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Elliptic triangles which are congruent to their polar triangles

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Abstract. We construct elliptic triangles which are congruent to their polar triangles. We present the elliptic version of Wallace-Simson lines (if a point projected onto a triangle has the three feet of its projections collinear, that line is called a Wallace-Simson line.) We prove that an elliptic triangle is congruent to its polar triangle if and only if six specific Wallace-Simson lines of the triangle are concurrent. The six lines come from projecting each vertex of both triangles onto the given triangle.

1 Elliptic geometry

Even a non-Euclidean geometry like elliptic geometry has congruent triangles and points projected onto lines. Being a space with curvature adds a twist to these ideas.

The Klein disk model of elliptic geometry starts with a unit disk. All the points of the disk, including the boundary, count as elliptic points. The points of the plane holding the disk are Euclidean points. Diameters of the disk and arcs of circles whose endpoints are the endpoints of a diameter are the elliptic lines. Such endpoints are called **antipodal points** and are treated as the same elliptic point. This model satisfies the negation of the Parallel Postulate which requires no parallel lines.

Elliptic geometry resembles a flat image of one hemisphere where the lines are great circles. An elliptic line has a unique **pole**, which is a point such that any line through the pole is perpendicular to the given line. Any line could be the Equator and its pole would be the North Pole. The line is called the **polar** of its pole.

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Useful fact: Three poles are collinear if and only if their polars are concurrent. In Figure 1A, lines $\overleftrightarrow{AP_1}, \overleftrightarrow{EA}$ and $\overleftrightarrow{OA}$ all meet at point A. Their poles are O, P_{EA} and P_1, respectively. The poles lie on $\overleftrightarrow{OE}$.

Useful fact: If two lines $\overleftrightarrow{AO}$ and $\overleftrightarrow{EA}$ are perpendicular to the same line $\overleftrightarrow{OE}$, then A must be the pole of $\overleftrightarrow{OE}$.

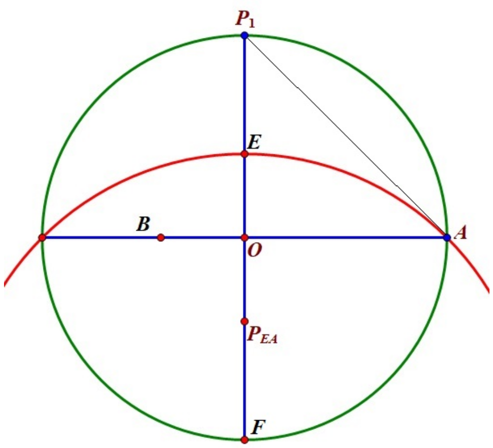

Figure 1A. Useful facts.

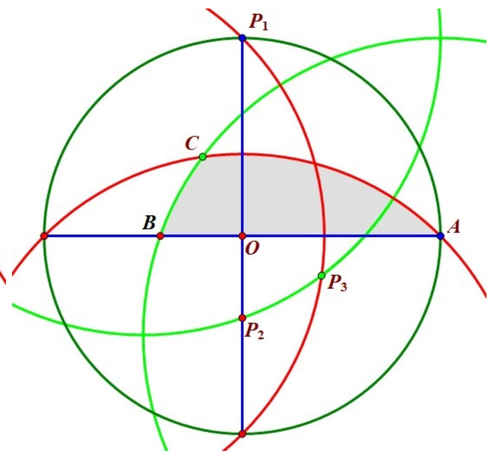

Figure 1B. Triangle ABC.

Three elliptic lines which are not concurrent must form a triangle with each intersection as a vertex, say triangle ABC. Each side has a pole. We name P_1 as the pole of $\overleftrightarrow{AB}$, P_2 as the pole of $\overleftrightarrow{AC}$ and P_3 as the pole of $\overleftrightarrow{BC}$. We will use the name of the line as a subscript for other poles, like P_{EA} is the pole of line $\overleftrightarrow{EA}$.

Triangle $P_1P_2P_3$ is the polar triangle of triangle ABC, and vice-versa.

1.1 Construct two triangles congruent to their polar triangles

A theorem from spherical geometry helps us right away: an angle of one triangle and its corresponding side from its polar triangle, like $\angle A$ and its corresponding side $\overline{P_1P_2}$, must be supplements.[1] An elliptic segment can be seen as an arc of a circle because elliptic lines are arcs of Euclidean circles. In order for triangle ABC to be congruent to its polar triangle, the angles and sides of triangle ABC have to be supplements as well. The triangle with three right angles is self-polar. We will focus on non-self-polar triangles because the self-polar triangle, being congruent to itself, is thus congruent to its polar triangle in the most boring way possible. The shaded triangle ABC in Figure 1B will introduce the elliptic concepts and it requires some preparation. We plan to place side $\overline{AB}$ on the horizontal axis with A on the boundary, which gives us pole P_1 for free. We choose a constructible $\angle A$, in this case $\frac{\pi}{4}$. We measure elliptic angles using tangents and it just so happens that the circle centered at F through point A in Figure 1A will make $\angle EAO = \frac{\pi}{4}$. (The tangent $\overline{P_1A}$ in Figure 1A proves this size is correct.) The angle size implies E is the elliptic midpoint of $\overline{OP_1}$, even though it is not the Euclidean midpoint.

We must discern between Euclidean lengths and elliptic lengths. Since the disk has Euclidean radius OA = 1, we work as if we are on a unit sphere, which means $\overline{OA}$ is a fourth of an Equator, so $\overline{OA}$ has elliptic length $\frac{\pi}{2}$. In fact, the distance from a pole to its polar is always $\frac{\pi}{2}$. For our triangle, we must have one side whose length is the supplement of $\frac{\pi}{4}$. The circle centered at A through P_1 will serve to make B the elliptic midpoint of its radius of circle O, giving us elliptic length of $\overline{AB}$ equal to $\dfrac{\pi}{2}+\dfrac{\pi}{4}=\dfrac{3\pi}{4}$, the required supplement. Now we have to incorporate an angle with measure $\dfrac{3\pi}{4}$ and a side length of $\dfrac{\pi}{4}$ into our triangle ABC because we intend for this triangle to be congruent to its polar triangle. The sketch in Figure 2 summarizes the situation.

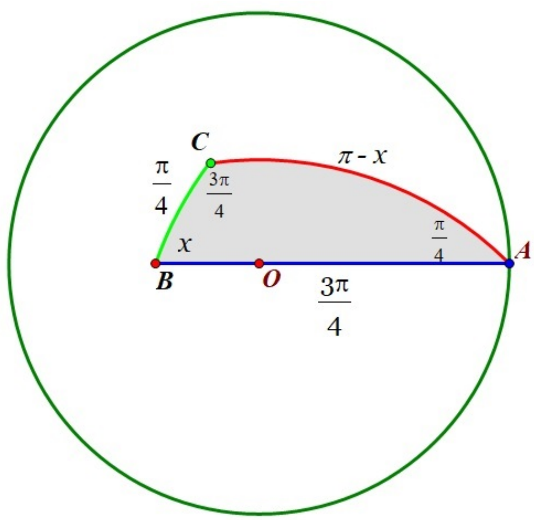

Figure 2. Sides and angles.

The experienced reader may be pleasantly surprised to find the proposed triangle in Figure 2 is not over-constrained, even though four parts of the triangle are determined and only one variable remains to calculate the other two parts. In this paper we only need one elliptic trigonometric formula, the spherical Law of Cosines:

$$\cos b = \cos a \cos c + \sin a \sin c \cos B \qquad (\mathbf{1})$$

Formula 1 gives the relationship between $\angle$A and $\angle$B (whose measure is x.)

$$\cos(\pi - x) = \cos A \cos(\pi - A) + \sin A \sin(\pi - A) \cos x$$

$$-\cos x = -\cos^2 A + \sin^2 A \cos x$$

$$\cos x = \frac{\cos^2 A}{1 + \sin^2 A}. \qquad (2)$$

For our specific construction, $\frac{1}{3} = \cos x$. This calculation explains the odd little circles in Figure 1C. We had to construct $\angle$OBM using a right triangle with the ratio of adjacent over hypotenuse being $\frac{1}{3}$. The Euclidean angle gave us the tangent we needed so that our elliptic angle would be the desired size. Formula 1 also explains how elliptic $\angle \mathrm{OP_1B}$ and elliptic segment $\overline{\mathrm{OB}}$ have the same size: triangle $\mathrm{OP_1B}$ has two lengths of $\frac{\pi}{2}$ and two right angles which simplify Formula 1 to $\cos\angle \mathrm{OP_1B} = \cos \mathrm{OB}$. Constructing an angle in a useful place gives us an elliptic segment with the angle's size as its elliptic length.

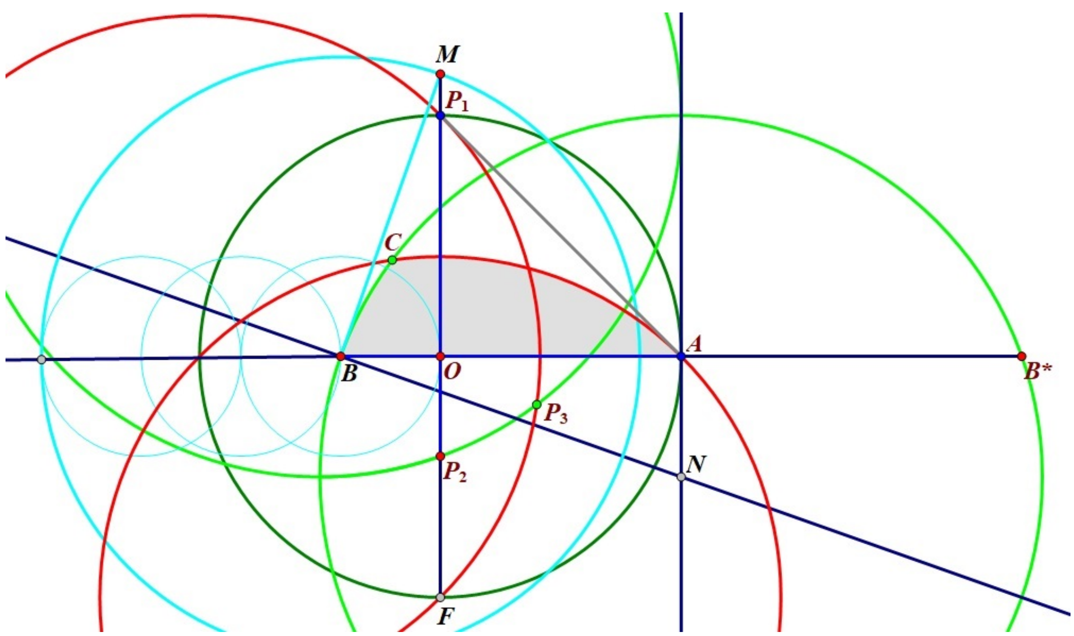

Figure 1C. Constructing $\angle$ABC.

The point B^* is the inverse of the reflection of point B across point O. (More specifically, on the line $\overleftrightarrow{\mathrm{OB}}$, we find the point the same distance from O as B, but on the other side of point O. We call this undrawn point Y. Then we construct the inverse of Y, B^* which has the property $OY \cdot OB^* = r^2$. The interested reader can find a nice construction of the inverse in [3]. Usually, we have two elliptic lines through B and we obtain B^* by finding the second Euclidean intersection of these two lines.) Any circle through B and B^* must pass through antipodal points, meaning the arc $\overline{BC}$ is on an elliptic line. [2] The perpendicular bisector of $\overline{BB^*}$ and the perpendicular to the tangent at B meet at the point N. The circle with center N through B contains side $\overline{BC}$. Our triangle ABC is complete and its polar triangle is the reflection of this triangle across the angle bisector of angle AOP_1.

Having the corresponding sides and angles being supplements is almost enough to verify the triangles are each other's polar triangle. Position also matters because we can't just pick up the undrawn triangle $P_1P_2P_3$, drop it somewhere else and expect that new triangle to be the polar of the triangle ABC. We had P_1 right from the start. The distance from B and P_2 to their polars was built to be $\frac{\pi}{2}$. We can place P_3 in only two places and P_3 certainly can't be in the third quadrant.

Since $\cos x \geq 0$ in (2), one of the angles of the triangle must be acute. (If we allow $\angle A = \frac{\pi}{2}$, we get the self-polar triangle.) Calculating the elliptic area of our Figure 2 triangle, (which is the sum of the angles minus π,) using $\angle A$ again, we find the area to be $\angle A + \pi - \angle A + x - \pi = x$. So the elliptic area of a triangle which is congruent to its polar triangle has to be less than $\frac{\pi}{2}$. Now we know any such triangle may always be positioned as in Figure 2. (Actually, we can construct our triangle less conveniently located, but we will not need those moves for this paper.) The calculation also guarantees we can construct triangles congruent to their polars for a constructible $\angle A$ because $\angle A$ returns a constructible size for x in Formula 2. Our construction method works for constructible angle A, even sizes like $\arctan\left(\frac{3}{4}\right)$, though Figure 1B has to be the most convenient version. We note the measure of angle A decides everything and knowing its measure is essential for the construction.

Step 1: construct $\angle A$ with its measure $\arctan\left(\frac{3}{4}\right)$. The Euclidean lengths $EA = 3$ and $ED = 4$ were obtained using little circles as in Figure 1B.

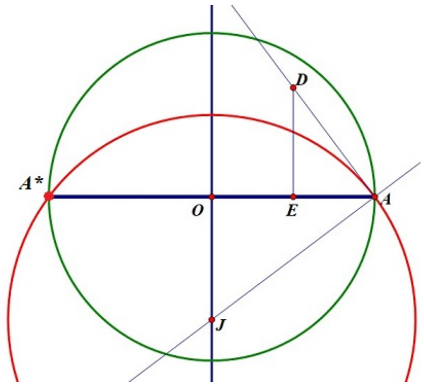

Figure 3A. $\angle A = \arctan(\frac{3}{4})$.

The line perpendicular to Euclidean line $\overleftrightarrow{AD}$ at A contains our desired center because we measure angles with tangents. An elliptic line through A must pass through its antipodal point, so we have the desired center at J. The circle with center J and radius $\overline{AJ}$ gives us an elliptic line through A and $\angle A$ has measure $\arctan(\frac{3}{4})$.

Step 2: locate point B. We need $AB = \pi - \arctan(\frac{3}{4})$, which forces $A^*B = \arctan(\frac{3}{4})$. The Formula 1 trick where right angles $\angle HA^*O$ and $\angle HBO$ imply $\cos\angle A^*HB = \cos A^*B$ tells us to place a Euclidean angle at H so that $\angle IHK = \arctan(\frac{3}{4})$. The perpendicular to $\overline{IH}$ at H intersects $\overleftrightarrow{AB}$ at G, the desired center for an elliptic line which locates point B. We note that $\overleftrightarrow{BH}$ does not contain a side of our triangle because we do not want a right angle at vertex B.

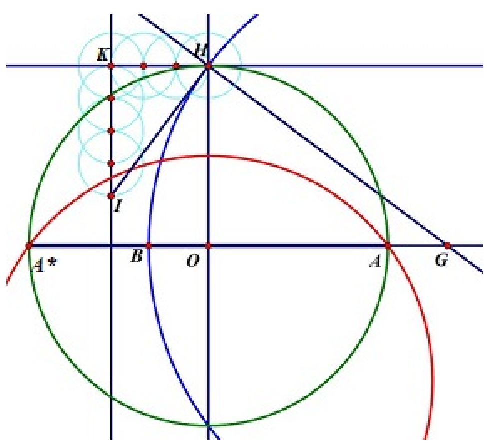

Figure 3B. Construct point B.

Step 3: calculate the measure of $\angle$ABC $= x$ from (2). We find $\cos x = \dfrac{\frac{16}{25}}{1+\frac{9}{25}} = \dfrac{8}{17}$, which is constructible in the same way we have constructed our other angles so far. The authors noticed that the 3, 4, 5 Pythagorean triple which started this process has generated another triple: 8, 15, 17. This is just a coincidence and does not generally occur for other triples defining $\angle$A. (When we apply Formula (2) to the 5, 12, 13 right triangle, $\cos x = \frac{144}{194}$, or $\cos x = \frac{25}{313}$, depending on the choice of acute angle A.) We construct the Euclidean angle at point B and find the elliptic line $\overleftrightarrow{BC}$ in Figure 3C. We had to construct the Euclidean $\angle$NBP $= \arctan\left(\frac{8}{15}\right)$. Even if we did not have a Pythagorean triple, we would have a constructible angle because, at worst, we would get square roots for side lengths.

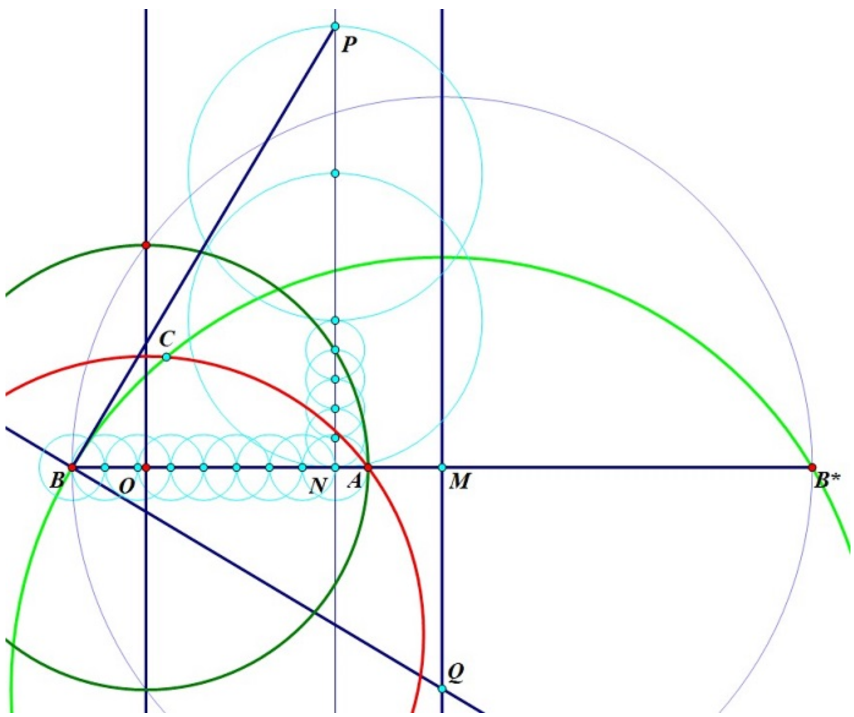

Figure 3C. Finish triangle ABC.

As before, the Euclidean $\angle$ABP and the perpendicular at B combine with the perpendicular bisector of $\overline{BB}^*$ to find the desired center Q of a circle which passes through antipodal points and has the tangent $\overline{BP}$. The intersection of this new elliptic line with our first elliptic line gives us point C and the triangle ABC is complete. Our trigonometric calculations guarantee triangle ABC is congruent to its polar triangle. Figure 3D shows both triangles with all construction marks removed except the line of symmetry.

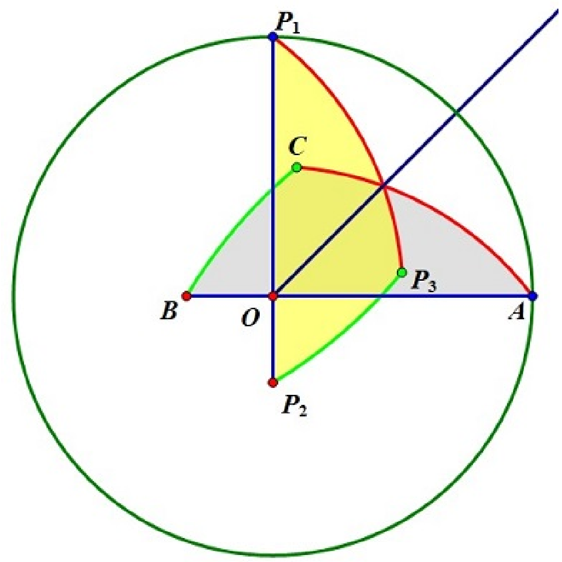

Figure 3D. Triangle ABC and its polar triangle.

The similarities between our two examples suggest some specific properties might occur for all triangles congruent to their polar triangles. A digression into projections will reward us with an unexpected characteristic of such triangles. The connection involves two new ideas in elliptic geometry: elliptic Wallace-Simson lines and these new triangles.

2 Wallace-Simson lines in elliptic geometry

In 1797, William Wallace published a theorem of Euclidean geometry which did not bear his name: the Simson line theorem. Wallace proved that any point P on the circumcircle of any triangle ABC projects onto the triangle in three collinear points and that the points on the circumcircle are the only such points. (All the other points in the plane project onto the triangle in the vertices of a pedal triangle.) We call the line on which these three points lie a **Wallace-Simson line**.

Wallace's proof used quadrilaterals with opposite angles supplementary but elliptic geometry has no such quadrilaterals; so it is no surprise that his theorem fails in elliptic geometry. Even though the theorem does not hold, we will see that for a triangle ABC it is possible to have a point such that the feet of the projections onto the sides of the triangle all lie on the same line. Such a point is called a **point of projection** and the line is a Wallace-Simson line. In Figure 4, we compare the Euclidean and elliptic situations.

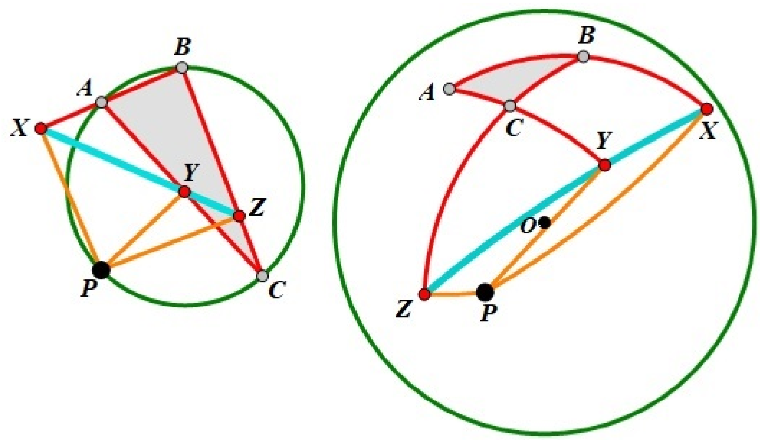

Figure 4. Wallace-Simson lines $\overleftrightarrow{XZ}$.

On the left is a Euclidean Wallace-Simson line as Wallace described. On the right, we have a corresponding example in elliptic geometry. The point P projects onto triangle ABC with feet X, Y, Z in both. We have right angles $\angle PXA$, $\angle PYC$, and $\angle PZB$. The circle O in the elliptic version is the boundary of elliptic space itself.

We can find twelve projection points with twelve Wallace-Simson lines in 4 sets of 3 each for non-self-polar triangle ABC. Briefly, each vertex projected onto each triangle gives a Wallace-Simson line and each side can be a Wallace-Simson line for each triangle, with a non-vertex projection point for each side. We now present examples of each type with details. Because we will do so many projections, we will employ a small table of steps for each projection so that the reader may check the ideas in a repetitive fashion.

In order to project a point onto a triangle, we use the poles and polars wisely. To project P_2 onto triangle ABC, we save the projection onto $\overleftrightarrow{AC}$ for last because any line through P_2 is perpendicular to $\overleftrightarrow{AC}$. This strategy pays off as soon as we use the line of projection $\overleftrightarrow{P_2P_1}$ because this line is perpendicular to both $\overleftrightarrow{AC}$ and $\overleftrightarrow{AB}$. Our notation matches Figure 5.

Proj Pt	triangle	proj lines	feet	WSL	3rd foot
P_2	ABC	$\overleftrightarrow{P_2P_1}, \overleftrightarrow{P_2P_3}$	R, T	$\overleftrightarrow{RT}$	$\overleftrightarrow{RT} \cap \overleftrightarrow{AC} = S$
B	$P_1P_2P_3$	$\overleftrightarrow{BA}, \overleftrightarrow{BC}$	R, T	$\overleftrightarrow{RT}$	$\overleftrightarrow{RT} \cap \overleftrightarrow{P_1P_3} = M$

Projecting a pole of a side of a triangle onto the triangle always gives a Wallace-Simson line and these lines do not have a special location as a side or an altitude, so we call such Wallace-Simson lines **pole-projected Wallace-Simson lines** (PPWSL).

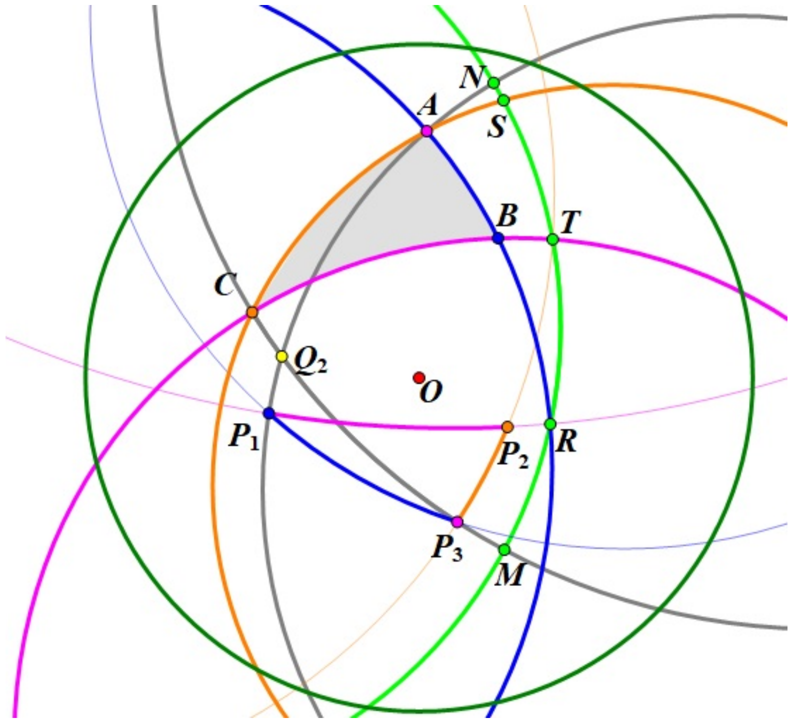

Figure 5. PPWSL $\overleftrightarrow{RT}$.

Since $\overleftrightarrow{AP_1}$ is perpendicular to both $\overleftrightarrow{P_1P_2}$ and $\overleftrightarrow{AB}$, T has to be the pole of $\overleftrightarrow{AP}$. Since $\overleftrightarrow{CP_3}$ is perpendicular to both $\overleftrightarrow{P_2P_3}$ and $\overleftrightarrow{BC}$, R is the pole of $\overleftrightarrow{CP_3}$. Then the intersection of $\overleftrightarrow{AP_1}$ and $\overleftrightarrow{CP_3}$ has to be the pole of $\overleftrightarrow{RT}$, and that point is Q_2, which turns out to be a projection point for both triangles, as well!

Proj Pt	triangle	proj lines	feet	WSL	3rd foot
Q_2	ABC	$\overleftrightarrow{Q_2P_1}, \overleftrightarrow{Q_2P_3}$	A, C	$\overleftrightarrow{AC}$	$\overleftrightarrow{AC} \cap \overleftrightarrow{Q_2P_2}$, (on $\overleftrightarrow{AC}$)
Q_2	$P_1P_2P_3$	$\overleftrightarrow{Q_2A}, \overleftrightarrow{Q_2C}$	P_1, P_3	$\overleftrightarrow{P_1P_3}$	$\overleftrightarrow{P_1P_3} \cap \overleftrightarrow{Q_2B}$, (on $\overleftrightarrow{P_1P_3}$)

We shall soon see that every projection point pulls double-duty as a pole of a Wallace-Simson line.

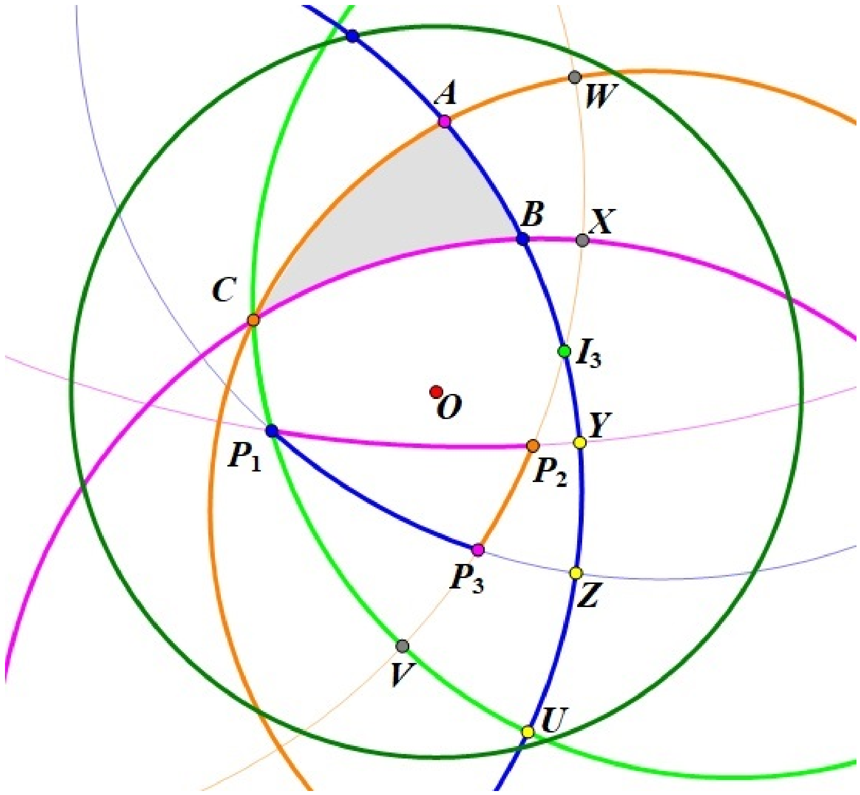

Figure 6. Altitude $\overleftrightarrow{\mathrm{CP_1}}$ with its pole $\mathrm{I_3}$.

It turns out that altitudes are Wallace-Simson lines. In Figure 6, note $\overleftrightarrow{\mathrm{CP_1}}$ is perpendicular to $\overleftrightarrow{\mathrm{AB}}$ at U because $\mathrm{P_1}$ is the pole of $\overleftrightarrow{\mathrm{AB}}$ and $\overleftrightarrow{\mathrm{CP_1}}$ is perpendicular to $\overleftrightarrow{\mathrm{P_2P_3}}$ at V because C is the pole of $\overline{\mathrm{P_2P_3}}$. The intersection of $\overleftrightarrow{\mathrm{AB}}$ and $\overleftrightarrow{\mathrm{P_2P_3}}$, $\mathrm{I_3}$, is the pole of $\overleftrightarrow{\mathrm{CP_1}}$ because $\overleftrightarrow{\mathrm{CP_1}}$ is the mutual perpendicular of these two lines. The pole of $\overleftrightarrow{\mathrm{CP_1}}$ is $\mathrm{I_3} = \overleftrightarrow{\mathrm{AB}} \cap \overleftrightarrow{\mathrm{P_2P_3}}$, which acts a projection point with Wallace-Simson lines for both triangles.

Proj Pt	triangle	proj lines	feet	WSL	3rd foot
C	ABC	$\overleftrightarrow{\mathrm{CP_1}}$	U, C	$\overleftrightarrow{\mathrm{CP_1}}$	C
$\mathrm{P_1}$	$\mathrm{P_1P_2P_3}$	$\overleftrightarrow{\mathrm{P_1C}}$	V, $\mathrm{P_1}$	$\overleftrightarrow{\mathrm{CP_1}}$	$\mathrm{P_1}$
$\mathrm{I_3}$	ABC	$\overleftrightarrow{\mathrm{I_3P_2}}, \overleftrightarrow{\mathrm{I_3P_3}}$	W, X	$\overleftrightarrow{\mathrm{P_2P_3}}$	$\mathrm{I_3}$
$\mathrm{I_3}$	$\mathrm{P_1P_2P_3}$	$\overleftrightarrow{\mathrm{I_3A}}, \overleftrightarrow{\mathrm{I_3B}}$	Z, Y	$\overleftrightarrow{\mathrm{AB}}$	$\mathrm{I_3}$

2.1 Table of projections points and Wallace-Simson lines

We summarize in the table below.

Projection points	triangle	W-S lines	Poles of W-S lines
P_1, P_2, P_3	ABC	P-P W-S L_i	Q_1, Q_2, Q_3
A, B, C	ABC	altitudes	I_1, I_2, I_3
P_1, P_2, P_3	$P_1P_2P_3$	altitudes	I_1, I_2, I_3
A, B, C	$P_1P_2P_3$	P-P W-S L_i	Q_1, Q_2, Q_3
Q_1, Q_2, Q_3	ABC	$\overleftrightarrow{AB}, \overleftrightarrow{AC}, \overleftrightarrow{BC}$	P_1, P_2, P_3
I_1, I_2, I_3	ABC	$\overleftrightarrow{P_1P_2}, \overleftrightarrow{P_1P_3}, \overleftrightarrow{P_2P_3}$	A, B, C
Q_1, Q_2, Q_3	$P_1P_2P_3$	$\overleftrightarrow{P_2P_3}, \overleftrightarrow{P_1P_3}, \overleftrightarrow{P_1P_2}$	A, B, C
I_1, I_2, I_3	$P_1P_2P_3$	$\overleftrightarrow{BC}, \overleftrightarrow{AC}, \overleftrightarrow{AB}$	P_1, P_2, P_3

More can be proved. Triangle ABC and its polar triangle share altitudes and altitudes are always concurrent in elliptic geometry. Then the concurrent polars (altitudes) imply the collinear poles (I points) because three poles are collinear if and only if their polars are concurrent. A theorem of Chasles also implies the I points are collinear because the I points occur from the same meetings of lines mentioned in his theorem. [1]

3 Main Theorem

Theorem 3.1. *: A suitable triangle* ABC *is congruent to its polar triangle if and only if the pole-projected Wallace-Simson lines are concurrent at the orthocenter. (By "suitable" we mean that triangle* ABC *must have the potential to be congruent to its polar triangle. That means triangle* ABC *requires an acute angle and an elliptic area less than* $\frac{\pi}{2}$*.)*

Proof. If triangle ABC is congruent to its polar triangle, we can place triangle ABC in the Figure 1 position. Triangle ABC and its polar triangle are symmetric across the angle bisector of $\angle BOP_2$.We claim this line of symmetry is $\overleftrightarrow{OH}$. We can construct the orthocenter H from the intersection of $\overleftrightarrow{BP_2}$ (an altitude) and the line of symmetry. Since the altitudes are concurrent at H, H must be on the line of symmetry.

The pole-projected Wallace-Simson lines we find when we project onto triangle ABC are the same as when we project A, B, C onto triangle $P_1P_2P_3$. When we project P_2 onto triangle ABC, we get a foot at point O and another at the intersection of $\overleftrightarrow{P_2P_3}$ and $\overleftrightarrow{BC}$. Both these feet are on the line of symmetry, which means the line of symmetry is the pole-projected Wallace-Simson line for this projection. The other two pole-projected Wallace-Simon lines must be symmetric across the line of symmetry.

These other two lines have to intersect each other and the line of symmetry in a way which obeys the symmetry and fulfills the way these Wallace-Simson lines are formed. If L_1 is the Wallace-Simson line from projecting P_1 onto triangle ABC, L_1 is the same line for projecting C onto the polar triangle. But the symmetry forces the projection of point A onto the polar triangle to be the reflection of L_1 across the line of symmetry. Then the three Wallace-Simson lines must be concurrent at a point we call W. We claim the point W must be point H.

Suppose $W \neq H$. Figure 7 illustrates both polars L_H and L_W. Both lines are perpendicular to the line of symmetry because their poles are on that line. A line through H and an I point and another line through W and a Q point lead to a contradiction. In the shaded quadrilateral IPQR, we have right angles at I and Q. The line of symmetry is perpendicular to both L_W and L_H. This forces $\angle IPQ$ to be acute. However, $\angle IRQ$ must also be acute. (lines intersecting $\overleftrightarrow{IH}$ on this side of L_H must have acute angles in that position.) Switching the relative positions of H and W does not change the situation.

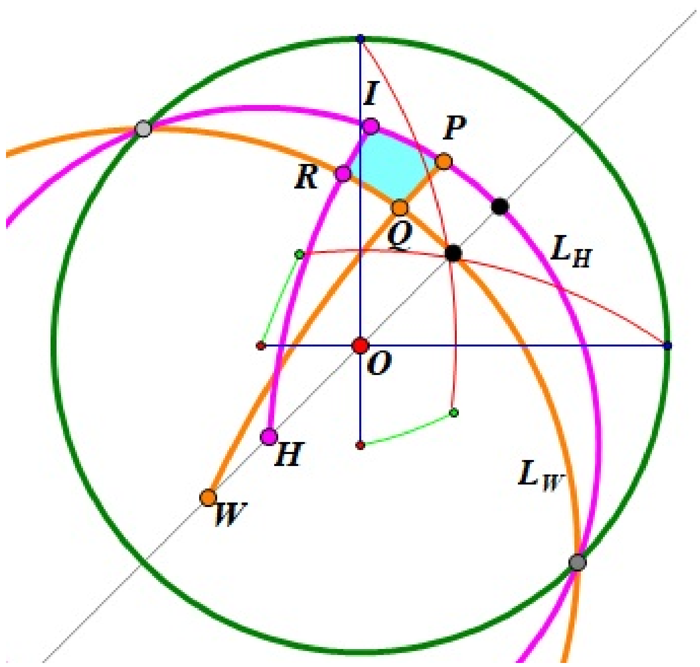

Figure 7. Quadrilateral IPQR.

Then the polars must be the same, giving us W = H. The three altitudes are concurrent at H and the pole-projected Wallace-Simson lines are concurrent at W. These are the required six concurrent lines.

A note to the reader: Figures 7 and 8 are the only figures in the paper which have not been constructed accurately because impossible situations needed to be illustrated.

For the converse, we do not know triangle ABC is congruent to its polar triangle. We have W = H. We get a lot of symmetry from this assumption. We can place suitable triangle ABC in the usual position because A is an acute angle. We get its polar triangle as usual, too. Figure 8 lays out what we know.

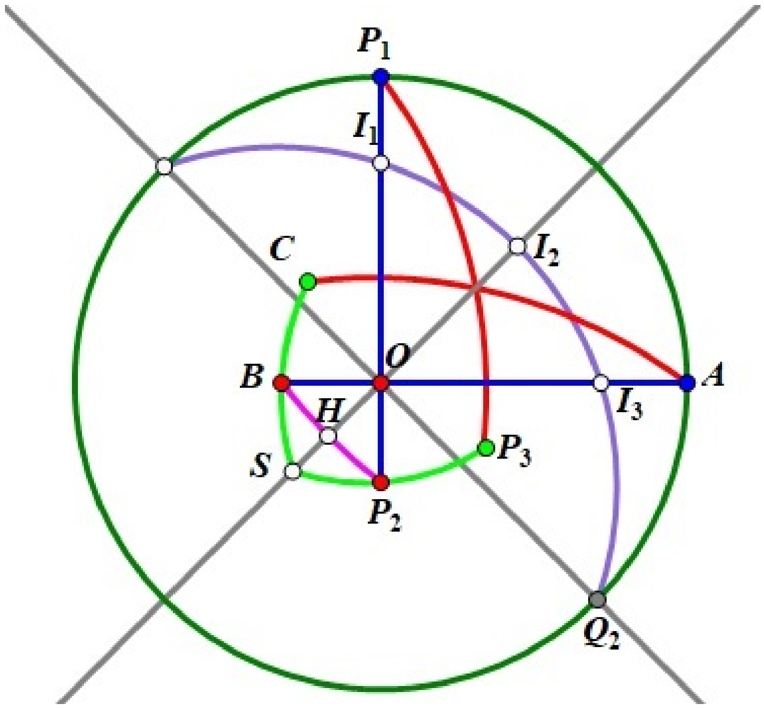

Figure 8. Given W = H.

Line $\overleftrightarrow{SO}$ is the Wallace-Simson line from projecting P_2 onto triangle ABC. The altitude $\overline{BP_2}$ intersects $\overleftrightarrow{SO}$ at H = W. We'll just call it H. Line $\overleftrightarrow{OS}$ is a Wallace-Simson line with pole Q_2. Segment $\overline{OQ_2}$ is perpendicular to $\overleftrightarrow{OS}$. Line $\overleftrightarrow{Q_2I_2}$ is $L_H = L_W$: all Q and I points are on this line. Because I_2 is the pole of $\overleftrightarrow{BP_2}$, I_2, O and H are collinear with S. Euclidean triangle OBP_2 has angle bisector $\overline{OH}$ because $\overline{BP_2}$ is a chord of the circle containing arc $\overline{BP_2}$ and its center is on $\overleftrightarrow{OH}$.

This implies $\angle P_2OH$ has to be $\frac{\pi}{4}$. The perpendicular lines at H give us Euclidean triangle OHP_2 congruent to triangle OHB by ASA. The Euclidean line $\overleftrightarrow{OH}$ bisects the chord $\overline{BP_2}$ and its arc. Now we can stick to elliptic objects. Elliptic segments $\overline{P_2H} \cong \overline{BH}$ and triangle BHS is congruent to triangle P_2HS by SAS. Lengths P_3S and CS are both $\frac{\pi}{2}$ because that is how far a pole is from its polar. Subtracting BS and P_2S, we get $CB = P_3P_2$.

The relationships between angles of one triangle and side lengths of its polar triangle give us $\pi - \angle A = P_1P_2$. We obtain $\overline{OB} \cong \overline{OP_2}$ from our congruent triangles, so $\overline{AB} \cong \overline{P_1P_2}$.

Our triangles also give us $\angle OP_2S \cong \angle OBS$; their supplements, $\angle P_3P_2P_1$ and $\angle ABC$ must be congruent. Our triangle ABC is congruent to its polar triangle $P_1P_2P_3$ by SAS. $\square$

We have incidentally proved a triangle is congruent to its polar triangle if and only if the corresponding vertices of the triangles are pair-wise equidistant from H. We could make such a statement about any point on the line of symmetry when the triangles are in Figure 1 position. It is best to make this statement about a point which will work no matter where triangle ABC is situated. For instance, point O is equidistant from corresponding pairs of vertices in Figure 1B. But if our triangles were off-center, point O might not have this property because it might not be on the line of symmetry.

Our theorem shows a connection between the elementary concepts of congruent triangles, polar triangles and the new concept of Wallace-Simson lines in elliptic geometry. A Euclidean theorem, modified to fit elliptic geometry, has joined with introductory ideas to give an insight. We have seen the same development when squaring the circle in non-Euclidean geometry [4] [5]. When Euclidean theorems fail in non-Euclidean geometry, geometers have their modifications to consider.

References

[1]. H. S. M. Coxeter, *Non-Euclidean Geometry*, 6th ed., Mathematical Association of America, 1998, p. 223, 53.

[2]. M. McDaniel, *Geometry by Construction*, Universal-Publishers, 2015, p. 100.

[3] C. Goodman-Strauss, Compass and straightedge in the Poincaré disk, *American Mathematical Monthly,* **108** 1, 2001.

[4] N. Davis, "Squaring the circle in hyperbolic geometry," *Rose-Hulman Undergraduate Math Journal,* 2014.

[5] K. Jansens and N. Davis, "Squaring the circle in non-Euclidean geometry," *Rose-Hulman Undergraduate Math Journal,* 2017.

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