Topology on BP-Algebras

V. Abisha*
N. Kandaraj†

Abstract

In this paper we will define topological BP-algebras and find some properties of these algebras. Also we show how to connect certain topologies with ideals of BP-algebras. Furthermore, we derived certain standard properties using topological concepts and we give a characterization of topological BP-algebras in terms of neighborhoods.

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*Research Scholar, PG and Researc h Department of Mathematics, Saiva Bhanu Kshatriya College, Aruppukottai-626 101,Affiliated to Madurai Kamaraj University, Madurai-625 021, Tamil Nadu, India. abisha1997@gmail.com.

†Associate Professor,PG and Research Department of Mathematics, Saiva Bhanu Kshatriya College, Aruppukottai-626 101, Affiliated to Madurai Kamaraj University, Madurai-625 021,Tamil Nadu, India. n.kandarajsbkc1998@gmail.com.

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1. Introduction

The two classes of abstract algebras namely BCK-algebras and BCI-algebras are introduced by Imai Y and Iseki K [5].It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. Hu Q.P and Li X [4] introduced a wide class of abstract algebras: BCH-algebras. Also it is known that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. Ahn S.S and Han J.S [1] introduced the concepts of BP-algebras and they discussed some relations with several algebras. Moreover they have discussed quadratic BP-algebras and shown that the quadratic BP-algebras is equivalent to several quadratic algebras. Alo R and Deeba E [3] attempted to study the topological concepts of the BCK-structure. Theories of topological groups, topological rings and topological modules are well known and still investigated by many Mathematicians. Topological universal algebraic structures have been studied by some authors.

Motivated by this, in this paper, we study the connection between BP-Algebras and topology. We will define topological BP-algebras (TBP-algebras) and find some properties of TBP-algebras. Also we show how to connect certain topologies with ideals of BP-algebras. Furthermore, we derive certain standard properties using topological concepts and we give a characterization of topological BP-algebras in terms of neighborhoods. Finally, we show that a TBP-algebra B is Hausdroff if and only if {0} is closed in B.

2. Preliminaries

Definition 2.1 [1]. Let X be a set with a binary operation $*$ and a constant 0.

Then (X, $*$, 0) is called a BP-algebra if it satisfies the following axioms.

1. $x * x = 0$
2. $x * (x * y) = y$
3. $(x * z) * (y * z) = x * y$ for any x, y, z $\in X$.

Example 2.2 [1]. Let X = {0, a, b, c} be a set with following table

*	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Table 1

Then (X, $*$, 0) is a BP-algebra.

Theorem 2.3 [1]. If (X, $*$, 0) is a BP-algebra, then the following results hold: for any x, y $\in X$

1. x$*$ 0 = x.
2. x$*$ y = 0 implies y $*$ x = 0.
3. 0 $*$ x = 0 $*$ y implies x = y.
4. 0 $*$ x = y implies 0 $*$ y = x.
5. 0 $*$ x = x implies x $*$ y = y $*$ x

Proposition 2.4 [1]. If (X, $*$, 0) is a BP-algebra with (x $*$ y) $*$ z = x $*$ (z $*$ y) for any x, y, z $\in X$, then 0 $*$ x = x for any x $\in X$.

Theorem 2.5 [1]. If (X, $*$, 0) is a BP-algebra with x $*$ y = 0 and y $*$ x = 0, then x = y.

Definition 2.6 [10]. Let S be a non-empty subset of a BP-algebra X, then S is called BP-subalgebra of X if x $*$ y $\in$ S for all x, y $\in$ S.

Definition 2.7 [10]. Let (X, $*$, 0) be a BP-algebra and I be a non-empty subset of X. Then I is called an ideal of X, if it satisfies the following conditions.

1. 0 $\in$ I.
2. x $*$ y $\in$ I and y $\in$ I $\Longrightarrow$ x $\in$ I.

Definition 2.8 [9]. Let X be a non-empty set. A Collection τ of subsets of X is called a Topology if it satisfies the following conditions.

1. ϕ and X is in τ.
2. Arbitrary union of the members of any finite sub collection of τ is in τ.
3. Intersection of the members of any finite sub collection of τ is in τ.

The tuple (X, τ) is called a topological space.

Definition 2.9 [9]. Let (X, τ) is a Topological space, we say that a subset U of X is an open set of X if U belongs to the collection τ.

Definition 2.10 [9]. The complement of an open set is called a closed set.

Definition 2.11 [9]. A subset A of X is said to be neighborhood of x$\in$ A if there exist an open set U such that x $\in$ U $\subseteq$ A.

Definition 2.12[9]. Let (X, τ) be a topological space. (X, τ) is called a T_i-space i$\in$ {0, 1, 2} if it satisfies the following axioms.

T_0: For every x, y $\in$ X and x $\neq$ y, there exist at least one open neighbourhood U that contains one point of the pair (either x or y) excluding the other point.

T_1: For each x, y $\in$ X and x $\neq$ y, there exist open neighbourhoods U_1 and U_2 such that U_1 contains x but not y and U_2 contains y but not x.

T_2 (Hausdroff): For each x, y $\in$ X and x $\neq$ y, both have disjoint open neighbourhoods U, V such that x $\in$ U and y $\in$ V

Definition 2.13 [9]. Let (X, τ) be a topological space and A $\subseteq$ X.

Define τ_A = { U $\cap$ A / U $\in$ τ}. Then (A, τ_A) is a topological space and the topology τ_A is called the subspace topology on A.

Definition 2.14 [9]. Let (X, τ) be a topological space and A $\subseteq$ X. Let a $\in$ A is called an interior point of A, if there exist an open set U of 'a' such that U $\subseteq$ A.

Remark 2.15 ([9]). 1. Restriction function of a continuous function is a continuous.
2. Every T_2-space is a T_1-space. Furthermore, every T_1-space is a T_0-space.

3. Topological BP-algebras

Definition 3.1: Let B be a BP-algebra and X, Y be any two non-empty subsets of B. We define a subset $X * Y$ of B as follows. $X * Y$ = {a $*$ b / a$\in X$, $b \in$ Y}.

Example 3.2: Let B= ({0, p, q, r}, $*$, 0) be a BP-algebra with the binary operation $*$ defined by the Cayley table.

*	0	p	q	r
0	0	p	q	r
p	p	0	r	q
q	q	r	0	p
r	r	p	q	0

Table 2

Let X= {0. p} and Y= {0, q} be two subsets of X,

$X * Y$ = {0 $*$ 0, 0 $*$ q, p $*$ 0, p $*$ q}

= {0, p, p, r}

= {0, p, r}$\subseteq$ B.

Definition 3.3: Let B be a BP-algebra. Let T be the collection of subsets of B, T is said to be a topology on B if the following conditions are satisfied.
1. B$\emptyset$ $\in$, T

2. Arbitrary union of members of T is in T.
3. Finite intersection of members of T is in T.

Definition 3.4: Let (B, $*$, 0) be a BP-algebra and T be a topology on B. Then (B, $*$, T) is called a topological BP-algebra if the operation $*$ is continuous. Equivalently, for a, b $\in$ B if there exist an open set U of a$*$b then there exist two open sets X and Y of a and b respectively such that X $* Y \subseteq$ U.

Example 3.5: Let B= ({0, p, q, r}, $*$, 0) be a BP-algebra with the binary operation $*$ defined by as on example 3.2.

Define a topology T= {B, $\emptyset$, {0}, {0, p, q}}

Let 0, p $\in$ B and U= {0, p, q}

Now a $*$ b = p$*$0 = p $\in$ U.

Then there exist an open set X= {0, p, q}, Y= {0} if p and 0 respectively such that X $* Y$ = {0 $*$ 0, p $*$ 0, q$*$ 0}

= {0, p, q} $\subseteq$ U.

Therefore every open set U for a $*$ b then there exist an open sets X and Y of a and b respectively such that X $* Y \in$ U for all a, b $\in$ B.

Hence (B, $*$, T) is a Topological BP-algebra and it is denoted by TBP-algebra.

Example 3.6: Let B= ({0, p, q, r},$*$, 0) be a BP-algebra with the binary operation $*$ defined in Example 3.2.

Define a topology T= {B, $\emptyset$, {q}, {r}}

Let 0, q $\in$ B, q$*$0 = q = {q} = U.

Let X={q} and Y={B}, then X $* Y$ = { q $*$ 0, q $*$ p, q $*$ q, q $*$ r} = {q, r, 0, p}$\not\subseteq$U.

Hence (B, $*$, T) is not a Topological BP-algebra.

Remark 3.7: 1. The member of T is known as open sets in B.
2. An open set of an element a $\in$ B is a member of T containing B.

3. The complement of C $\in$ T that is B\C is called a closed set in B.

Example 3.8: Let B= ({0, p, q, r}, $*$, 0) be a BP-algebra with the binary operation $*$ defined in Example 3.5. Define a topology T= {B, $\emptyset$, {0, p, q }, {r}}.

Here {0, p, q} is open in B. Now {r} is closed in B. An open set of p ∈ B. is {0, p, q}.

Theorem 3.9: Let (B, ∗, T) be a TBP-algebra. If {0} is an open set in TBP-algebra, then T is a discrete topology in TBP-algebra.

Proof. Let (B, ∗, T) be a TBP-algebra. For every a ∈ B. a∗a = 0.

Since {0} is an open in TBP-algebra, there exist two neighborhoods X and Y of a such that $X * Y \subseteq \{0\} \Rightarrow X * Y = \{0\}$ ---------------------(1)

Let U = X ∩ Y. Clearly U∗ U = {0} by (1)

Since a ∗ b = 0 and b ∗ a = 0 ⇒ a =b, which implies U = {a}. Therefore T is discrete.

4. Properties of topological BP-algebras

Definition 4.1: Let (B, ∗, T) be a TBP-algebra. Then B is said to be T_0-space BP-algebra if it satisfies the following axiom:

For every a, b ∈ B and a ≠ b, there exist at least one open neighborhood U that contains one point of the pair (either a or b) excluding the other point.

Definition 4.2: Let (B, ∗, T) be a TBP-algebra. Then B is said to be T_1-space BP-algebra if it satisfies the axiom:

For each a, b ∈ B and a ≠ b, there exist an open neighborhoods X_1 and X_2 such that X_1 contains a but not b and X_2 contains b but not a.

Definition 4.3: Let (B, ∗, T) be a TBP-algebra. Then B is said to be T_2-space (Hausdroff) BP-algebra if it satisfies the axiom:For each a, b ∈ B and a ≠ b, both have disjoint open neighborhoods X, Y such that a ∈ X and b ∈ Y.

Example 4.4: Consider the TBP-algebra (B = {0, p, q},∗, T) with the following Cayley table.

∗	0	p	q
0	0	q	p
p	p	0	q
q	q	p	0

Table 3

Consider the topology T = {∅, B, {0, p }, {q}, {q, 0}, {0}}.

Then (B, ∗, T) is called a T_0-space BP-algebra.

Example 4.5: From the TBP-algebra given in Example 3.13.
Consider the topology Ton B is given by T={∅, B, {0, p }, {q}, {q, 0}, {0}, {p}, {q, p}}. Here T satisfies the T_1 and T_2 axioms.
Thus (B, ∗, T) is both T_1-space BP-algebra and T_2-space BP-algebra.

Theorem 4.6: Let B be a TBP-algebra and {0} is closed set on B if and only if B is Hausdroff.
Proof. Let (B, ∗, T) be a TBP-algebra.

Let a, b ∈ B. Assume that a and b are different in B, then a ∗ b ≠0 or b∗ a ≠0

We can assume that a ∗ b ≠0, then there exist some open sets X and Y of a and b respectively such that X ∗ Y ⊆ B \ {0}, since {0} is closed and B is a TBP-algebra.

Suppose, let a∈ X ∩ Y. Then a ∈ X and a ∈ Y implies a ∗ a = 0 ∈ X ∗ Y.

This contradicts to the fact that X ∗ Y ⊆ B \ {0} ⇒ X ∩ Y= ∅.

Therefore B is Hausdroff.
Conversely, Assume that B is Hausdroff
Claim: {0} is closed in TBP-algebra
We want to prove B \ {0}, there exist two neighborhoods X and Y of a and 0 respectively such that X ∩ Y= ∅ .

Since B is Hausdroff, then X ⊆ B \ {0}

⇒ B \ {0} is an open so that {0} is closed in TBP-algebra.

Theorem 4.7: Let B be a TBP-algebra and {0} be closed, then $\cap \mathbb{N}_0$ = {0}

where $\mathbb{N}_0$ is the neighborhood system of 0.

Proof. Let B be a TBP-algebra
Since {0} is closed. By the above theorem 3.15, B is Hausdroff.
Given an element a∈ B \ {0}, 0 has a neighborhood X such that a∉ X and

a∉$\cap \mathbb{N}_0$.Hence $\cap \mathbb{N}_0$ = {0}.

Definition 4.8: Let (B, ∗, T) be a TBP-algebra and I be a non-empty subset of B. Then I is called an ideal of B if it satisfies the following conditions.
1. 0 ∈ I.

2. If a ∗ b ∈ I and b ∈ I⟹a∈ I.

Example 4.9: Let (B = {0, p, q, r},∗, T) be a TBP-algebra with the binary operation ∗ defined by the example 3.2. I = {0} is an ideal.
Theorem 4.10: Let B be a TBP-algebra. If D is an open set in B which is also a BP-ideal of B, then it is closed set in B.

Proof. Let D be a BP-ideal which is an open set in B and let a ∈ B \ D. Then there exist a neighborhood X of a such that X∗X ⊆ D.

Since a∗a = 0 ∈ D and D is open.

Claim: X ⊆ B \ D.

If X ⊄ B \ D, then there exists b ∈ X ∩ D.

Note that c ∗ b ∈ X∗X ⊆ D foe all c ∈ X

Since b∈ D and D is a BP-ideal, it follows that c ∈ D.

Which show that X ⊆ D, ⟹a contradiction.Hence D is closed.

Definition 4.11: Let (B, ∗, T) be a TBP-algebra. Let A ⊆ B, x ∈ A is called an interior point of A, if there exist a neighborhood X of x contained in A.

Example 4.12: Let (B= {0, p, q},∗, T) be a TBP-algebra with the binary operation ∗ defined by the example 4.4.

Define a topology T = {B, ∅, {p, 0 }, {q} }.

Let A = {p, q}. Here, r is an interior point of A.

Theorem 4.13: Let A be a BP-ideal of a TBP-algebra B. If 0 is an interior point of A, then A is an open.

Proof. Let a∈ A.

Since a∗ a = 0 ∈ A and 0 is an interior point of A, there exist a neighborhood X of 0 which is contained in A.

Then there exist neighborhoods G and H of a such that G ∗ H ⊆ X ⊆ A

On the other hand for every b ∈ G, b ∗ a ∈ G ∗ H ⊆ A.

Since A is a BP-ideal and a ∈ A, it follows that b ∈ A so that a ∈ G ⊆ A

Hence A is open.

Definition 4.14: Let B and C be a TBP-algebra. A map f: B ⟶ C is continuous. If for every a ∈ B and any open set Y of f (a) there exist an open set X containing a such that f(X) ⊆ Y.

Example 4.15: Consider a TBP-algebra (B = {0, p, q}, ∗, T) as an example 3.13.

Define a topology T = {B, ∅, {0, p }}.

Define a function f: B ⟶ B by f (a) = a for all a ∈ B.

Clearly, f is continuous.

Notation 4.16: Let B be TBP-algebra and a, b ∈ B, then {b ∈ A / b ≡ a mod I}

$= \{b \in A \,/\, a * b,\ b * a \in I\}$.

Theorem 4.17: Let B be a TBP-algebra. If I is an (closed) open ideal, then for each $a \in B$, then $\{b \in B \,/\, b \equiv a \bmod I\}$ is closed (open).

Proof. Define a map G_a: $B \longrightarrow B$ such that $G_a(b) = a * b$.

Since B is TBP-algebra, then G_a is continuous.
Define a map H_a: $B \longrightarrow B$ such that $H_a(b) = b * a$.

Since B is a TBP-algebra, then H_a is continuous.
$G_a^{-1}(I) \cap H_a^{-1}(I) = \{b \in B \,/\,/\, a * b,\ b * a \in I\}$.

$\Longrightarrow$ Inverse image of an (closed) open set is an closed (open)

$\Longrightarrow$ $G_a^{-1}(I) \cap H_a^{-1}(I)$ is an closed (open)

Therefore $\{b \in B \,/\, b \equiv a \bmod I\}$ is an closed (open).

Theorem 4.18: Suppose that $\{0\}$ is closed (open), then $\{a\}$ is closed (open) for all $x \in B$.

Proof. Define a map g: $B \times B \longrightarrow B$ such that $g(x, y) = x * y$.

Since B is a TBP-algebra '$*$' is continuous.

which implies g is continuous.
Define a map h: $B \times B \times B \longrightarrow B \times B$ such that $h(x, y, z) = (x * y,\ y * z)$

which implies h is continuous. (Since g is continuous and Remark 2.15)
Since $\{0\}$ is closed, then $\{(0, 0)\} \in B \times B$ is closed.

Let $x \in B$ and let f: $B \longrightarrow B \times B$ such that $f(y) = h(x, y, x) = (x * y,\ y * x)$

Here f is the restriction of h to $\{x\}$ (since h to $\{x\} \times B \times \{x\}$)

and hence f is continuous (since h is continuous and Remark 2.15)
$f^{-1}(0, 0) = \{y \,/\, x * y = 0 \text{ and } y * x = 0\} = \{x\}$. Thus $\{x\}$ is also closed.

Similarly, for the other case where $\{0\}$ is an open.
Theorem 4.19: B is T_1-space BP-algebra if and only if B is T_2-space BP-algebra.
Proof. Let B be a T_1-space BP-algebra.
Let $a, b \in A$ and $a \neq b$, there exist an open neighborhoods X_1 and X_2 such that

X_1 contains a but not b and X_2 contains b but not a.
$\Longrightarrow$ $\{a\}$ is closed, for all $a \in B$

$\Longrightarrow$ $\{0\}$ is closed.

By theorem 3.15, we have B is a TBP-algebra.
Conversely, Let B be a T_2-space BP-algebra.

Let a, b $\in$ B and a $\neq$ b both have disjoint open neighborhoods X and Y such that a $\in$ X and b$\in$ Y and a $\notin$ Y and b $\notin$ X.$\Longrightarrow$ B is T_1-space BP-algebra.

Theorem 4.20: Let (B, $*$, T) be a TBP-algebra. If S is a subset of B such that foreach a, b $\in$ S, a$*$ b $\in$ S, then (S, $*$, T_S) is a TBP-algebra, where T_S is the subspace topology from B.

Proof. Let a, b $\in$ S $\Longrightarrow$ a $*$ b $\in$ S

Let U be an open set of a$*$ b in B.

$\Longrightarrow$ U $\cap$ S is an open set for a $*$ b in S.

Since B is a TBP-algebra, there exist an open sets X and Y of a and b respectively such that X $*$ Y $\in$ U

$\Longrightarrow$ There exist an open sets X $\cap$ S and Y $\cap$ S of a and b in S such that

(X $\cap$ S) $*$ (Y $\cap$ S) $\subseteq$ U $\cap$ S $\Longrightarrow$ (S, $*$, T_S) is a TBP-algebra.

4 Conclusion and Discussion

In this paper, we have introduced the topological concepts on BP-algebra, which is induced by the binary operation * of the BP-algebra and provided some important properties. We have studied the connection of the topologies of the BP-algebra with the ideals. In addition, we have shown that if 0 is an interior point of a BP-ideal of a TBP-algebra B, then the BP-ideal of a TBP-algebra B is open.

Finally, studying a topological properties on other algebras (BF/BH/TM-algebras) is an interesting open problem.

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