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Many thanks to Helen Grundman and the anonymous reviewers.

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Further Generalizations of Happy Numbers

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Abstract. In this paper we generalize the concept of happy numbers in several ways. First we present known results of Grundman and Teeple and establish further results, not given in that work. Then we construct a similar function expanding the definition of happy numbers to negative integers. Using this function, we compute and prove results extending those regarding higher powers and sequences of consecutive happy numbers that El-Sidy and Siksek and Grundman and Teeple proved to negative integers. Finally, we consider a variety of special cases, in which the existence of certain fixed points and cycles of infinite families of generalized happy functions can be proven.

1 Introduction

Happy numbers have been studied for many years, although the origin of the concept is unclear. Consider the sum of the square of the digits of an arbitrary positive integer. If repeating the process of taking the sums of the squares of the digits of an integer eventually gets us 1, then that integer is happy.

The goals of this paper are to provide an overview of existing research into happy numbers, extend that research into the quartics, construct a new function to generalize happy numbers into the negatives, and examine several patterns that became evident during the research.

In Section 2 we describe basic happy numbers. Then, in Section 3, we summarize existing work on generalizations of the concept to other bases and extend some earlier results for the cubic case to the quartic case.

In Section 4, we extend the happy function S to negative integers, constructing a function $Q : \mathbb{Z} \to \mathbb{Z}$ which agrees with S over $\mathbb{Z}^+$ for this purpose. We then generalize Q to various bases and higher powers. In Section 5 we consider consecutive sequences of happy numbers, as is traditional in the study of special numbers, and generalize this study to Q, including studying sequences of consecutive 0-attracted and −1-attracted numbers.

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Finally, in Section 6, we study several special cases suggested by patterns in the fixed points and cycles of S and Q. First we consider two patterns that occur under S, and then we examine two patterns which occur under Q in the negative numbers.

2 Traditional Happy Numbers

Definition 2.1. Any positive integer A can be expressed as $\sum_{i=0}^{n} a_i 10^i$ where, for each i, a_i is the i^{th} digit of A in base 10. We define the function $S : \mathbb{Z}^+ \to \mathbb{Z}^+$ by

$$S(A) = S\left(\sum_{i=0}^{n} a_i 10^i\right) = \sum_{i=0}^{n} a_i^2,$$

the sum of the squares of the digits of A. For $m \in \mathbb{Z}^+$, define S^m to be the m^{th} iteration of S. A is defined to be *happy* if there exists some $m \in \mathbb{Z}^+$ such that $S^m(A) = 1$.

There are infinitely many happy numbers. An intuitive proof for this is that there are infinitely many integers of the form 10^n, which have one digit equal to one and some number of digits equal to zero. These numbers are all happy.

There are also infinitely many numbers that are not happy. Consider integers of the form $2 \cdot 10^n$ with $n \geq 0$. These integers each enter the cycle

$$\mathbf{4} \to 16 \to 37 \to 58 \to 89 \to 145 \to 42 \to 20 \to \mathbf{4}$$

under the iteration of S. In fact, for any number A that is not happy, there is some m such that $S^m(A) = 4$, after which iterations of S move through the above cycle.

Thus S has exactly one fixed point and one cycle.

Theorem 2.2. *Given $a \in \mathbb{Z}$, there exists some $m \in \mathbb{Z}^+$ such that $S^m(a) = 1$ or $S^m(a) = 4$.*

Theorem 2.2 follows from the more general Theorem 2.5, which we state and prove below.

Grundman and Teeple (2001) [2] generalized the definition of happy numbers to bases other than 10.

Definition 2.3. Any positive integer A can be expressed in a base b as $\sum_{i=0}^{n} a_i b^i$, with $0 \leq a_i < b$ for each i and $a_n > 0$. Define $S_b : \mathbb{Z}^+ \to \mathbb{Z}^+$ by

$$S_b\left(\sum_{i=0}^{n} a_i b^i\right) = \sum_{i=0}^{n} a_i^2,$$

the sum of the square of the digits in the base b expansion of A. An integer for which there exists some $m \in \mathbb{Z}$ such that $S_b^m(A) = 1$ is called a *b-happy* number.

Not only are different numbers happy in different bases, in some bases there are fixed points other than 1. There are three possible outcomes of iterating S_b over a positive integer A. Either there can be some m for which $\mathrm{S}_b^m(\mathrm{A}) = 1$, $\mathrm{S}_b(\mathrm{A})$ can enter a cycle, or, for some integers F and r, $\mathrm{S}_b^i(\mathrm{A}) = \mathrm{F} \quad \forall i \geq r$. In the last case, A is referred to as being F−attracted. Table 1, taken from Grundman and Teeple (2001) [2] and verified using Mathematica (2015) [5], displays all fixed points and a representative of each cycle of S_b for $2 \leq b \leq 10$, each written in the relevant base.

Base	Fixed Points	Cycle Representatives
10	1	**4**
9	1, 45, 55	**58**, **82**
8	1, 24, 64	**4**, **5**, **15**
7	1, 13, 34, 44, 63	**2**, **16**
6	1	**5**
5	1, 23, 33	**4**
4	1	∅
3	1, 12, 22	**2**
2	1	∅

Table 1: Fixed Points and Cycle Representatives of S_b

Grundman and Teeple (2001) [2] used the following lemma to show that Table 1 is complete.

Lemma 2.4. *If* $b \geq 2$ *and* $\mathrm{A} \geq b^2$*, then* $\mathrm{S}_b(\mathrm{A}) < \mathrm{A}$.

Proof. Let A be an arbitrary integer greater than or equal to b^2 with $n+1$ base b digits. Then $\mathrm{A} = \sum_{i=0}^{n} a_i b^i$ with $0 \leq a_i \leq b-1$ for each i and $a_n \neq 0$. Then

$$\mathrm{A} - \mathrm{S}_b(\mathrm{A}) = \sum_{i=0}^{n} a_i b^i - \sum_{i=0}^{n} a_i^2 = \sum_{i=0}^{n} a_i (b^i - a_i).$$

Note that, for each $i \neq 0$, because $0 \leq a_i < b$, we have $b^i - a_i > 0$.

The minimum possible value of $a_0(b^0 - a_0)$ occurs when $a_0 = b-1$, and is $(b-1)(1-(b-1)) = (b-1)(1-b+1) = -b^2+3b-2$. For $0 < i < n$, the least possible value of $a_i(b^i - a_i)$ is 0. The minimum possible value of a_n is 1, and so the least possible value of $a_n(b^n - a_n)$ is $b^n - 1$. Since $\mathrm{A} \geq b^2$, we know that $n \geq 2$. Thus, $\mathrm{A} - \mathrm{S}_b(\mathrm{A}) > (b^2-1) + (-b^2+3b-2) = 3b-3$. Since $b \geq 2$, we see that $\mathrm{A} - \mathrm{S}_b(\mathrm{A}) > 0$. □

Lemma 2.4 implies that, for any integer $\mathrm{A} \geq b^2$, there is some r such that $\mathrm{S}_b^r(\mathrm{A}) < b^2$. This allows us to use the Mathematica program presented below to calculate all fixed points and cycles of S_b.

```
S[x_] := Total[(IntegerDigits[x, b])^2]
T = {1}
For[a = 1, a < b^2 + 1, a++, U = {}; Print[a];
For[d = a, FreeQ[T, d], d = S[d],
If[MemberQ[U, d], AppendTo[T, S[d]]; Print[T], AppendTo[U, d]]]]
Print[T].
```

This Mathematica formula first defines the function S_b in terms Mathematica can understand, then makes a set containing 1, which we know is a fixed point in all bases. Then we begin the actual calculations with the For loops. The outer For loop chooses the number on which to iterate S_b, calling it a, and repeating the iteration for all values of $a < b^2+1$, increasing a by one every time the inner For loop hits a stop condition. The outer For loop also defines an empty set U and prints a.

The inner For loop iterates $S_b(a)$ until it reaches a stop condition. It is here that U and T come into play – these sets are used to define the stop conditions. The inner For first checks that $a \notin T$, and stops if this is not satisfied, that is, if $a \in T$. If $a \notin T$, then the program finds $S_b(a)$. If $S_b(a)$ is also not in T, and not in U, the program adds it to U, finds $S_b(S_b(a))$, and repeats the process. If the result is in U, this is a stop condition, and $S_b(S_b(a))$ is added to T. Then the program repeats with a+1 and the expanded T.

When this program has completed calculations for all integers $0 < A < b^2$, T contains all the fixed points of S_b, and a representative of each of the cycles. The other elements of the cycles are found by direct calculation.

This calculation completes the proof of the following theorem.

Theorem 2.5. *Table 1 lists all fixed points of* S_b*, and a representative of each cycle.*

3 Higher Powers

Happy numbers are defined in terms of squaring, but Grundman and Teeple (2001) [2] consider the parallel construction using cubing. The function $S_{3,b}$ is defined for any $A \in \mathbb{Z}^+$ by $S_{3,b}(A) = \sum_{i=0}^{n} a_i^3$, where a_i is the i^{th} digit of A in base b. Grundman and Teeple (2001) [2] refer to numbers for which there exists some n such that $S_{3,10}^n(A) = 1$ as *cubic happy numbers.*

Definition 3.1. More generally, for $p \geq 2$, $S_{p,b}$ is defined by $S_{p,b}(A) = \sum_{i=0}^{n} a_i^p$, with a_i defined as above. A positive integer A for which there exists some integer m such that $S_{p,b}^m(A) = 1$ is called a *p-power b-happy number.*

At this point, it is useful to state and prove the following lemma, as presented by Grundman and Teeple (2001) [2].

Lemma 3.2. *For all powers* $p \geq 2$*, every positive integer is a p-power 2-happy number.*

Proof. Fix p, and let $A = \sum_{i=0}^{n} a_i 2^i$ be an integer with $n+1$ digits base 2. Then, for each i, $0 \le a_i < 2$, and $a_n = 1$. Hence, for each i, $a_i^p = a_i$. Thus,

$$A - S_{p,2}(A) = \sum_{i=0}^{n} a_i 2^i - \sum_{i=0}^{n} a_i^p = \sum_{i=0}^{n} a_i 2^i - \sum_{i=0}^{n} a_i = \sum_{i=0}^{n} a_i (2^i - 1) \ge 1. \tag{1}$$

Suppose $A \in \mathbb{Z}^+$ is not p-power 2-happy. Then $S_{p,2}^m(A) \ne 1$ for all m. By above Equation 1, $S_{p,2}(A) < A$, and so $S_{p,2}^m(A), S_{p,2}^{m+1}(A)$ is a decreasing sequence of positive integers, which must decrease infinitely and never reach 1, a contradiction. Thus all positive integers are p-power 2-happy.

□

This gives us of the all cycles and fixed points of $S_{p,2}$ for any power p. Recall that we found the fixed points and cycles for various bases in traditional happy numbers by finding a value N for which, for each $A \ge N$, $S_{2,b}(A) < A$, and then calculating $S_{2,b}^n(A)$ for increasing values of n and for all $A < N$. Similarly, we need an $N_3 \in \mathbb{Z}$ so that $S_{3,b}(A) < A$, for all $A \ge N_3$. Grundman and Teeple (2001) [2] prove that is $2b^3$ such a bound. As in Section 2, this bound allows us to generate the cycles and fixed points of $S_{3,b}$ by calculation. These fixed points, and a representative of each cycle, are presented in Table 2.

Theorem 3.3. *For $b > 2$, if $A \ge 2b^3$, $S_{3,b}(A) < A$.*

Base	Fixed points	Cycle Representatives
10	1, 153, 370, 371, 407	**55, 136, 160, 919**
9	1, 30, 31, 150, 151, 570, 571, 1388	**38, 152, 638, 818**
8	1, 134, 205, 463, 660, 661	**662**
7	1, 12, 22, 250, 251, 305, 505	**2, 13, 23, 51, 160, 161, 466, 516**
6	1, 243, 514, 1055	**13**
5	1, 103, 433	**14**
4	1, 20, 21, 130, 131, 203, 223, 313, 332	∅
3	1, 122	**2**
2	1	∅

Table 2: Fixed Points and Cycle Representatives of $S_{3,b}$

Beginning our original work, we now consider $S_{4,b}$, defined by $S_{4,b}(A) = \sum_{i=0}^{n} a_i^4$ where a_i is the ith digit of the base b expansion of A.

To find all the cycles and fixed points of $S_{4,b}$, we must generalize Lemma 2.4. That is, we need to find some bound N_4 in terms of b such that, for all A greater than or equal to N_4, $S_{4,b}(A) < A$. Continuing the pattern from square and cubic happy numbers, we conjectured that $3b^4$ will serve as such a bound.

Theorem 3.4. *For all bases $b \geq 2$, and any $A \geq 3b^4$, $S_{4,b}(A) < A$.*

It follows from this theorem that for each $A > 0$, there is some $m \in \mathbb{Z}^+$ such that $S_{4,b}^m(A) < 3b^4$.

Proof. By Lemma 4, all numbers are 4-power 2-happy, so we may consider only $b \geq 3$.

Let $A \geq 3b^4$ be given, and let $n+1$ be the number of digits of A in base b. For $0 \leq i \leq n$, let $a_i \in \mathbb{Z}$ with $0 \leq a_i < b$ such that $A = \sum_{i=0}^{n} a_i b^i$. Then

$$A - S_{4,b}(A) = \sum_{i=0}^{n} a_i b^i - \sum_{i=0}^{n} a_i^4 = \sum_{i=0}^{n} a_i (b^i - a_i^3).$$

Thus, to prove that $S_{4,b}(A) < A$, it suffices to show that, taking the minimum over all possible values of the a_i,

$$\min\left(\sum_{i=0}^{n} a_i (b^i - a_i^3)\right) > 0.$$

Since we are working in $\mathbb{Z}$ and the values of a_i, for distinct i, are independent, the minimum of the sum is the sum of the minima of the summands. The summands of $\sum_{i=0}^{n} a_i (b^i - a_i^3)$ can be viewed as functions of one variable, $f_i(a) = a(b^i - a^3)$, $0 \leq a < b$. For each i, the second derivative of $f_i(a)$, $f_i''(a) = -12a^2$, is less than 0. So f_i, for $i \neq n$, is concave down on the closed interval $[0, b-1]$, and f_n is concave down on the closed interval $[1, b-1]$, and thus must achieve its minimum at one of the end points. Hence to determine the minimum of f_i for any i we calculate the value of the f_i at both endpoints and take the smaller.

We consider several cases.

Case 1: Let $b \geq 4$. Recall that $f_i(a_i) = (a_i)(b^i - (a_i)^3)$, so $f_i(a_i) < 0$ if $a_i < 0$ and $b^i - (a_i)^3 > 0$ or $(a_i) > 0$ and $b^i - (a_i)^3 < 0$. Since a_i is always greater than or equal to 0, $f_i(a_i) < 0$ if $b^i - (a_i)^3 < 0$. Recall that f_i reaches its minimum at $a_i = 0$ or $a_i = b-1$. Thus, if f_i can be negative, $f_i(b-1) < 0$, and so $b^i - (b-1)^3 = b^i - b^3 + 3b^2 - 3b + 1 = b^i - b^3 + 3b(b-1) + 1 < 0$. Since $3(b-1)$ and 1 are positive, $b^i - (b-1)^3$ is negative only if $b^i < b^3$, and thus only if $i = 0, 1, 2$. By calculation, for bases 4 or greater, $f_i(b-1) \leq 0$ if $i = 0, 1, 2$.

Case 1a: $b \geq 4, n = 4$.
Note that $f_i(b-1)$ can be negative for $i = 0, 1, 2$, and $f_i(a) \geq 0$ for all a for all $i > 2$. As

Base	Fixed points	Cycles
10	1, 1634, 8208, 9474	**2178** → 6514 → **2178**, **4338** → 4514 → 1138 → 4179 → 9219 → 13139 → 6725 → **4338**
9	1, 432, 2446	**5553** → 2613 → 1818 → 12214 → 352 → 882 → 12223 → 136 → 1801 → **5553**, **137** → 3358 → 6625 → 4382 → 6083 → 7451 → 4447 → 4311 → 472 → 2115 → 786 → 10233 → 218 → 5570 → 5006 → 2564 → 3006 → 1800 → 5552 → 2531 → 883 → 12312 → **137**
8	1, 20, 21, 400, 401, 420, 421	∅
7	1	**22** → 44 → 1331 → 323 → 343 → 1135 → 2031 → 200 → **22**, **2544** → 3235 → 2225 → 1651 → 5415 → 4252 → 2443 → 1530 → 166 → 10363 → 4153 → **2544**, **5162** → 5436 → 6404 → **5162**, **516** → 5414 → 3214 → 1014 → **516**, **613** → 4006 → 4345 → 3360 → 4152 → 2422 → **613**
6	1	**3** → 213 → 242 → 1200 → 2545 → 1201 → 112 → 4 → 1104 → 1110 → **3**, **10055** → 5443 → 5350 → **10055**, **4243** → 2453 → 4310 → 1322 → 310 → 214 → 1133 → 432 → 1345 → **4243**
5	1, 2124, 2403, 3134	**2323** → 1234 → 2404 → 4103 → **2323** **2324** → 2434 → 4414 → **2324** **3444** → 11344 → 4340 → 4333 → **3444**
4	1, 1103, 3303	**3** → 1101 → **3**
3	1	**121** → 200 → **121** **122** → 1020 → **122**

Table 3: Fixed Points and Cycles of $S_{4,b}$

before, the minimal values of all f_i occur at $a = b-1$ or $a = 0$. For $i = 0,1,2$ the minimal value occurs at $a = b-1$, but the minimal value for $i = 3,4$ occurs at $a = 0$, and thus is zero. Since the bound we proposed is $3b^4$, however, $a_4 \geq 3$. Thus

$$\min(\sum_{i=0}^{n} a_i(b^i - a_i^3)) = f_0(b-1) + f_1(b-1) + f_2(b-1) + f_4(3),$$

which makes $\min(\sum_{i=0}^{4} a_i(b^i - a_i^3)) = b^2(13b-18)+12b-85$. Since $b \geq 4$, $\min(\sum_{i=0}^{4} a_i(b^i - a_i^3)) \geq \sum_{i=0}^{4} a_i(4^i - a_i^3) = 507$. Since this is much larger than 0, for any $\mathrm{A} \geq 3b^4$, $\mathrm{S}_{4,b}(\mathrm{A}) < \mathrm{A}$ for $n = 4$, as desired.

Case 1b: $b \geq 4, n > 4$.

As above, $\min(f_i(a)) = f(b-1)$ for $i = 0,1,2$, and $\min(f_i(a)) = f_i(0) = 0$ for $i = 3,\dots n$. Since A has $n+1$ digits and $a_n \neq 0$ $\min(f_n(a))$ is either $f_n(1) = 1(b^n - 1^3)$ or $f_n(b-1) = (b-1)(b^n - (b-1)^3)$ where $n \geq 5$. Since $n \geq 5$, for all a and $i > 2$, $f_n(a_i) \geq f_5(a_i)$, so we can consider the minima of f_5. $f_5(1) = b^5 - 1$ and $f_5(b-1) = (b-1)(b^5 - (b-1)^3) = (b-1)(b^5 - b^3 + 3b^2 - 3b + 1)$, so, by basic algebra, $f_5(b-1) = b^6 - b^5 - b^4 + 2b^2(2b-3) + 4b - 1$. Note that, since $b > 4$, $2b-3 > 0$ and $4b-1 > 0$, so $f_5(b-1) > b^6 - b^5 - b^4$. In turn, $b^6 - b^5 - b^4 > b^6 - 2b^5$, so $f_5(b-1) > b^6 - 2b^5 = b^5(b-2)$. Since $b \geq 4$, $b-2 > 0$, so $f_5(b-1) > b^5(b-2) > b^5 - 1 = f_5(1)$. Then

$$\min(\sum_{i=0}^{5} a_i(b^i - a_i^3)) = f_0(b-1) + f_1(b-1) + f_2(b-1) + f_5(1),$$

which implies $\min(\sum_{i=0}^{n} a_i(b^i - a_i^3)) = b^4(b-3) + b^2(13b-18) + 12b - 5$. Since $b \geq 4$, $(b-3) \geq 0$ and $(13b-18) \geq 0$ as well. Thus $\min(\sum_{i=0}^{n} a_i(b^i - a_i^3)) > 0$, so, for any $\mathrm{A} \geq 3b^4$, $\mathrm{S}_{4,b}(\mathrm{A}) < \mathrm{A}$ for $n > 4$ and bases greater than or equal to 4, as desired.

Since $\mathrm{S}_{4,b}(\mathrm{A}) < \mathrm{A}$ for $n = 4$ and $n > 4$ for bases greater than or equal to 4, $3b^4$ is an adequate bound for these bases.

Case 2: $b = 3$ As we are here in a specific base, we may consider specific numbers. Recall that the bound we are considering is $3b^4$. Since $3 = b$ in this case, $3b^4 = b^5$.

The closed interval over which we evaluate f_i is [0,2]. Consider $a_i = b-1 = 2$ for $i = 0,1,2$. Then $f_0(a_0) = 2(1-2^3) = -14$, $f_1(a_1) = 2(3-2^3) = -10$, and $f_2(a_2) = 2(3^2 - 2^3) = 2$. As above, $f_i(b-1) \leq 0$ when $i = 0,1$. However, $f_2(b-1)$ is greater than 0. Thus, $\min(f_2(a_2))$ occurs when $a_2 = 0$, and

$$\min(\sum_{i=0}^{n} a_i(b^i - a_i^3)) = a_n(b^n - a_n^3) + (-10) + (-14).$$

Recall that $n \geq 5$ Thus, we consider $f_n(a_n) + f_1(a_1) + f_0(a_0)$. As A has $n+1$ digits, $\min(a_n) = 1$. $f_5(1) = 1(3^5 - 1)$ and $f_5(2) = 2(3^5 - 2) = 2(3^5) - (4)$. Since $3^5 - 1 + 4 < 2(3^5)$, $f_5(1) < f_5(2)$. Hence $\min(\sum_{i=0}^{n} a_i(b^i - a_i^3)) \geq 242 + -10 + -14 > 0$, and so for any $\mathrm{A} \geq 3b^4$, $\mathrm{S}_{4,b}(\mathrm{A}) < \mathrm{A}$ for base 3, as desired. □

The cycles and fixed points generated in Table 3 are complete.

4 Generalization to the Negatives

We now define a function Q that extends S to all integers. Any integer $A \neq 0$ can be expressed as $A = \pm\sum_{i=0}^{n} a_i b^i$ where $0 \leq a_i < b$ are the digits of the base b expansion of A. We define the function $Q_{2,b}$ by $Q_{2,b}(0) = 0$ and

$$Q_{2,b}(A) = sgn(A)a_n^2 + \sum_{i=0}^{n-1} a_i^2$$

for $A \neq 0$.

Note that if $A > 0$, $Q_{2,b}(A) = S_{2,b}(A)$, so all fixed points and cycles of $S_{2,b}$ are also fixed points and cycles of $Q_{2,b}$. Further, by Lemma 2.4, for each $A > 0$, there is some $k \in \mathbb{Z}^+$ such that $Q_{2,b}^k(A) < b^2$.

Thus, to prove that calculating over a finite interval will determine all possible cycles and fixed points of Q we need only find a bound for negative values of A. That is, we need to find a value $B < 0$ such that $Q_{2,b}(A) > A$ for all $A \leq B$.

Theorem 4.1. *For* $A < -b$, $Q_{2,b}(A) > A$.

Proof. Let A be an integer $A < -b$. $Q_{2,b}(A) > A$ is equivalent to $Q_{2,b}(A) - A > 0$. Represent A as $-\sum_{i=0}^{n} a_i b^i$, with $0 \leq a_i \leq b-1$ and $a_n \neq 0$. Then, using the definition of $Q_{2,b}(A)$, $Q_{2,b}(A) - A = sgn(A)(a_n)^2 + \sum_{i=0}^{n-1} a_i^2 - (-\sum_{i=0}^{n} a_i b^i)$. Since $A < 0$,

$$Q_{2,b}(A) - A = -(a_n^2) + a_n b^n + \sum_{i=0}^{n-1} a_i^2 + \sum_{i=0}^{n} a_i b^i.$$

As all summands of $\sum_{i=0}^{n-1} a_i^2 + \sum_{i=0}^{n-1} a_i b^i$ are nonnegative, the minimum value of these sums is 0, and we need only concern ourselves with $\min(-(a_n^2) + a_n b^n)$. Recall from the proof of Theorem 6 that, since $0 < a_n < b$, $\min(-(a_n^2) + a_n b^n)$ occurs when $a_n = 1$ or $a_n = (b-1)$. Then

$$\min(-(a_n^2) + a_n b^n) = -(1^2) + 1b^n = b^n - 1$$

or

$$\min(-(a_n^2) + a_n b^n) = -(b-1)^2 + (b-1)b^n = -b^2 + 2b - 1 + b^{n+1} - b^n.$$

Since $A < -b$, $n \geq 1$. Thus, $\min(-(a_n^2) + a_n b^n) = b - 1$ or $-b^2 + 2b + b^2 - b - 1 = b - 1$, and so $\min(-(a_n^2) + a_n b^n) > 0$, as desired. □

Thus calculating over the interval $(-b, b^2)$ yields all fixed points and cycles of $Q_{2,b}$.

Base	Fixed Points	Cycles
10	-1, 0, 1	**4** → 16 → 37 → 58 → 89 → 145 → 42 → 20 → **4**
9	-1, 0, 1, 45, 55	**58** → 108 → 72 → **58** **75** → 82 → **75**
8	-1, 0, 1, 24, 64	**-7** → −61 → −43 → **-7** **-4** → −20 → **-4** **4** → 20 → **4** **5** → 31 → 12 → **5** **15** → 32 → **15**
7	-1, 0, 1, 13, 34, 44, 63	**2** → 4 → 22 → 11 → **2** **16** → 52 → 41 → 23 → **16**
6	-1, 0, 1	**5** → 41 → 25 → 45 → 32 → 21 → **5**
5	-1, 0, 1, 23, 33	**4** → 31 → 30 → **4**
4	-1, 0, 1	**-3** → −21 → **-3**
3	-1, 0, 1, 12, 22	**2** → 11 → **2**
2	-1, 0, 1	∅

Table 4: Fixed points and Cycles of $Q_{2,b}$

4.1 Higher Powers

As with S, Q can be generalized to higher powers. Define the function $Q_{p,b} : \mathbb{Z} \to \mathbb{Z}$ by $Q_{p,b}(0) = 0$ and

$$Q_{p,b}(A) = sgn(A)a_n^p + \sum_{i=0}^{n-1} a_i^p$$

for $A = \pm\sum_{i=0}^{n} a_i b^i$. For $A > 0$, $Q_{p,b}(A) = S_{p,b}(A)$, so all fixed points and cycles of $S_{p,b}$ are also fixed points or cycles of $Q_{p,b}$. To find all fixed points and cycles of $Q_{p,b}$, however, we need to find those generated by negative numbers as well. We use a proof parallel to that for $Q_{2,b}$ to prove that we find all such cycles and fixed points by calculating $Q_{p,b}$ for numbers greater than $-b^{p-1}$ and less than 0.

Theorem 4.2. *For all* $A < -b^{p-1}$, $Q_{p,b}(A) > A$.

Proof. Fix $A = -\sum_{i=0}^{n} a_i b^i$, an integer less than $-b^{p-1}$. Thus

$$Q_{p,b}(A) - A = -a_n^p + \sum_{i=0}^{n-1} a_i^p - (-\sum_{i=0}^{n} a_i b^i) \geq -a_n^p + a_n b^n.$$

Then $Q_{p,b}(A) - A \geq a_n(b^n - a_n^{p-1})$. Since $A < -b^{p-1}$, $n \geq p-1$, and so $Q_{p,b}(A) - A \geq a_n(b^{p-1} - a_n^{p-1})$. Thus, since $0 < a_n < b$, $a_n(b^{p-1} - a_n^{p-1}) > 0$. Hence $Q_{p,b}(A) > A$, as desired. □

Base	Fixed points	Cycle Representatives
10	-1, 0, 1, 153, 370, 371, 407	**55, 136, 160, 919**
9	-30, -1, 0, 1, 30, 31, 150, 151, 570, 571, 1388	**38, 152, 638, 818**
8	-1, 0, 1, 134, 205, 463, 660, 661	**662**
6	1, 243, 514, 1055	**13**
7	-1, 0, 1, 12, 22, 250, 251, 305, 505,	**2, 13, 23, 51, 160, 161, 466, 516**
5	-1, 0, 1, 103, 433	**14**
4	-20, -1, 0, 1, 20, 21, 130, 131, 203, 223, 313, 332	∅
3	-21, -1, 0, 1, 122	**2**
2	-1, 0, 1	∅

Table 5: Fixed Points and Cycle Representatives of $Q_{3,b}$

While the lower bound for the interval over which $Q_{p,b}$ must be calculated is generally defined, the upper bound is not. We know that $S_{3,b}(A) < A$ for $A \geq 2b^3$, and $S_{4,b}(A) < A$ for $A \geq 3b^4$, but we do not have such a bound for $S_{5,b}$. Since $Q_{p,b} = S_{p,b}$, we only have upper bounds for $Q_{p,b}$ where $p < 5$. For $Q_{3,b}$ and $Q_{4,b}$, we have the following:

Corollary 4.3. *For all* $A < -b^2$ *or* $A > 2b^3$, $|Q_{3,b}(A)| < |A|$.

Corollary 4.4. *For all* $A < -b^3$ *or* $A > 3b^4$, $|Q_{4,b}(A)| < |A|$.

These theorems gives us finite intervals over which to calculate to find all cycles and fixed points of $Q_{3,b}$ and $Q_{4,b}$. For $Q_{3,b}$ the interval is $(-b^2, 2b^3)$, and for $Q_{4,b}$ the interval is $(-b^3, 3b^4)$. Thus Table 5 and Table 6 are complete.

5 Consecutive Sequences

5.1 Traditional Happy Numbers

In the second edition of *Unsolved Problems in Number Theory*, 2004 [4], Guy raised the question, "How many consecutive happy numbers can there be?" Since there is an infinite number of numbers that aren't happy, there cannot be an infinite number of consecutive happy numbers. However, El-Sidy and Siksek [1] proved in 2000 that there can be arbitrarily long finite strings of consecutive happy numbers. As $Q = S$ for

Base	Fixed points	Cycles
10	-1, 0, 1, 1634, 8208, 9474	**2178** → 6514 → **2178**, **4338** → 4514 → 1138 → 4179 → 9219 → 13139 → 6725 → **4338**
9	-1, 0, 1, 432, 2446	**5553** → 2613 → 1818 → 12214 → 352 → 882 → 12223 → 136 → 1801 → **5553**, **137** → 3358 → 6625 → 4382 → 6083 → 7451 → 4447 → 4311 → 472 → 2115 → 786 → 10233 → 218 → 5570 → 5006 → 2564 → 3006 → 1800 → 5552 → 2531 → 883 → 12312 → **137**
8	-400, -20, -1, 0, 1, 20, 21, 400, 401, 420, 421	∅
7	-21, -1, 0, 1	**22** → 44 → 1331 → 323 → 343 → 1135 → 2031 → 200 → **22**, **2544** → 3235 → 2225 → 1651 → 5415 → 4252 → 2443 → 1530 → 166 → 10363 → 4153 → **2544**, **5162** → 5436 → 6404 → **5162**, **516** → 5414 → 3214 → 1014 → **516**, **613** → 4006 → 4345 → 3360 → 4152 → 2422 → **613**
6	-423, -1, 0, 1	**3** → 213 → 242 → 1200 → 2545 → 1201 → 112 → 4 → 1104 → 1110 → **3**, **10055** → 5443 → 5350 → **10055**, **4243** → 2453 → 4310 → 1322 → 310 → 214 → 1133 → 432 → 1345 → **4243**
5	-310, -1, 0, 1, 2124, 2403, 3134	**2323** → 1234 → 2404 → 4103 → **2323** **2324** → 2434 → 4414 → **2324** **3444** → 11344 → 4340 → 4333 → **3444**
4	-1, 0, 1, 1103, 3303	**3** → 1101 → **3**
3	-1, 0, 1	**121** → 200 → **121** **122** → 1020 → **122**
2	-1, 0, 1	∅

Table 6: Fixed Points and Cycles of $Q_{4,b}$

all positive integers, there are also arbitrarily long finite strings of consecutive positive happy numbers for Q. Grundman and Teeple (2007) [3] proved that there exist arbitrarily long finite strings of consecutive b-happy numbers for even bases. Noting that there are no even b-happy numbers, and thus there are no strings of consecutive b-happy numbers for odd bases, they also proved that there are, however, arbitrarily long finite strings of consecutive odd b-happy numbers for all bases under $S_{2,b}$. Both of these results generalize to produce arbitrarily long finite strings of consecutive or consecutive odd r-attracted numbers for any fixed point r of $S_{2,b}$. However, neither the proof El-Sidy and Siksek used nor the proof Grundman and Teeple (2007) used for S applies to non-positive fixed points of Q.

5.2 1-attracted Numbers

Recall that a number A is said to be F-attracted if there is some positive integer k such that $Q^i(A) = F$ for all $i \geq k$. Thus a 1-attracted number is a happy number. We can use the fact that there are arbitrarily long finite strings of consecutive or consecutive odd positive b-happy numbers to construct arbitrarily long finite strings of consecutive [odd] negative numbers that are 1-attracted.

Theorem 5.1. *For every $n \in \mathbb{Z}^+$ and even [odd] $b \geq 2$ there is a string of consecutive [odd] 1-attracted negative numbers of length n under $Q_{2,b}$.*

Proof. Let $A_1, A_2, ..., A_n$ be a string of consecutive [odd] b-happy numbers. Let m denote the number of digits of the largest A_n. Then we construct $B_1, B_2, ..., B_n$ by $B_i = -(10^{m+2} + 10^{m+1} + A_i)$. Then $Q_{2,b}(B_i) = sgn(B_i)(1)^2 + 1^2 + Q_{2,b}(A_i)$. Since $B_i < 0$ for all i,

$$Q_{2,b}(B_i) = -(1^2) + 1^2 + Q_{2,b}(A_i) = Q_{2,b}(A_i) = S_{2,b}(A_i).$$

For each i, as A_i is b-happy, there exists some $r \in \mathbb{Z}^+$ such that $S^r_{2,b}(A_i) = 1$, and so $Q^r_{2,b}(B_i) = Q^r_{2,b}(A_i) = S^r_{2,b}(A_i) = 1$. Thus $B_1, B_2, ..., B_n$ is a string of n consecutive [odd] 1-attracted negative numbers of length n. □

5.3 −1-attracted Numbers

We also consider the other fixed points of $Q_{2,10}$. One of these fixed points is −1. Since, for $A > 0$, $Q_{2,10}(A) > 0$, all −1-attracted numbers are negative. There are not, in fact, arbitrarily long strings of consecutive −1-attracted numbers under $Q_{2,10}$. Quite the contrary: there cannot be more than two −1-attracted numbers in a row. A direct search shows that the greatest numbers forming such a string are −6135, −6134.

Theorem 5.2. *Any consecutive pair of −1-attracted numbers is of the form* $A-1 = -a_n a_{n-1}...5$, $A = -a_n a_{n-1}...4$.

Proof. Recall that $Q_{2,10}(A) = sgn(A)a_n^2 + \sum_{i=0}^{n-1} a_i^2$ with a_i the digits of A in base 10, $a_n > 0, 0 \le a_i \le 9$. Thus, $Q_{2,10}(A) \ge -81$ for any integer A, so if A is -1-attracted, $Q_{2,10}(A)$ is a -1-attracted integer greater than or equal to -81. By direct calculation, the only such integers are -10 and -1. Thus, for any consecutive string of -1-attracted numbers $A-1, A$, we have

$$\{Q_{2,10}(A), Q_{2,10}(A-1)\} \subseteq \{-1,-10\}.$$

Hence, $|Q_{2,10}(A-1) - Q_{2,10}(A)| \in \{9,0\}$. Let A be a negative integer. If $a_0 = 9$, $Q_{2,10}(A) \ge -9^2 + 9^2 = 0$, so A is not -1-attracted.

Let $A-1, A$ be -1-attracted integers. Then a_0, the last digit of A, is not 9, so $A-1 = -\sum_{i=1}^{n} a_i b^i - (a_0+1)$. If the difference $Q_{2,10}(A-1) - Q_{2,10}(A)$ is 0, then

$$sgn(A)a_n^2 + \sum_{i=1}^{n-1} a_i^2 + a_0^2 = Q_{2,10}(A-1) = Q_{2,10}(A) = sgn(A)a_n^2 + \sum_{i=1}^{n-1} a_i^2 + (a_0+1)^2.$$

Thus $a_0^2 = (a_0+1)^2$, which implies $a_0 = -\frac{1}{2}$, a contradiction. Hence $|Q_{2,10}(A-1) - Q_{2,10}(A)| = 9$.

So

$$\begin{aligned} 9 =& |Q_{2,10}(A-1) - Q_{2,10}(A)| \\ =& \left| -a_n^2 + \sum_{i=1}^{n-1} a_i^2 + (a_0+1)^2 + a_n^2 - \sum_{i=1}^{n-1} a_i^2 - a_0^2 \right| \\ =& 2a_0 + 1. \end{aligned}$$

Thus $a_0 = 4$.

$\{Q_{2,10}(A-1), Q_{2,10}(A)\} = \{-1,-10\}$ if and only if $a_0 = 4$, so $A-1, A$ is only a pair of consecutive -1-attracted numbers if and only if $a_0 = 4$.

□

Corollary 5.3. *There are no more than two consecutive -1-attracted numbers under* $Q_{2,10}(A)$.

Proof. The only possible form for a string of two consecutive -1-attracted numbers is $A = a_n a_{n-1} \ldots 4$, $A - 1 = a_n a_{n-1} \ldots 5$. Thus there do not exist any three consecutive -1-attracted numbers. □

5.4 0-attracted

Note that 0 is also a fixed point of $Q_{2,10}$. As with -1, all 0-attracted numbers are non-positive and there are no strings of 0-attracted numbers longer than two in a row. It is easy to see that -65, and -66 are the greatest numbers that form such a string. Strings of 0-attracted numbers may be of the form ...6, ...5, the form ...5, ...4, the form ...2, ...1, or the form ...1, ...0.

Theorem 5.4. *There does not exist any string of three consecutive* 0*-attracted numbers.*

Proof. As with -1-attracted numbers, if an integer $A = -\sum_{i=0}^{n} a_i 10^i$ is 0-attracted, $Q_{2,10}(A)$ is a 0-attracted number greater than -81. By direct calculation, the set of such integers is

$$p = \{-77, -74, -66, -65, -55, -44, -33, -22, -11, 0\}.$$

Thus, for any two consecutive 0-attracted numbers,

$$|Q_{2.10}(A) - Q_{2,10}(A-1)| \in \{0, 1, 3, 8, 9, 10, 11, 12, 19, 21, 22, 30, 32, 33, 41, 43, 44, 52, 54, 55, 63, 65, 66, 74, 77\}.$$

Since, for $a_0 \neq 9$, $|Q_{2,10}(A) - Q_{2,10}(A-1)| = 2a_0 + 1$, and $\max(2a_0 + 1) = 17$, however, $|Q_{2,10}(A) - Q_{2,10}(A-1)|$ must be odd and less than 17. Thus

$$|Q_{2,10}(A) - Q_{2,10}(A-1)| \in \{1, 3, 9, 11\}.$$

By calculation, the values of a_0 for which this can be true are $0, 1, 4$, and 5.

Note that in a string of three 0-attracted numbers, the first two numbers must be the first numbers in a pair of 0-attracted numbers. Thus, the first two numbers of a string must end with the digits $0, 1, 4, 5$, or 9.

There are then three possibilities for strings of three consecutive 0-attracted numbers:,

$$A = -a_n a_{n-1} ... a_1 9 \to A - 1 = -a_n a_{n-1} ... (a_1 + 1)0 \to A - 2 = -a_n a_{n-1} ... (a_1 + 1)1,$$

$$A = -a_n a_{n-1} ... 0 \to A - 1 = -a_n a_{n-1} ... 1 \to A - 2 = -a_n a_{n-1} ... 2$$

and

$$A = -a_n a_{n-1} ... 4 \to A - 1 = -a_n a_{n-1} ... 5 \to A - 2 = -a_n a_{n-1} ... 6.$$

Recall that $Q_{2,10}(A) = sgn(A) a_n^2 + \sum_{i=0}^{n-1} a_i^2$.

Case 1: Let $A = a_n a_{n-1} ... 9$. Then $Q_{2,10}(A) \geq -a_n^2 + 81$. Thus, $Q_{2,10}(A) \geq 0$, and is equal to 0 if and only if $a_n = 9$, so any 0-attracted number with $a_0 = 9$ is of the form $A = -9a_{n-1} ... a_1 9$ with $a_i = 0$ for all $0 < i < n$. Assume that A is a 0-attracted number with $n + 1$ digits.

Let $n = 1$. Then $A = -99$, and so $A - 1 = -100$. $Q_{2,10}(A-1) = -1$, so $A - 1$ is not 0-attracted.

Let $n > 1$. Then $A = -9 \cdot 10^n - 9$, and so $A - 1 = -9 \cdot 10^n - 10$. Thus $Q(A-1) = -9^2 + 1 = -80$. Since -80 is not 0-attracted, $A - 1$ is not 0-attracted. Hence $A, A - 1$ is not a string of 0-attracted numbers, and so $A, A - 1, A - 2$ is not a string of consecutive 0-attracted numbers.

Case 2: Let $\mathrm{A} = -a_n a_{n-1} ... 0$, $\mathrm{A} - 1 = -a_n a_{n-1} ... 1$, and $\mathrm{A} - 2 = a_n a_{n-1} ... 2$. Consider

$$\begin{aligned}
\mathrm{Q}_{2,10}(\mathrm{A}) &= sgn(\mathrm{A}) a_n^2 + \sum_{i=1}^{n-1} a_i^2 + 0^2 \\
\mathrm{Q}_{2,10}(\mathrm{A}-1) &= sgn(\mathrm{A}) a_n^2 + \sum_{i=1}^{n-1} a_i^2 + 1^2 \\
\mathrm{Q}_{2,10}(\mathrm{A}-2) &= sgn(\mathrm{A}) a_n^2 + \sum_{i=1}^{n-1} a_i^2 + 2^2
\end{aligned}$$

Let $\mathrm{B} = sgn(\mathrm{A}) a_n^2 + \sum_{i=1}^{n-1} a_i^2$. We need $\mathrm{B}+0, \mathrm{B}+1, \mathrm{B}+4 \in p$, so B must be an element of $p-0, p-1$, and $p-4$, where

$$\begin{aligned}
p &= \{-77, -74, -66, -65, -55, -44, -33, -22, -11, 0\}, \\
p-1 &= \{-78, -75, -67, -66, -56, -45, -34, -23, -12, -1\}, \\
p-4 &= \{-81, -78, -70, -69, -59, -48, -37, -26, -15, -4\}.
\end{aligned}$$

There are no elements shared by all three sets, so there is no possible value for B such that $\mathrm{A}, \mathrm{A}-1$, and $\mathrm{A}-2$ are all 0-attracted.

Case 3: Let $\mathrm{A} = -a_n a_{n-1} ... 4$, $\mathrm{A} - 1 = -a_n a_{n-1} ... 5$, and $\mathrm{A} - 2 = -a_n a_{n-1} ... 6$. As with Case 1, consider

$$\begin{aligned}
\mathrm{Q}_{2,10}(\mathrm{A}) &= sgn(\mathrm{A}) a_n^2 + \sum_{i=1}^{n-1} a_i^2 + 4^2 \\
\mathrm{Q}_{2,10}(\mathrm{A}-1) &= sgn(\mathrm{A}) a_n^2 + \sum_{i=1}^{n-1} a_i^2 + 5^2 \\
\mathrm{Q}_{2,10}(\mathrm{A}-2) &= sgn(\mathrm{A}) a_n^2 + \sum_{i=1}^{n-1} a_i^2 + 6^2.
\end{aligned}$$

Let $\mathrm{B} = sgn(\mathrm{A}) a_n^2 + \sum_{i=1}^{n-1} a_i^2$. We need $\mathrm{B}+16, \mathrm{B}+25, \mathrm{B}+36 \in p$, so B must be an element of $p-16, p-25$, and $p-36$, where

$$\begin{aligned}
p-16 &= \{-93, -90, -82, -81, -71, -60, -49, -38, -27, -16\} \\
p-25 &= \{-102, -99, -91, -90, -80, -69, -58, -47, -36, -25\} \\
p-36 &= \{-113, -110, -102, -101, -91, -80, -69, -58, -47, -36\}
\end{aligned}$$

Again, there are no elements shared by all three sets, so there is no value of B such that $\mathrm{A}, \mathrm{A}-1$, and $\mathrm{A}-2$ are all 0-attracted.

Thus, the longest possible string of 0-attracted numbers is two. □

The above proof leads to the following corollary:

Corollary 5.5. *Any consecutive pair of* 0*-attracted numbers are of the form* $A-1 = -a_n a_{n-1}...6$, $A = -a_n a_{n-1}...5$,
$A-1 = -a_n a_{n-1}...1$, $A = -a_n a_{n-1}...0$, *or*
$A-1 = -a_n a_{n-1}...2$, $A = -a_n a_{n-1}...1$.

Proof. In the beginning of the proof of Theorem 14, we found possibilities for the final digit of the first of a pair of consecutive 0-attracted numbers. The possibilities were $0,1,4,5$, and 9. Case 3 of the above proof proves that there are no string of consecutive numbers $A, A-1$ where $a_0 = 9$. To see that there are no strings with $a_0 = 4$ we may consider the sets $p-x$ as in the previous proof.

Let us consider $A = a_n a_{n-1}...4$, $A-1 = a_n a_{n-1}...5$. Then $B = sgn(A)a_n^2 + \sum_{i=1}^{n-1} a_i^2$ must be an element of $p-16$ and $p-25$:

$$p-16 = \{-93,-90,-82,-81,-71,-60,-49,-38,-27,-16\}$$
$$p-25 = \{-102,-99,-91,-90,-80,-69,-58,-47,-36,-25\}$$

There is a shared element of these sets, -90. However, $B = sgn(A)a_n^2 + \sum_{i=1}^{n-1} a_i$ can never equal -90 – the minimum value of B is -81. Therefore there are no consecutive 0-attracted numbers of the form $a_n a_{n-1}...4$, $a_n a_{n-1}...5$.

□

6 Special Cases

6.1 Statements Involving Only Positive Integers

6.1.1 Bases of the form 2^ℓ in $S_{\ell+1,2^\ell}$. The functions $S_{3,4}$ and $S_{4,8}$ share the fixed points 1, 20, and 21. In both of these cases, the bases in question can be expressed as 2^ℓ for some $\ell \in Z^+$: $4 = 2^2$, and $8 = 2^3$. Rewriting $S_{3,4}$ as $S_{3,2^2}$ and $S_{4,8}$ as $S_{4,2^3}$, it becomes clear that the functions can be expressed as $S_{\ell+1,2^\ell}$. Other functions of this structure, $S_{3,9}$ and $S_{5,16}$ display a similar pattern, as shown in Table 7. This suggests the following results.

Theorem 6.1. *For a base* $b = a^\ell$, $\ell \geq 2$, *and* $k < \ell$, *the number* $(ab)^k$ *is a fixed point for* $S_{\ell+1,b}$.

Proof. Given $a^k < b$, $(ab)^k = a^k b^k$ has one nonzero digit, a^k, so

$$S_{\ell+1,b}((ab)^k) = (a^k)^{\ell+1}.$$

Recall that $b = a^\ell$. Then,

$$(ab)^k = a^k b^k = a^k a^{\ell k} = (a^k)^{\ell+1} = S_{\ell+1,b}((ab)^k).$$

Hence, $S_{\ell+1,b}((ab)^k) = (ab)^k$, and so $(ab)^k$ is a fixed point of $S_{\ell+1,b}$. □

Theorem 6.2. *For a base $b = a^\ell, \ell \geq a$, let* $\mathrm{P} = \{0, 1, 2, \ldots, \ell - 1\}$, *and let I be a nonempty subset of P. Then* $\sum_{i\in \mathrm{I}}(ab)^i$ *is a fixed point of* $\mathrm{S}_{\ell+1,b}$.

Proof. Note that $\sum_{i\in\mathrm{I}}(ab)^i = \sum_{i\in\mathrm{I}}(a^i b^i)$. For all $i \in \mathrm{I}$, $i < \ell$, so the nonzero digits of $\sum_{i\in\mathrm{I}}(ab)^i$ base b are $\{a^i | i \in \mathrm{I}\}$. Thus,

$$\mathrm{S}_{\ell+1,b}\left(\sum_{i\in\mathrm{I}}(ab)^i\right) = \sum_{i\in\mathrm{I}}(a^i)^{\ell+1}.$$

Recall that $b = a^\ell$. Thus

$$(ab)^i = a^i b^i = a^i (a^\ell)^i = a^{i\ell+i} = (a^i)^{\ell+1}.$$

Hence, $\sum_{i\in\mathrm{I}}(ab)^i = \sum_{i\in\mathrm{I}}(a^i)^{\ell+1}$. So we have

$$\mathrm{S}_{\ell+1,b}\left(\sum_{i\in\mathrm{I}}(ab)^i\right) = \sum_{i\in\mathrm{I}}(a^i)^{\ell+1} = \sum_{i\in\mathrm{I}}(ab)^i,$$

and so $\sum_{i\in\mathrm{I}}(ab)^i$ is a fixed point of $\mathrm{S}_{\ell+1,b}$. □

Table 7 displays several base and power combinations which satisfy the requirements for Theorems 6.1 and 6.2. Note that, as we have not found the interval in which all fixed points and cycles of $\mathrm{S}_{5,b}$ are represented, those presented here for $\mathrm{S}_{5,16}$ are not necessarily all possible cycles and fixed points of the function. The extra digits required for base 16 are as provided in standard hexidecimal, so a=10, b=11, c=12, d=13, e=14, and f=15.

Base	Power	Fixed points	Cycles
16	5	1, 20, 21, 400, 401, 420, 421, c80e1, 8000, 8001, 8020, 8021, 8400, 8401, 8420, 8421	**135**, **a354**, **c5a9**, **234bdf**
8	4	1, 20, 21, 400, 401, 420, 421	∅
9	3	1, 30, 31, 150, 539, 570, 571, 151, 755, 1388	**38**, **152**, **638**
4	3	1, 20, 21, 130, 131, 203, 223, 313, 332	∅

Table 7: Fixed points and cycles of $\mathrm{S}_{\ell+1,a^\ell}$

6.1.2 Bases with exponents of the form 2^n+1.

Theorem 6.3. *Let $a \geq 2$, and $n \in \mathbb{Z}^+$. Then $S_{2,a^{2^n+1}}$ has a cycle of length $2n$ with representative a^2.*

Proof. Let $a \geq 2$ and $n \in \mathbb{Z}^+$, and let the base $b = a^{2^n+1}$.

We will show that, for $0 \leq k \leq n-1$, $S^k_{2,a^{2^n+1}}(a^2) = a^{2^{k+1}}$. Since $a^2 < a^{2^n+1}$, $S_{2,a^{2^n+1}}(a^2) = a^4$. Observe that $a^{2^{0+1}} = a^2$ and $a^4 = a^{2^{1+1}}$, so the statement holds for the base case. For $k \leq n-1$, $a^{2^{k+1}} < a^{2^n+1}$, and thus has one digit.

Assume $S^k_{2,a^{2^n+1}}(a^2) = a^{2^{k+1}}$ for some $k \leq n-1$. Then

$$S^{k+1}_{2,a^{2^n+1}}(a^2) = (a^{2^{k+1}})^2 = a^{2\cdot 2^{k+1}} = a^{2^{(k+1)+1}},$$

as desired.

If $n \leq k \leq 2n-1$, then express $S^k_{2,a^{2^n+1}}(a^2)$ as $S^{n+j}_{2,a^{2^n+1}}(a^2)$. We will show that, for $0 \leq j \leq n-1$, $S^{n+j}_{2,a^{2^n+1}}(a^2) = a^{2^n-(2^{j+1}-1)}(a^{2^n+1})$. By the argument for $0 \leq k \leq n-1$, $S^{n-1}_{2,a^{2^n+1}} = a^{2^n}$, and so

$$S^n_{2,a^{2^n+1}} = (a^{2^n})^2 = a^{2^{n+1}} = a^{2^{n+1}-2^n-1}(a^{2^n+1}) = a^{2^n-1}(a^{2^n+1}) = a^{2^n-(2^{0+1}-1)}(a^{2^n+1}).$$

Thus the statement holds for the base case.

Let $S^{n+j}_{2,a^{2^n+1}}(a^2) = a^{2^n-(2^{j+1}-1)}(a^{2^n+1})$. Then

$$S^{n+j+1}_{2,a^{2^n+1}}(a^2) = (a^{2^n-(2^{j+1}-1)})^2 = a^{2^{n+1}-2^{j+2}+2} = a^{2^{n+1}-2^n-2(2^{j+1})+2-1}a^{2^n+1} = a^{2^n-(2^{j+2}-1)}a^{2^n+1},$$

as desired.

Then $S^{n+n}_{2,a^{2^n+1}}(a^2) = (a^{2^n-(2^n-1)})^2 = a^2$.

□

6.2 Statements Involving the Negative Integers

6.2.1 Bases that are powers of 2 in $Q_{2,b}$. As with S, the behavior of $Q_{2,4}$ and $Q_{2,8}$ is unusual. In both of these bases, Q has a cycle of negative integers. $Q_{2,4}$ has the cycle $-3 \to -2121 \to -3$, and $Q_{2,8}$ has the cycle $-7 \to -61 \to -43 \to -7$. Each cycle has a representative of the form $-b+1$, and each base is a power of 2. This gives the following theorem.

Theorem 6.4. *For a base $b = 2^n$, $n > 1$, $Q_{2,2^n}$ has a cycle of length n with representative $-(2^n-1)$.*

Proof. Fix $b = 2^n, n > 1$. We will show that, for all $1 \le i < n$,

$$Q^i_{2,2^n}(-2^n+1) = -((2^n-2^i)2^n+(2^i-1)).$$

Consider $Q_{2,2^n}(-2^n+1)$. Since $-2^n < -2^n+1 < 2^n$, it has one digit. Thus

$$Q_{2,2^n}(-2^n+1) = -(-2^n+1)^2 = -2^{2n}+2^{n+1}-1 = -((2^n-2)2^n+1),$$

Then

$$Q^2_{2,2^n}(-2^n+1) = -(2^n-2)^2+1^2 = -(2^{2n}-2^{n+2}+2^2)+1 = -((2^n-2^2)2^n+(2^2-1)).$$

. Thus the assertion holds for $i = 2$, the base case.

Let there be some $1 < k < n$ such that

$$Q^k_{2,2^n}(-2^n+1) = -((2^n-2^k)2^n+(2^k-1)).$$

Then $Q^{k+1}_{2,2^n}(-2^n+1) = -(2^n-2^k)^2+(2^k-1)^2$, so

$$\begin{aligned}
Q^{k+1}_{2,2^n}(-2^n+1) &= -(2^{2n}-2^{n+k+1}+2^{2k})+2^{2k}-2^{k+1}+1\\
&= -(2^n-2^{k+1})2^n-2^{2k}+2^{2k}-2^{k+1}+1\\
&= -((2^n-2^{k+1})2^n-(2^{k+1}-1)).
\end{aligned}$$

Thus the assertion holds for all $1 < i < n$.

Consider $i = n$. $Q^n_{2,2^n}(-2^n+1) = Q_{2,2^n}(Q^{n-1}_{2,2^n}(-2^n+1))$. By the above, then,

$$\begin{aligned}
Q^n_{2,2^n}(-2^n+1) &= Q_{2,2^n}(-((2^n-2^{n-1})2^n-(2^{n-1}-1))\\
&= -(2^n-2^{n-1})^2+(2^{n-1}-1)^2\\
&= -(2^{2n}-2^{n-1+n}-2^{n-1+n}+2^{2n-2})+(2^{2n-2}-2^{n-1}-2^{n-1}+1)\\
&= -(2^{2n}-2^{2n-1+1}+2^{2n-2})+(2^{2n-2}-2^n+1)\\
&= -2^{2n-2}+2^{2n-2}-2^n+1\\
&= -2^n+1
\end{aligned}$$

Thus -2^n+1 is a cycle representative of $Q_{2,2^n}$ for all n. □

6.2.2 Bases that are perfect squares in $Q_{3,b}$.

In $Q_{3,b}$ bases 4 and 9 have negative fixed points which are not -1. -20 is a fixed point of $Q_{3,4}$ – as is 20 – and -30 and 30 are fixed points of $Q_{3,9}$. This suggests the following theorem:

Theorem 6.5. *For any $a \ge 2$ and base $b = a^2$, ab and $-ab$ are fixed points of* $Q_{3,b}$.

Proof. Let $b = a^2$. Since $-ab$ has the single non-zero digit a, $Q_{3,b}(-ab) = -a^3+0^3 = -a^3$.As $b = a^2$, $a^3 = ab$, so $Q_{3,b}(-ab) = -ab$. Identically,$Q_{3,b}(ab) = a^3+0^3 = a^3 = ab$. □

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