}\frac{1}{\mathfrak{h}_{us}}\sum_{(w,\lambda)\in I_{us},(s,t)\in\mathfrak{I}_{us}}|\varphi(\xi,\Upsilon_{w\lambda},\varpi)-\varphi(\xi,\Upsilon_{st},\varpi)|=0,$$

for all $w,\lambda,s,t>u(r)$.

So, it follows that for each $\xi\in\Omega$ and $(w,\lambda),(s,t)\in\chi$. To show these, let $\varepsilon\in(0,1)$ and $r\in\mathbb{N}$ such that $>\frac{2}{\varepsilon}$. If $(w,\lambda),(s,t)\in\chi$, then $\chi\backslash\chi_r$ is a finite set, therefore, there is $u=u(r)$ so that

$$\left.\begin{array}{c}\frac{1}{\mathfrak{h}_{us}}\sum\limits_{(w,\lambda)\in\mathfrak{I}_{us}}|\psi(\xi,\Upsilon_{w\lambda},\varpi)-\psi(\xi,\Upsilon_{s_rt_r},\varpi)|>1-\frac{1}{r},\\ \frac{1}{\mathfrak{h}_{us}}\sum\limits_{(s,t)\in\mathfrak{I}_{us}}|\psi(\xi,\Upsilon_{w\lambda},\varpi)-\psi(\xi,\Upsilon_{s_rt_r},\varpi)|>1-\frac{1}{r},\\ \frac{1}{\mathfrak{h}_{us}}\sum\limits_{(w,\lambda)\in\mathfrak{I}_{us}}|\varrho(\xi,\Upsilon_{w\lambda},\varpi)-\varrho(\xi,\Upsilon_{s_rt_r},\varpi)|<\frac{1}{r},\\ \frac{1}{\mathfrak{h}_{us}}\sum\limits_{(s,t)\in\mathfrak{I}_{us}}|\varrho(\xi,\Upsilon_{w\lambda},\varpi)-\varrho(\xi,\Upsilon_{s_rt_r},\varpi)|<\frac{1}{r},\\ \frac{1}{\mathfrak{h}_{us}}\sum\limits_{(w,\lambda)\in\mathfrak{I}_{us}}|\varphi(\xi,\Upsilon_{w\lambda},\varpi)-\varphi(\xi,\Upsilon_{s_rt_r},\varpi)|<\frac{1}{r},\\ \frac{1}{\mathfrak{h}_{us}}\sum\limits_{(s,t)\in\mathfrak{I}_{us}}|\varphi(\xi,\Upsilon_{w\lambda},\varpi)-\varphi(\xi,\Upsilon_{s_rt_r},\varpi)|<\frac{1}{r},\end{array}\right\}$$

$$\frac{1}{\mathfrak{h}_{us}}\sum_{(w,\lambda),(s,t)\in\mathfrak{I}_{us}}|\psi(\xi,\Upsilon_{w\lambda},\varpi)-\psi(\xi,\Upsilon_{st},\varpi)|$$

$$\geq \frac{1}{\mathfrak{h}_{us}} \sum_{(w,\lambda)\in\mathfrak{I}_{us}} |\psi(\xi,\Upsilon_{w\lambda},\varpi) - \psi(\xi,\Upsilon_{s_rt_r},\varpi)|$$
$$+ \frac{1}{\mathfrak{h}_{us}} \sum_{(s,t)\in\mathfrak{I}_{us}} |\psi(\xi,\Upsilon,\varpi) - \psi(\xi,\Upsilon_{s_rt_r},\varpi)|$$
$$> \left(1-\frac{1}{r}\right) + \left(1-\frac{1}{r}\right) > 1-\varepsilon,$$
$$\frac{1}{\mathfrak{h}_{us}} \sum_{(w,\lambda)\in\mathfrak{I}_{us}} |\varrho(\xi,\Upsilon_{w\lambda},\varpi) - \varrho(\xi,\Upsilon_{st},\varpi)|$$
$$\leq \frac{1}{\mathfrak{h}_{us}} \sum_{(w,\lambda)\in\mathfrak{I}_{us}} |\varrho(\xi,\Upsilon_{w\lambda},\varpi) - \varrho(\xi,\Upsilon_{s_rt_r},\varpi)|$$
$$+ \frac{1}{\mathfrak{h}_{us}} \sum_{(s,t)\in\mathfrak{I}_{us}} |\varrho(\xi,\Upsilon_{w\lambda},\varpi) - \varrho(\xi,\Upsilon_{s_rt_r},\varpi)|$$
$$< \frac{1}{r} + \frac{1}{r} < \varepsilon.$$
$$\frac{1}{\mathfrak{h}_{us}} \sum_{(w,\lambda)\in\mathfrak{I}_{us}} |\varphi(\xi,\Upsilon_{w\lambda},\varpi) - \varphi(\xi,\Upsilon_{st},\varpi)|$$
$$\leq \frac{1}{\mathfrak{h}_{us}} \sum_{(w,\lambda)\in\mathfrak{I}_{us}} |\varphi(\xi,\Upsilon_{w\lambda},\varpi) - \varphi(\xi,\Upsilon_{s_rt_r},\varpi)|$$
$$+ \frac{1}{\mathfrak{h}_{us}} \sum_{(s,t)\in\mathfrak{I}_{us}} |\varphi(\xi,\Upsilon_{w\lambda},\varpi) - \varphi(\xi,\Upsilon_{s_rt_r},\varpi)|$$
$$< \frac{1}{r} + \frac{1}{r} < \varepsilon.$$

Therefore, for each $\varepsilon \in (0,1)$, there exists $u = u(\varepsilon)$, and $(w,\lambda),(s,t) \in \chi \in \Upsilon(\mathfrak{I}_2)$, we get

$$\left\{ \begin{array}{c} \frac{1}{\mathfrak{h}_{us}} \sum\limits_{(w,\lambda)\in\mathfrak{I}_{us}} |\psi(\xi,\Upsilon_{w\lambda},\varpi) - \psi(\xi,\Upsilon_{st},\varpi)| \leq 1-\varepsilon \\ \text{or} \frac{1}{\mathfrak{h}_{us}} \sum\limits_{(w,\lambda)\in\mathfrak{I}_{us}} |\varrho(\xi,\Upsilon_{w\lambda},\varpi) - \varrho(\xi,\Upsilon_{st},\varpi)| \geq \varepsilon \\ \text{and} \frac{1}{\mathfrak{h}_{us}} \sum\limits_{(w,\lambda)\in\mathfrak{I}_{us}} |\varphi(\xi,\Upsilon_{w\lambda},\varpi) - \varphi(\xi,\Upsilon_{st},\varpi)| \geq \varepsilon \end{array} \right\} \in \mathfrak{I}_2.$$

This implies that $\{\Upsilon_{w\lambda}\}$ is $\mathfrak{WS}\mathfrak{I}_2^*\mathfrak{LC}a$ sequence. □

Definition 3.4. *A sequence $\{\Upsilon_{w\lambda}\}$ in $\mathfrak{NMS}$ is called to be Wijsman lacunary convergent to Υ with regard to $\mathfrak{NM}(\psi,\varrho,\varphi)$ if, for every $\varpi > 0$ and $\varepsilon \in (0,1)$, there is $m_0, n_0 \in \mathbb{N}$ such that*

$\frac{1}{\mathfrak{h}_{us}} \sum_{(w,\lambda)\in\mathfrak{I}_{us}} |\psi(\xi,\Upsilon_{w\lambda},\varpi)-\psi(\xi,\Upsilon,\varpi)| > 1-\varepsilon,$
$\frac{1}{\mathfrak{h}_{us}} \sum_{(w,\lambda)\in\mathfrak{I}_{us}} |\varrho(\xi,\Upsilon_{w\lambda},\varpi)-\varrho(\xi,\Upsilon,\varpi)| < \varepsilon$ *and*
$\frac{1}{\mathfrak{h}_{us}} \sum_{(w,\lambda)\in\mathfrak{I}_{us}} |\varphi(\xi,\Upsilon_{w\lambda},\varpi)-\varphi(\xi,\Upsilon,\varpi)| < \varepsilon$, *for all* $u > m_0$ *and* $\mathfrak{S} > n_0$.
We write $(\psi,\varrho,\varphi)^{\omega_2} - \lim \Upsilon_{w\lambda} = \Upsilon$.

Definition 3.5. *Take* $(\Omega,\psi,\varrho,\varphi,*,\diamond,\otimes)$ *as a seperable* $\mathfrak{NMS}$ *and take* $\{\Upsilon_{w\lambda}\} \in \Omega$.

- *(i)* $\Upsilon \in \Omega$ *is known as Wijsman Lacunary* $\mathfrak{I}_2$ $(\mathfrak{WLI}_2)$*-limit point of* $\{\Upsilon_{w\lambda}\}$ *if there is set* $\mathfrak{Q} = \{(w_1,\lambda_1) < (w_2,\lambda_2) < \ldots (w_u,\lambda_s) < \ldots\} \subset \mathbb{N}\times\mathbb{N}$ *such that the set*
  $\mathfrak{M}' = \{(w_u,\lambda_s) \in \mathfrak{I}_{us}\} \neq \mathfrak{I}_2$ *and* $(\psi,\varrho,\varphi)^{\omega_2} - \lim \Upsilon_{w_u\lambda_s} = \Upsilon$.
- *(ii)* $\Upsilon \in \Omega$ *is known as* $\mathfrak{WLI}_2$*-cluster point of* $\{\Upsilon_{w\lambda}\}$ *if, for every* $\varpi > 0$ *and* $\epsilon \in (0,1)$, *we get*

$$\left\{\begin{array}{l} \frac{1}{\mathfrak{h}_{us}} \sum_{(w,\lambda)\in\mathfrak{I}_{us}} |\psi(\xi,\Upsilon_{w\lambda},\varpi)-\psi(\xi,\Upsilon,\varpi)| > 1-\varepsilon \\ \text{and } \frac{1}{\mathfrak{h}_{us}} \sum_{(w,\lambda)\in\mathfrak{I}_{us}} |\varrho(\xi,\Upsilon_{w\lambda},\varpi)-\varrho(\xi,\Upsilon,\varpi)| < \varepsilon \\ \text{and } \frac{1}{\mathfrak{h}_{us}} \sum_{(w,\lambda)\in\mathfrak{I}_{us}} |\varphi(\xi,\Upsilon_{w\lambda},\varpi)-\varphi(\xi,\Upsilon,\varpi)| < \varepsilon \end{array}\right\} \notin \mathfrak{I}_2.$$

Here, $\wedge^{\mathfrak{I}_{\omega_2}}_{(\psi,\varrho,\varphi)}(\Upsilon_w\lambda)$ *denotes the set of all* $\mathfrak{WLI}_2$*-limit points and* $\Gamma^{\mathfrak{I}_{\omega_2}}_{(\psi,\varrho,\varphi)}(\Upsilon_{w\lambda})$ *indicates the set of all* $\mathfrak{WLI}_2$*-cluster points in* $\mathfrak{NMS}$.

Theorem 3.15. *For each sequence* $\{\Upsilon_{w\lambda}\}$ *in* $\mathfrak{NMS}$, *we have,*
$\wedge^{\mathfrak{I}_{\omega_2}}_{(\psi,\varrho,\varphi)}(\Upsilon_w\lambda) \subseteq \Gamma^{\mathfrak{I}_{\omega_2}}_{(\psi,\varrho,\varphi)}(\Upsilon_{w\lambda})$.

Proof. Let $\Upsilon \in \wedge^{\mathfrak{I}_{\omega_2}}_{(\psi,\varrho,\varphi)}(\Upsilon_w\lambda)$. So, there is a set $\mathfrak{Q} \subset \mathbb{N}\times\mathbb{N}$ such that $\mathfrak{M}' \neq \mathfrak{I}_2$, where $\mathfrak{Q}$ and $\mathfrak{M}'$ are as in Definition (3.5), satisfying $(\psi,\varrho,\varphi)^{\omega_2} - \lim \Upsilon_{w_u\lambda_s} = \Upsilon$. Hence, for every $\varpi > 0$ and $\varepsilon \in (0,1)$, there are $m_0, n_0 \in \mathbb{N}$ such that
$\frac{1}{\mathfrak{h}_{us}} \sum_{(w,\lambda)\in\mathfrak{I}_{us}} |\psi(\xi,\Upsilon_{w_u\lambda_s},\varpi)-\psi(\xi,\Upsilon,\varpi)| > 1-\varepsilon,$
$\frac{1}{\mathfrak{h}_{us}} \sum_{(w,\lambda)\in\mathfrak{I}_{us}} |\varrho(\xi,\Upsilon_{w_u\lambda_s},\varpi)-\varrho(\xi,\Upsilon,\varpi)| < \varepsilon$ and
$\frac{1}{\mathfrak{h}_{us}} \sum_{(w,\lambda)\in\mathfrak{I}_{us}} |\varrho(\xi,\Upsilon_{w_u\lambda_s},\varpi)-\varrho(\xi,\Upsilon,\varpi)| < \varepsilon$, for all $u \geq m_0$ and $s \geq n_0$. Therefore,

$$\mathfrak{Z} = \left\{ \begin{array}{c} \frac{1}{\mathfrak{h}_{us}} \sum\limits_{(w,\lambda)\in\mathfrak{I}_{us}} |\psi(\xi, \Upsilon_{w\lambda}, \varpi) - \psi(\xi, \Upsilon, \varpi)| > 1-\varepsilon, \\ \frac{1}{\mathfrak{h}_{us}} \sum\limits_{(w,\lambda)\in\mathfrak{I}_{us}} |\varrho(\xi, \Upsilon_{w\lambda}, \varpi) - \varrho(\xi, \Upsilon, \varpi)| < \varepsilon \\ \text{and } \frac{1}{\mathfrak{h}_{us}} \sum\limits_{(w,\lambda)\in\mathfrak{I}_{us}} |\varphi(\xi, \Upsilon_{w\lambda}, \varpi) - \varphi(\xi, \Upsilon, \varpi)| < \varepsilon \end{array} \right\}$$

$\supseteq \mathfrak{M}'\{(w_1, \lambda_1), (w_2, \lambda_2), \ldots, (w_{m_0}, \lambda_{n_0})\}$.
Now, with $\mathfrak{I}_2$ being admissible, we must have
$\mathfrak{M}'\{(w_1, \lambda_1), (w_2, \lambda_2), \ldots, (w_{m_0}, \lambda_{n_0})\} \neq \mathfrak{I}_2$ and as such $\neq \mathfrak{I}_2$.
Hence $\Upsilon \in \Gamma^{\mathfrak{I}_{\omega 2}}_{(\psi,\varrho,\varphi)}(\Upsilon_{w\lambda})$. □

4 Conclusion

In this investigation, researchers looked at the Wijsman lacunary ideal combination of the double sets collections, a kind of ideal union. We looked at several novel $\mathfrak{NMG}$ concepts for two-set groups, and we got some verifying results. Binary sets recurrence in $\mathfrak{NMG}$ have been characterised, together with their corresponding Wijsman lacunary $\mathfrak{I}_2$ - limit as well as cluster foci. While confirmation typically employ an alternate strategy, a few of the findings given in the current work have almost similar to the research focused on the pertinent topic. Only when $\mathfrak{I}$ and $\mathfrak{I}^*$ are admissible Ideals some of the results are true. We can apply all the results of the current paper and introduce new theories in different spaces like neutrosophic normed linear space, locally solid Riesz space and so on. Once we have proved the completeness of the space, easily we can obtain a fixed point theories in the respectiive space.

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