10-1973

Higher Mathematics Applied to Business Decisions

John W. Gillet

HIGHER MATHEMATICS APPLIED TO

BUSINESS DECISIONS

by
John W. Gillett

Bachelor of Science, University of North Dakota, 1967

An Independent Study

Submitted to the Faculty

of the

University of North Dakota

in partial fulfillment of the requirements

for the degree of

Master of Science

Grand Forks, North Dakota

October
1973

TABLE OF CONTENTS

INTRODUCTION

The use of mathematics in the study of business management, economics and accounting is apparently becoming more and more relevant to the students of these chosen fields. Although many faculty members in these fields learned their mathematics in graduate school and therefore have seen the need for higher mathematics, students wonder why they have to take a course in higher mathematics.

In most instances the answer is not forthcoming in the math course itself. Mathematicians are not economists or accountants. The consequence of these simple facts is that the students conclude that mathematics is just an intellectual exercise that they must endure as a sort of initiation fee into their major field.

This paper is designed to be used by students of business management, accounting and economics in conjunction with the current textbook for Mathematics 203 which is Linear Algebra, Calculus, and Probability: Fundamental Mathematics for the Social and Management Sciences by Lloyd S. Emerson and Laurence R. Paquette. Hopefully, it will provide a medium of motivation for them and will kindle their interest in mathematics that can be applied in the fields of business management, economics and accounting.

In preparing the problems of each chapter, there has been one goal in mind. It is to illustrate, the application of the particular mathematical concepts presented in the 203 textbook to business management, economics and accounting.

The form employed to achieve this goal is to group the 12 chapters in the 203 course into 6 major areas in this paper. At the beginning of each of the 6 areas the corresponding 203 chapters are revealed with a short listing of the math concepts contained in those chapters. The problems will follow arranged in order.

CHAPTER 1

CHAPTER 1 ELEMENTARY ALGEBRA AND BREAK-EVEN ANALYSIS

The Real Number System · Functions · Linear Functions · Solution of Systems of Linear Equations by Elimination · Subscripts and Summation Notation · Cost and Revenue Functions · Break-Even Analysis · Inequalities · Systems of Linear Inequalities

Example 1.

The total operating cost, C, of a certain firm is given as a function of the production level, x, by the linear function $C = 0.72x + 1000$. C is in dollars and x is in units produced. Discuss the variable cost and the fixed cost for this firm. Interpret this situation geometrically.

Solution: We see from definitions that the fixed cost is \$1000. Also, the variable cost is 0.72x. The variable cost per unit can be obtained by dividing the variable cost by the number of units produced. This involves dividing 0.72x by x, therefore, the variable cost per unit is 0.72. We note that the variable cost per unit is a constant in this case.

This cost function is of the form $C = mx+b$, where $m = 0.72$ and $b = 1{,}000$. This means that the variable cost per unit can be interpreted geometrically as the slope of the straight line and fixed cost can be interpreted as its C intercept.

Example 2.

The "break-even point" is defined as the level of sales at which the total profit is zero. If costs are considered to consist only of a fixed cost which does not vary with the level of production and of a constant cost for production of each unit, usually called variable cost, the total cost of production may be expressed as

Total production cost =

fixed cost + (variable cost)(number units produced)

Profit is defined as total revenue minus total production cost where total revenue is sales price per unit multiplied by the number of units sold.

Letting FC be the fixed cost, VC the variable cost, p the sales price per unit and x the number of units sold, the total cost function C and revenue function R are defined by

$$C(x) = FC + (VC)(x)$$
$$R(x) = p \cdot x.$$

Since profit is defined as $R(x) - C(x)$ and the break-even point is that level of sales at which profit is zero, we find the break-even point by letting

$$R(x) - C(x) = 0.$$

That is,

$$p \cdot x - (FC + VC \cdot x) = 0,$$

$$x = \frac{FC}{p - VC}$$

The last equation defines $\underline{x}$ as a function of $\underline{p}$; that is,

$$x = f(p) = \frac{FC}{p - VC}.$$

The difference p - VC is called the <u>incremental return per unit of sales</u>.

Thus the quotient $\frac{FC}{p - VC}$ states how many times this increment must be earned in order to pay the fixed costs.

If, rather than computing the sales break-even point, we wish to determine the break-even sales price if the production quantity is at some fixed level we compute the inverse function f^{-1}. Solving $x = \frac{FC}{p - VC}$ for p, we obtain

$$p = \frac{FC + (VC)(x)}{x}.$$

Thus

$$f^{-1}(x) = \frac{FC + (VC)(x)}{x}.$$ [1]

---

[1] Beuan Youse, Calculus for Students of Business and Management (Scranton, Pa.: International Textbook Company, 1967).

CHAPTER 2

CHAPTER 2-4 INTRODUCTION TO VECTORS AND MATRICES

Definition, Equality, and Addition of Vectors · Multiplication of a Vector by a Scalar and by a Vector · Definition, Equality and Addition of Matrices · Multiplication of a Matrix by a Scalar and by a Matrix · Systems of Equations as Single Matrix Equations · Business Applications of Vector and Matrix Multiplication · Additional Applications of Matrix Multiplication

ALGEBRA OF SQUARE MATRICES

Introduction to the Algebra of Square Matrices · Some Algebraic Laws for Square Matrices · Failure of the Commutative Law for Matrix Multiplication · The Inverse of a Square Matrix · The Application of the Inverse of a Matrix to Solve Systems of Equations · An Algorithm for Finding the Inverse of a Matrix

DETERMINANTS

Definition of the Determinant of n X n Matrix · Evaluation of Determinants of 2 X 2 and 3 X 3 Matrices · The Method of Cofactors · Cramer's Rule · The Use of Determinants for Finding the Inverse of a Matrix · Summary of Methods for Solving Systems of Equations

Example 1.

Baker is a hot dog salesman who owns a stand near a baseball stadium. Each game he buys Q hot dogs. He makes a profit of 10 cents on each hot dog sold. The unsold hot dogs, if any, are returned to the manufacturer at a loss of 2 cents each to Baker. From his past records, Baker estimates the following probabilities for selling hot dogs:

d = number of hot dogs demanded	10	20	30	40	50
p = probability of that demand	.1	.2	.4	.2	.1

(For simplicity we assume his sales are always a multiple of 10.) How many hot dogs should Baker buy to maximize his net profit?

It is clear that Baker's net profit is

$$10Q \quad \text{if } Q \leq d$$

i.e., if the demand is sufficient to sell all hot dogs; and it is

$$10d - 2(Q - d) = 12d - 2Q, \quad \text{if } Q > d$$

i.e., if supply exceeds demand, since he makes 10 cents on those he sells and loses 2 cents on those he does not sell. If we compute his profit in cents for each of the ways he can order from 10 to 50 hot dogs and sell from 10 to 50 hot dogs we obtain the matrix M of Figure 11.

number of hot dogs demanded

$$M = \begin{array}{c} \text{number of} \\ \text{hot dogs} \\ \text{ordered} \end{array} \begin{array}{c} \\ 10 \\ 20 \\ 30 \\ 40 \\ 50 \end{array} \begin{array}{c} \begin{array}{ccccc} 10 & 20 & 30 & 40 & 50 \end{array} \\ \begin{pmatrix} 100 & 100 & 100 & 100 & 100 \\ 80 & 200 & 200 & 200 & 200 \\ 60 & 180 & 300 & 300 & 300 \\ 40 & 160 & 280 & 400 & 400 \\ 20 & 140 & 260 & 380 & 500 \end{pmatrix} \end{array}$$

Figure 11

If we let p be the vector

$$p = \begin{pmatrix} .1 \\ .2 \\ .4 \\ .2 \\ .1 \end{pmatrix}$$

then we see that Mp is a column vector whose entries are Baker's expected profits for each number of hot dogs that he can order. We have

$$Mp = \begin{pmatrix} 100 \\ 188 \\ 252 \\ 268 \\ 260 \end{pmatrix}$$

The action that we then expect Baker to take is to order 40 hot dogs to obtain the maximum profit of

$$\max Mp = 268.$$ [2]

Example 2.

Suppose that a building contractor has accepted orders from five ranch style houses, seven Cape Cod houses, and twelve Colonial style houses. We can represent his orders by means of a row vector x = (5, 7, 12). The contractor is familiar, of course, with the kinds of "raw materials" that go into each type of house. Let us suppose that these raw materials are steel, wood, glass, paint, and labor. The numbers in the matrix below give the amounts of each raw material going into each type of house, expressed in convenient units. (The numbers are put in arbitrarily, and are not meant to be realistic.)

	Steel	Wood	Glass	Paint	Labor
Ranch:	5	20	16	7	17
Cape Cod:	7	18	12	9	21
Colonial:	6	25	8	5	13

= R

Observe that each row of the matrix is a five-component row vector which gives the amounts of each raw material needed for a given kind of house.

[2]John Kemeny, and others, Finite Mathematics with Business Applications (Englewood Cliffs, N.J.: Prentice Hall Inc., 1962), p. 306-307.

Similarly, each column of the matrix is a three-component column vector which gives the amounts of a given raw material needed for each kind of house. Clearly, a matrix is a succinct way of summarizing this information.

Suppose now that the contractor wishes to compute how much of each raw material to obtain in order to fulfill his contracts. Let us denote the matrix above by R; then he would like to obtain something like the product xR, and he would like the product to tell him what orders to make out. The product should have the following form:

$$xR = (5,\ 7,\ 12)\begin{pmatrix} 5 & 20 & 16 & 7 & 17 \\ 7 & 18 & 12 & 9 & 21 \\ 6 & 25 & 8 & 5 & 13 \end{pmatrix}$$

$$= (5\cdot5 + 7\cdot7 + 12\cdot6,\ 5\cdot20 + 7\cdot18 + 12\cdot25,\ 5\cdot16 + 7\cdot12 + 12\cdot8,\ 5\cdot7 + 7\cdot9 + 12\cdot5,\ 5\cdot17 + 7\cdot21 + 12\cdot13)$$

$$= (146,\ 526,\ 260,\ 158,\ 388).$$

Thus we see that the contractor should order 146 units of steel, 526 units of wood, 260 units of glass, 158 units of paint, and 388 units of labor. Observe that the answer we get is a five-component row vector and that each entry in this vector is obtained by taking the vector product of x times the corresponding column of the matrix R.

The contractor is also interested in the prices that he will have to pay for these materials. Suppose that steel costs $15 per unit, wood costs $8 per unit, glass costs $5 per unit, paint costs $1 per unit, and labor costs $10 per unit. Then we can write the cost as a column vector as follows:

$$y = \begin{pmatrix} 15 \\ 8 \\ 5 \\ 1 \\ 10 \end{pmatrix}$$

Here the product Ry should give the costs of each type of house, so that the multiplication should have the form

$$Ry = \begin{pmatrix} 5 & 20 & 16 & 7 & 17 \\ 7 & 18 & 12 & 9 & 21 \\ 6 & 25 & 8 & 5 & 13 \end{pmatrix} \begin{pmatrix} 15 \\ 8 \\ 5 \\ 1 \\ 10 \end{pmatrix}$$

$$= \begin{pmatrix} 5\cdot 15 + 20\cdot 8 + 16\cdot 5 + 7\cdot 1 + 17\cdot 10 \\ 7\cdot 15 + 18\cdot 8 + 12\cdot 5 + 9\cdot 1 + 21\cdot 10 \\ 6\cdot 15 + 25\cdot 8 + 8\cdot 5 + 5\cdot 1 + 13\cdot 10 \end{pmatrix}$$

$$= \begin{pmatrix} 492 \\ 528 \\ 465 \end{pmatrix}$$

Thus the cost of materials for the ranch style house is \$492, for the Cape Cod house is \$528, and for the Colonial house \$465.

The final question which the contractor might ask is what is the total cost of raw materials for all the houses he will build. It is easy to see that this is given by the vector xRy. We can find it in two ways as shown below.

$$xRy = (xR)y = (146, 526, 260, 158, 388)\cdot \begin{pmatrix} 15 \\ 8 \\ 5 \\ 1 \\ 10 \end{pmatrix} = 11{,}736$$

$$xRy = x(Ry) = (5, 7, 12)\cdot \begin{pmatrix} 492 \\ 528 \\ 465 \end{pmatrix} = 11{,}736.$$

The total cost is then \$11,736.[3]

---

[3] Ibid., p. 240-242.

Depreciation Schedules

Example 3 contains an accounting problem that requires an ordinary transformation matrix. Here it is necessary to prepare a depreciation lapse schedule, and the elements in the transformation vector measure rates of usage (depreciation) for each time period. Input space has been represented by unit vectors, but the residual values of each asset have been subtracted from each unit vector to leave the depreciation base values. This type of logical problem arises frequently in accounting. It may be referred to as a branching problem, because it involves the segregation of input data into two or more groups, each of which requires different transformation (or no transformation, as in the residual book values in the depreciation illustration). Here the branching is treated by taking from 100% (represented by the unit vectors in the identity matrix) the residual 20% (represented by a scalar multiplication of the identity matrix). Alternatively, the branching problem can be treated without involving matrices by separating residual values and depreciable bases.

Example 3.

Accounting Problem Employing
an Ordinary Transformation Matrix

Accounting area:	Depreciation lapse schedule preparation.
Input data:	Three machines acquired at costs of $75,000, $37,500, $112,500, respectively.
Transformation data:	Usage (depreciation) is to be measured by the sum-of-the-years'-digits method using economic life of 3 years and 20% residual value.
Output data:	Usage is to be allocated to yearly time periods.

Input Space Transformation Matrix Output Space (lapse schedule)

$$\underset{(I - 20\%I)}{\begin{bmatrix}0.8 & 0 & 0\\ 0 & 0.8 & 0\\ 0 & 0 & 0.8\end{bmatrix}} \underset{\lambda}{\begin{bmatrix}75,000\\ 37,500\\ 112,5000\end{bmatrix}} \underset{r}{\begin{bmatrix}1/2 & 1/3 & 1/6\end{bmatrix}} = O$$

	Year 1	Year 2	Year 3
	30,000	20,000	10,000
	15,000	10,000	5,000
	45,000	30,000	15,000
Total Depreciation	90,000	60,000	30,000

I = identity matrix (an ordered collection of unit vectors)
λ = vector containing cost of asset i, where i = 1, 2, 3
r = rate vector containing elements r_j, which show the rate of depreciation for period j = 1, 2, 3. Usage rates were determined according to the sum-of-the-years'-digits depreciation method. KEY
0 = output matrix containing elements 0_{ij}, which show the depreciation charge for asset i in period j

Then, the depreciable bases [i.e., $(I - 20\%I)\lambda$] are entered directly into a vector, thereby reducing the transformation to

$$\begin{bmatrix}60,000\\ 30,000\\ 90,000\end{bmatrix}\begin{bmatrix}1/2 & 1/3 & 1/6\end{bmatrix} = \begin{bmatrix}30,000 & 20,000 & 10,000\\ 15,000 & 10,000 & 5,000\\ 45,000 & 30,000 & 15,000\end{bmatrix}$$

This is the same result as shown in Example 3.[4]

Example 4 shows a transformation requiring the use of an inverse matrix. Here the elements a_{ij} in matrix A show the interest of expense i in expense J, where i, j = B (loans), F (franchise tax), and T (federal income tax). The interests are determined in accord with given statutory-contractual rates.

<u>Computing Bonuses</u>

Example 4.

Accounting Problem Requiring
an Inverse Transformation Matrix

---

[4] Wayne A. Corconen, <u>Mathematical Applications in Accounting</u> (New York and Chicago: Harcourt, Brave and World, Inc., 1968), p. 158-159.

KEY

B = executives' bonus
F = franchise tax
T = federal tax on income
$100,000 = profits before B, F, and T

Original system of equations:

$$B = 0.1\ (\$100{,}000 - F - T)$$
$$F = 0.05\ (\$100{,}000 - B)$$
$$T = 0.50\ (\$100{,}000 - B - F)$$

Rearranged system:

$$10{,}000 = B + 0.1F + 0.1T$$
$$5{,}000 = 0.05B + F + 0T$$
$$50{,}000 = 0.5B + 0.5F + T$$

Restated system:

$$\underset{g}{\begin{bmatrix} 10{,}000 \\ 5{,}000 \\ 50{,}000 \end{bmatrix}} = \underset{A}{\begin{bmatrix} 1 & 0.1 & 0.1 \\ 0.05 & 1 & 0 \\ 0.5 & 0.5 & 1 \end{bmatrix}} \underset{x}{\begin{bmatrix} B \\ F \\ T \end{bmatrix}}$$

KEY g = vector of constants in the rearranged system
A = matrix of coefficients of the rearranged system
x = solution vector

Solution — Input Space — Transformation Matrix — Output Space

$$\underset{x}{\begin{bmatrix} B \\ F \\ T \end{bmatrix}} = \underset{A^{-1}}{\begin{bmatrix} 1.05541 & -0.05277 & -0.10554 \\ -0.05277 & 1.00263 & 0.00528 \\ -0.50132 & -0.47493 & 1.05013 \end{bmatrix}} \underset{g}{\begin{bmatrix} 10{,}000 \\ 5{,}000 \\ 50{,}000 \end{bmatrix}} = \begin{bmatrix} 5{,}013 \\ 4{,}749 \\ 45{,}119 \end{bmatrix}$$ [5]

Example 5.

Secondary Overhead Allocation

In the process of determining overhead rates, the accountant is concerned with, among other matters,

1. the primary (or direct) allocation of itemized overhead costs to various service and production departments, and

---

[5] *Ibid.*, p. 159-160.

2. the secondary allocation of individual service department (or better, service rendering department) primary overhead costs to service consuming departments (i.e., other service departments and production departments).

The criterion governing secondary allocation is that of usage. How much of a given service will each consuming department use? Here usage is meant in two senses: actual usage and potential usage. Given some measurement indices, such as number of employees, floor area, man hours, and kilowatt hours, actual usage is measured by estimating consuming department responsibility (in terms of a specific index) for increasing rendering department variable costs. Potential usage is measured (again in terms of a specific index) by calculating consuming department responsibility for bringing about the fixed costs of the rendering departments. In either case, consuming department usage may be expressed as a set of percentages, each percentage weighted according to rendering department proportions of fixed and variable costs.

Let us see how matrices may be applied to secondary overhead allocation. Following are the data for an example. It is assumed that the percentages in the table have been determined by observing the usage criterion.

KEY S_i = service department i

P_j = production department j

Consuming department / Rendering department	S_1	S_2	P_1	P_2	P_3
S_1	0	40%	10%	30%	20%
S_2	25%	0	45%	20%	10%

	S_1	S_2	P_1	P_2	P_3
Primary overhead allocation totals (000 omitted)	$90	$180	$377	$307	$246
Standard machine hours (estimated, 000 omitted)			200	50	150

The system of equations requiring solution is

$S_1 = 90 + 0.25S_2$

$S_2 = 180 + 0.40S_1$

The system may be stated in matrices as follows:

$$\overset{A}{\begin{bmatrix} 1 & -0.25 \\ -0.40 & 1 \end{bmatrix}} \overset{x}{\begin{bmatrix} S_1 \\ S_2 \end{bmatrix}} = \overset{b}{\begin{bmatrix} 90 \\ 180 \end{bmatrix}}$$

$$\text{Total} = 270$$

The solution is

$$\overset{x}{\begin{bmatrix} S_1 \\ S_2 \end{bmatrix}} = \overset{A^{-1}}{\begin{bmatrix} 1/0.9 & 0.25/0.9 \\ 0.40/0.9 & 1/0.9 \end{bmatrix}} \overset{b}{\begin{bmatrix} 90 \\ 180 \end{bmatrix}} = \overset{x}{\begin{bmatrix} 150 \\ 240 \end{bmatrix}}$$

The amounts in vector x must be allocated to the production departments. Accordingly, we form matrix P by transposing the percentages shown under P_i and use this matrix to obtain our ultimate amounts for redistribution (shown in vector r).

$$\overset{P}{\begin{bmatrix} 0.10 & 0.45 \\ 0.30 & 0.20 \\ 0.20 & 0.10 \end{bmatrix}} \overset{x}{\begin{bmatrix} 150 \\ 240 \end{bmatrix}} = \overset{r}{\begin{bmatrix} 123 \\ 93 \\ 54 \end{bmatrix}}$$

$$\text{Total} = 270$$

The amounts in vector r must then be added to the primary allocation amounts for the production departments (say, vector d) to obtain the total overhead costs (vector t).

$$\overset{r}{\begin{bmatrix}123\\93\\54\end{bmatrix}} + \overset{d}{\begin{bmatrix}377\\307\\246\end{bmatrix}} = \overset{t}{\begin{bmatrix}500\\400\\300\end{bmatrix}}$$

Because matrices may be multiplied and added, it is possible to "link up" several stages of allocation. In our secondary overhead allocation example, for instance, we could proceed as follows:

$$t = d + PA^{-1}b$$

Let us first form PA^{-1}. It would always make sense to do this where the departmental interrelationships can be expected to remain stable, as they might for planning purposes.

$$\overset{P}{\begin{bmatrix}0.10 & 0.45\\0.30 & 0.20\\0.20 & 0.10\end{bmatrix}} \overset{A^{-1}}{\begin{bmatrix}1/0.9 & 0.25/0.9\\0.40/0.9 & 1/0.9\end{bmatrix}} = \overset{PA^{-1}}{\begin{bmatrix}0.3111 & 0.5278\\0.4222 & 0.3055\\0.2667 & 0.1667\end{bmatrix}}$$

We see that the equation for t holds.

$$\overset{t}{\begin{bmatrix}500\\400\\300\end{bmatrix}} = \overset{d}{\begin{bmatrix}377\\307\\246\end{bmatrix}} + \overset{PA^{-1}}{\begin{bmatrix}0.3111 & 0.5278\\0.4222 & 0.3055\\0.2667 & 0.1667\end{bmatrix}} \overset{b}{\begin{bmatrix}90\\180\end{bmatrix}} = \begin{bmatrix}377\\307\\246\end{bmatrix} + \begin{bmatrix}123\\93\\54\end{bmatrix}$$

The overhead rates are, therefore,

$\frac{\$500}{200}$ for P_1, $\frac{\$400}{50}$ for P_2, and $\frac{\$300}{150}$ for P_3[6]

---

[6] Ibid., p. 174-176.

CHAPTER 3

CHAPTER 5-6 LINEAR PROGRAMMING

Introduction · The Graphical Method · Table 1 of the Simplex Method · Simplex Method Continued · The Simplex Method--A Maximization Problem and a Summary

EXPONENTS, LOGARITHMS, AND SOME IMPORTANT NONLINEAR FUNCTIONS

Laws of Exponents · Definition and Laws of Logarithms · Use of Tables of Logarithms · Computations with Logarithms · Log Functions · Exponential and Power Functions · Semilog and Log-Log Paper

Example 1.

An automobile manufacturer makes automobiles and trucks in a factory that is divided into two shops. Shop 1, which performs the basic assembly operation, must work 5 man-days on each truck but only 2 man-days on each automobile. Shop 2, which performs finishing operations, must work 3 man-days for each automobile or truck that it produces. Because of men and machine limitations Shop 1 has 180 man-days per week available while Shop 2 has 135 man-days per week. If the manufacturer makes a profit of $300 on each truck and $200 on each automobile, how many of each should he produce to maximize his profit?

To state the problem mathematically we set up the following notation: Let x_1 be the number of trucks and x_2 the number of automobiles to be produced per week. Then these quantities must satisfy the following restrictions:

$$5x_1 + 2x_2 \leq 180$$

$$3x_1 + 3x_2 \leq 135.$$

We want to maximize the linear function $300x_1 + 200x_2$, subject to these inequality constraints, together with the obviously necessary constraints that $x_1 \geq 0$ and $x_2 \geq 0$.

To further simplify notation we define the quantities

$$A = \begin{pmatrix} 5 & 2 \\ 3 & 3 \end{pmatrix}, \quad b = \begin{pmatrix} 180 \\ 135 \end{pmatrix} \quad \text{and} \quad c = (300,200).$$

Then we can state this linear programming problem as follows.

<u>Maximum problem</u>: Determine the vector x so that the weekly profit, given by the quantity cx, is a maximum subject to the inequality constraints $Ax \leq b$ and $x \geq 0$. The inequality constraints insure that the weekly number of available man-hours is not exceeded and that nonnegative quantitites of automobiles and trucks are produced.

The graph of the convex set of possible x vectors is pictured in Figure 6. Clearly this is a problem of the kind discussed in the previous section.

The extreme points of the convex set C are

$$T_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 36 \\ 0 \end{pmatrix}, T_3 = \begin{pmatrix} 0 \\ 45 \end{pmatrix} \quad \text{and } T_4 = \begin{pmatrix} 30 \\ 15 \end{pmatrix}.$$

Following the solution procedure outlined in the previous section we test the function $cx = 300x_1 + 200x_2$ at each of these extreme points. The values taken on are 0, 10800, 9000, and 12000. Thus the maximum weekly profit is \$12,000 and is achieved by producing 30 trucks and 15 automobiles per week.[7]

---

[7]Kemeny, and others, <u>Finite Mathematics</u>, p. 379-380.

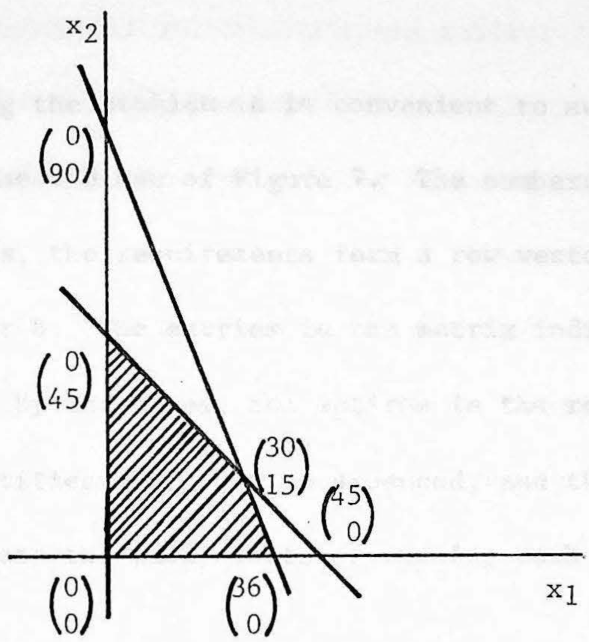


Figure 6

Example 2.

A mining company owns two different mines that produce a given kind of ore. The mines are located in different parts of the country and hence have different production capacities. After crushing, the ore is graded into three classes: high-grade, medium-grade, and low-grade ores. There is some demand for each grade of ore. The mining company has contracted to provide a smelting plant with 12 tons of high-grade, 8 tons of medium-grade, and 24 tons of low-grade ore per week. It costs the company \$200 per day to run the first mine and \$160 per day to run the second. However, in a day's operation the first mine produces 6 tons of high-grade, 2 tons of medium-grade, and 4 tons of low-grade ore, while the second mine produces daily 2 tons of high-grade, 2 tons of medium-grade, and 12 tons of low-grade ore. How many days a week should each mine be operated in order to fulfill the company's orders most economically?

Before solving the problem it is convenient to summarize the above information as in the tableau of Figure 7. The numbers in the tableau form a 2-by-3 matrix, the requirements form a row vector c, and the costs form a column vector b. The entries in the matrix indicate the production of each kind of ore by the mines, the entires in the requirements vector c indicate the quantities that must be produced, and the entries in the cost vector b indicate the daily costs of running each mine.

	High Grade Ore	Medium Grade Ore	Low Grade Ore	b
Mine 1	6	2	4	$200
Mine 2	2	2	12	$160
c	12	8	24	

Figure 7

Let $w = (w_1\ w_2)$ be the 2-component row vector whose component w_1 gives the number of days per week that mine 1 operates and w_2 gives the number of days per week that mine 2 operates. If we define the quantities

$$A = \begin{pmatrix} 6 & 2 & 4 \\ 2 & 2 & 12 \end{pmatrix}, \quad c = (12,\ 8,\ 24), \text{ and } b = \begin{pmatrix} 200 \\ 160 \end{pmatrix},$$

we can state the above problem as a minimum problem.

Minimum problem: Determine the vector w so that the weekly operating cost, given by the quantity wb, is a minimum subject to the inequality restraints $wA \geq c$ and $w \geq 0$. The inequality restraints insure that the weekly output requirements are met and the limits on the components of w are not exceeded.

It is clear that this is a minimum problem of the type discussed in detail in the preceding section. In Figure 8 we have graphed the convex polyhedral set C defined by the inequalities $wA \geq c$.

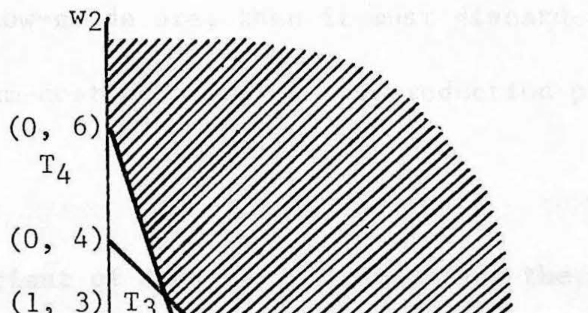


Figure 8

The extreme points of the convex set $\underline{C}$ are

$$T_1 = (6, 0),\ T_2 = (3, 1),\ T_3 = (1, 3),\ T_4 = (0, 6).$$

Testing the function $wb = 200w_1 + 160w_2$ at each of these extreme points we see that it takes on the values 1200, 760, 680, and 960, respectively. We see that the minimum operating cost is \$680 per week and it is achieved at T_3, i.e., by operating the first mine one day per week and the second mine three days a week.

Observe that if the mines are operated as indicated, then the combined weekly production will be 12 tons of high-grade ore, 8 tons of medium-grade ore, and 40 tons of low-grade ore. In other words, for this solution low-grade ore is overproduced. If the company has no other demand for the low-grade ore, then it must discard 16 tons of it per week in this minimum-cost solution of its production problem.[8]

Example 3.

As a variant of Example 2, assume that the cost vector is $b = \begin{pmatrix}160\\200\end{pmatrix}$; in other words the first mine now has a lower daily cost than the second. By the same procedure as above we find that the minimum cost level is again \$680 and is achieved by operating the first mine three days a week and the second mine one day per week. In this solution 20 tons of high-grade ore, instead of the required 12 tons, are produced, while the requirements of medium- and low-grade ores are exactly met. Thus eight tons of high-grade ore must be discarded per week.[9]

---

[8] Ibid., p. 380-381.

[9] Ibid., p. 382.

Example 4.

As another variant of Example 2, assume that the cost vector is $b = \begin{pmatrix} 200 \\ 200 \end{pmatrix}$; in other words, both mines have the same production costs. Evaluating the cost function wb at the extreme points of the convex set we find costs of \$1200 on two of the extreme points (T_1 and T_4) and costs of \$800 on the other two extreme points (T_2 and T_3). Thus the minimum cost is attained by operating either one of the mines three days a week and the other one day a week. But there are other solutions, since if the minimum is taken on at two distinct extreme points it is also taken on at each of the points on the line segment between. Thus any vector w where $1 \leq w_1 \leq 3$, $1 \leq w_2 \leq 3$, and $w_1 + w_2 = 4$ also gives a minimum-cost solution. For example, each mine could operate two days a week.[10]

Example 5.

With the formula given for the sum of n terms of an arithmetic sequence as $n/2 \left[2\ t_1 + (n-1)\ d\right]$, where t_1, is the first term and d is the common difference.

An incentive plan for executives permits individuals to purchase 100 shares of the company's stock when the executive first becomes eligible for participation in the plan. Thereafter, if he has exercised the initial option, the participating executive may purchase 110 shares at the end of the first year, 120 shares the next year, 130 the next, and 140 at the end of the fourth year, etc., until he has accumulated a maximum total of 1,000 shares. How many years will be required to reach

[10]*Ibid.*, p. 382.

the maximum?

Solution: Here $S_n = 1{,}000$, $t_1 = 100$, $d = 10$ and substituting

$$1{,}000 = \frac{n}{2}\left[2(100) + (n - 1)10\right]$$

$$1{,}000 = 100n + 5n^2 - 5n$$

$$5n^2 + 95n - 1{,}000 = 0$$

or

$$n^2 + 19n - 200 = 0.$$

Solving by use of the quadratic formula,

$$n = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where $a = 1$, $b = 19$, and $c = -200$,

$$n = \frac{-19 \pm \sqrt{1161}}{2}$$

Thus

$$n \approx 7.54 \quad \text{or} \quad n \approx -26.54.$$

Since a negative or noninteger solution is meaningless for this example, the maximum will be reached on the eighth purchase, or at the end of the seventh year. To determine how many shares may be purchased on the last option we calculate the sum of the first seven purchases:

$$S_7 = \frac{7}{2}\left[2(100) + (6)(10)\right] = 910.$$

Thus the last purchase is 1,000 - 910 = 90 shares.[11]

Example 6.

(a) In an attempt to decrease his inventory a paint manufacturer offers to sell the first case for \$12. If two cases are ordered, a dis-

[11]Youse, Calculus for Students, p. 46.

count of \$0.50 is applied to the second case, and an additional \$0.50 discount on each succeeding case. What is the total cost of an order for 20 cases?

Solution: For this example $n = 20$, $t_1 = 12$, $d = -0.50$, and

$$S_{20} = \frac{20}{2}\left[2(12) + (19)(-0.50)\right] = \$145,$$

the total cost of 20 cases. The price of the 20th case is

$$t_{20} = 12.00 + (19)(-0.5) = \$2.50.$$

(b) Obviously, the manufacturer should put an upper bound on the number of cases a customer can buy so that the buyer will not be able to claim "free cases." To find the limit on purchases that the manufacturer should set, we want to find the largest n such that $t_n > 0$. Thus

$$t_n = 12.00 + (n - 1)(-0,50) > 0$$

$$12.00 > (0.50)(n - 1)$$

$$1,200 > 50n - 50$$

$$50n < 1,250$$

$$n < 25.$$

A maximum of 24 cases should be sold to a customer.[12]

Example 7.

Federal laws require banks subject to regulation by the government to maintain as a reserve a certain proportion of the bank's current deposits. Cash in excess of this reserve may be loaned to the bank's borrowers. Assuming the reserve rate is 25 percent and that the bank requires its borrowers to deposit the amount loaned, find the total

[12] *Ibid.*, p. 46-47.

amount made available for loans as a result of a \$500 deposit.

Solution: From the original deposit, the bank may loan (500) (0.75) = 375. Assuming unlimited demand for loans, the bank may now loan $(375)(0.75) = [(500)(0.75)](0.75) = (500)(0.75)^2 = 281.25$, and then $(500)(0.75)^3 = 211.31$. This is clearly a geometric sequence. Since $(500) \cdot (0.75)^n \neq 0$ for any integer n, there is no upper bound of n. We note that the sequence is decreasing and can be made arbitrarily close to zero by choosing n large enough; thus, assuming that no loan will be made for less than \$1, we find the number of loans required to reach this minimum by letting

$$r^{n-1}t_1 = 1.$$

Thus

$$(0.75)^{n-1}(500) = 1$$

and taking logarithms of both sides,

$$(n - 1) \log 0.75 + \log 500 = 0$$

$$(n - 1)(9.87506 - 10) = -2.69877$$

$$(n - 1)(-0.12494) = -2.69877$$

$$0.12494n = 2.82371$$

$$n \approx 22.6.$$

Since n must be an integer, we find the total amount available for loan for the first 22 repetitions of the cycle. The total amount available is

$$S_{22} = \frac{500 - 500(0.75)^{22}}{1 - 0.75}$$

$$= \frac{500\left[1 - (3/4)^{22}\right]}{1/4}$$

$$= 2{,}000\left[1 - (3/4)^{22}\right]$$

$$\approx 2{,}000.$$

The total amount is just under \$2,000. It should be evident that since $(3/4)^n$ can be made arbitrarily "close" to 0 for "large" n the total amount could not exceed \$2,000 even with an unlimited number of loans.[13]

Example 8.

Capital assets, such as a factory, which will be used for several years cannot be considered as an expense at the time they are purchased. Instead, periodic charges call <u>depreciation</u> are made for the use of such assets. Many methods are available for computing depreciation charges, the simplest being the <u>straight-line method</u>. If C is the total cost of a capital asset and a fixed integer n is the estimated life, in years, of the asset, the annual straight-line depreciation charge is C/n. Thus, the <u>total</u> depreciation charge at the end of the first year is $S_1 = C/n$, at the end of the second year is $S_2 = 2C/n$, and at the end of the nth year $S_n = nC/n = C$. The total depreciation is directly proportional to time and the constant of proportionality is C/n. The points on the graph of this arithmetic series $(1, C/n)$, $(2, 2C/n)$, $(3, 3C/n)$,... lie on a line through the origin with slope C/n.

As an alternative, one may use a double-declining method. For this method, the annual rate of depreciation is taken as twice the straight-line rate on the remaining value of the assets, if certain conditions are met. Under this method, depreciation for the first year is $C\left(\frac{2}{n}\right)$; for the

---

[13] <u>Ibid</u>., p. 48.

second,

$$\left[C - C\frac{2}{n}\right]\frac{2}{n} = C\left(\frac{2}{n}\right)\left(1 - \frac{2}{n}\right);$$

for the third,

$$\left[C - C\left(\frac{2}{n}\right) - C\left(\frac{2}{n}\right)\left(1 - \frac{2}{n}\right)\right]\frac{2}{n} = C\left[\left(1 - \frac{2}{n}\right) - \frac{2}{n}\left(1 - \frac{2}{n}\right)\right]\frac{2}{n}$$

$$= C\left(1 - \frac{2}{n}\right)^2\frac{2}{n};$$

and for the kth year the allowable charge is

$$C\left(1 - \frac{2}{n}\right)^{k-1}\frac{2}{n}.$$

The series representing the total depreciation allowed for the first k years is

$$S_k = C\frac{2}{n} + C\left(1 - \frac{2}{n}\right)\frac{2}{n} + C\left(1 - \frac{2}{n}\right)^2\frac{2}{n} + \ldots + C\left(1 - \frac{2}{n}\right)^{k-1}\frac{2}{n},$$

a geometric series with common ratio $r = \left(1 - \frac{2}{n}\right)$ and $t_1 = C\left(\frac{2}{n}\right)$. Thus

$$S_k = \frac{C\frac{2}{n} - C\frac{2}{n}\left(1 - \frac{2}{n}\right)^k}{1 - \left(1 - \frac{2}{n}\right)} = C\left[1 - \left(1 - \frac{2}{n}\right)^k\right],$$

can be used to calculate any year.[14]

---

[14]*Ibid.*, p. 49.

CHAPTER 4

CHAPTER 7-9 INTRODUCTION TO DIFFERENTIAL CALCULUS

The Rate of Change of a Function · The Limit Concept · Definition of the Derivative · Interpretations of the Derivative · The Derivative as a New Function · The Power Rule and Some Properties of the Derivative. Differentiation of Exponential and Logarithmic Functions · Geometric and Economic Applications of the Derivative

ADDITIONAL DIFFERENTIATION TECHNIQUES AND APPLICATIONS

The Behavior of the Dependent Variable in a Differentiable Function · The Chain Rule · Some General Differentiation Formulas · Summary of Differentiation Techniques · Applications of the Derivative · Definition and Application of Differentials

MAX-MIN THEORY AND APPLICATION

The First Derivative Test for Maxima and Minima · The Second Derivative and Inflection Points · The Second Derivative Test for Maxima and Minima · Summary of Maxima, Minima, and Inflection Points · Applied Max-Min Problems

Example 1.

Assume, for example, that the regional sales manager of a firm has reached the following judgment about assigning salesmen to a given territory: Each salesman assigned can exploit one half of the untapped sales potential of the territory. This judgment is based on the manager's knowledge of the potential customers, their locations throughout the territory, and the attractiveness of customers in the territory to his competitors. The expected contribution of each additional salesman is presented in Table 1. With the assignment of enough salesmen, the regional manager can be assured of realizing essentially all the sales potential of the territory. But there is a limit to the sales he can expect; he can expect no more than 100 percent of the potential. Furthermore he can never (quite) attain that level, since each new salesman adds only one half the remaining potential. Thus, no matter how many salesmen are added, some small percentage of potential cannot be obtained; 100% of potential cannot be reached.[15]

Table 1

Proportion of Sales Potential Contributed by Each Additional Salesman and Cumulative Total (Percent)

Salesman Number	Proportion of Total Potential Added by the Salesman	Cumulative Proportion of Potential Sales[a]
1	.5000000	.5000000
2	.2500000	.7500000
3	.1250000	.8750000
4	.0625000	.9375000
5	.0312500	.9687500
6	.0156250	.9843750
7	.0078125	.9921875
⋮		
n	$(0.5)^n$	$\sum_{i=1}^{n}(0.5)^n$
⋮		

[15]Alan K. McAdams, Mathematical Analysis for Management Decisions: Introduction to Calculus and Linear Algebra (London: The Macmillian Company, 1970).

Example 2.

In this example a very interesting function is presented. Assume that the capacity output of a given machine is 1000 units per month, and that each machine costs \$500 to purchase. The function that describes the machine-costs for different output levels each month is the function f with the following rule:

$$y = f(x) = \begin{cases} \$\,500 \text{ for } 0 < x \leq 1000 \\ \$1000 \text{ for } 1000 < x \leq 2000 \\ \$1500 \text{ for } 2000 < x \leq 3000 \\ \text{and so on.} \end{cases}$$

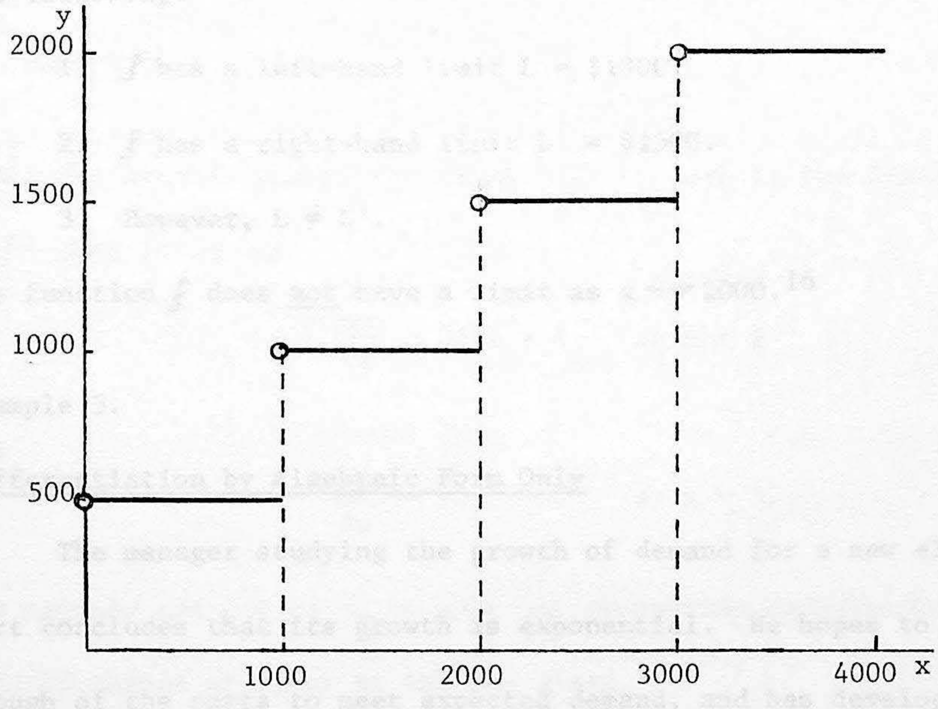


Figure 2

Cost of Machines for Different Output Levels.

Does the function have a limit as $x \to 2000$? The function is shown graphically in Figure 2. As $x \to 2000$ from the left, the function f has a value of \$1000 for all values up to and including 2000 units; thus the candidate limit of \$1000 is established. It meets the requirements for a left-hand limit (it is equal to \$1000 everywhere in the interval $1000 < x \leq 2000$); however, there is <u>no</u> value of x to the right of <u>x</u> = 2000 for which the value $f(x) = \$1000$. In fact, for every value $2000 < x \leq 3000$, the value of f equals \$1500, so that has a right-hand limit L' equal to \$1500. The results of the limit test would show the following.

1. f has a left-hand limit L = \$1000.
2. f has a right-hand limit L' = \$1500.
3. However, $L \neq L'$.

The function f does <u>not</u> have a limit as $x \to 2000$.[16]

Example 3.

<u>Differentiation by Algebraic Form Only</u>

The manager studying the growth of demand for a new electronic part concludes that its growth is exponential. He hopes to produce enough of the parts to meet expected demand, and has developed a procedure for doing so which depends on predictions of the rate at which sales are changing at key times. The two times that are of most interest to him are the second and third years. The rule for the function f that he feels is appropriate is $y - x^2$. He needs to find the rule for f' and evaluate

[16] <u>Ibid</u>., p. 48-49.

it when x = 2 and x = 3.

Solution:

Substituting $x + \Delta x$ for x gives $y + \Delta y$. Then

1. $y + \Delta y = (x + \Delta x)^2$.

Subtracting the value of y from $y + \Delta y$ gives

2. $y + \Delta y - y = \Delta y = (x^2 + 2x\Delta x + \Delta x^2) - x^2$.

Dividing both sides of the equation by Δx gives

3. $\dfrac{\Delta y}{\Delta x} = \dfrac{2x\Delta x + \Delta x^2}{\Delta x} = 2x + \Delta x$.

Taking the limit of both sides as $\Delta x \to 0$ gives

4. $\lim\limits_{\Delta \to 0} \dfrac{\Delta y}{\Delta x} = \lim\limits_{\Delta \to 0} (2x + \Delta x) = 2x$.

since the second term on the right goes to zero in the limit. The value of f' then is 2x and

$$\frac{dy}{dx} = 2(2) = 4 \quad \text{at } x = 2$$

and

$$\frac{dy}{dx} = 2(3) = 6 \quad \text{at } x = 3.$$

The manager can now go on to make his production decision based on the rate of change of sales at the key times.[17]

---

[17] Ibid., p. 66-67.

Example 4.

Assume that in the absence of advertising for its product a company faces a demand function given by

$$q = 200 - 4p$$

where q = quantity sold and p = price per unit, and additional net revenues r can be generated through advertising expenditures A. Assume that the (unlikely) equation

$$r = (5pA - 3A^2 + 50A)$$

expresses the net change in revenues resulting from changes in the advertising budget A. The negative term $-3A^2$ reflects the cost implications of the advertising expenditures. The cost directly associated with production (and sale) of the product exclusive of advertising costs is given by

$$c = 300 + 3q$$

then

$$\text{Total (basic) revenue } r = p \cdot q = p(200 - 4\,p)$$

$$\text{Total cost} \qquad c = (300 + 3q)q$$

$$c = 300q + 3q^2.$$

Since q = 200 - 4p, substituting for q gives

$$c = 60{,}000 - 1200p + 120{,}000 - 4800p + 48p^2$$

$$c = 180{,}000 - 6000p + 48p^2.$$

The profits of the firm π can be stated as the difference between revenues and costs, including net revenues associated with advertising; that is,

$$\pi = \text{revenue} - \text{cost} + \text{advertising}$$

$$\pi = (200p - 4p^2) + (-180{,}000 + 6000p - 48p^2) + 5pA - 3A^2 + 50A.$$

To find the maximum level of π, the first partial derivatives must be taken and set equal to zero:

$$\frac{\partial \pi}{\partial p} = 200 - 8p + 6000 - 96p + 5A = 0$$

$$\frac{\partial \pi}{\partial A} = 5p - 6A + 50 = 0.$$

Simplifying gives

$$6200 - 104p + 5A = 0$$

$$50 + 5p - 6A = 0.$$

Multiplying the first equation above by 6 and the second equation by 5 and adding gives

$$\begin{array}{r} 37{,}200 - 624p + 30A = 0 \\ 250 + 25p - 30A = 0 \\ \hline 37{,}450 - 599p \qquad = 0 \end{array}$$

then

$$p = \frac{37{,}450}{599}$$

$$p = \$62.50$$

$$A = \frac{50 + 312.50}{6} = \frac{362.50}{6} = \$60.42.$$

Thus \$62.50 appears to be the profit-maximizing price—when used in conjunction with an expenditure of approximately \$60 on advertising. The two straight second partial derivatives are:

$$\frac{\partial^2 \pi}{\partial p^2} = -104, \ \frac{\partial^2 \pi}{\partial A^2} = -6$$

and both are negative, therefore, meeting one condition. The second cross-partial derivative is

$$\frac{\partial^2 \pi}{\partial p\, \partial A} = 5.$$

Therefore, the second condition is also met:

$$\left[\frac{\partial^2 \pi}{\partial p^2}\right]\left[\frac{\partial^2 \pi}{\partial A^2}\right] > \left[\frac{\partial^2 \pi}{\partial p\, \partial A}\right]^2$$

$$(-104)(-6) > (5)^2$$

$$624 > 25.$$

The second-order conditions show the figures to be a maximum and not a minimum or a saddle point.[18]

Example 5.

The total cost curve of a commodity is $y = 2x - 2x^2 + x^3$, where y represents total cost and x represents quantity. Suppose that market conditions indicate that between 3 and 10 units should be produced (that is, $3 \leq x \leq 10$). For what quantity in this interval is average cost a minimum (see Fig. 3)?

---

[18] *Ibid.*, p. 129-130.

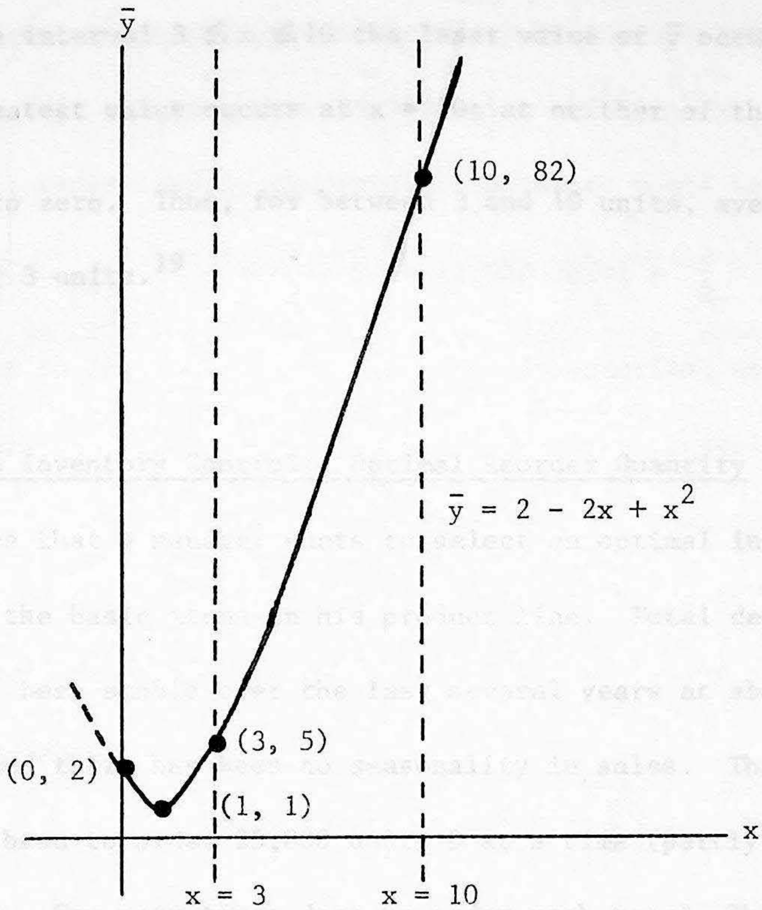


Figure 3

Average cost $= \frac{y}{x} = \bar{y} = 2 - 2x + x^2$.

$$\frac{d\bar{y}}{dx} = -2 + 2x$$

$$= 0 \quad \text{if } x = 1$$

$$\left.\begin{array}{l} < 0 \quad \text{if } x < 1 \\ > 0 \quad \text{if } x > 1 \end{array}\right\} \text{ so minimum at } x = 1$$

But $x = 1$ is outside the interval $3 \leq x \leq 10$.

If $x = 3$, $\bar{y} = 5$

if $x = 10$, $\bar{y} = 82$

Thus in the interval $3 \leq x \leq 10$ the least value of $\bar{y}$ occurs at $x = 3$ and the greatest value occurs at $x = 10$; at neither of these points is $\frac{d\bar{y}}{dx}$ equal to zero. Thus, for between 3 and 10 units, average cost is minimum for 3 units.[19]

Example 6.

Problems in Inventory Control: Optimal Reorder Quantity

Assume that a manager wants to select an optimal inventory policy for one of the basic items in his product line. Total demand Q for the product has been stable over the last several years at about 100,000 units per year, and there has been no seasonality in sales. The manager's policy has been to order 25,000 units D at a time (partly because he hates paper work). How many times does he order each year? This can be calculated from:

$$\frac{\text{Total demand}}{\text{Amount delivered each time}} = \frac{Q}{D} = \frac{100{,}000}{25{,}000} = 4 \text{ times.}$$

To decide on the number of times he should order each year, the manager must know the costs that are influenced by his inventories and the magnitudes by which these costs vary. The relevant costs are: the costs of carrying the inventory once he has it, and the costs of ordering when he runs out.

1. Carrying costs. If demand for an item is stable, the carrying costs can be approximated by the cost per unit multiplied by the average number of units held in the inventory. These costs consist of the ware-

---

[19] Jean E. Draper, and Jane S. Klingman, Mathematical Analysis: Business and Economic Applications (Harper and Row, 1972), p. 235.

If k were equal to \$8 per thousand units, then the carrying cost would be

$$(12.5)(\$8) = \$100.$$

2. Order costs. The number of orders currently placed during the year is 4 = 100/25. More generally, if Q units are to be sold during the year and D units are delivered for each order, the required number of deliveries is Q/D. Suppose that the cost of ordering is related to the amount delivered by the expression a + bD, where a = \$60 and b = \$3 per thousand units. Here b may be interpreted as the shipping cost per thousand units so that the cost of sending D items is b(D) dollars, that is, \$3 x 25 = \$75; a represents costs such as bookkeeping and long-distance telephoning for orders—in other words, costs whose magnitude is not affected by the amount involved in the shipment. With a = \$60 and b = \$3 per thousand, the cost per delivery, a + bD, would total \$135.

The total annual ordering costs equals the number of deliveries (Q/D) multiplied by the cost per delivery a + bD;[1]

$$\text{Total annual ordering cost} = \frac{(a + bD)Q}{D} = \frac{aQ}{D} + \frac{bQD}{D} = \frac{aQ}{D} + bQ.$$

The total cost C, which results from the manager's inventory policy is the sum of the two costs: the carrying costs and the ordering costs. It is equal to

$$C = \frac{kD}{2} + \frac{aQ}{D} + bQ.^{20} \qquad (4.1)$$

---

[20] McAdams, Mathematical Analysis, p. 100-101.

Example 7.

Finding the Minimum Cost Inventory Policy

The only unknown in equation (4.1) for the total annual cost associated with inventory is the value of D, the number of items to be delivered per shipment. Once the number of items that leads to the minimum cost for the year is determined, all other factors are determined, because the corresponding average inventory level D/2 is immediately available and the number of shipments per year must be Q/D.

Equation (4.1) is the rule for the function that specifies the total cost for each value of the variable D. For example, given Q = 100 (thousand), k = 8, a = 60, and b = 3, then equation (4.1) becomes

$$C = \frac{8D}{2} + \frac{60 \times 100}{D} + 3 \times 100$$

or

$$C = 4D + \frac{6000}{D} + 300. \qquad (4.2)$$

If D were equal to 25 (thousand), then

$$C = 100 + 240 + 300 = 640 \qquad \text{(that is, } 100 + 540\text{)}.$$

There is no assurance that D = 25 is the minimum cost for the year's inventory. However, with equation (4.2), the value of D that minimizes costs can be approximated. Thus, determining the inventory-cost equation provides the means for solving the problem. The objective is to minimize cost. The function that specifies cost is thus called the <u>objective function</u>. It shows how the firm's costs are affected by different values of the relevant variables. This is illustrated by several methods; first, by a series of trial calculations; second, graphically; and then through the use of the calculus. A formula for the least cost order quantity is derived from the calculus solution.

To find the approximate value of minimum costs one can simply take a number of alternative values of D, substitute them in turn into the equation, compute the corresponding values of C, and thus find, roughly, the value of D that gives the lowest cost. This is done in Table 4.1.

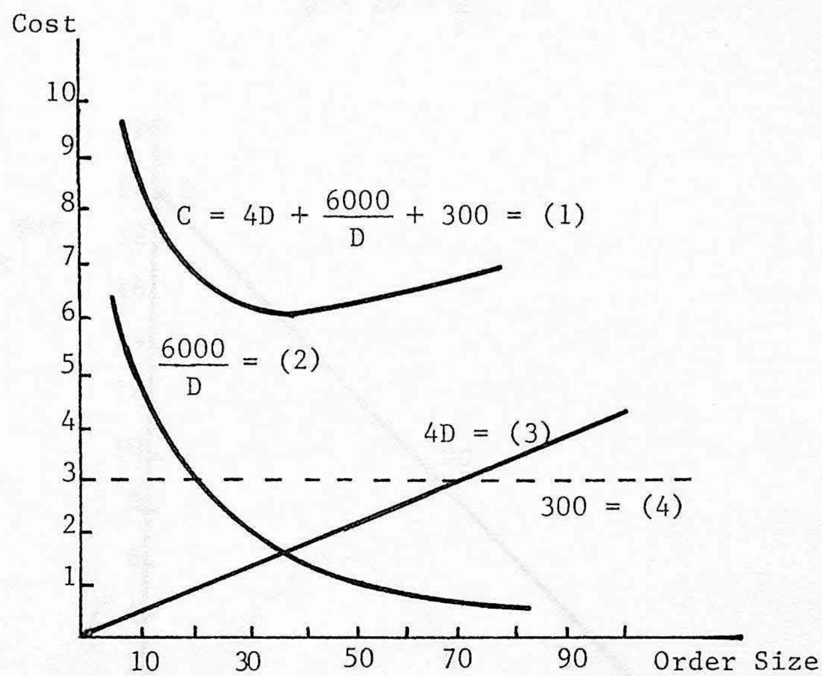
Table 4.1

Total Cost as a Function of Amount Ordered

D	10	20	30	40	50	60	70	80	...
C	940	680	620	610	620	640	666	695	...

From this table, it can be seen that a value of D of approximately 40 (thousand) units per delivery minimizes costs. In effect, then, the inventory problem has now been solved. However, some additional analysis will make it possible to extract a great deal of additional information from the equation (and perhaps clarify some of the concepts presented in the previous examples).

The objective function, equation (4.2) and its components, can be analyzed graphically as shown in Figure 4.2. The carrying costs are represented by the straight line 4D in this diagram. They increase in direct proportion to the size of D. The order costs are made up of two components: a component that does not vary with D, at $300, and a continually decreasing total, variable ordering cost, indicated by the (hyperbolic) function 6000/D. The total costs from Table 4.1 are also plotted. Note that the minimum total cost falls just slightly to the left of the point at which D = 40.



(1) Total Inventory Costs
(2) Total Variable Ordering Costs
(3) Total Carrying Costs per Year
(4) Total Fixed Ordering Costs[21]

Figure 4.2

Cost Components for Inventories: (1) Total Inventory Costs. (2) Total Variable Ordering Costs. (3) Total Carrying Costs. (4) Total Fixed Ordering Costs.

---

[21] *Ibid.*, p. 102-103.

Example 8.

Problem of Profit Maximizing: Quantity to Order

The demand curve DD relates the price of goods to the quantity of goods that would be purchased at a given price, as specified in equation (4.6) where p is the price of the product and q is the quantity demanded

$$p = 20 - \frac{q}{5}. \qquad (4.6)$$

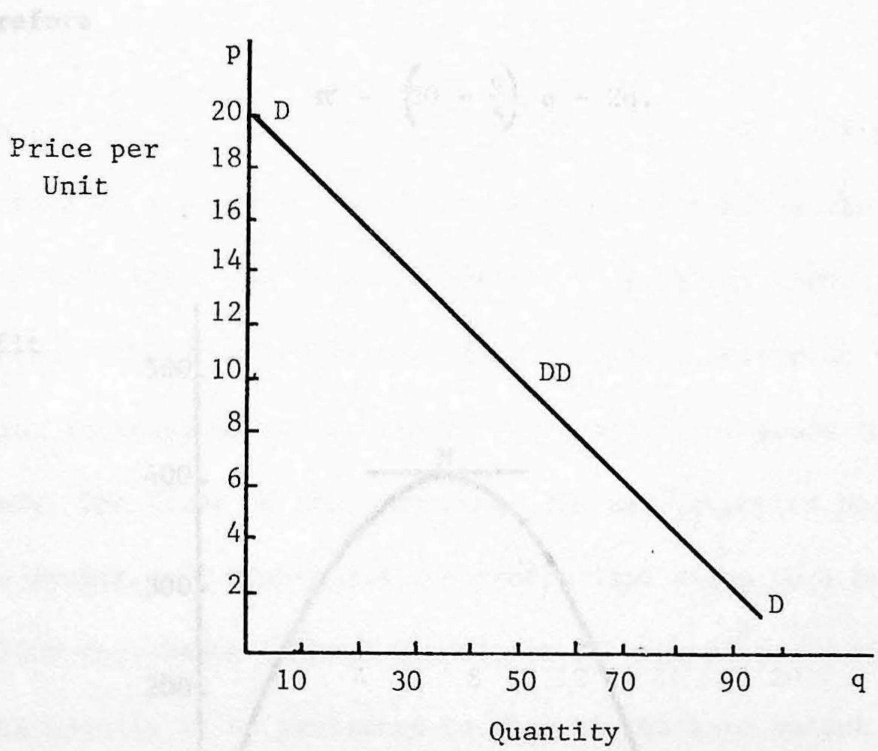


Figure 4.3

Demand Curve DD for a Given Product.

The demand curve is the line DD in Figure 4.3. It has the negative slope characteristic of a monopolist's demand curve; with a decrease in price goes an increase in the quantity demanded.

Assume that the product costs \$2 per unit to produce. Since profit is equal to revenue—cost, an equation for the total profit of the firm can be written as

$$\pi = r - c$$

where π is profit, r is revenue, and c is cost. Since

$$r = p \times q \quad \text{and} \quad c = 2q$$

but

$$p = 20 - \frac{q}{5}$$

therefore

$$\pi = \left(20 - \frac{q}{5}\right) q - 2q. \tag{4.7}$$

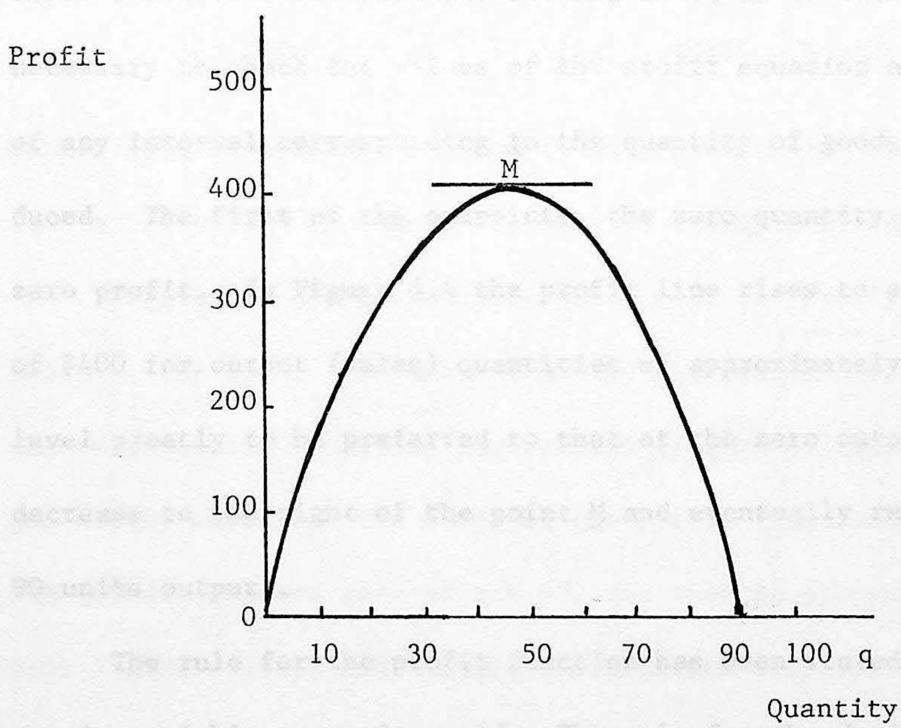


Figure 4.4

Profits in Relation to Total Quantity Sold.

The total-profit equation (4.7) has been developed in terms of total revenue minus total costs, with the quantity sold being a function of the price at which the product is sold. It can be simplified further to

$$= 20q - \frac{q^2}{5} - 2q,$$

and

$$= 18q - \frac{q^2}{5}. \qquad (4.8)$$

This profit equation (4.8) can be plotted as shown in Figure 4.4. Its relative maxima and minima can be located by finding the rule for the first derivative function and setting it equal to zero. It is also necessary to check the values of the profit equation at the endpoints of any interval corresponding to the quantity of goods that can be produced. The first of the endpoints, the zero quantity point, results in zero profit. In Figure 4.4 the profit line rises to a height in excess of $400 for output (sales) quantities of approximately 40-50 units, a level greatly to be preferred to that at the zero output point. Profits decrease to the right of the point M and eventually reach zero again (at 90 units output).

The rule for the profit function has been stated in terms of the domain variable, quantity sold. The rule for the derivative function is

$$\frac{d}{dq}(\pi) = \frac{d}{dq}\left(18q - \frac{q^2}{5}\right) = 18 - \frac{2}{5}q. \qquad (4.9)$$

This is set equal to zero:

$$18 - \frac{2}{5}q = 0$$
$$q = (18)\left(\frac{5}{2}\right) = 45.$$

There is a single value of q at which the value of the first derivative is zero; this value is q = 45. From Figure 4.4 the function is seen to reach a maximum value at q = 45, the point M. This can also be demonstrated mathematically through the use of the second derivative function. The rule for the second derivative function is

$$\frac{d}{dq}\left(\frac{d}{dq}\right) = \frac{d}{dq}\left(18 - \frac{2}{5}q\right) = -\frac{2}{5}. \qquad (4.10)$$

Thus for <u>all</u> values of q, including q = 45, the second derivative is negative; that is, the rate of change of the slope of the function is everywhere negative. This means that the extreme point at q = 45 is a maximum point. The value of the profit equation (4.8) at q = 45 is

$$\pi = 18(45) - \frac{(45)^2}{5} = 18(45) - 9(45) = 9(45) = 405.$$

Since, from equations (4.9) and (4.10), profit must be everywhere decreasing for values of $q > 45$, then (45, 405) is the (global) maximum profit level.

Figure 4.5 shows the three functions, the profit function (Figure 4.5(a)), the first derivative function for profits (Figure 4.5(b)) and the second derivative function for profits (Figure 4.5(c)). Figures 4.5(a) and 4.5(b) show that the slope of the profit function is positive for $0 \leq q \leq 45$, becomes zero at q = 45, and then is everywhere negative. From Figures 4.5(b) and 4.5(c), the negative slope of the first derivative function is verified. Implicit in the foregoing discussion is the economist's widely known decision rule: produce (and sell) that output for which the marginal cost equals the marginal revenue.

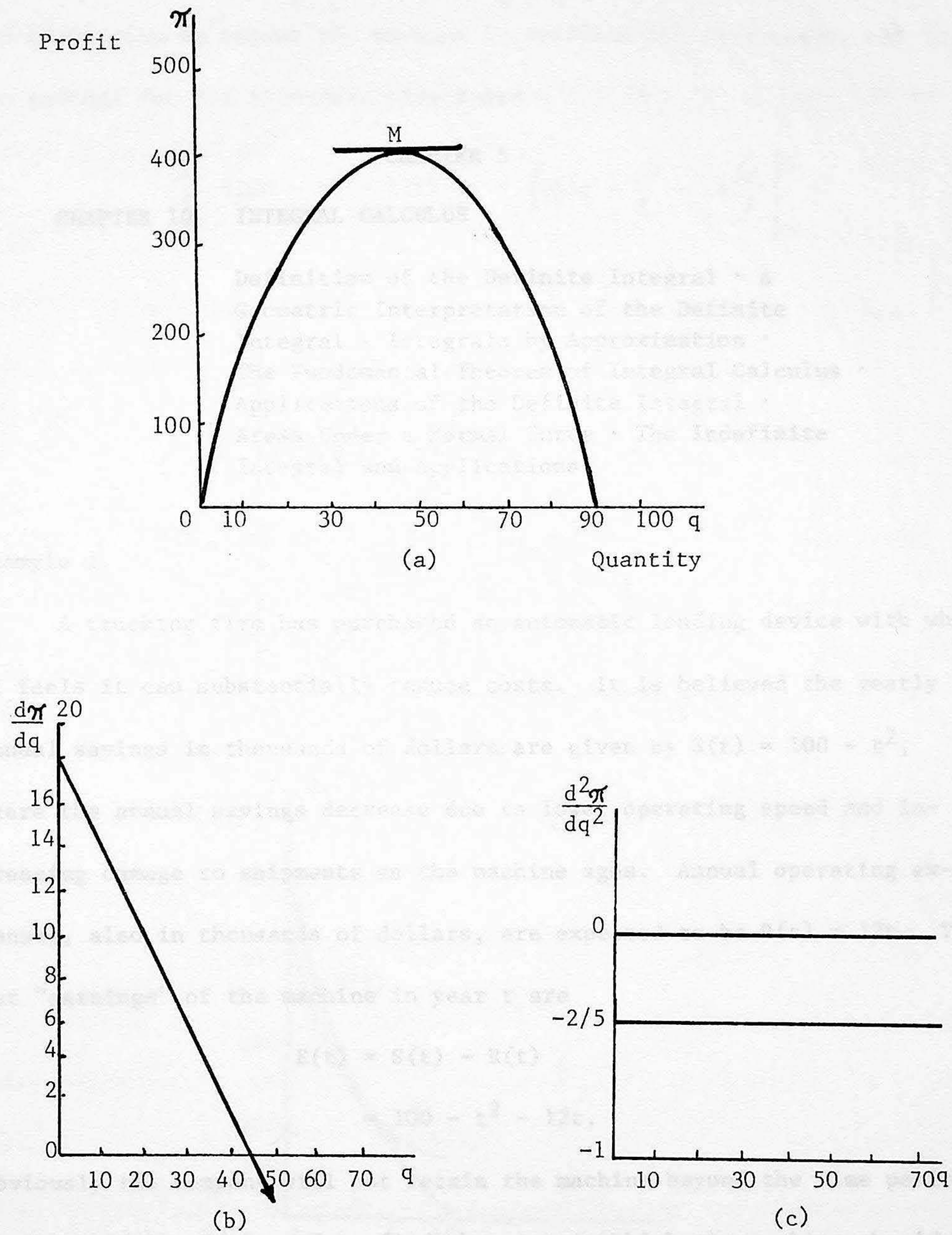


Figure 4.5

(a) Profits in Relation to Total Quantity Sold. (b) Rate of Change of Profits with Changes in q. (c) Second Derivative Function.[22]

---

[22] Ibid., p. 106-109.

CHAPTER 5

CHAPTER 10 INTEGRAL CALCULUS

Definition of the Definite Integral · A Geometric Interpretation of the Definite Integral · Integrals by Approximation · The Fundamental Theorem of Integral Calculus · Applications of the Definite Integral · Areas Under a Normal Curve · The Indefinite Integral and Applications

Example 1.

A trucking firm has purchased an automatic loading device with which it feels it can substantially reduce costs. It is believed the yearly annual savings in thousands of dollars are given by $S(t) = 100 - t^2$, where the annual savings decrease due to lower operating speed and increasing damage to shipments as the machine ages. Annual operating expenses, also in thousands of dollars, are expected to be $R(t) = 12t$. The net "earnings" of the machine in year t are

$$E(t) = S(t) - R(t)$$

$$= 100 - t^2 - 12t.$$

Obviously the company will not retain the machine beyond the time period in which $S(t) = R(t)$, and to find the year in which the machine should be scrapped, we solve

$$100 - t^2 = 12t$$

$$t = 5.65.$$

For simplicity we assume the machine is retained for five years, and total net savings for the five-year period are

$$\int_0^5 (100 - t^2 - 12t)\,dt = \left(100t - \frac{t^3}{3} - 6t^2\right)\Bigg|_0^5$$

$$= 308.333.$$ [23]

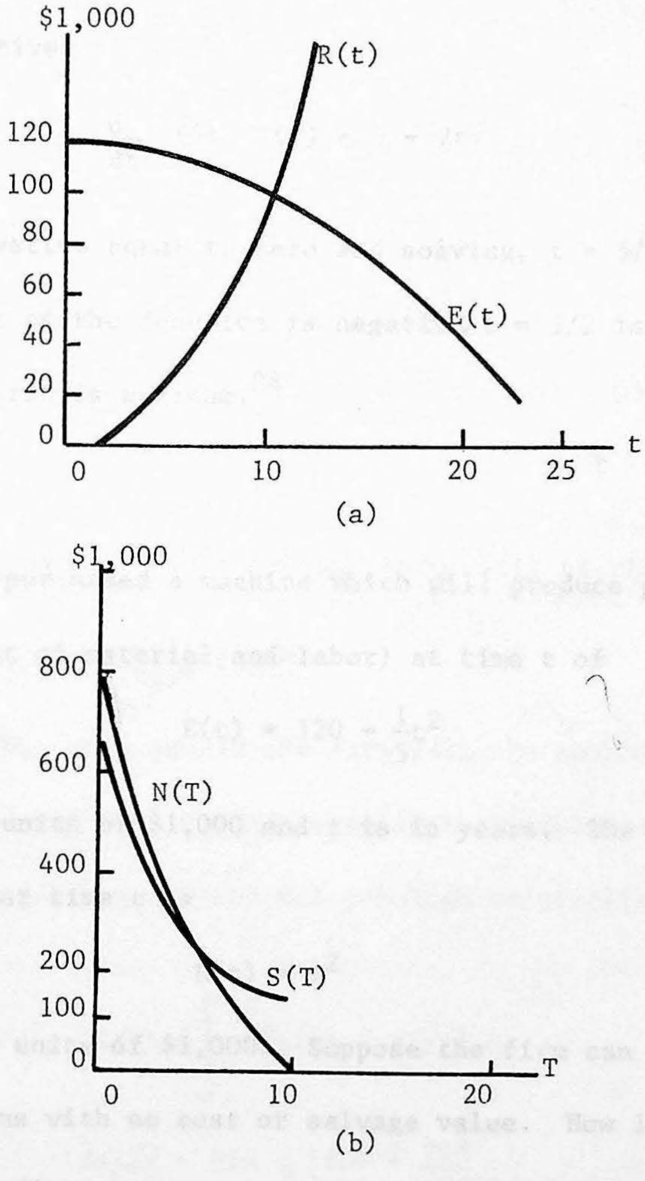


Figure 1

(a) Earnings function E(t) and repair costs R(t) for data in Example 3. (b) Total future earnings N(T) and salvage values S(T) for data in Example 3.

---

[23] Youse, Calculus for Students, p. 190.

Example 2.

The total earnings from the introduction of a new product are expected to be given by

$$\int_0^T (-2x + 5)\,dx = (-x^2 + 5x)\Big|_0^T = 5t - t^2$$

To find the time period in which profit is maximum we apply the techniques of derivatives

$$\frac{d}{dt}\,(5t - t^2) = 5 - 2t.$$

Setting the derivative equal to zero and solving, $t = 5/2$. Since the second derivative of the function is negative, $t = 5/2$ is the point in time at which profit is maximum.[24]

Example 3.

A firm has purchased a machine which will produce gross earnings (revenue less cost of material and labor) at time t of

$$E(t) = 120 - \frac{1}{5}t^2$$

where E(t) is in units of $1,000 and t is in years. The repair and maintenance cost at time t is

$$R(t) = t^2$$

again measured in units of $1,000. Suppose the firm can dispose of the machine at any time with no cost or salvage value. How long should the machine be operated?

The two functions E(t) and R(t) are shown in Figure 1. E(t) is a decreasing concave function for $t > 0$ and R(t) is an increasing convex

---

[24] Ibid., p. 190-191.

function. The value of t at which the curves cross is the solution of

$$E(t) = R(t)$$

$$120 - \frac{1}{5}t^2 = t^2 \quad \text{or} \quad t = \pm 10$$

Since after ten years the repair costs exceed the gross earnings, the firm will maximize net earnings if the machine is disposed of at the end of ten years. The total net earnings are

$$\int_0^{10} [E(t) - R(t)]\, dt = \int_0^{10} 120 - \frac{1}{5}t^2 - t^2 \; dt = \int_0^{10} [120 - \frac{6}{5}t^2]\, dt$$

$$= \left[120t - \frac{6}{5}\frac{t^3}{3}\right]_0^{10}$$

$$= 1{,}200 - \frac{6000}{15} = 1{,}200 - 400 = 800$$

or \$800,000.

Suppose the machine has a salvage value S(t) at time t where

$$S(t) = \frac{2{,}000}{3 + t}$$

in units of \$1,000. When should the firm sell the machine? The firm will maximize its total net earnings if it sells the machine at a time T when the salvage value just equals the net earnings obtainable after T; i.e., when

$$S(T) = \int_T^{10} [E(t) - R(t)]\, dt$$

$$\frac{2{,}000}{3 + T} = 800 - 120T + \frac{2T^3}{5}$$

The solution is T = 5. S(T) is a monotonically decreasing function with S(0) = 666.67 and S(10) = 153.84. Similarly, the net earnings from year T to year 10; i.e.,

$$N(T) = 800 - 120T + \frac{2T^3}{5}$$

is a monotonically decreasing function with N(0) = 800 and N(10) = 0. Before five years the salvage value will be less than the potential future net earnings and after five years the salvage value will be greater. The optimum time to sell, therefore, is at the end of five years.[25]

Example 4.

Given the functions c with rule $c(q) = 4 + (q - 4)^2$ for the marginal cost and r with rule $r(q) = 20 - 2q$ for the marginal revenue of a firm where q is the quantity of product produced, find:

(a) The profit-maximizing output for the firm.

(b) The total profit for the firm at that output.

(c) The profit that would be achieved if output were increased by two units beyond the profit maximizing output.

(d) The rule for the demand function for the firm.

(e) The rule for the average cost function of the firm.

(a) Profit maximization requires that r = c, or r - c = 0. Because profit equals total revenue minus total cost, as derived in Chapter 4 from the profit function, then

$$r - c = 0$$

$$(20 - 2q) - [4 + (q - 4)^2] = 0$$

Solving for q, this gives:

---

[25]Daniel Teichroew, An Introduction to Management Science (New York: John Wiley and Sons, Inc., 1964), p. 216-218.

$$20 - 2q - (4 + q^2 - 8q + 16) = 0$$

$$6q - q^2 = 0$$

$$q(6 - q) = 0$$

$$q = 0, 6.$$

The maximum profit output occurs at one (or both) of these points. Equation (7.22) is the first derivative $d\pi/dq$ of the profit function. The second derivative is found by taking its derivative in turn:

$$\frac{d^2\pi}{dq^2} = \frac{d}{dq}(6q - q^2)$$

$$\frac{d^2\pi}{dq^2} = 6 - 2q.$$

At $q = 0$, it equals 6 identifying this as a minimum. At $q = 6$, it equals -6; an output of 6 units will provide maximum profit.

(b) Total revenue is found from the integral of marginal revenue, and total cost is found from the integral of marginal cost. Profit is the difference between total revenue and total cost. Graphically it is the area between the curve for marginal profit and the curve for marginal revenue shown in Figure 2.

Total profit π_1 is thus given by the integral of (area under) r minus the integral of (area under) c for the appropriate limits of integration, in Figure 2 from $a = 0$ to $b = 6$. Thus,

$$\pi_1 = \int_a^b r\,dq - \int_a^b c\,dq = \int_0^6 (20 - 2q)\,dq - \int_0^6 [4 + (q - 4)^2]\,dq.$$

Since the limits are the same for both integrals and both involve the variable q, the integrals can be combined and simplified before integrating as follows:

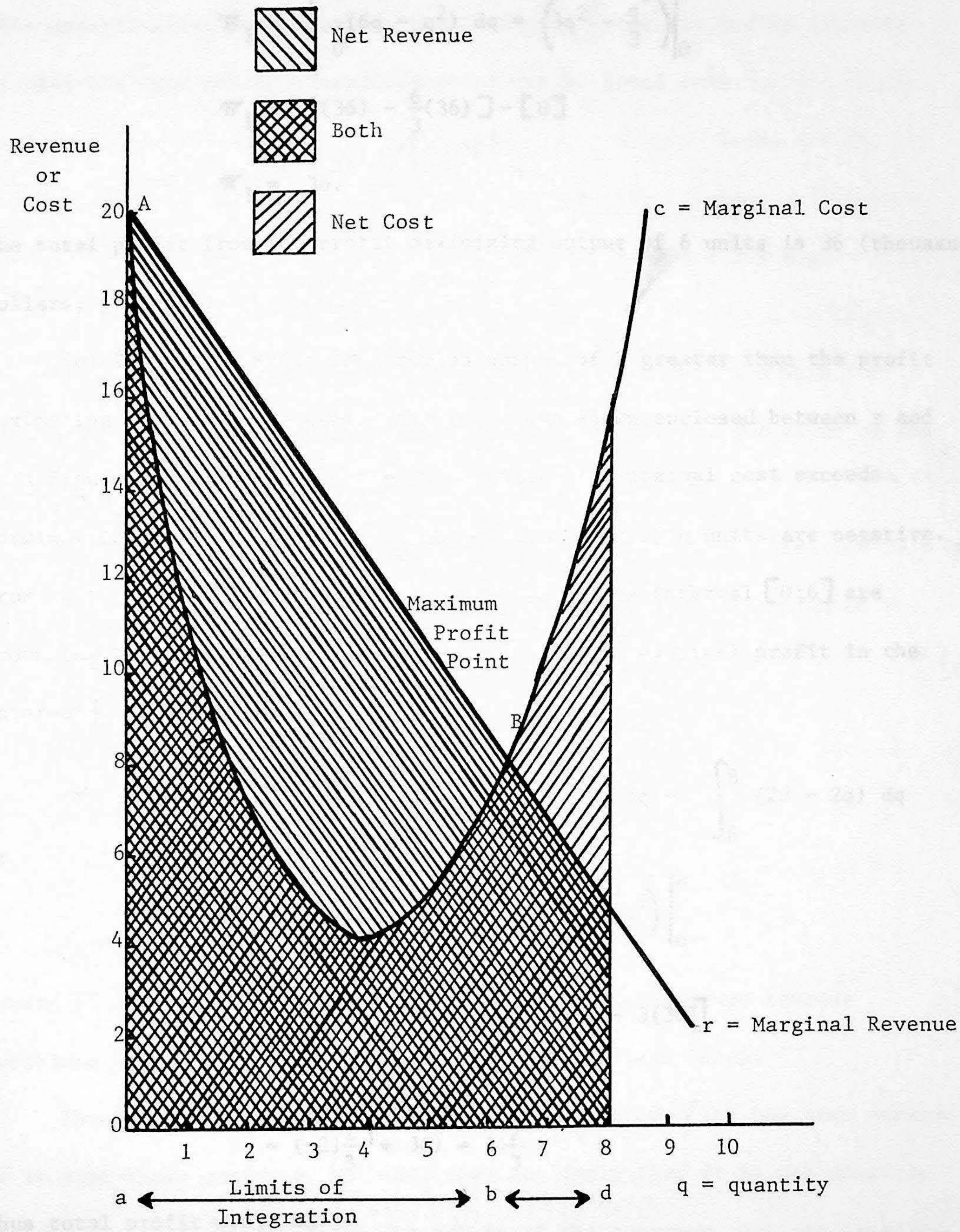


Figure 2.

The Marginal Revenue (r) and Marginal Cost (c) Functions and the Areas between Them.

$$\pi_1 = \int_0^6 (6q - q^2)\, dq = \left(3q^2 - \frac{q^3}{3}\right)\Big|_0^6$$

$$\pi_1 = \left[3(36) - \frac{6}{3}(36)\right] - [0]$$

$$\pi_1 = 36.$$

The total profit from the profit maximizing output of 6 units is 36 (thousand dollars, perhaps).

(c) The total profit π_2 from an output of 2 greater than the profit maximizing output is the sum of the two areas shown enclosed between r and c in Figure 2. From the point B on in Figure 2, marginal cost exceeds marginal revenue, and profits for output in excess of 6 units are negative. From part (b) of this example, total profits in the interval $[0;6]$ are known to be 36 (thousand dollars, perhaps). The (negative) profit in the interval $[6;8]$ is given by

$$\pi_2 = \int_b^d c\,dq - \int_b^d r\,dq = \int_6^8 [4 + (q-4)^2]\, dq - \int_6^8 (20 - 2q)\, dq$$

or

$$\pi_2 = \int_6^8 (q^2 - 6q)\, dq = \left(\frac{q^3}{3} - 3q^2\right)\Big|_6^8$$

$$= \left[\frac{8}{3}(64) - 3(64)\right] - \left[\frac{6}{3}(36) - 3(36)\right]$$

$$= \left[-\frac{1}{3}(64)\right] - \left[-36\right]$$

$$= (-21\tfrac{1}{3} + 36) = 14\tfrac{2}{3}.$$

Thus total profit would be

$$36 - 14\tfrac{2}{3} = 21\tfrac{1}{3}$$

if output were increased from 6 to 8 units.

(d) The demand function D for a firm is an average revenue function

AR. Average revenue is equal to total revenue $\int r\,dq$ divided by quantity q; thus the rule for the demand function can be found from

$$D = AR = \left(\int r dq\right) \div q$$

$$= \left[\int (20 - 2q)\, dq\right] \div q = (20q - q^2) \div q$$

$$= 20 - q.$$

Observe that the rule for the demand curve is a straight line with the same intercept as the rule for r̲, but with slope only 1/2 the slope of the rule for r.

(e) The rule for the average cost function AC is found from the rule for the total cost function divided by the quantity produced; thus AC = $\int cdq) \div q$ or

$$AC = \left(\int [4 + (q - 4)^2]\, dq\right) \div q$$

$$= \left[\int (20 - 8q + q^2) dq\right] \div q$$

$$= \left(20q - 4q^2 + \frac{q^3}{3}\right) \div q$$

$$= 20 - 4q + \frac{q^2}{3}.$$

Profit could be determined from the average cost and average revenue functions which can also be graphed (but have not been here).

Throughout the preceding discussion the function $f(x)$ has been assumed to be everywhere positive, but this does not imply that it is not possible to find the area under a curve for values of the function that are negative. The question arises when an area that is positive is bounded by the endpoints of an interval, the x axis, and a <u>negative</u> value of the function in the interval. From the definition of an integral as the limit of a sum of products, the negative value of $f(x)$ in each product leads to a negative number that would imply a negative area, an illogical result. The usual convention for dealing with this problem is also adopted here; for those domain intervals over which the function is negative, the integral representing the area is preceeded by a negative sign, thus reversing the sign of the product in that interval. The situation is illustrated in Figure 3.

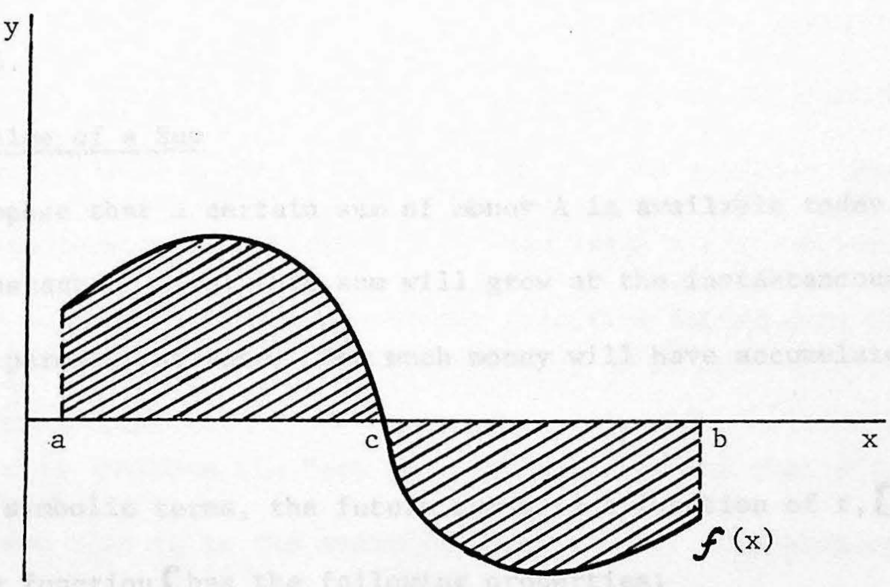


Figure 3

Area under the Graph of the Function f with Rule $y = f(x)$ for Values of f that Are Both Positive and Negative.

The area under f(x) in the interval [a;b] is the sum of the two shaded areas. By the convention, this area is represented by

$$\int_a^c (x)dx - \int_c^b (x)dx.$$ [26]

One very important group of applications of the integral and of the exponential function is in computing what are called present values and future values. These concepts will be encountered very frequently in subsequent work, notable in capital budgeting. These examples provide a mathematical basis for such applications, and prove some frequently used results.

Example 5.

Future Value of a Sum

Suppose that a certain sum of money A is available today. In each future instant of time, this sum will grow at the instantaneous rate of (10 = r) percent per year. How much money will have accumulated t years later?

In symbolic terms, the future sum y is a function of t, $[y = f(t)]$ where the function f has the following properties:

$$(1)\quad f(0) = A$$

$$(2)\quad \frac{dy}{dt} = ry.$$

Condition (2) is satisfied by the function $y = f(t) = Ae^{rt}$ since $(d/dt)\ Ae^{rt} = rAe^{rt} = ry$.

Condition (1) is also satisfied since $f(0) = Ae^{r \cdot 0} = A$.

---

[26]McAdams, Mathematical Analysis, p. 238-241.

Example 5A.

Initial sum is \$100. The continuous growth rate is .125 = 12.5 percent. The future period is eight years.

$$y = \$100e^{.125t} = 100e^{(.125)8}$$

if t = 8,

$$y = \$100e^{1} = 100(2.718) = \$271.80.$$ [27]

Example 6

Present Value of a Future Sum

The future value can be computed if a present sum is given, and the question asked: How much will it be worth at some future date assuming the amount increased at an instantaneous rate of r per period? The inverse of this problem occurs if a future sum, say B, is given which will become available at a specified future date. The question then arises, at an instantaneous growth rate of r: How large a present sum would be needed so that its future value at the specified future date would be equal to B?

In this instance the fact that $(dy/dt) = ry$ and that $y(t_n) = B$ are given. Here time t_n is the specified future date. The problem is to find the value of y at $t = 0$. Let A be the required present sum. The given condition $(dy/dt) = ry$ is satisfied by $y = y(t) = Ae^{rt}$ since t_n, $y(t_n) = Ae^{rtn} = B$. Solving this equation for A, we have $A = Be^{-rtn}$. A lender would like to earn 6 percent on any loan he makes. The borrower agrees to pay a lump sum of \$1500 ten years from now. How much can the

---

[27] Ibid., p. 251.

lender lend today and still satisfy his objectives?

$$x = (\$1,500)e^{-(.06)(10)}$$

$$= \frac{1}{e^{.60}} (\$1,500).$$

The antilog of -.6 is .549. The amount that could be lent is (.549) (1,500) = \$824.[28]

Example 7.

Present Value of an Annuity

A series of equal future receipts is called an annuity. Suppose that a piece of equipment has a useful life of 10 years. It can be rented at \$10,000 per year (in equal weekly installments), or purchased by an outright cash payment of \$65,000. The salvage value after 10 years is expected to be zero. Is the present value of the rental payments at an 8 percent interest rate greater or less than the cash outlay?

The present value of the annuity, the series of rental payments, can be approximated by the following integral (since the integral is for continuous compounding and the problem calls for compounding 52 times per year).

$$A = \int_0^n Re^{-rt_n}\, dt \quad \text{where R is the annual payment}$$

$$= \int_0^{10} (10,000)e^{-(.08)t}\, dt.$$

The indefinite integral has the following solution:

$$R\int e^{-rt}\, dt = \left(\frac{-e^{-rt}}{r} + C\right) R.$$

---

[28] Ibid., p. 251-252.

Applying the fundamental theorem for r = .08, t = 8, and R = 10,000 we have

$$\int (10,000)e^{-(.08)t}\, dt = (10,000)\left[\frac{-e^{-.8}}{.08} - \left(\frac{-1}{.08}\right)\right]_0^{10}$$

Since

$$e^{-.8} = \frac{1}{e^{.8}} = \frac{1}{2.23} = .448; \frac{1}{.08} = \frac{100}{8} = 12.5,$$

the final bracket is equal to:

$$[-.448(12.5) + (12.5)] = (12.5)(1 - .448).$$

Therefore

$$\int (10,000)e^{-.08t}\, dt = (\$10,000)(12.5)(.552) = \$69,000.$$

The present value of the annuity is $4000 greater than the present value of the cash purchase price.

In case the annuity has an indefinite life, the greatest value of the annuity has a simple expression:

$$\int_0^{\infty} e^{-rt}\, dt = \frac{-e^{-rt}}{r}\Bigg|_0^{\infty} = 0 - \left(\frac{-1}{r}\right) = \frac{1}{r}.$$

Remember this result. It is especially useful in economics.[29]

---

[29] _Ibid._, p. 252.

CHAPTER 6

CHAPTER 11-12 INTRODUCTION TO PROBABILITY

Sets and Subsets · Intersections and Unions · Sample Spaces · Assignment of Probabilities · Events · An Application · Some Counting Formulas

ADDITIONAL TOPICS IN PROBABILITY

Univariate vs Bivariate Data · Joints and Marginal Probabilities · Conditional Probabilities · Independence · Tree Diagrams · Probability Vectors, Markov Analysis

Example 1.

Seven different machining operations are to be performed on a part, but they may be performed in any sequence. We may then consider 7! = 5040 different orders in which the operations may be performed.[30]

Example 2.

Ten workers are to be assigned to 10 different jobs. In how many ways can the assignments be made? The first worker may be assigned in 10 possible ways, the second in any of the 9 remaining ways, the third in 8, and so forth: there are 10! = 3,628,800 possible ways of assigning the workers to the jobs.[31]

---

[30] Kemeny, and others, Finite Mathematics, p. 99.

[31] Ibid.

Example 3.

A company has n directors. In how many ways can they be seated around a circular table at a board meeting if two arrangements are considered different only if at least one person has a different person sitting on his right in the two arrangements. To solve the problem, consider one director in a fixed position. There are $(n - 1)!$ ways in which the other people may be seated. We have now counted all the arrangements we wish to consider different. Thus there are also $(n - 1)!$ possible seating arrangements.[32]

Example 4.

The number of permutations of n distinct objects is a special case of this principle. If we were to list all the possible permutations, there would be n possibilities for the first, for each of these $n - 1$ for the second, etc., until we came to the last object, and for which there is only one possibility. Thus there are $n(n - 1) \ldots 1 = n!$ possibilities in all.[33]

Example 5.

An automobile manufacturer produces 4 different models; models A and B can come in any of four body styles—sedan, hardtop, convertible, and station wagon—while models C and D come only as sedans or hardtops. Each car can come in one of 9 colors. Thus models A and B each have

---

[32] Ibid., p. 99.

[33] Ibid., p. 100.

$4 \cdot 9 = 36$ distinguishable types, while C and D have $2 \cdot 9 = 18$ types, so that there are in all

$$2 \cdot 36 + 2 \cdot 18 = 108$$

different car types produced by the manufacturer.[34]

Example 6.

Suppose there are n applicants for a certain job. Three interviewers are asked independently to rank the applicants according to their suitability for the job. It is decided that an applicant will be hired if he is ranked first by at least two of the three interviewers. What fraction of the possible reports would lead to the acceptance of some candidate? We shall solve this problem by finding the fraction of the reports which do not lead to an acceptance and subtract this answer from 1. Frequently an indirect attack of this kind on a problem is easier than the direct approach. The total number of reports possible is $(n!)^3$ since each interviewer can rank the men in n! different ways. If a particular report does not lead to the acceptance of a candidate, it must be true that each interviewer has put a different man in first place. This can be done in $n(n - 1)(n - 2)$ different ways by our general principle. For each possible first choice, there are $[(n - 1)!]^3$ ways in which the remaining men can be ranked by the interviewers. Thus the number of reports which do not lead to acceptance is

$$n(n - 1)(n - 2)[(n 0 1)!]^3.$$

Dividing this number by $(n!)^3$ we obtain

---

[34] Ibid., p. 100.

$$\frac{(n - 1)(n - 2)}{n^2}$$

as the fraction of reports which fail to accept a candidate. The fraction which leads to acceptance is found by subtracting this fraction from 1 which gives

$$\frac{3n - 2}{n^2}.$$

For the case of three applicants, we see that $\frac{7}{9}$ of the possibilities lead to acceptance. Here the procedure might be criticized on the grounds that even if the interviewers are completely ineffective and are essentially guessing, there is a good chance that a candidate will be accepted on the basis of the reports. For n equal to ten, the fraction of acceptances is only .28, so that it is possible to attach more significance to the interviewers' ratings, if they reach a decision.[35]

Example 7.

"Key sort" cards are cards containing a series of punched holes along the upper edge. A card may contain information about an employee, an account, a part number, a stockholder, a sales order, etc. A set of cards is kept in a box, and each card is classified by assigning it to a cell of one or more partitions of the set. To make it easy to find all the members of a cell of a particular partition, or of a cross-partition, the following mechanical procedure is used. A subset of the set of holes along the upper edge of the card is assigned to each partition. The holes so assigned are called a "field." The cells of a partition can be dis-

[35] *Ibid.*, p. 101.

tinguished by cutting slots through some of the holes in the field. A rod can be inserted through the set of cards at a particular hole location and then lifted. The cards that are slotted at that location will remain in the box and those that are not will be lifted out with the rod.

The number of hole locations assigned to a field depends on the number of cells in the partition that the field represents. We can think of each hole location as a bit, letting a slot represent a 0 and an unslotted hole a 1. If there are n hole locations in a field, they can be made to correspond to as many as 2^n cells, since with n bits we can represent the 2^n decimal integers 0 through $2^n - 1$. Thus a four-bit field can represent a 16-cell partition.

Suppose, for example, we have a card for each worker in a factory. The workers are classified by sex, job classification, and department number. A single bit can be arbitrarily assigned to indicate a worker's sex; for example, let 0 represent female, 1 represent male. If there are 13 job classifications, we need a four-bit field for them; we may assign 0001 for classification 1, up to 1101 for classification 13. Let there be 25 departments; then we need a five-bit field for them, assigning 00001 to department 1 up to 11001 for department 25.

Now suppose we want to find all the male workers in job classification 6 working in department 11. These will be identified by the binary number 1011001011, the first bit identifying sex, the next four bits the job, and the last five bits the department. We first partition the set by withdrawing those cards with unslotted holes in hole location 1. These are all the male workers in the factory. We withdraw from this subset all cards with unslotted holes in location 2; these correspond to workers in job classification 8 and higher, while those that are left represent job classifications 7 or lower. Of those which are left, we now withdraw those with unslotted holes in locations 3 and 4, and remove from this subset the cards with slotted holes in location 5. We now have the subset of male workers in job classification 6. The procedure is continued in this fashion to find the subset of this subset which contains workers in department 11.

Observe that the procedure works independently of the original order of the cards in the box.[36]

Example 8.

An electronic component is mass-produced and then tested unit by unit on an automatic testing machine. According to the electrical characteristics of each component, the machine automatically classifies it as "good" or "defective." If the same unit is tested twice, the machine should, theoretically, classify it in the same way both times. We assume, however, that the machine has a certain probability q of misclassifying a part on any given trial, because of electrical or mechanical failure on the part of the testing machine. To improve the accuracy of our classification we may have the machine test the same unit not just once but r times, and finally classify a unit according to the classification which a majority of the tests give. To avoid ties we assume that r is odd. Let us see how this process decreases the probability of classification error.

---

[36]McAdams, Mathematical Analysis, p. 86-87.

Consider r experiments on each unit, where the jth experiment results in success if the jth test classifies the unit without error. The probability of success is then $p = 1 - q$. The majority decision will classify a unit correctly if we have more than r/2 successes. Suppose, for example, that we test each unit five times, and that the probability of misclassification on any single test is .1. Then the probability for success is .9, and the probability that the majority of the test results will correspond with the true state of the unit is

$$b(3;\ 5,\ .9) + b(4;\ 5,\ .9) + b(5;\ 5,\ .9)$$

which is found to be approximately .991.

Thus the above procedure decreases the probability of misclassification from .1 in the case of one test to .009 in the case of five.[37]

Example 9.

A company is applying statistical quality control procedures in an attempt to control the costs of its five branches. The branches are the same size and are situated in like communities, and the business is not characterized by seasonal fluctuations. To obtain estimates of the mean of a certain cost, the company has averaged quarterly reports of the cost for each branch. Thus, one branch had the following readings for this cost: \$20,000; \$18,200; \$18,000; and \$21,400. The arithmetic mean for this branch was, therefore,

$$\bar{x} = \frac{\sum x}{n} = \frac{\$77,600}{4} = \$19,400$$

---

[37] Kemeny, and others, <u>Finite Mathematics</u>, p. 172.

Similarly computed averages for the other four branches were \$20,800; \$19,000; \$20,200; and \$18,600. Consequently, the population mean may be determined as follows:

$$\mu \cong \bar{\bar{x}} = \frac{\sum \bar{x}}{N} = \frac{\$98,000}{5} = \$19,600$$

The standard deviation σ may be calculated by constructing a table similar to

$\bar{x}_i$	μ	$\bar{x}_i - \mu$	$(\bar{x}_i - \mu)^2$
\$19,400	\$19,600	\$ -200	\$ 40,000
20,800	19,600	1,200	1,440,000
19,000	19,600	-600	360,000
20,200	19,600	600	360,000
18,600	19,600	-1,000	1,000,000
$\sum \bar{x}$ = \$98,000			$\sum(\bar{x} - \mu)^2$ = \$3,200,000

$$\sigma = \sqrt{\frac{\sum(\bar{x} - \mu)^2}{N - 1}} \quad \frac{3,200,000}{5 - 1} = \$894.50$$

1. One of the important attributes of σ is that $\mu \pm 3\sigma$ can be shown to include 99.73% of the normally distributed population. COMMENT

2. In practice, the values μ and σ are rarely known because most populations are infinite; hence, the values of samples are used as proxies.

Setting Control Limits

With these values of μ and σ at hand, management now wishes to choose decision rules in order that acceptable ranges of performance may be determined. In other words, when a branch submits a quarterly report of this cost, management wants to have predetermined figures that designate whether performance is acceptable or whether it requires further investigation.

Two things could happen at any time:

1. The performance could be in control, but it falls outside the acceptable range and is needlessly investigated (this is known as a type I or α error); or

2. The performance could actually be out of control—in the sense that the population mean has shifted—but because it falls in the acceptable range, we fail to detect the change (this is known as a type II or β error).

Management must decide what risks of these errors it is willing to take. Furthermore, the risks work inversely to one another; if the α risk decreases, the β risk increases, and vice versa. It is these decisions that determine how tight the standard or budget will be. Let us continue our example.

Management decides that it is willing to risk a type I error (investigating acceptable performance needlessly) 10% of the time (i.e., 5% when performance is on the low side and 5% when it is on the high side). Management also wants to know what percentage the population mean would have to change while retaining the same standard deviation in order that it could have 80% confidence that the change would be detected. (Observe that 80% is the probability of <u>not</u> making a β error.)

The upper and lower control limits corresponding to the α risk may be determined as follows. First turn a normal curve on its side in order to understand why the limits are called "upper" and "lower." Any performance falling within the limits is acceptable, and any performance outside the limits requires further investigation. See Figure 11. We consult a

Table to find the number of standard deviations corresponding to a probability of 0.4500. We see, by interpolation, that 1.645 standard deviations is our answer. Therefore, our limits become

$$\mu \pm 1.645$$

$$19{,}600 \pm 1.645(894.50)$$

$$UCL = 21{,}071$$

$$LCL = 18{,}129$$

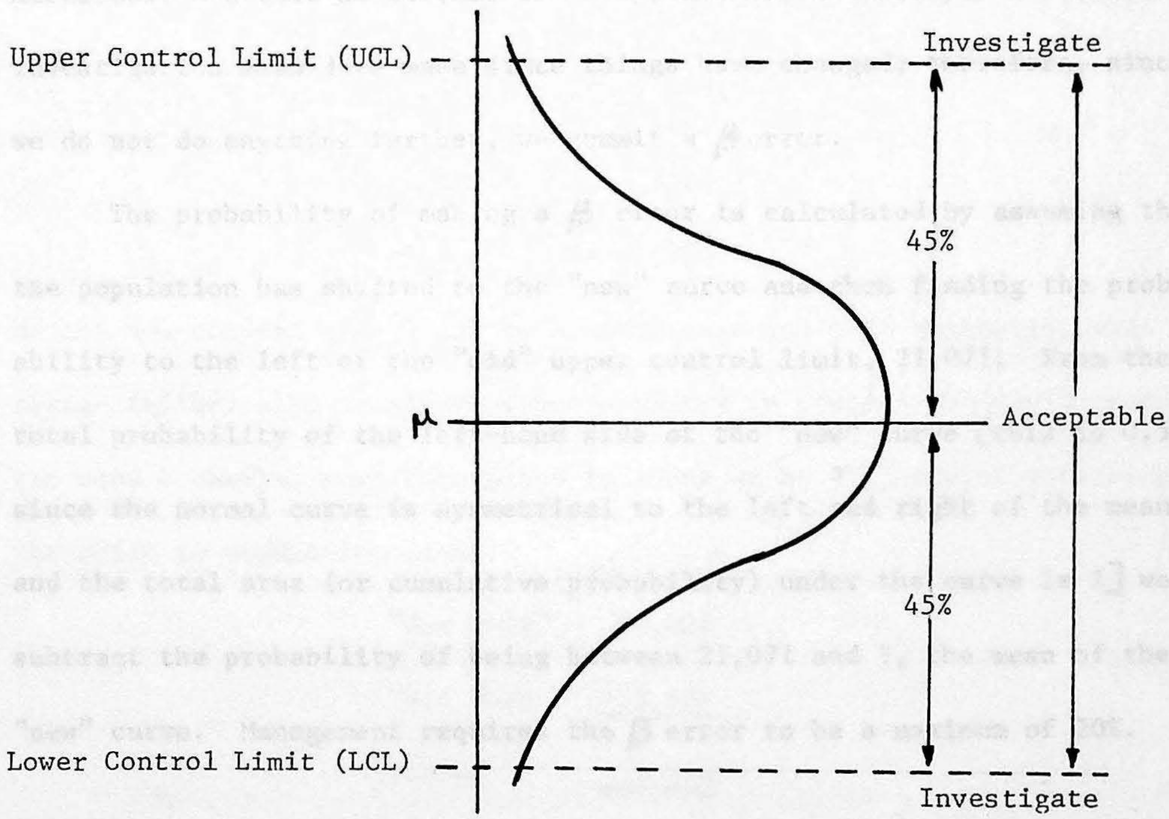


Figure 11

Now, any quarterly cost x such that $(18{,}129 \leq x \leq 21{,}071)$ is true will be considered to be in control, but any x outside these limits would call for investigation.

To understand what the β error is all about, study Figure 12. As matters presently stand, (the "old" curve) μ = 19,600, and the upper control limit is 21,071. If the quarterly cost x were greater than 21,071, we would say that further investigation was necessary. But suppose μ had actually changed so that the state of affairs found us in the "new" curve. Further suppose that x was less than 21,071. Now, because of our set limits based on the "old" curve, we would have no suspicions that μ had changed, and we would say that performance was acceptable and that no further investigation was necessary. But further investigation should be made since things have changed; therefore, since we do not do anything further, we commit a β error.

The probability of making a β error is calculated by assuming that the population has shifted to the "new" curve and then finding the probability to the left of the "old" upper control limit, 21,071. From the total probability of the left-hand side of the "new" curve [this is 0.5000, since the normal curve is symmetrical to the left and right of the mean and the total area (or cumulative probability) under the curve is 1] we subtract the probability of being between 21,071 and ?, the mean of the "new" curve. Management requires the β error to be a maximum of 20%.

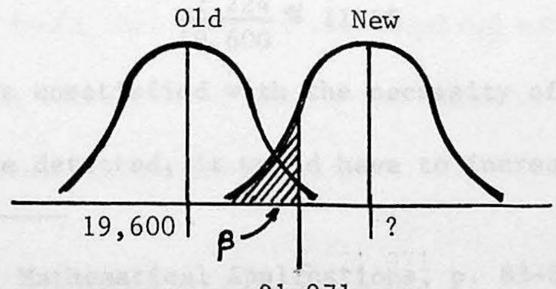


Figure 12

$$0.5000 - p(21{,}071 \leq x \leq ?) = 0.2000$$

$$p(21{,}071 \leq x \leq ?) = 0.3000$$

We wish to find the number (z) of standard deviations from the mean that will yield a probability of 0.3000. Actually, we apply the formula

$$z = \frac{x - \mu}{\sigma}$$

However, tables tabulate only the right side of the normal curve, and we are interested in the left. To adjust for this, we can change the formula to $z = (\mu - x)/\sigma$. We therefore have

$$p(z) = p\left(\frac{? - 21{,}071}{894.50}\right) = 0.3000$$

For the p(z) to equal 0.3000, z must be 0.8418.

$$\frac{? - 21{,}071}{894.50} = 0.8418$$

$$? = 21{,}824$$

We can now convert this value to a percentage and tell management what change (either plus or minus—since symmetry is present and would produce the same % change) must take place in order to be 80% sure of detecting the shift in population means.

"New mean"	21,824
"Old mean	-19,600
Change	2,224

$$\frac{2{,}224}{19{,}600} \cong 11.3\%$$

If management were unsatisfied with the necessity of an 11.3% change before it could be detected, it would have to increase the α risk.[38]

---

[38] Corconen, Mathematical Applications, p. 83-86.

CHAPTER 7

SUMMARY AND CONCLUSIONS

The purpose of this study was to attempt to dispense with the business students common conclusions that mathematics is just an intellectual exercise that they must endure as a sort of initiation fee into their major field. It was felt that the examples of the applications of the particular mathematical concepts presented in applied mathematics textbooks to business situations as presented here would do this.

A further purpose was to arrive at the author's recommedations to the extent of the amount of mathematics that should be required for the undergraduate business student.

The author feels that an applied mathematics course that actually shows the student the business application should be required. The author did not find any textbook that had the necessary applications, but did find many textbooks that had the proper mathematics.

The author feels that much of the required mathematics for undergraduates could be incorporated into a two semester course with equal emphasis on business and mathematics.

REFERENCES

Allen, Roy, 1950, Mathematical Analysis for Economists: London, Macmillan and Company.

Arkin, Herbert, 1963, Handbook of Sampling for Auditing and Accounting: New York and London, McGraw-Hill Book Company, Inc.

Bowen, Earl, 1963, Mathematics with Applications in Management and Economics: Homewood, Illinois, R. D. Irwin.

Buck, Robert Creighton, 1965, Advanced Calculus: New York, McGraw-Hill.

Cissell, Robert, and Cissell, Helen, 1964, Mathematics of Finance: Boston, Houghton Mifflin.

________, and Bruggeman, Thomas J., 1962, Mathematics for Business and Economics: Boston, Houghton Mifflin.

Corconen, A. Wayne, 1968, Mathematical Applications in Accounting: New York and Chicago, Harcourt, Brace and World, Inc.

Dano, Sven, 1965, Linear Programing in Industry: Wien, New York, Springen-Verlag.

Draper, Jean E., and Klingman, Jane S., 1972, Mathematical Analysis: Business and Economic Applications: Harper and Row.

Ellerby, G., 1964, Graphs and Calculus: New York, Pergamon Press.

Federer, Herbert, 1961, Analytic Geometry and Calculus: New York, Ronald Press Company.

Hilborn, Claire Egbert, 1948, Mathematics for use in Business: Boston, Houghton Mifflin.

Howell, James Edwin, and Teichroew, Daniel, 1963, Mathematical Analysis for Business Decisions: Homewood, Illinois, R. D. Irwin.

Kells, Lyman Morse, 1950, Analytic Geometry and Calculus: New York, Prentice-Hall.

Kemeny, John; Schleifer, Arthur; Snell, J.; and Thompson, Gerald, 1962, Finite Mathematics with Business Applications: Englewood Cliffs, N.J., Prentice-Hall, Inc.

Locke, Flora, 1969, College Mathematics for Business: New York, Wiley.

McAdams, Alan K., 1970, Mathematical Analysis for Management Decisions: Introduction to Calculus and Linear Algebra: London, The Macmillian Company.

Silverman, Richard, 1969, Limits and Continuity: New York, Gordon and Breach.

Smith, Frank, and Davies, George, 1956, Calculus for Business and Related Fields: Dubuque, Iowa, W. C. Brown.

Teichroew, Daniel, 1964, An Introduction to Management Science: New York, John Wiley and Sons, Inc.

Thompson, Howard, 1973, Applications of Calculus in Business and Economics: Menlo Park, California, W. A. Benjamin, Inc.

Widder, David, 1947, Advanced Calculus: New York, Prentice-Hall, Inc.

Youse, Beuan, 1967, Calculus for Students of Business and Management: Scranton, Pa., International Textbook Company.