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New existence results on Caputo Fractional derivative with non-linear Integral conditions via Fixed Point theorems

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Abstract

In this article, we investigate the sufficient conditions for the existence of solutions to a Caputo fractional derivative with a class of boundary value problem dependence on the lipschitz first derivative conditions in Banach Space. Our main tool is a fixed point theorem. An numerical example is given to clarify the results.
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1 Introduction

In the past decades, Fractional differential equations extensively appear in the study of many real world phenomena. They have used to evolve variety of different areas of applied mathematics and physics such as heat conduction, chemical engineering, underground water flow, thermo-elasticity,and plasma physics can be reduced to nonlocal problems with integral boundary conditions. Boundary value problems with integral boundary conditions constitute a very interesting and important class of problems. we refer the reader to the articles (7; 8). Recently, most of the researchers have applying various fixed point theorems and get many interesting results of the existence of solutions for fractional differential equations (9; 11). Hilal et al. (12) studied the existence of an impulsive fractional integro-differential equations in a non-compact semigroup. In (13), Zhang et al. gave the existence and uniqueness of mild solutions for impulsive fractional equations with nonlocal conditions.

Motivated by the above mentioned works, this article investigate the existence of solutions for the boundary value problem with Caputo fractional differential equations and nonlinear integral conditions in Banach space as follows

$$\begin{cases} {}^{c}D^{\alpha}u(t) = f(t, u(t), u^{'}(t, u(t))), \text{for all t} \in J = [0, \mathfrak{T}], 1 < \alpha \leq 2, \\ u(0) - u^{'}(0) = \int_0^{\mathfrak{T}} p(s, u(s))ds, \\ u(\mathfrak{T}) + u^{'}(\mathfrak{T}) = \int_0^{\mathfrak{T}} q(s, u(s))ds, \end{cases} \tag{1}$$

Where ${}^{c}D^{\alpha}u(t)$is the Caputo derivative, $f \in (J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $p, q \in (J \times R, R)$ are given continuous functions. $f(t, u(t), u^{'}(t, u(t))) = 0$ Consider $\mathfrak{D}u(t) = u^{'}(t, u(t))$. Then (1) becomes

$$\begin{cases} {}^{c}D^{\alpha}u(t) = f(t, u(t), \mathfrak{D}u(t)), \text{for all t} \in J = [0, \mathfrak{T}], 1 < \alpha \leq 2, \\ u(0) - u^{'}(0) = \int_0^{\mathfrak{T}} p(s, u(s))ds, \\ u(\mathfrak{T}) + u^{'}(\mathfrak{T}) = \int_0^{\mathfrak{T}} q(s, u(s))ds, \end{cases} \tag{2}$$

The main aim of the paper is to explore the existence of solutions to the equation (2) by using various fixed point theorems. This paper is organized as five parts. In Section 2, we review few preliminaries and lemmas of fractional calculus that we require in the sequel. The existence of solutions for initial value problem for an fractional derivative results relying on various fixed point theorems for the problem (2) are proved in Section 3. Finally we present an illustrative example.

2 Facts

Here, we familarize the some notation, definitions and basic lemmas and theorems. These are used in the further section in this paper. We denote $C(J,\mathbb{R})$ which means set of all continuous functions from J into $\mathbb{R}$ in the Banach space with the norm

$$\|u\|_{\infty} = \sup\{|u(t)| : t \in J\}$$

Let $L^1(J, R)$ denote the Banach space of functions u: $J \to R$ which is the Lebesgue integrable with norm

$$\|u\|_{L^1} = \int_0^{\mathfrak{T}} |u(t)|dt,$$

Also the second derivative $u''$of the differentiable function u is continuous which is belongs to the space $C^2(J, R)$. (4; 5) Integral of the Function $f \in L^1([a, b], \mathbb{R}_+)$ with the Fractional order $\alpha \in \mathbb{R}_+$ is defined by

$$I_a^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)ds$$

When $a = 0$, we get $I^{\alpha} f(t) = [f * \varphi_{\alpha}](t)$, where $\varphi_{\alpha}(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t > 0$, and $\varphi_{\alpha}(t) = 0$ for $t \leq 0$, and $\varphi_{\alpha} \to \delta(t)$ as $\alpha \to 0$, where δ is the delta function. (4; 5) Function $f \in [a, b]$, the αth R-L fractional order derivative of f, is given as

$$(D_{a+}^{\alpha} f)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} f(s)ds.$$

Here $n = [\alpha] + 1$ and $[\alpha]$ is the integer part of α. (3) Function $f \in [a, b]$, the Caputo fractional derivative of order α of f, is defined as

$$({}^C D_{a+}^{\alpha} f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^n(s)ds,$$

where $n = [\alpha] + 1$.

Proposition 2.4 ((1; 2)) $f'(u) \in D$ satisfy the Lipschitz condition.
i.e., $\exists$ a +ve constant η such that

$$\|f'(u) - f'(v)\| \leq \eta\,(\|u - v\|),\ \ \forall\, u, v \in D.$$

3 Main Results

In this section, we give the existence of solutions via fixed point theorems for the problem (2)
Function $u \in C^2(J,R)$ whose α-derivative exists on J is said to be a solution of (2) if u satisfies the equation ${}^cD^\alpha u(t) = f(t,u(t),\mathfrak{D}u(t))$ on J and the conditions

$$u(0)-u^{'}(0) = \int_0^{\mathfrak{T}} p(s,u(s))ds, \qquad and \qquad u(\mathfrak{T})+u^{'}(\mathfrak{T}) = \int_0^{\mathfrak{T}} q(s,u(s))ds,$$

we use the following auxiliary lemmas to prove the existence solution for the equation (2). (6) Let $\alpha > 0$; then the differential equation

$${}^cD^\alpha g(t) = 0$$

has solutions $g(t) = c_0+c_1t+c_2t^2+\cdots+c_0t^{n-1}, c_i \in R, \quad i = 0,1,2,\ldots,n, \quad n = [\alpha]+1$ (6) Let $\alpha > 0$, then

$$I^{\alpha c}D^\alpha g(t) = g(t)c_0 + c_1t + c_2t^2 + \cdots + c_0t^{n-1}$$

for some $c_i \in R, \quad i = 0,1,2,\ldots,n, \quad n = [\alpha]+1,$

Follow up the Lemma (3) and (**??**) the result is useful in pursuing result. Let $1 < \alpha \leq 2$ and $\eta, \Theta_1, \Theta_2 : J \to R$ be continuous. A function u is a solution of the fractional integral equation

$$u(t) = \mathfrak{H}(t) + \int_0^{\mathfrak{T}} G(t,s)\eta(s)ds \tag{3}$$

with

$$\mathfrak{H}(t) = \frac{(\mathfrak{T}+1-t)}{\mathfrak{T}+2}\int_0^{\mathfrak{T}} \Theta_1(s)ds + \frac{(t+1)}{\mathfrak{T}+2}\int_0^{\mathfrak{T}} \Theta_2(s)ds \tag{4}$$

and

$$G(t,s) = \begin{cases} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(1+t)(\mathfrak{T}-s)^{\alpha-1}}{(\mathfrak{T}+2)\Gamma(\alpha)} - \frac{(1+t)(\mathfrak{T}-s)^{\alpha-2}}{(\mathfrak{T}+2)\Gamma(\alpha-1)}, & 0 \leq s \leq t, \\ -\frac{(1+t)(\mathfrak{T}-s)^{\alpha-1}}{(\mathfrak{T}+2)\Gamma(\alpha)} - \frac{(1+t)(\mathfrak{T}-s)^{\alpha-2}}{(\mathfrak{T}+2)\Gamma(\alpha-1)} & t \leq s \leq \mathfrak{T}. \end{cases} \tag{5}$$

if and only if u is a solution of the fractional BVP

$${}^cD^\alpha u(t) = \eta(t), t \in J \tag{6}$$

$$u(0) - u^{'}(0) = \int_0^{\mathfrak{T}} \Theta_1(s)ds \tag{7}$$

$$u(\mathfrak{T}) + u^{'}(\mathfrak{T}) = \int_0^{\mathfrak{T}} \Theta_2(s)ds \tag{8}$$

Proof Assume that (6); and Lemma (**??**) we get that

$$u(t)=c_0+c_1t+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\eta(s)ds. \tag{9}$$

From(7) and (8), we obtain

$$c_0-c_1=\int_0^{\mathfrak{T}}\Theta_1(s)ds, \tag{10}$$

and

$$c_0+c_1(\mathfrak{T}+1)+\frac{1}{\Gamma(\alpha)}\int_0^{\mathfrak{T}}(\mathfrak{T}-s)^{\alpha-1}\eta(s)ds+\frac{1}{\Gamma(\alpha-1)}\int_0^{\mathfrak{T}}(\mathfrak{T}-s)^{\alpha-2}\eta(s)ds=\int_0^{\mathfrak{T}}\Theta_2(s)ds. \tag{11}$$

From (10) and (11) implies that

$$\begin{aligned} c_1 &= \frac{1}{\mathfrak{T}+2}\int_0^{\mathfrak{T}}\Theta_2(s)ds-\frac{1}{\mathfrak{T}+2}\int_0^{\mathfrak{T}}\Theta_1(s)ds-\frac{1}{(\mathfrak{T}+2)\Gamma(\alpha)}\int_0^{\mathfrak{T}}(\mathfrak{T}-s)^{\alpha-1}\eta(s)ds \\ &\quad -\frac{1}{(\mathfrak{T}+2)\Gamma(\alpha-1)}\int_0^{\mathfrak{T}}(\mathfrak{T}-s)^{\alpha-2}\eta(s)ds \end{aligned} \tag{12}$$

and

$$\begin{aligned} c_0 &= \frac{\mathfrak{T}+1}{\mathfrak{T}+2}\int_0^{\mathfrak{T}}\Theta_1(s)ds+\frac{1}{\mathfrak{T}+2}\int_0^{\mathfrak{T}}\Theta_2(s)ds-\frac{1}{(\mathfrak{T}+2)\Gamma(\alpha)}\int_0^{\mathfrak{T}}(\mathfrak{T}-s)^{\alpha-1}\eta(s)ds \\ &\quad -\frac{1}{(\mathfrak{T}+2)\Gamma(\alpha-1)}\int_0^{\mathfrak{T}}(\mathfrak{T}-s)^{\alpha-2}\eta(s)ds \end{aligned} \tag{13}$$

From (9),(12) and (13) and we utilize the fact that $\int_0^{\mathfrak{T}}=\int_0^t+\int_t^{\mathfrak{T}}$,we obtain

$$u(t)=\mathfrak{H}(t)+\int_0^{\mathfrak{T}}G(t,s)\eta(s)ds \tag{14}$$

Where

$$\mathfrak{H}(t)=\frac{(\mathfrak{T}+1-t)}{\mathfrak{T}+2}\int_0^{\mathfrak{T}}\Theta_1(s)ds+\frac{(t+1)}{\mathfrak{T}+2}\int_0^{\mathfrak{T}}\Theta_2(s)ds \tag{15}$$

and

$$G(t,s)=\begin{cases}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}-\frac{(1+t)(\mathfrak{T}-s)^{\alpha-1}}{(\mathfrak{T}+2)\Gamma(\alpha)}-\frac{(1+t)(\mathfrak{T}-s)^{\alpha-2}}{(\mathfrak{T}+2)\Gamma(\alpha-1)}, & 0\le s\le t,\\ -\frac{(1+t)(\mathfrak{T}-s)^{\alpha-1}}{(\mathfrak{T}+2)\Gamma(\alpha)}-\frac{(1+t)(\mathfrak{T}-s)^{\alpha-2}}{(\mathfrak{T}+2)\Gamma(\alpha-1)} & t\le s\le\mathfrak{T}.\end{cases}$$

Hence we have (3). Conversely, it is clear that if u satisfies Equation (3), then (6)-(8) hold.

We need the following assumptions to prove Banach fixed point theorem. Let

$$G^*=\sup_{(t,s)\in J\times J}|G(t,s)|.$$

$\mathbf{A_1}$ Let $u \in C[J, R]$ and $g \in (C[a, b] \times \Re \times \Re, \Re)$ is continuous function and there exist a positive constants $\mathfrak{M}_1, \mathfrak{M}_2$ and $\mathfrak{M}$ such that

$$\|f(t, u_1, v_1) - f(t, u_2, v_2)\| \leq \mathfrak{M}_1(\|u_1 - u_2\| + \|v_1 - v_2\|)$$

for all u_1, v_1, u_2, v_2 in Y, $\mathfrak{M}_2 = \max_{t\in\Re} \|f(t, 0, 0)\|$ and $\mathfrak{M} = \max\{\mathfrak{M}_1, \mathfrak{M}_2\}$. Let $Y = C[\Re, X]$ be the set continuous functions on $\Re$ with values in the Banach spaces X .

$\mathbf{A_2}$ Let $u' \in C[a, b]$ satisfy the Lipschitz condition. i.e.,There exist a positive constants $\mathfrak{N}_1, \mathfrak{N}_2$ and $\mathfrak{N}$ such that

$$\|\mathfrak{D}(t, u) - \mathfrak{D}(t, v)\| \leq \mathfrak{N}_1(\|u - v\|),$$

for all u, v in Y. $\mathfrak{N}_2 = \max_{t\in D} \|\mathfrak{D}(t, 0)\|$ and $\mathfrak{N} = \max\{\mathfrak{N}_1, \mathfrak{N}_2\}$.

Assume If $\mathbf{A_1}$ and $\mathbf{A_2}$ are hold, then the estimate $\|\mathfrak{D}u(t)\| \leq t(\mathfrak{N}_1\|u\| + \mathfrak{N}_2)$, $\|\mathfrak{D}u(t) - \mathfrak{D}v(t)\| \leq \mathfrak{N}t\|u - v\|$, $\forall\, t \in \mathbb{R}$, and $u, v \in Y$ are satisfied. Take ϱ is $(\mathfrak{M} + \mathfrak{N}t)$.

Assume that

$\exists$ a +ve constant ϱ_0 such that

$$\|p(t, u_1) - p(t, u_2)\| \leq \varrho_0\|u_1 - u_2\|, \forall\, t \in J,\ \forall\, u_1, u_2 \in \mathbb{R}.$$

$\exists$a +ve constant ϱ_1 such that

$$\|q(t, u_1) - q(t, u_2)\| \leq \varrho_1\|u_1 - u_2\|, \forall\, t \in J,\ \forall\, u_1, u_2 \in \mathbb{R}.$$

if

$$\left[\frac{\mathfrak{T}(\mathfrak{T} + 1)}{\mathfrak{T} + 2}(\varrho_0 + \varrho_1) + \mathfrak{T}\varrho G^*\right] < 1 \tag{16}$$

then the BVP (2) has a unique solution on J.

Proof For the fixed point problem we reform the problem (2). Now let us consider the operator

$$F : C(J, R) \to C(J, R)$$

defined by

$$(Fu)(t) = \mathfrak{H}(t) + \int_0^{\mathfrak{T}} G(t, s)f(s, u(s), \mathfrak{D}(u(s)))ds. \tag{17}$$

where

$$\mathfrak{H}(t) = \frac{(\mathfrak{T}+1-t)}{\mathfrak{T}+2}\int_0^{\mathfrak{T}} p(s,u(s))ds + \frac{(t+1)}{\mathfrak{T}+2}\int_0^{\mathfrak{T}} q(s,u(s))ds$$

we have the value of the function G(t,s) by (5). the soluton of the problem (2) is the fixed point operator F. we have to prove that the operator F has a fixed point for that we will use the Banach contraction principle. Here we have to prove that F is a contraction.

Take u,v $\in$ C(J,R). For everey t $\in$ J we have

$$\begin{aligned}
|(Fu)(t)-(Fv)(t)| &\leq \frac{\mathfrak{T}+1}{\mathfrak{T}+2}\int_0^{\mathfrak{T}} |p(s,u(s))-p(s,v(s))|ds \\
&\quad + \frac{\mathfrak{T}+1}{\mathfrak{T}+2}\int_0^{\mathfrak{T}} |q(s,u(s))-q(s,v(s))|ds \\
&\quad + \int_0^{\mathfrak{T}} G(s,t)|f(s,u(s),\mathfrak{D}(u(s)))-f(s,v(s),\mathfrak{D}(v(s)))|ds \\
&\leq \frac{\mathfrak{T}(\mathfrak{T}+1)}{\mathfrak{T}+2}\varrho_0\|u-v\|_\infty + \frac{\mathfrak{T}(\mathfrak{T}+1)}{\mathfrak{T}+2}\varrho_1\|u-v\|_\infty \\
&\quad + \mathfrak{T}G^*(\mathfrak{M}\|u-v\|_\infty + \mathfrak{N}_t\|u-v\|_\infty) \\
&\leq \frac{\mathfrak{T}(\mathfrak{T}+1)}{\mathfrak{T}+2}\varrho_0\|u-v\|_\infty + \frac{\mathfrak{T}(\mathfrak{T}+1)}{\mathfrak{T}+2}\varrho_1\|u-v\|_\infty \\
&\quad + \mathfrak{T}G^*\varrho\|u-v\|_\infty \\
&\leq \left[\frac{\mathfrak{T}(\mathfrak{T}+1)}{\mathfrak{T}+2}\varrho_0 + \frac{\mathfrak{T}(\mathfrak{T}+1)}{\mathfrak{T}+2}\varrho_1 + \mathfrak{T}\varrho G^*\right]\|u-v\|_\infty
\end{aligned}$$

Consequently

$$\|F(u)-F(v)\|_\infty \leq \left[\frac{\mathfrak{T}(\mathfrak{T}+1)}{\mathfrak{T}+2}(\varrho_0+\varrho_1) + \mathfrak{T}\varrho G^*\right]\|u-v\|_\infty$$

Hence, by (17) F is a contraction. Here we conclude that based on the fixed point theorem, the operator F has a fixed point. Hence, we get the solution of the problem (2). □

Assume that

The function f: J $\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous.

$\exists$ a +ve constant N such that

$$|f(t,u(t),\mathfrak{D}(u(t)))| \leq \text{N} \ \forall \text{ t} \in \text{J}, \forall \text{ u} \in \mathbb{R}$$

$\exists$ a +ve constant N_0 such that

$$|p(t,u(t))| \leq N_0 \ \forall \text{ t} \in \text{J}, \forall \text{ u} \in \mathbb{R}$$

$\exists$ a +ve constant N_1 such that

$$|q(t,u(t))| \leq N_1 \ \forall \text{ t} \in \text{J}, \forall \text{ u} \in \mathbb{R}$$

Then the BVP (2) has atleast one solution on J.

Proof We shall prove $\mathcal{F}$ has a fixed point using Schaefer's fixed point theorem. This proof have four steps.

Step 1 : F is continuous. Let u_n be a sequence such that $u_n \rightarrow$ u in C(J,R). Then for each t $\in$ J,

$$\begin{aligned} |(Fu_n)(t)-(Fu)(t)| &\leq \frac{\mathfrak{T}+1}{\mathfrak{T}+2}\int_0^{\mathfrak{T}}|p(s,u(s))-p(s,v(s))|ds \\ &+\frac{\mathfrak{T}+1}{\mathfrak{T}+2}\int_0^{\mathfrak{T}}|q(s,u(s))-q(s,v(s))|ds \\ &+G^*\int_0^{\mathfrak{T}}|f(s,u(s),\mathfrak{D}(u(s)))-f(s,v(s),\mathfrak{D}(v(s)))|ds \end{aligned}$$

Here the functions f ,p and q are continuous, we have

$$\|F(u_n)-F(u)\|_\infty \rightarrow 0 \text{ as n} \rightarrow \infty.$$

Step 2 : The function operator $\mathcal{F}$ maps into bounded sets in $J \times \mathbb{R} \times \mathbb{R}$.

It suffices to prove that for any $\gamma > 0$, $\exists$ a + constant Λ such that for each value u $\in B_\gamma$ =$\{u \in C(J,R) : \|u\|_\infty \leq \gamma\}$ we have $\|F(u)\|_\infty \leq \Lambda$ by (A_6)-A_8 we have

$$\begin{aligned} |(Fu)(t)| &\leq \frac{\mathfrak{T}+1}{\mathfrak{T}+2}\int_0^{\mathfrak{T}}|p(s,u(s))|ds+\frac{\mathfrak{T}+1}{\mathfrak{T}+2}\int_0^{\mathfrak{T}}|q(s,u(s))|ds \\ &+\int_0^{\mathfrak{T}}G(s,t)|f(s,u(s),\mathfrak{D}(u(s)))|ds \\ &\leq \frac{\mathfrak{T}(\mathfrak{T}+1)}{\mathfrak{T}+2}N_0+\frac{\mathfrak{T}(\mathfrak{T}+1)}{\mathfrak{T}+2}N_1+N\mathfrak{T}G^*. \end{aligned}$$

for all t $\in$ J, thus

$$\|F(u)\|_\infty \leq \left[\frac{\mathfrak{T}(\mathfrak{T}+1)}{\mathfrak{T}+2}\right][N_0+N_1]+N\mathfrak{T}G^* = \Lambda$$

Step 3: The operator F maps bounded sets into equicontinuous sets in C(J,R).

Consider $t_1, t_2 \in J$,$t_1 < t_2$, and let B_γ be a bounded set in C(J,R) as in Step 2. If $u \in B_\gamma$, then

$$\begin{aligned}
&|(Fu)(t_2)-(Fu)(t_1)| \\
&= \frac{(t_2-t_1)}{\mathfrak{T}+2}\int_0^{\mathfrak{T}}|p(s,u(s))|ds + \frac{(t_2-t_1)}{\mathfrak{T}+2}\int_0^{\mathfrak{T}}|q(s,u(s))|ds \\
&+ \int_0^{\mathfrak{T}}|G(t_2,s)-G(t_1,s)||f(s,u(s),\mathfrak{D}u(s))|ds. \\
&\leq \frac{(t_2-t_1)\mathfrak{T}}{\mathfrak{T}+2}N_0 + \frac{(t_2-t_1)\mathfrak{T}}{\mathfrak{T}+2}N_1 + \mathfrak{T}N\sup_{s\in J}|G(t_2,s)-G(t_1,s)|.
\end{aligned}$$

since $t_1 \to t_2$, the above inequality tends to zero. Theorefore the operator $F : C(J,R) \to C(J,R)$ is continuous and completely continuous, since the result of Step 1 to Step 3 with the Arzela-Ascoli theorem.

Step 4 A priori bounds.

Here we have to prove the below set is bounded.

$$\lambda = \{u \in C(J,R) : u = \omega F(u) for any 0 < \omega < 1\}$$

Consider u $\in \lambda$; then u=ω F(u) for any $0 < \omega < 1$. Thus, for all $t \in J$ we have

$$u(t) = \omega\left[\mathfrak{H}(t) + \int_0^{\mathfrak{T}} G(t,s)g(s,u(s),\mathfrak{D}u(s))ds\right].$$

from the assumption (A_7) -(A_9) implies that

$$\begin{aligned}
|u(t)| \leq\ & \frac{\mathfrak{T}+1}{\mathfrak{T}+2}\int_0^{\mathfrak{T}}|p(s,u(s))|ds + \frac{\mathfrak{T}+1}{\mathfrak{T}+2}\int_0^{\mathfrak{T}}|q(s,u(s))|ds \\
& + \int_0^{\mathfrak{T}} G(t,s)|f(s,u(s),\mathfrak{D}(u(s)))|ds \\
& \leq \frac{\mathfrak{T}(\mathfrak{T}+1)}{\mathfrak{T}+2}N_0 + \frac{\mathfrak{T}(\mathfrak{T}+1)}{\mathfrak{T}+2}N_1 + N\mathfrak{T}G^*
\end{aligned}$$

for all $t \in J$, we have

$$\|u\|_{\infty} \leq \left[\frac{\mathfrak{T}(\mathfrak{T}+1)}{\mathfrak{T}+2}\right][N_0 + N_1] + N\mathfrak{T}G^* = \Lambda$$

Therefore the set λ is bounded. As a conclusion of Schaefer's fixed point theorem. F has a fixed point that is a solution of the problem (2) □

Next we confer an existence result for the problem (2) based on application of the non-linear alternative of Leray-Schauder type. Here we can cripple the conditions (A_7) -(A_9) . If the following conditions and (A_6) hold

There exists $\psi_f \in L^1(J, R^+)$ and a continuous and a non-decreasing functions $\tau : [0, \infty) \to (0, \infty)$ and $\theta : [0, \infty) \to (0, \infty)$ such that

$$|f(t, u, \mathfrak{D}(u(t))| \leq \psi_f(t)\tau(|u|)\theta(|\mathfrak{D}u|), \ \ \forall\, t \ \in J, u \in \mathbb{R}.$$

$\exists\ \psi_p \in L^1(J, R^+)$ and a continuous and a non-decreasing function $\tau_0 : [0, \infty) \to (0, \infty)$ such that

$$|p(t, u| \leq \psi_p(t)\tau_0(|u|), \ \ \forall\, t \ \in J, u \in \mathbb{R}.$$

$\exists\ \psi_q \in L^1(J, R^+)$ and a continuous and a non-decreasing function $\tau_1 : [0, \infty) \to (0, \infty)$ such that

$$|q(t, u| \leq \psi_q(t)\tau_1(|u|), \ \ \forall\, t \ \in J, u \in \mathbb{R}.$$

$\exists$ a number $N > 0$

$$\frac{\|u\|_{\infty}}{\mathtt{x}\tau_0(\|u\|_{\infty}) + \mathtt{y}\tau_1(\|u\|_{\infty}) + \mathtt{z}\mathfrak{N}_1 G^*\tau(\|u\|_{\infty})\theta(\|u\|_{\infty}) + \mathtt{z}\mathfrak{N}_2 G^*\theta^0\tau(\|u\|_{\infty})} > 1 \tag{18}$$

where

$$\mathtt{x} = \frac{\mathfrak{T}+1}{\mathfrak{T}+2}\int_0^{\mathfrak{T}} \psi_p(s)ds \qquad\qquad \mathtt{y} = \frac{\mathfrak{T}+1}{\mathfrak{T}+2}\int_0^{\mathfrak{T}} \psi_q(s)ds$$

$$\mathtt{z} = \int_0^{\mathfrak{T}} \psi_f(s)ds$$

let $\theta^0 = sup\,\{\theta(t) : t \in J\}$ then the BVP (2) has atleast one solution on J.

Proof It is easy to shown that the operator defined by (17) is continuous and completely continuous. For $\omega \in [0,1]$and let the value u satisfy $u(t) = \omega(Fu)(t)$ for all $t \in J$. by the assumption A_9-A_{11}, we have

$$\begin{aligned}
|y(t)| &\leq \frac{\mathfrak{T}+1}{\mathfrak{T}+2}\int_0^{\mathfrak{T}} \psi_p(s)\tau_0(|u(s))|ds + \frac{\mathfrak{T}+1}{\mathfrak{T}+2}\int_0^{\mathfrak{T}} \psi_q(s)\tau_1(|u(s))|ds \\
&\quad \int_0^{\mathfrak{T}} G(t,s)\psi_f(s)\tau(|u(s)|)\theta(|\mathfrak{D}u(s)|)ds \\
&\leq \tau_0(\|u\|_\infty)\frac{\mathfrak{T}+1}{\mathfrak{T}+2}\int_0^{\mathfrak{T}} \psi_p(s)ds + \tau_1(\|u\|_\infty)\frac{\mathfrak{T}+1}{\mathfrak{T}+2}\int_0^{\mathfrak{T}} \psi_q(s)ds + \\
&\quad \tau(\|u\|_\infty)\theta(t(\mathfrak{N}_1\|u\|_\infty + \mathfrak{N}_2))G^*\int_0^{\mathfrak{T}} \psi_f(s)ds \\
&\leq \tau_0(\|u\|_\infty)\frac{\mathfrak{T}+1}{\mathfrak{T}+2}\int_0^{\mathfrak{T}} \psi_p(s)ds + \tau_1(\|u\|_\infty)\frac{\mathfrak{T}+1}{\mathfrak{T}+2}\int_0^{\mathfrak{T}} \psi_q(s)ds + \\
&\quad \mathfrak{N}_1\theta\tau(\|u\|_\infty)\theta(\|u\|_\infty)G^*\int_0^{\mathfrak{T}} \psi_f(s)ds + \mathfrak{N}_2\theta^0\tau(\|u\|_\infty)G^*\int_0^{\mathfrak{T}} \psi_f(s)ds
\end{aligned}$$

for all $t \in J$, thus,

$$\frac{\|u\|_\infty}{\mathtt{x}\tau_0(\|u\|_\infty) + \mathtt{y}\tau_1(\|u\|_\infty) + \mathtt{z}\mathfrak{N}_1G^*\tau(\|u\|_\infty)\theta(\|u\|_\infty) + \mathtt{z}\mathfrak{N}_2G^*\theta^0\tau(\|u\|_\infty)} \leq 1$$

by condition (18) , $\exists$ N such that $\|u\| \neq$ N.
Let

$$Y = \{u \in C(J,R) : \|u\|_\infty < N\}$$

The operator F: $\overline{Y} \to C(J,R)$ is continuous and completely continous. From the choice of Y,there is no $u \in \partial u$ such that $u = \omega F(u)$ for some $\omega \in (0,1)$. As a consequence of the non-linear altenative of Leray-Schauder type (**?**), we see that F has a fixed point u in Y that is a solution of the problem (2). □ The last existence result is based on the following fixed point theorem due to Burton and Kirk (**?**). . Let X be a Banach space, and let $C, D : X \to X$ be two operators satisfying:

(i) C is contraction

(ii) D is completely continuous

then either

1. the operator equation $u = C(u) + D(u)$ has a solution, or
2. the set $\lambda = \{l \in X : l = \omega C(l/\omega) + \omega D(l)\}$ is unbounded for $\omega \in (0,1)$

Assume that $(A_4), (A_5) and (A_{10})$ hold if

$$\frac{\mathfrak{T}(\mathfrak{T}+1)}{\mathfrak{T}+2}(\varrho_0+\varrho_1) < 1 \tag{19}$$

and

$$\limsup_{l\to+\infty} \frac{(1-(\mathfrak{T}(\mathfrak{T}+1)(\varrho_0+\varrho_1)/(\mathfrak{T}+2)))l}{\mathfrak{N}_1 \mathrm{z} G^* \tau(l)\theta(l) + \mathfrak{N}_2 \mathrm{z} G^* \theta^0 \tau(l) + ((\mathfrak{T}(\mathfrak{T}+1)(\overline{p}+\overline{q}))/\mathfrak{T}+2)} > 1 \tag{20}$$

hold, where $\overline{p} = \sup_{s\in J}|p(s,0)|$ and $\overline{q} = \sup_{s\in J}|q(s,0)|$, then the BVP has atleast one solution on J.

Proof Consider the operators $C, D : C(J,R) \to C(J,R)$ as defined by

$$(Cu)(t) = \frac{(\mathfrak{T}+1-t)}{\mathfrak{T}+2}\int_0^{\mathfrak{T}} p(s,u(s))ds + \frac{(t+1)}{\mathfrak{T}+2}\int_0^{\mathfrak{T}} q(s,u(s))ds$$

and

$$(Du)(t) = \int_0^{\mathfrak{T}} G(t,s) f(s,u(s),\mathfrak{D}(u(s)))ds,$$

where G is defined in equation (6). From A_4,A_5 and (19) we can simply prove that C is a contraction. From the assumption A_{10} the operator D is continuous. To complete the existence of a fixed point of the operator C, D, it suffices to prove that the set λ in Theorem (3) is bounded. Let $u \in \lambda$, then for each $t \in J$.

$$u(t) = \omega C\Big(\frac{l}{\omega}\Big)(t) + \omega D(l)(t).$$

from A_4,A_5 and A_{10} we have

$$\begin{aligned}
|u(t)| &\leq \frac{\omega(\mathfrak{T}+1)}{\mathfrak{T}+2}\int_0^{\mathfrak{T}}|p(s,\frac{u(s)}{\omega})|ds+\frac{\omega(\mathfrak{T}+1)}{\mathfrak{T}+2}\int_0^{\mathfrak{T}}|q(s,\frac{u(s)}{\omega})|ds \\
&\quad +\omega\int_0^{\mathfrak{T}}|G(t,s)|f(s,u(s),\mathfrak{D}(u(t)))|ds \\
&\leq \frac{(\mathfrak{T}+1)}{\mathfrak{T}+2}\int_0^{\mathfrak{T}}\varrho_0|u(s)|ds+\frac{(\mathfrak{T}+1)}{\mathfrak{T}+2}\int_0^{\mathfrak{T}}|p(s,0)|ds \\
&\quad +\frac{(\mathfrak{T}+1)}{\mathfrak{T}+2}\int_0^{\mathfrak{T}}\varrho_1|u(s)|ds+\frac{(\mathfrak{T}+1)}{\mathfrak{T}+2}\int_0^{\mathfrak{T}}|q(s,0)|ds \\
&\quad +G^*\int_0^{\mathfrak{T}}\psi_g(s)\tau(|u(s)|)\theta(|\mathfrak{D}u(s)|)ds \\
&\leq \frac{\mathfrak{T}(\mathfrak{T}+1)(\varrho_0+\varrho_1)}{\mathfrak{T}+2}\|u\|_\infty+\frac{\mathfrak{T}(\mathfrak{T}+1)(\overline{p}+\overline{q})}{\mathfrak{T}+2} \\
&\quad +\mathrm{z}G^*\tau(\|u\|_\infty)\theta(\|\mathfrak{D}u(t)\|_\infty) \\
&\leq \frac{\mathfrak{T}(\mathfrak{T}+1)(\varrho_0+\varrho_1)}{\mathfrak{T}+2}\|u\|_\infty+\frac{\mathfrak{T}(\mathfrak{T}+1)(\overline{p}+\overline{q})}{\mathfrak{T}+2} \\
&\quad +\mathrm{z}G^*\tau(\|u\|_\infty)\theta(t(\mathfrak{N}_1\|u\|_\infty+\mathfrak{N}_2)) \\
&\leq \frac{\mathfrak{T}(\mathfrak{T}+1)(\varrho_0+\varrho_1)}{\mathfrak{T}+2}\|u\|_\infty+\frac{\mathfrak{T}(\mathfrak{T}+1)(\overline{p}+\overline{q})}{\mathfrak{T}+2} \\
&\quad +\mathfrak{N}_1\mathrm{z}G^*\tau(\|u\|_\infty)\theta(\|u\|_\infty)+\mathfrak{N}_2\mathrm{z}G^*\theta^0\tau(\|u\|_\infty)
\end{aligned}$$

thus

$$\frac{(1-(\mathfrak{T}(\mathfrak{T}+1)(\varrho_0+\varrho_1)/(\mathfrak{T}+2)))\|u\|_\infty}{\mathfrak{N}_1\mathrm{z}G^*\theta^0\tau(\|u\|_\infty)\theta(\|u\|_\infty)+\mathfrak{N}_2\mathrm{z}G^*\theta^0\tau(\|u\|_\infty)+((\mathfrak{T}(\mathfrak{T}+1)(\overline{p}+\overline{q}))/\mathfrak{T}+2)}\leq 1 \tag{21}$$

From (20) it follows that,$\exists$ an +ve constant Λ such that $\|u\|_\infty>\Lambda$, $\forall\, u\in\lambda$, the above condition (21)is violated. Hence $\|u\|_\infty\leq\Lambda,\ \forall\, u\in\lambda$ in order that the set λ is bounded. From theorem (3), we can conclude. □

4 An Example

Let us consider the following fractional BVP to illustrate our main results.

$$\begin{cases} {}^cD^\alpha u(t)=\frac{t}{3\sqrt{(\pi)}}sin(u(t)+u^{'}(t)), \forall\, t\in[0,1], 1<\alpha\leq 2 \\ u(0)-u^{'}(0)=\sum_{i=0}^{\infty}c_iu(t_i) \\ u(T)+u^{'}(T)=\sum_{j=0}^{\infty}d_ju(t_j) \end{cases} \tag{22}$$

where $0 < t_0 < t_1 < t_2 < \cdots < 1$, $0 < \overline{t_0} < \overline{t_1} < \overline{t_2} < \cdots < 1$, and c_i and d_j, $i, j = 0, \ldots,$ are given positive constants with

$$\textstyle\sum_{i=0}^{\infty} c_i < \infty \qquad\qquad \sum_{j=0}^{\infty} d_j < \infty$$

Let

$$f(t, u(t), \mathfrak{D}u(t)) = \frac{t}{3\sqrt{(\pi)}} sin(u(t) + u^{'}(t))$$

Let $u, v \in [0, \infty)$ and $t \in [0, 1]$. then we have

$$\begin{aligned} |f(t, u_1, v_1) - f(t, u_2, v_2)| \;\; &\leq \;\; \frac{t}{3\sqrt{(\pi)}}(|u_1 - u_2| + |v_1 - v_2|), \\ &\leq \frac{1}{3\sqrt{(\pi)}}|u - v| \\ &\leq \frac{1}{5}|u - v|. \end{aligned}$$

hence the condition A_1 and A_2 holds with $\varrho = (1/5)$ clearly A_4 and A_5 now we check the condition (16) is satisfied with $\mathfrak{T} = 1$, $\varrho_0 = \sum_{i=0}^{\infty} c_i$, $\varrho_1 = \sum_{i=0}^{\infty} d_i$, and G is given by

$$G(t,s) = \begin{cases} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(1+t)(1-s)^{\alpha-1}}{3\Gamma(\alpha)} - \frac{(1+t)(1-s)^{\alpha-2}}{3\Gamma(\alpha-1)}, & 0 \leq s \leq t, \\ -\frac{(1+t)(1-s)^{\alpha-1}}{3\Gamma(\alpha)} - \frac{(1+t)(1-s)^{\alpha-2}}{3\Gamma(\alpha-1)} & t \leq s \leq 1. \end{cases} \tag{23}$$

from above equation we have

$$G^* < \frac{5}{3\Gamma(\alpha)} + \frac{2}{3\Gamma(\alpha - 1)}$$

Then,

$$\frac{\mathfrak{T}(\mathfrak{T}+1)}{\mathfrak{T}+2}\varrho_0 + \frac{\mathfrak{T}(\mathfrak{T}+1)}{\mathfrak{T}+2}\varrho_1 + \mathfrak{T}\varrho G^* = \frac{2}{3}(\sum_{i=0}^{\infty} c_i + \sum_{j=0}^{\infty} d_j) + \frac{5}{3\Gamma(\alpha))} + \frac{2}{15(\Gamma(\alpha - 1)} < 1 \tag{24}$$

the equation is contented for relevent values of c_i, d_j,and $\alpha \in (1, 2]$. We get from Theorem (3) that the problem (22) has a unique solution on $[0, 1]$ for such values of $\alpha \in (1, 2]$

Remark 1 we can choose, for instance the constants c_i and d_j to be

$$c_i = \tfrac{2}{25}(\tfrac{1}{5})^i \qquad\qquad d_j = \tfrac{2}{75}(\tfrac{1}{5})^j$$

in this case

$$\sum_{i=0}^{\infty} c_i = \frac{1}{10} \qquad \sum_{j=0}^{\infty} d_j = \frac{1}{30}$$

then the equation gives

$$\frac{1}{3\Gamma(\alpha))} + \frac{2}{15(\Gamma(\alpha-1)} < \frac{41}{45}$$

which is satisfied for each $\alpha \in (1, 2]$. Indeed the numerical calculation shows that the function

$$W(\alpha) = \frac{1}{3\Gamma(\alpha))} + \frac{2}{15\Gamma(\alpha-1)} - \frac{41}{45}$$

is negative for all $\alpha \in (1, 2]$.

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