Integration method of non-elementary exponential functions using iterated Fubinni integrals

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ABSTRACT

The present work presents a new method of integration of non-elementary exponential functions where Fubinni's iterated integrals were used. In this research, some approximations were used in order to generalize the results obtained through mathematical series, in addition to integration methods and double integrals. In addition to the integration methods, the Taylor series was used, where the value found and compatible with the values of the power series that are used to calculate the value of the exponential function demonstrated in the work was verified. In addition to the methods described, a comparison of the values obtained by the series and the values described in the method was improvised, where it was noticed that the higher the value of the variable, the closer the results show a stability for the variable *greater than the value 4, described in table 01. The conclusions point to a great improvement, mainly for solving elliptic differential equations and statistical functions.*

Keywords: Non-elementary exponential functions; Approaches; Integrals; Taylor series.

1. Introduction

It has long been known that there are functions that do not have primitives, or if they do, methods to demonstrate them have not yet been discovered, and that only the area under the curve defined by the given function can be found through the techniques used by numerical analysis. Like the trapeze rule, Simpson's rule, the Newton-Cotes technique, among others. Therefore, the objective of this work is to demonstrate a technique for integrating non-elementary exponential functions through the iterative Fubinni method.

Since, this article was developed based on bibliographical research, we have pertinent ones such as: history of mathematics, pure and applied mathematics and on numerical series and some articles that contained the referred theme of this present work, contemplated. For the presentation of graphics, Geogebra will be used and for the calculation of series, Symbolab. While the other discussions involving the general solution of the primitive of the non-elementary exponential function through differential equations, will not be discussed in this present work.

2. Methodology

This research is classified as quantitative. We chose this research model because it shows, in a simple way, the relevant and current factors and aspects of the discipline of Calculus in the Brazilian educational environment, as well as its impact on the mathematical scientific environment.

This work is a diagnostic research and approach (quantitative), The study involved a quantitative approach that uses quantitative methods, Pereira, et al. (2018), was carried out considering the period from May 1, 2020 to March 20, 2021 of exploratory nature.

3. Demonstration of the method used through iterated integrals to find the primitive of a non-elementary function.

Although, one of the best known non-elementary exponential functions, such as the exponential function with a negative sign in the exponent, found in an important step on the resolution of the Normal Error Curve attributed to the French mathematician Abraham de Moivre, this article proposes to solve a function similar.

According to Boyer, (2003) and Eves (2011), the work of Carl Friedrich Gauss on the use of double integrals to solve an exponential function, applied in statistics, was taken into account. Even though, the method that Gauss uses is, through changing coordinates, and in a domain that differs from the one exposed in this article. There is another method, which takes into account some manipulations with the Gamma function, which can be conveniently used to solve the same problem attacked by Gauss, although it is more laborious.

And, in an attempt to find a solution, apart from numerical analysis, Gauss (1798), in fact for the integral of non-elementary functions, was made some hypotheses that corroborate the use of interacted integrals to attack the problem. on the primitive of non-elementary functions. Which will be detailed below:

$$\int_0^x e^{t^2} dt = te^{t^2}\Big|_0^x - \int_0^x t \cdot 2e^{t^2} dt$$

Whereby,

$$te^{t^2}\Big|_0^x = xe^{x^2}.$$

Then,

$$\int_0^x e^{t^2} dt = xe^{x^2} - \int_0^x 2t^2 \cdot e^{t^2} dt$$

Applying the method described by Guidorizzi, (2005); Gonçalves & Flemming (2012) have:

$$\int_0^x e^{t^2} dt = xe^{x^2} + \int_0^x 2t^2 \cdot e^{t^2} dt$$

$$\int_0^x e^{t^2} dt = xe^{x^2} - \left(2t^2 \int_0^x e^{t^2} dt\right)\Big|_0^x + \int_0^x 4t \int_0^x 2t \cdot e^{t^2} dtdt$$

Detailing the demonstration and using the properties of double integrals step-by-step:

$$\int_0^x e^{t^2} dt = xe^{x^2} - \left(2t^2 \int_0^x e^{t^2} dt\right)\Big|_0^x + \int_0^x 4t \int_0^x 2t \cdot e^{t^2} dtdt$$

By observing the properties of interacted integrals, Stewart (2013), sees that:

$$\int_0^x 4t \int_0^x 2t \cdot e^{t^2} dtdt = 2\int_0^x \int_0^x 2t \cdot e^{t^2} dtdt$$

Now the expression:

$$\left(2t^2 \int_0^x e^{t^2} dt\right)\Big|_0^x$$

Can be interpreted this way, therefore:

$$\left(2t^2 \int_0^x e^{t^2} dt\right)\Big|_0^x = 2t^2 \cdot \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)\cdot n!}\Big|_0^x =$$

$$2x^2 \cdot \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)\cdot n!} - 0 = 2x^2 \cdot \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)\cdot n!}$$

Therefore,

$$\left(2t^2\int_0^x e^{t^2}dt\right)\Big|_0^x = 2x^2\cdot\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)\cdot n!} = 2x^2\int_0^x e^{t^2}dt$$

Since the integral below is a number, it can therefore be in its integral form.

$$\left(2t^2\int_0^x e^{t^2}dt\right)\Big|_0^x = 2x^2\cdot\int_0^x e^{t^2}dt$$

Soon, it comes that:

$$\int_0^x e^{t^2}dt = xe^{x^2} - 2x^2\int_0^x e^{t^2}dt + 2\int_0^x\int_0^x 2t\cdot e^{t^2}dtdt$$

Like,

$$2\int_0^x\int_0^x 2t\cdot e^{t^2}dtdt = 2\int_0^x e^{t^2}dt$$

Follow that,

$$\int_0^x e^{t^2}dt = xe^{x^2} - 2x^2\int_0^x e^{t^2}dt + 2\int_0^x e^{t^2}dt$$

$$\int_0^x e^{t^2}dt + 2x^2\int_0^x e^{t^2}dt - 2\int_0^x e^{t^2}dt = xe^{x^2}$$

Placing the non-elementary exponential function, as a common factor, on the left side of the equality, has:

$$\int_0^x e^{t^2}dt + 2x^2\int_0^x e^{t^2}dt - 2\int_0^x e^{t^2}dt = xe^{x^2}$$

$$(2x^2-1)\int_0^x e^{t^2}dt = xe^{x^2} desde\ \ que\ \ x \neq \pm\sqrt{\frac{1}{2}}.$$

Finally, we have the integral of the function sought, as shown below:

$$\int_0^x e^{t^2}dt = \frac{xe^{x^2}}{(2x^2-1)}$$

Although the defined integral found above is obtained by techniques using differential and integral calculus, it is noteworthy that if the integral value at the zero point is undetermined, it can be done in a new interval, as follows:

$$\int_a^x e^{t^2}dt = \frac{xe^{x^2}}{(2x^2-1)} + \lambda, tal\ \ que\ \ \lambda = cte\ \ ,\ \ a \approx 0\ \ e\ \ x \neq \pm\sqrt{\frac{1}{2}}.$$

If λ = f(x), it will be analyzed further after discussing the derivative of the non-elementary exponential function. Once the derivative test is necessary to verify the authenticity of the mathematical method employed, or if it serves as an approximation of the value of the function, infinite series are used.

4. Differentiation of the primitive from the non-elementary function found.

Using anti-derivative functions and processes, Dana (1993), we have:

$$\frac{d}{dx}\left(\frac{xe^{x^2}}{(2x^2-1)}\right) = e^{x^2}\frac{(4x^4-4x^2-1)}{(4x^4-4x^2+1)}$$

Although,

$$\frac{d}{dt}\left(\frac{xe^{x^2}}{(2x^2-1)}\right) \neq e^{x^2}$$

It's got to,

$$\frac{(4x^4-4x^2-1)}{(4x^4-4x^2+1)} \approx 1$$

Soon,

$$e^{x^2} \approx e^{x^2}\frac{(4x^4-4x^2-1)}{(4x^4-4x^2+1)}$$

Once,

$$(4x^4-4x^2-1) = (4x^4-4x^2+1-2) \;\; e \; (4x^4-4x^2-1) = (2x^2-1)^2-2$$

What in an infinite space we will have:

$(4x^4-4x^2-1) = (2x^2-1)^2-2 \approx (2x^2-1)^2$

It has to:

$$e^{x^2} \approx e^{x^2} - \frac{2e^{x^2}}{(2x^2-1)^2}$$

Thus:

$$\frac{d}{dt}\left(\frac{xe^{x^2}}{(2x^2-1)}\right) = e^{x^2} - \frac{2e^{x^2}}{(2x^2-1)^2} \approx e^{x^2}$$

Evidently, although the desired function has not been acquired, it can be assumed that there is a function λ(x), such that:

$$\frac{d}{dx}(\lambda(x)) = \frac{2e^{x^2}}{(2x^2-1)^2}$$

Therefore, it can be said that:

$$\int_a^x e^{t^2}dt = \frac{xe^{x^2}}{(2x^2-1)} + \lambda(x), desde\ \ que\ \ \frac{d}{dx}(\lambda(x)) = \frac{2e^{x^2}}{(2x^2-1)^2}$$

The existence of the function λ(x), as well as the method using differential equations will not be addressed in this work.

5. A brief analysis of the non-elementary exponential function in relation to the Taylor series

We can obtain a representation of an exponential function through a Taylor series, or power series, soon known and demonstrated by, Nóbrega (2012); Joseph (1991); Simmons (2009) the following relationship of function and series, as follows:

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}$$

We have to,

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} = 1 + \frac{t}{1!} + \frac{t^2}{2!} + \cdots + \frac{t^n}{n!} + \cdots$$

Therefore, we can replace t by making $t = x^2$, so the series by Taylor, Guidorizzi (2005), is:

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$$

Therefore,

$$\sum_{n=0}^{\infty} \frac{x^{2n}}{n!} = 1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \cdots + \frac{x^{2n}}{n!} + \cdots$$

Now, integrating the two sides of equality, comes that:

$$\sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)\cdot n!}\Big|_0^x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)\cdot n!}$$

Therefore,

$$\int_0^x e^{t^2}dt = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)\cdot n!} = \frac{xe^{x^2}}{(2x^2-1)} + \lambda(x)$$

Although the value of λ(x) is still unknown, the value found by the method described in this work has to be verified with the values of the power series that is used to calculate the value of the non-elementary exponential function. As will be illustrated below.

$$\int_0^x e^{t^2}\,dt \approx \sum_{n=0}^{k} \frac{x^{2n+1}}{(2n+1)\cdot n!} \approx \frac{xe^{x^2}}{(2x^2-1)}$$

Soon,

$$\sum_{n=0}^{k} \frac{x^{2n+1}}{(2n+1)\cdot n!} \approx \frac{xe^{x^2}}{(2x^2-1)}$$

6. Numerical comparison between the Taylor series and the function approximated by the double integral method.

Now we will compare the values of the series and the approximated function for the primitive of the non-elementary exponential. As shown in table 1. However, it is noteworthy that the values obtained by the series and by the function by the method, described in this article, are placed in the nearest decimal place.

We can observe that from valuation 4 onwards, we perceive stability in relation to the comparable results of the two calculation columns presented, with a very small error, almost non-existent.

TABELA 1.

x	$\sum_{n=0}^{10^7} \frac{x^{2n+1}}{(2n+1)\cdot n!}$	$\frac{xe^{x^2}}{(2x^2-1)}$
1	$1,46265$	$2,71828$
2	$16,4526$	$15,5994$
3	$1,1445\cdot 10^5$	$1,4299\cdot 10^5$
4	$1,149\cdot 10^6$	$1,146\cdot 10^6$
5	$7,354\cdot 10^9$	$7,347\cdot 10^9$
6	$3.64483\cdot 10^{14}$	$3,6432\cdot 10^{14}$
7	$1,37674\cdot 10^{20}$	$1,37643\cdot 10^{20}$
8	$3,92816\cdot 10^{26}$	$3,92765\cdot 10^{26}$
9	$8,41984\cdot 10^{33}$	$8,41917\cdot 10^{33}$
10	$1,35088\cdot 10^{42}$	$1,35081\cdot 10^{42}$
11	$1,61817\cdot 10^{51}$	$1,61810\cdot 10^{51}$
12	$1,44449\cdot 10^{61}$	$1,44445\cdot 10^{61}$

Source: authors

7. Final Considerations

I hope that with this work I can contribute and open the way, effectively and efficiently, for the advancement of pure and applied mathematics regarding the use of this technique developed and used by the author of this respective article to find the integrals of functions in full. elementary education in general.

That will be of great value for solving elliptic differential equations, statistical functions, among many other important applications.

References

Boyer, C. B. (2003).História da Matemática. São Paulo: Edgard Bencher Ltda.

Dana, P. W. (1993). Non Elementary Functions Antiderivatives, Article.

Eves, H. (2011). Introdução à história da matemática / Howard Eves; tradução Hygino H. Domingues. 5a ed. – Campinas, SP: Editora da Unicamp.

Gauss, C. F. (1798). Disquisitiones Arithmeticae.

Gonçalves, M. B.& Flemming, D. M. (2012). Funções de várias variáveis, integrais múltiplas, integrais curvilíneas e de superfície.2ª edição. Ed. Pearson.

Guidorizzi, H. L. (2005). Um Curso de Cálculo, Vol. 1 2, 3 e 4. 5ª ed., Ed. LTC.

Joseph, G. G. (1991). The crest of the peacock: Non-European roots of mathematics.
Princeton, NJ: Princeton University Press.

Pereira, A. S., et al. (2018). *Metodologia da pesquisa científica.* [e-book]. Santa Maria. Ed. UAB/NTE/UFSM.

Nóbrega, B. S. (2012). Análise Numérica para Integrais Não elementares.

Roque, T. (2012). Uma visão crítica: Desfazendo mitos e lendas, Editora Zahar.

Simmons, G. F. (2009). Cálculo com Geometria Analítica, Vol. 2, McGraw-Hill.

Stewart, J.(2013). Cálculo, volume II / James Stewart, 7ª edição. São Paulo: Cengage Learning.