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SPECIAL REPORT

PERS 80

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A COMPARISON BY ARTIFICIAL EXPERIMENT OF METHODS OF ESTIMATION IN FACTOR ANALYSIS

NATIONAL INSTITUTE FOR PERSONNEL RESEARCH
COUNCIL FOR SCIENTIFIC AND INDUSTRIAL RESEARCH

C.S.I.R. Special Report, no. 80

UDC 519.24

Johannesburg, South Africa, June 1964

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ACKNOWLEDGEMENT

This report describes a study carried out during the period which I spent at the I. B. M. Research Center in 1963. Some preliminary work was done at Educational Testing Service.

My thanks go to the I. B. M. Research Center for the provision of computing facilities. To members of staff of the I. B. M. Research Center and Educational Testing Service I am indebted for encouragement and advice.

In particular, I wish to express my sincere appreciation and gratitude to Dr. Rolf Bargmann of the I. B. M. Research Center who gave up much precious time for the discussion of problems and offered some very valuable suggestions. His contributions, particularly with respect to the method for generating random Wishart matrices, are gratefully acknowledged. It should not be construed, however, that Dr. Bargmann necessarily agrees with all opinions expressed in this report.

My thanks go also to Mr. R. S. Hall, Head of the Statistics Department of the N. I. P. R. for his continued interest and encouragement. The difficult task of typing the manuscript was undertaken by Mrs. E. L. Murray to whom I am grateful.

M. W. BROWNE.

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INTRODUCTION

Factor analysis is a statistical technique which has been widely used, particularly in psychological research. Many methods of estimation of factor loadings have been proposed, but the majority are of an approximate and non statistical nature and little is known of the properties of the estimates. By contrast, a statistically sound technique for the estimation of factor loadings was first developed by Lawley (24), using the method of maximum likelihood. These estimates of factor loadings are theoretically preferable to other estimates which have been proposed, as they are asymptotically efficient and there is a corresponding likelihood ratio test for assessing the fit of the factor analysis model. However, a great amount of computation is involved in solving the maximum likelihood equations, and the earlier computational procedures suggested have been known to break down in certain cases (29), so that other methods for estimating factor loadings have been used in the majority of reported studies.

The problem of estimating the minimum number of factors for which the factor analysis model fits the population under consideration has never been solved satisfactorily. A sequence of likelihood ratio tests may be used to provide an estimate of number of factors (3, 36), but little is known of the properties of this estimate apart from the fact that the probability of obtaining an overestimate of the number of factors is less than or equal to the significance level used. A vast amount of computation is involved, as the

lengthy /....

lengthy maximum likelihood estimation procedure has to be repeated several times. In addition, many rule of thumb procedures for determining the number of factors have been proposed, some of which have been examined in previous empirical studies (33, 39).

The object of this paper is to compare by artificial experiment, the effectiveness of some methods of factor analysis and procedures for estimating the number of factors. A description will be given of an investigation in which sample correlation matrices, generated from a population satisfying the factor analysis model, were subjected to various methods of factor analysis and procedures for estimating number of factors. The results will be evaluated, taking into consideration both the accuracy of the estimates and the amount of computation involved. The quick and effective method which was used for generating the sample correlation matrices will be described.

There was no attempt in this study to obtain accurate confidence limits for estimates of factor loadings as the amount of computation required would have been prohibitive, and the results of limited value, being specific to the population parameters chosen and the identification conditions imposed on the sample factor matrices. However, a rough impression will be given of the variation which can occur in estimates of factor loadings. The effect on estimates, of increasing the sample size, and of increasing the number of variables while holding constant the number of factors, will be examined.

A number of empirical studies on various aspects of factor analysis have been carried out (5, 6, 17, 27, 30, 31, 32, 39, 45, 47) but methods of factor analysis have not previously been compared by means of artificial experiments.

Before proceeding further it will be convenient to describe the factor analysis model and review the methods of estimation of factor loadings and of number of factors which will be investigated.

2. FACTOR ANALYSIS MODEL AND METHODS OF ESTIMATION

2.1 Notation

A matrix will be denoted by a capital letter and a vector by a small letter underlined. For example, A (p x m) stands for a matrix with p rows and m columns and $\underline{x}$ (p x 1) for a column vector with p elements. The transpose of a matrix or vector will be indicated by a prime. The identity matrix of order p will be represented by I (p) and the null vector with p elements by $\underline{0}$ (p x 1). A diagonal matrix with the i^{th} diagonal element equal to a function $f(x_i)$ of x_i, i = 1, 2........, will be denoted by $D_{f(x)}$ while Diag (B) will stand for a diagonal matrix formed from the diagonal elements of a square matrix B.

2.2 Model

Suppose that $\underline{x}$ (p x 1) is a random vector distributed according to $N(\underline{\mu}\ \ \Sigma)$, the multivariate normal distribution with mean vector μ (p x 1) and covariance matrix Σ (p x p)

Let P be the intercorrelation matrix between the variates $x_1 \ldots\ldots x_p$ so that

$$P = D_{1/\sqrt{\sigma_{ii}}} \ \Sigma \ D_{1/\sqrt{\sigma_{ii}}}$$

We /....

We postulate that it is possible to construct, artificially, m ($m<p$) "factor variables" $z_i \ldots\ldots z_m$ distributed according to $N(\underline{0}'(m \times 1), I(m))$, such that the matrix of partial correlations between pairs of the "observed variables" $x_i \ldots\ldots x_p$, after eliminating the effect of the m factor variables is the identity matrix (19, 4).

i.e. $$D_{1/\sqrt{\delta}}(P-\Phi\Phi')D_{1/\sqrt{\delta}} = I(p) \qquad (1)$$

so that

$$P = \Phi\Phi' + D_{\delta} \qquad (2)$$

where Φ ($p \times m$) is the "factor matrix" or matrix of correlations between the observed variables and the factor variables and $D_{\delta} = \text{Diag}(P - \Phi\Phi')$

The elements δ_i will be referred to as the "uniquenesses". It may be shown (10) that $(1-\delta_i)$, commonly known as the i^{th} "communality", is equal to the square of the multiple correlation coefficient of the i^{th} observed variable x_i with the m factor variables $z_i \ldots\ldots z_m$.

The number, m, of factor variables is defined to be the smallest number of factor variables for which a factor matrix, Φ, satisfying equation (1), can be constructed. The factor matrix, Φ will not be unique, as an equivalent solution to (1) may be obtained by multiplying Φ on the right by any orthogonal matrix. Additional restrictions must be imposed on the factor matrix to eliminate this indeterminacy.

Let /....

Let y denote the vector of observed variables in standardised form.

i.e. $\underline{y} = D_{\frac{1}{\sqrt{\sigma}}} (\underline{x} - \mu)$

The relationship between the observed variables and factor variables may then be expressed (19) in the form

$$\underline{y} = \Phi \underline{z} + \underline{e} \qquad (3)$$

where $\underline{e}$ (p x 1) is distributed according to $N(\underline{0} \text{ (p x 1)}, D_\delta)$

At this point it is important to distinguish between this factor analysis model and an alternative factor analysis model which may be expressed in the form of equation (3) but where $\underline{z}$ is considered to be a vector of non-random quantities or parameters to be estimated separately for each observation on $\underline{y}$, and not as a vector of random variables following a specified distribution (26, 44, 20). The second model will not be used for the present study, but some artificial experiments based on this model have been carried out by Wold (45).

In the following description of methods of factor analysis, sample estimates of Φ and D_δ obtained by any of the methods to be investigated, will be denoted by F and D_u respectively. The sample product moment correlation matrix will be denoted by R and the number of cases in the sample by N.

2.3 Methods of factor analysis

2.3.1 Maximum likelihood

Three different approaches to the problem of obtaining estimates of the elements of the factor matrix, or "factor loadings", have yielded equivalent results.

Lawley /....

Lawley (24) used the method of maximum likelihood, and Rao (36) maximised the canonical correlation coefficient between observed variables and factor variables to derive estimates of factor loadings. Bargmann (4) and Howe (19) obtained estimates of factor loadings by maximising the determinant of the matrix of sample partial correlations between pairs of observed variables after eliminating the effect of the factor variables. Although apparently different, the equations derived from these three approaches are equivalent.

Different iterative computational procedures were proposed for solving these three equivalent sets of equations. In order to ensure a unique solution, additional identification conditions, namely that $F' D_{1/u} F$ be diagonal, were included in the sets of maximum likelihood and canonical factor equations. The maximum determinant computing procedure does not yield a unique solution, but this is no disadvantage, as the factor matrix obtained can be subjected to an orthogonal transformation resulting in a matrix which satisfies specified identification conditions, and thus is unique. Provided that the maximum likelihood equations have only one solution, the factor matrices yielded by Lawley's and by Rao's computational procedures should be identical. The same matrix should result if the maximum determinant factor matrix is transformed to satisfy the same identification conditions.

Howe /....

Howe (19) found that the maximum determinant computational procedure converged more rapidly than Lawley's method (25). The canonical factor computing procedure was applied to a correlation matrix consisting of the first 12 variables of the population correlation matrix given in Table 5. This computational procedure was found to converge far too slowly for use even on a computer the size of the I.B.M. 7094, which was used for all computation in the present study. Furthermore, it was found that inaccuracies in the communalities obtained were very much greater than the tolerance limit for differences between communalities on successive iterations.

The maximum determinant computational procedure (19, pp. 33-40) was used for the remainder of the study and was found to be satisfactory.

The solution of the maximum determinant equations (4)

$$(R - FF')\, D_{1/u}\, F = F$$

$$D_u = \text{Diag}\,(R - FF')$$

will maximise the determinant of the partial correlation matrix

$$R^* = D_{1/\sqrt{u}}\,(R - FF')\, D_{1/\sqrt{u}} \qquad ④$$

This determinant may be evaluated (4) using the relationship

$$|R^*| = |R|\;|I - F'R^{-1}F|\;|D_u|^{-1}$$

The maximum of unity is attained when R^* is the identity matrix.

The determinant $|R^*|$ was evaluated after each complete iteration of the maximum determinant computational procedure,

and /....

and the direction of the change in factor loadings from one iteration to the next was reversed whenever there was a decrease in this determinant. It was found that if this was not done, cases occurred where the determinant continued decreasing so that the matrix obtained was not the maximum likelihood factor matrix.

Iteration was continued until the maximum difference between corresponding uniquenesses, u_i, on successive iterations was less than .0001. When the maximum determinant method was applied to the population correlation matrix and this convergence limit was used, all results were accurate to three decimal places. In order to avoid inaccuracy due to rounding error in the calculation of $1/\sqrt{u_i}$, no estimate of uniqueness was permitted to become less than .0004. If, on any iteration, a uniqueness became less than .0004 and there was simultaneously a decrease in the determinant of the partial correlation matrix, computation was terminated prematurely, and the factor matrix obtained on the previous iteration was used as the final solution.

A likelihood ratio statistic for testing the null hypothesis that m factors are sufficient for equation ① to hold against the alternative that more than m factors are required, was provided by Lawley (24). This statistic may be expressed (19) in the form

$$\lambda_m = -k \log_e |R^*| \qquad ⑤$$

where /....

where R* is defined in equation ④, the factor matrix F being of order (p x m). The multiplier, given by Bartlett (7), is

$$k = N - \frac{2p + 11}{6} - \frac{2m}{3}$$

This statistic is asymptotically distributed as Chi Square with $\frac{1}{2}\left((p - m)^2 - p - m \right)$ degrees of freedom.

Lawley (25) gave the easily computed approximation for the likelihood ratio statistic.

$$\lambda_m \doteq k \sum_{j=1}^{p} \sum_{i=1}^{j} r_{ij}^{*2}$$

where r^*_{ij} is the element in the i^{th} row and j^{th} column of the partial correlation matrix R*.

The following decision procedure using a sequence of likelihood ratio tests may be used to obtain an estimate of the number of factors (3, 36):-

The likelihood ratio test given in equation ⑤ is applied for 0, 1, 2 .... factors consecutively until the null hypothesis is not rejected at a specified significance level. The first value of m for which the null hypothesis is not rejected is taken as an estimate of the number of factors. This procedure will tend to underestimate the number of factors, the probability of an overestimate being obtained being less than or equal to the significance level used.

2.3.2 <u>Initial approximation to the maximum likelihood solution. Weighted principal factors.</u>

When the population correlation matrix satisfies equation ① for a specified number of factors m ($m \leq p - 1$), the

square /....

square of the population multiple correlation coefficient (S. M. C.) between the i^{th} observed variable and the remaining (p - 1) observed variables is a lower bound to the population communality of the i^{th} observed variable (10). If the number of variables is increased while the number of factors is held constant the differences between population S. M. C.s and population communalities are reduced (14).

The maximum likelihood estimate of the S. M. C.

$$r^2_{i.1\ldots(i-1)(i+1)\ldots p} = 1 - 1/r^{ii}$$

where r^{ii} is the i^{th} diagonal element of R^{-1}, is often used as an approximation for communality (22, 16, 46). In order to obtain an initial approximation to the factor matrix with which to initiate the maximum determinant computing procedure, the Rao form of the maximum likelihood equations with communalities replaced by S. M. C.s as suggested by Harris (16) were used.

$$D_{1/\sqrt{u}}(R - D_u) D_{1/\sqrt{u}} = V D_{\ell}$$

$$F = D_{1/\sqrt{u}} V D_{1/\sqrt{\ell}} \qquad ⑥$$

where $u_i = 1 - (1 - 1/r^{ii}) = 1/r^{ii}$ $(i = 1\ldots p)$

ℓ_i $(i = 1\ldots m)$ are the m largest latent roots of $D_{1/\sqrt{u}}(R-D_u)D_{1/\sqrt{u}}$

V (p x m) is a matrix formed from the latent vectors of unit length corresponding to the ℓ_i.

The Jacobi method (12) for obtaining latent roots and vectors of a matrix was used throughout this study.

The likelihood ratio test statistic for number of factors

may /....

may be expressed as a function of the latent roots ℓ_i, (36) but this test is not appropriate here as approximations are used for the communalities. Joreskog (20), making the assumption that

$$\delta_i = \Theta \left(1 - \rho^2_{i \cdot 1 \ldots (i-1)(i+1) \ldots p} \right) \qquad (i = 1, p)$$

where $\rho^2_{i \cdot 1 \ldots (i-1)(i+1) \ldots p}$ denotes the i^{th} population S. M. C. and Θ is a constant, obtained a test statistic which was similar in form to the likelihood ratio statistic, but different degrees of freedom were given for the asymptotic chi square distribution.

$$\lambda^{(J)}_m = -\left[N - {}^1\!/_6 (2p + 4m + 7 + \frac{2}{p-m}) \right] \left[\sum_{i=m+1}^{p} \log(\ell_i + 1) - (p-m) \log \frac{\sum_{i=m+1}^{p} (\ell_i + 1)}{p-m} \right]$$

The distribution given for this statistic was the chi square distribution with $\frac{1}{2}(p - m + 2)(p - m - 1)$ degrees of freedom. Joreskog justified the assumption about the relationship between population uniquenesses and S. M. C. s on the grounds that this assumption will hold sufficiently closely for practical purposes when the ratio of number of factors to number of observed variables is small. He suggested that a sequence of these tests may be used to obtain an estimate of the number of factors if the number of observed variables is greater than 15.

Joreskog obtained estimates of factor loadings using the factor analysis model where factor scores are treated as parameters which vary from one individual to another. The

columns /....

columns of the Joreskog factor matrix are proportional to the columns of the factor matrix obtained from equation ⑥. No assumption about the nature of the factor scores was made in the derivation of the test statistic.

2.3.3 Principal factors using S.M.C.s as approximations to the communalities

The principal factor method, or Hotelling's method of principal components (18) applied to the correlation matrix with diagonal elements replaced by communality estimates, is the most popular method of factor analysis among psychologists and is generally used when an electronic computer is available (15, 21, 48). In this study, S.M.C.s were used as initial approximations to the communalities.

The equations are

$$(R - D_u) V = V D_\ell$$
$$F = V D_{1/\sqrt{\ell}} \quad ⑦$$

where $u_i = 1/r^{ii}$

ℓ_i ($i = 1 \ldots m$) are the m largest latent roots of $(R - D_u)$

V ($p \times m$) is a matrix formed from the latent vectors of unit length which are associated with the ℓ_i.

2.3.4 Thomson's modification of the principal factor method

Thomson (43) suggested iterating on the communalities obtained from the principal factor method in order to improve the estimates. After a factor matrix F has been obtained from ⑦ a new estimate of the matrix D_u is obtained from

$$D_u = \text{Diag}(R - F F')$$

This /....

This estimate of D_u is substituted in ⑦ and the process continued until all differences between the u_i on successive iterations fall below a specified limit.

It can easily be shown that this procedure will result in a matrix F for which the sum of squares of non diagonal elements of the residual matrix $(R - FF')$ is a minimum. In the population, where all partial correlations between variables after eliminating the effect of the factors are specified to be zero, maximising the determinant of the partial correlation matrix $D_{1/\sqrt{\delta}} (P - \Phi \Phi') D_{1/\sqrt{\delta}}$ is equivalent to minimising the sum of squares of non diagonal elements of the residual matrix $(P - \Phi \Phi')$. As the elements of R are consistent estimates of the corresponding elements of P, Thomson's solution and the maximum likelihood (maximum determinant) solution are asymptotically equivalent.

There is no statistical test of significance for number of factors applicable to the principal factor estimates and rules of thumb are generally used for estimating the number of factors. Guttmann (13) proved that the number of latent roots greater than or equal to one of the population correlation matrix and the number of non negative latent roots of the population correlation matrix with diagonal elements replaced by S. M. C. s, are both lower bounds to the number of

factors /....

factors, the second being better than the first. Kaiser (21) uses the number of latent roots greater or equal to one of the sample correlation matrix as an estimate of the number of factors.

Saunders (38) takes, at each iteration of Thomson's method, the number of positive latent roots greater than the absolute value of the smallest latent root as a criterion for the number of factors to be used for the following iteration.

In this investigation, iteration of the principal factor method was continued until all differences between the communalities $(1 - u_i)$ on successive iterations were less than .0001. When this procedure was applied to the population correlation matrix, all factor loadings obtained were correct to at least three decimal places. Iteration was terminated before convergence of the communalities within the limit of .0001 if any one of the communalities exceeded one. In this case the factor matrix from the previous iteration was taken as the final solution.

The latent vectors obtained on each iteration of Thomson's method were used to reduce the matrix $(R - D_u)$ for the next iteration to near diagonal form, as described by Appel (1). This decreased the number of iterations required for the Jacobi method for obtaining latent roots to converge, and halved computing time.

2.3.5 Centroid method

Before the advent of electonic computers a great many factor analytic studies were carried out using Thurstone's centroid method (42) which yields an easily computed approximation for the principal factor matrix.

In the present investigation the highest correlation of each variable with the remaining variables was used as an approximation for its communality. This approximation is rough but it has been widely used.

3. GENERATION OF SAMPLE CORRELATION MATRICES FROM A GIVEN POPULATION

3.1 Procedure

Sample correlation matrices from a given population could be obtained by generating samples of scores from a multivariate normal distribution which has the specified population correlation matrix (22) and calculating the correlation matrices in the usual manner. A considerable amount of computation is involved, particularly if the samples are large, so that it is advisable to find a more economical procedure for generating the sample correlation matrices.

The sample correlation matrix R (p x p) for a sample of size N from a population with correlation matrix

$$P = D_{1/\sqrt{\sigma_{ii}}} \, \Sigma \, D_{1/\sqrt{\sigma_{ii}}}$$

satisfies /....

satisfies the relationship

$$R = D_{1/\sqrt{a_{ii}}} \; A \; D_{1/\sqrt{a_{ii}}} \qquad (8)$$

where A is distributed according to $W(\Sigma, N-1)$, the Wishart distribution with population covariance matrix Σ and $(N-1)$ degrees of freedom (2). The sample covariance matrix is $^1/_N\, A$.

The population covariance matrix Σ may be taken to be equal to the population correlation matrix P without affecting the distribution of the sample correlation matrix R.

Let Ω (p x p) be a matrix such that

$$\Sigma = \Omega\, \Omega'$$

In practice it is convenient to choose Ω to be a lower triangular matrix which may be obtained by means of the square root method (11).

Let C (p x p) be a random symmetric matrix distributed according to W (I, N-1) so that

$$C = \sum_{i=1}^{N-1} \underline{\nu}_i \; \underline{\nu}_i'$$

where $\underline{\nu}_i$ (p x 1) is distributed according to $N(\underline{0}, I)$.

Then

$$\begin{aligned} \Omega\, C\, \Omega' &= \Omega \left(\sum_{i=1}^{N-1} \underline{\nu}_i \, \underline{\nu}_i' \right) \Omega' \\ &= \sum_{i=1}^{N-1} \underline{\nu}_i^{*} \; \underline{\nu}_i^{*\prime} \end{aligned}$$

where $\underline{\nu}_i^{*}$ is distributed according to $N(O, \Sigma)$, o that

$\Omega\, C\, \Omega'$ is distributed as $W(\Sigma, N-1)$ (2)

Thus the matrix A may be obtained from

$$A = \Omega\, C\, \Omega' \qquad (9)$$

Let /....

Let T (p x p) be the lower triangular matrix with non negative diagonal elements which satisfies

$$C = TT' \qquad (10)$$

The density function of T is (37 p. 33; 35)

$$f(T) = \frac{\exp\left(-\frac{1}{2}\,\text{trace}\,(I^{-1}TT')\right)\ \prod_{i=1}^{p} t_{ii}^{N-1-i}}{2^{\frac{p(N-1)}{2}}\ \pi^{\frac{p(p-1)}{4}}\ |I|^{\frac{N-1}{2}}\ \prod_{i=1}^{p}\Gamma\left(\frac{N-i}{2}\right)}$$

This reduces to

$$\prod_{i=1}^{p}\left[\frac{t_{ii}^{(N-i)-1}\ \exp\left(-\frac{1}{2}t_{ii}^{2}\right)}{2^{\frac{N-i}{2}-1}\ \Gamma\left(\frac{N-i}{2}\right)}\right]\cdot\prod_{j=1}^{p}\prod_{i=1}^{j-1}\left[\frac{1}{\sqrt{2\pi}}\ e^{-\frac{t_{ij}^{2}}{2}}\right]$$

so that the non zero elements of T are independently distributed and

t_{ij} is distributed as N (0, 1) (i < j) (11)

t_{ii} is distributed as Chi with N - i degrees of freedom.

t_{ij} = 0 (i > j)

Random matrices C distributed as W (I, N - 1) may then be generated by using (10) and (11). An alternative method is given by Teichnoew (41).

The sample correlation matrices may be computed from

$$T^* = \Omega T$$

$$A = T^*T^{*\prime}$$

$$R = D_{1/\sqrt{a_{ii}}}\ A\ D_{1/\sqrt{a_{ii}}}$$

This method for obtaining a sample correlation matrix directly without the need for individual scores, reduces the number of random

variables /....

variables to be generated from (N x p) random normal deviates to p (p - 1) /2 random normal deviates and p random variates from Chi distributions. In addition there is a considerable reduction in the computation involved in calculating the correlation matrix.

A number of techniques exist for generating pseudo random numbers from a rectangular distribution (40) and transforming these into the random normal deviates (34) and the random numbers from Chi distributions (41) required for the matrix T. For the present study pseudo random numbers α_j from the rectangular distribution were obtained (40) from the multiplicative congruential generator

$$\alpha_{j+1} = 5^{13} \alpha_j \pmod{2^{35}}$$

Pseudo random normal deviates β_k were obtained (9) from the transformations

$$\beta_k = (-2 \log_e \alpha_j)^{\frac{1}{2}} \text{ Cos } 2\Pi \alpha_{j+1}$$

$$\beta_k = (-2 \log_e \alpha_j)^{\frac{1}{2}} \text{ Sin } 2\Pi \alpha_{j+1}$$

and pseudo random numbers χ_i (f) from a chi distribution with f degrees of freedom (9) were obtained from

$$\chi_i \ (2f) = (-2 \sum_{j=1}^{f} \log_e \alpha_j)^{\frac{1}{2}}$$

$$\chi_i \ (2f+1) = (-2 \sum_{j=1}^{f} \log_e \alpha_j + \beta_k^2)^{\frac{1}{2}}$$

3.2 Tests on distributions of random matrices

A series of tests was carried out on the random matrices C, generated from W (I, N - 1) by the method described above.

It is difficult to test directly the goodness of fit of the distribution of a series of observations on several variates to a

theoretical /....

theoretical multivariate distribution. Because of this, the fit of the distribution of the elements the matrices C to the Wishart distribution was tested indirectly. This was accomplished by testing the goodness of fit of the distributions of three different functions of the elements of C to the theoretical univariate distributions of these functions assuming a Wishart distribution for C. The three functions used were the following three likelihood ratio statistics for testing different hypotheses about the form of the population covariance matrix, given a random matrix from a Wishart distribution.

a. <u>Identity Statistic</u>

This is the likelihood ratio test statistic for testing whether C is a sample matrix from a population with covariance matrix I, or, equivalently, whether A is a sample matrix from a population with covariance matrix Σ (8, 2).

$$\tau = -k_1 \left(\log_e |C| - p \log_e N + p - {}^1\!/_N \operatorname{trace} C\right)$$

$$= -k_1 \left(2 \sum_{i=1}^{p} \log_e t_{ii} - p \log_e N + p - {}^1\!/_N \sum_{j=1}^{p} \sum_{i=1}^{j} t_{ij}^2\right)$$

where

$$k_1 = \left[N - {}^1\!/_6 \left(2p+7 - \frac{2}{p+1}\right)\right]$$

The exact distribution of τ_1 under the null hypothesis is not known, but, if N is large, τ_1, is approximately distributed as chi square with $\frac{p(p+1)}{2}$ degrees of freedom

$$\text{Prob}\ (\tau_1 \leq \tau_1^*) \doteq \text{Prob}\ (\chi^2_{f_1} \leq \tau_1^*)$$

where /....

where

$$f_1 = \frac{p(p+1)}{2}$$

b. Sphericity Statistic

This is the likelihood ratio statistic for testing whether C is a sample matrix from a population with covariance matrix δI where δ is a constant, or, equivalently, whether A is a sample matrix from a population with covariance matrix $\delta \Sigma$ (2).

$$\tau_2 = -k_2 \left[\log_e |C| - p \log_e \left({}^1\!/_p \text{ trace } C \right) \right]$$

where

$$k_2 = (N-1) \left(1 - \frac{2p^2 + p + 2}{6p\,(N-1)} \right)$$

A good approximation to the cumulative distribution function of τ_2 is given (2) by the chi square series

$$\text{Prob}\,(\tau_2 \leq \tau_2^*) \doteq \text{Prob}\,(\chi^2_{f_2} \leq \tau_2^*)$$

$$+ \frac{(p+2)(p-1)(p-2)(2p^3 + 6p^2 + 3p+2)}{288p^2\,k_2^2} \left[\text{Prob}\,(\chi^2_{f_2+4} \leq \tau_2^*) - \text{Prob}\,(\chi^2_{f_2} \leq \tau_2^*) \right]$$

where

$$f_2 = \frac{p(p+1)}{2} - 1$$

c. Independence Statistic

This is the likelihood ratio statistic for testing whether C is a sample matrix from a population with a diagonal covariance matrix (2).

/....

$$\tau_3 = -k_3 \left[\log_e |C| - \sum_{i=1}^{p} \log_e c_{ii} \right]$$

where

$$k_3 = N - \frac{2p + 11}{6}$$

A good approximation to the cumulative distribution function of τ_3 is given (2) by

$$\text{Prob } (\tau_3 \leqslant \tau_3^*) \doteq \text{Prob } (\chi^2_{f_3} \leqslant \tau^*_3)$$

$$+ \frac{p(p-1)}{288 k_3^2} (2p^2 - 2p - 13) \left[\text{Prob } (\chi^2_{f_3+4} \leqslant \tau^*_3) - \text{Prob } (\chi^2_{f_3} \leqslant \tau^*_3) \right]$$

where

$$f_3 = \frac{p(p-1)}{2}$$

The null hypothesis about the form of the population covariance matrix is most stringent for the identity test, less stringent for the sphericity test and least stringent for the independence test. Provided that the random matrices C are distributed according to the Wishart distribution with identity population covariance matrix, all three test statistics should follow the specified distributions.

Three tests of goodness of fit were applied to each series of values of a statistic. Firstly, the values of the statistic were grouped into 10 classes with equal expected frequencies, and the fit of observed frequencies to expected frequencies tested by means of the chi square goodness of fit test (23).

The second test which was used is sensitive to deviations in the lower tail of the distribution. Let $\tau^{(i)}$ denote the value of the statistic under consideration for the i^{th} random matrix of a series of s

random /....

random matrices. The test statistic is

$$\eta_1 = -2 \sum_{i=1}^{s} \log_e \left[\text{Prob}(\tau \leq \tau^{(i)}) \right]$$

If the $\tau^{(i)}$ are independently distributed according to the hypothesised distribution, the values Prob $(\tau \leq \tau^{(i)})$ will have a rectangular distribution and η_1 will be distributed as Chi Square with 2s degrees of freedom.

The cumulative distribution functions of the likelihood ratio statistics were evaluated using the approximations given above. Subroutines prepared by S.P. Ghosh for the I.B.M. Statistical Computer Language were used to evaluate the necessary chi square cumulative distribution functions.

The third test statistic

$$\eta_3 = -2 \sum_{i=1}^{s} \log_e \left[1 - \text{Prob} \quad (\tau \leq \tau^{(i)}) \right]$$

will also have a Chi Square distribution with 2s degrees of freedom if the $\tau^{(i)}$ have the hypothesised distribution, but will be sensitive to deviations in the upper tail of the distribution.

Three sets of pseudo random Wishart matrices with different values of the parameters N and p were generated and tested. The first set consisted of 100 matrices with N = 100 and p = 16, the second of 100 matrices with N = 15 and p = 4, and the third of 50 matrices with N = 1500 and p = 16. Each test of fit was applied to the set of complete p x p matrices, and to each set of the square submatrices of order p - 1, p - 2 ...., 3, 2 formed by deleting rows and columns from the end of the matrix.

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The results of the tests of fit are given in Tables 1, 2 and 3. These results are generally satisfactory. The few significant values may be attributed to chance as in most cases they occur for only one of the tests of fit on one of the likelihood ratio statistics, and are not corroborated by other results for the same set of matrices. We may then conclude that the pseudo random Wishart matrix generator is satisfactory, and that the approximations used for the cumulative distribution functions of the likelihood ratio statistics are close enough for practical purposes.

TABLE 1 Goodness of Fit Tests Applied to Identity, Sphericity and Independence Statistics for 100 Random Matrices from W (I, 100-1)

Order of Submatrix	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Chi Square Goodness of Fit Test Distributed as χ^2 with 9 d.f.															
Identity Statistic	7.6	5.8	6.4	7.4	10.0	19.4*	12.4	11.8	3.4	8.4	5.2	10.8	12.2	7.6	11.0
Sphericity Statistic	6.4	8.6	14.8	7.0	10.8	8.8	4.8	7.0	7.4	14.8	5.2	14.8	8.2	11.6	6.2
Independence Statistic	4.2	6.0	12.6	9.2	5.8	9.2	5.0	4.0	6.8	6.4	6.8	12.2	7.4	5.6	8.4
$-2\ \Sigma \log [\text{Prob}\ (\tau \leq \tau^{(i)})]$ Distributed as χ^2 with 200 d.f.															
Identity Statistic	193	202	204	183	176	188	179	189	196	204	212	211	210	201	201
Sphericity Statistic	193	205	205	185	180	193	185	195	200	207	217	216	216	206	207
Independence Statistic	172	181	194	179	176	192	188	202	200	207	218	216	217	209	204
$-2\ \Sigma \log [1-\text{Prob}\ (\tau \leq \tau^{(i)})]$ Distributed as χ^2 with 200 d.f.															
Identity Statistic	194	205	185	188	196	192	201	204	201	203	188	187	180	183	189
Sphericity Statistic	199	210	186	187	196	190	197	200	198	201	186	184	177	178	184
Independence Statistic	223	238	206	204	217	206	210	210	211	210	194	190	182	183	183

* Significant at 5% level

TABLE 2 **Goodness of Fit Tests Applied to Identity, Sphericity and Independence Statistics for 100 Random Matrices from W (I, 15-1)**

Order of Submatrix	2	3	4
Chi Square Goodness of Fit Test **Distributed as χ^2 with 9 d.f.**			
Identity Statistic	11.6	4.2	20.8*
Sphericity Statistic	9.4	4.8	6.6
Independence Statistic	12.2	7.6	4.4
$-2 \Sigma \log \left[\text{Prob } (\tau \leq \tau^{(i)}) \right]$ **Distributed as χ^2 with 200 d.f.**			
Identity Statistic	174	209	189
Sphericity Statistic	204	205	195
Independence Statistic	203	215	204
$-2 \Sigma \log \left[1 - \text{Prob } (\tau \leq \tau^{(i)}) \right]$ **Distributed as χ^2 with 200 d.f.**			
Identity Statistic	206	177	183
Sphericity Statistic	198	176	177
Independence Statistic	193	174	167

* Significant at 5% level

TABLE 3 Goodness of Fit Tests Applied to Identity, Sphericity and Independence Statistics for 50 Random Matrices from W (I, 1500-1)

Order of Submatrix	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Chi Square Goodness of Fit Test Distributed as χ^2 with 9 d.f.															
Identity Statistic	9.2	6.8	5.6	9.6	8.0	5.6	5.2	6.0	7.6	10.4	7.6	13.2	15.6	13.6	10.4
Sphericity Statistic	14.8	10.4	10.4	9.2	6.4	5.2	5.2	6.8	4.8	8.4	5.6	12.8	17.6*	12.4	10.0
Independence Statistic	1.2	16.8	8.0	4.0	6.4	7.2	5.6	3.2	5.2	17.6*	24.0**	16.8	18.4*	16.4	16.4
$-2\,\Sigma \log \left[\text{Prob}\,(\tau \leqslant \tau^{(i)})\right]$ Distributed as χ^2 with 100 d.f.															
Identity Statistic	81	81	103	99	99	104	98	102	94	90	104	111	109	113	121
Sphericity Statistic	88	74	105	99	99	101	94	97	92	89	104	111	109	115	123
Independence Statistic	95	102	125*	110	106	114	112	108	102	98	113	122	120	126*	131*
$-2\,\Sigma \log \left[1-\text{Prob}\,(\tau \leqslant \tau^{(i)})\right]$ Distributed as χ^2 with 100 d.f.															
Identity Statistic	102	100	91	101	109	115	109	104	102	107	99	90	87	79	74
Sphericity Statistic	96	96	82	95	104	113	109	104	102	106	98	88	85	77	73
Independence Statistic	96	99	84	100	108	107	94	92	92	94	89	83	80	72	68

* Significant at 5% level
** Significant at 1% level

4. EXPERIMENTAL PROCEDURE

4.1 Data

Three sets of sample correlation matrices were generated from a population correlation matrix satisfying the factor analysis model with four factors. The first set consisted of 20 correlation matrices of order 12 for samples of 100 cases. Four factors were extracted from each of these sample correlation matrices, using each method of factor analysis described in Chapter 2 in turn. Each procedure for estimating the number of factors was applied. The estimates provided by the different techniques were then compared with the population parameters and their accuracy evaluated. The second set consisted of 10 of the sample correlation matrices from the first set with four variables added, and the third set consisted of 10 sample correlation matrices with sample size of 1500 and 12 variables. The weighted principal factor and maximum likelihood estimates of factor loadings were obtained for each matrix from the second and third sets. These estimates were compared with the corresponding estimates obtained from the first set in order to demonstrate the effect on the estimates of increasing the number of variables while holding constant the number of factors, and the effect of increasing sample size.

The population factor matrix in simple structure form (42) is given in Table 4, and the population correlation matrix, computed using equation ②, is given in Table 5. The first 12 variables only were used for the first and third sets of sample matrices while all 16 variables were used for the second set.

If any row of the population factor matrix is deleted, there remain two disjoint submatrices of rank 4 (= m). This is a sufficient condition for the population communalities to be unique (3), and is satisfied when the first 12 variables only are used as well as for all 16 variables.

The population communalities were chosen so as to cover a wide range in order to emphasize the difference between the maximum likelihood solution which minimises a function of the non diagonal elements of the matrix $D_{1/\sqrt{U}} (R - FF') D_{1/\sqrt{U}}$, where the reciprocals of the uniquenesses are used as weights, and Thomson's solution which minimises a function of the non diagonal elements of the unweighted matrix $(R - FF')$.

The Wishart matrices which were used for constructing the sample correlation matrices were selected from the matrices obtained while testing the generation procedure as described in the previous chapter. In an attempt to simulate the distribution of the correlation matrix as closely as possible while using a small sample of correlation matrices, the Wishart matrices used were selected such that the values of the cumulative distribution function of the identity statistic for the 12 x 12 submatrix were more or less evenly spaced over the interval from 0 to 1. The values of the cumulative distribution functions of the identity, sphericity and independence statistics for the selected matrices with sample size of 100 and 12 variables, and the matrices with sample size of 1,500 and 12 variables, are given in Tables 6 and 7 respectively.

The sample correlation matrices of order 12 and with sample size of 100 are given in Appendix A.

TABLE 4 Population Factor Matrix

Variable \ Factor	I	II	III	IV	Communality
1	.0	.80	.51	.0	.9001
2	.0	.70	.0	.60	.8500
3	.76	.0	.0	.48	.8080
4	.0	.0	.84	.0	.7056
5	.78	.0	.0	.0	.6084
6	.71	.0	.0	.0	.5041
7	.0	.0	.0	.64	.4096
8	.40	.0	.0	.38	.3044
9	.0	.45	.0	.0	.2025
10	.0	.0	.39	.0	.1521
11	.0	.0	.35	.0	.1225
12	.0	.32	.0	.0	.1024
13	.70	.0	.0	.0	.4900
14	.0	.70	.0	.0	.4900
15	.0	.0	.70	.0	.4900
16	.0	.0	.0	.70	.4900

TABLE 5 Population Correlation Matrix

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1.0	.5600	.0	.4284	.0	.0	.0	.0	.3600	.1989	.1785	.2560	.0	.5600	.3570	.0
2		1.0	.2880	.0	.0	.0	.3840	.2280	.3150	.0	.0	.2240	.0	.4900	.0	.4200
3			1.0	.0	.5928	.5396	.3072	.4864	.0	.0	.0	.0	.5320	.0	.0	.3360
4				1.0	.0	.0	.0	.0	.0	.3276	.2940	.0	.0	.0	.5880	.0
5					1.0	.5538	.0	.3120	.0	.0	.0	.0	.5460	.0	.0	.0
6						1.0	.0	.2840	.0	.0	.0	.0	.4970	.0	.0	.0
7							1.0	.2432	.0	.0	.0	.0	.0	.0	.0	.4480
8								1.0	.0	.0	.0	.0	.2800	.0	.0	.2660
9									1.0	.0	.0	.1440	.0	.3150	.0	.0
10										1.0	.1365	.0	.0	.0	.2730	.0
11											1.0	.0	.0	.0	.2450	.0
12												1.0	.0	.2240	.0	.0
13													1.0	.0	.0	.0
14														1.0	.0	.0
15															1.0	.0
16																1.0

4.2 Measures of accuracy of estimated factor matrices

A reasonable measure of the accuracy of an estimate $\hat{\varphi}$ of a population parameter φ, is the square of the difference between the estimate and the parameter. The expected value of this quantity is the mean square error which is equal to the variance of the estimator plus the square of its bias (23). When the estimator has been applied to a number of random samples from the same population, the average value of the squared differences is an estimate of the mean square error and could be used as a criterion for comparing estimators.

When comparing methods of factor analysis, it is more convenient to have a single measure of the accuracy of a matrix of estimates of factor loadings, than to have a separate measure of accuracy for each loading. The magnitude of this measure should not be influenced by the arbitrary restrictions imposed on the factor matrix to eliminate indeterminacy due to rotation. (See Section 4.3)

Two different measures of the accuracy of the estimated factor matrices were used in the present study. The first measure was

$$c_1 = \sum_{i=1}^{p} \sum_{j=1}^{i} b_{ij}^2$$

where

$$B = (\Phi \Phi' - FF')$$

The magnitude of this quantity is not affected by orthogonal rotation either of the population factor matrix Φ, or of the sample factor matrix F.

The /....

TABLE 6 Values of the cumulative distribution functions of the likelihood ratio statistics

Set 1 (N = 100 p = 12)

Matrix Number	Identity	Sphericity	Independence	L. R. test for 4 factors
1	.05	.03	.04	.11
2	.10	.09	.06	.56
3	.16	.16	.15	.24
4	.20	.21	.39	.28
5	.24	.27	.13	.49
6	.29	.28	.24	.26
7	.37	.40	.41	.15
8	.40	.38	.47	.71
9	.45	.47	.57	.73
10	.50	.52	.48	.68
11	.55	.45	.33	.18
12	.59	.55	.55	.66
13	.66	.66	.63	.59
14	.70	.71	.88	.94
15	.73	.75	.80	.91
16	.81	.83	.78	.76
17	.85	.79	.88	.65
18	.89	.90	.96	.87
19	.96	.96	.62	.46
20	.99	.99	.99	.63

TABLE 7 Values of the cumulative distribution functions of likelihood ratio statistics

Set 3 (N = 1500 p = 12)

Matrix Number	Identity	Sphericity	Independence	L. R. test for 4 factors
1	.04	.03	.00	.00
2	.16	.17	.32	.60
3	.25	.26	.28	.07
4	.34	.35	.24	.28
5	.46	.47	.57	.75
6	.52	.54	.24	.60
7	.65	.65	.69	.18
8	.76	.77	.65	.75
9	.84	.84	.68	.80
10	.96	.95	.87	.47

The second measure used was the minimum possible value of the sum of squares of differences between elements of the simple structure population factor matrix, and the corresponding elements of an oblique rotation of the sample factor matrix.

i.e. $c_2 = \text{trace} \; (\Phi - F\Lambda)\;(\Phi - F\Lambda)'$

where Λ (m + m) is an oblique transformation matrix chosen so as to minimise c_2. The transformation matrix Λ may be obtained by solving the following equations for Λ and the diagonal matrix D_β (33).

$$F'\Phi - F'F\Lambda = \Lambda D_\beta$$

$$\text{Diag}\;(\Lambda'\Lambda) = I$$

The computational procedure used for solving these equations is given in Appendix C.

The use of an oblique transformation implies that the resulting factor variables are correlated over the sample, although these factor variables are uncorrelated over the population. This is justified if the non zero sample correlations between factors are regarded as due to random sampling fluctuation.

For both measures of accuracy, a value of zero implies perfect agreement between the population factor matrix and the sample factor matrix.

4.3 Identification of sample factor loadings

In order to make possible a comparison of the distributions of individual factor loadings obtained by different methods, all sample

factor /....

factor matrices were rotated orthogonally to a common form with six zero elements determining the factors uniquely. The zero sample factor loadings correspond to the zero population factor loadings which are underlined in Table 4. A triangular matrix may be constructed by selecting rows 5, 9, 4, and 7, of a factor matrix in this form.

The following procedure was used to transform the original factor matrix to the required form.

1) The rows of the original factor matrix F^* were rearranged in the order 5, 9, 4, 7, 1, 2, 3, 6, 8, 10, 11, 12 to form the matrix G^* (12 x 4), and the square matrix $(G^*G^{*\prime})$ was computed.

2) The square root method of triangulating a matrix (11) was applied to $(G^*G^{*\prime})$ to obtain a matrix G(12 x 4) with 6 zero elements

i.e. $g_{ij} = 0 \qquad (j > i)$

3) The rows of G were rearranged to form the factor matrix F with zero elements in the specified positions.

The mean, standard deviation and range of estimates of each factor loading were computed for each of the methods of factor analysis. These values are specific to the identification conditions used. A different choice of zero elements for the sample factor matrices would have resulted in different distributions for the estimated factor loadings.

4. RESULTS

4.1 Accuracy of estimated factor matrices

Columns 2 to 6 of Table 8 and Table 9 give the values of the two measures of accuracy of every estimated factor matrix based on each sample correlation matrix in Set 1 (N = 100, p = 12). When the population correlation matrix was factored, both the maximum likelihood solution and Thomson's solution were exact and had zero values (to three decimal places) for both measures of accuracy. The three other methods of factor analysis provided approximate solutions when applied to the population correlation matrix. When the sample correlation matrices were factored, no single method consistently provided the most accurate result for every sample. Each method yielded the most accurate estimate of the factor matrix for at least one of the twenty samples, but the maximum likelihood method was the most accurate for more samples than any of the other methods.

Using the mean of the twenty values of the first measure of accuracy for each method as criterion, the methods of factor analysis may be ranked in the following order:-

1) Maximum likelihood solution.
2) Thomson's solution.
3) Principal factor solution with S.M.C.s as approximations for communalities.
4) Weighted principal factor solution. (Solution to canonical factor equations with communalities replaced by S.M.C.s (16))
5) Centroid solution.

The ranking obtained when using means of the second measure of accuracy as criterion was similar, except that the positions of the principal factor solution

and /....

TABLE 8. Values of Accuracy Measure 1

Sets 1 and 2. N = 100 m = 4

	12 / 12 Variables					12 / 16 Variables	
Method / Population	Maximum Likelihood	Thomson's Method	Principal F.(S.M.C.)	Weighted P.F.(S.M.C.)	Centroid	Maximum Likelihood	Weighted P.F.(S.M.C.)
Population	.000	.000	.162	.187	.257	.000	.070
1	.466	.408	.562	.602	.998	.469	.522
2	.327	.404	.401	.419	.553		
3	.617	.561	.627	.656	.917	.549	.632
4	.686	.658	.695	.729	.805		
5	.600	.602	.657	.661	1.137		
6	.720	.750	.708	.696	1.047	.377	.401
7	.988	.997	1.009	1.038	1.304	.824	.833
8	.524	.554	.605	.620	.877		
9	.477	.717	.648	.604	.837	.377	.434
10	.826	.982	.926	.933	1.157		
11	.604	.456	.571	.577	.730	.444	.480
12	.653	.702	.714	.740	.992		
13	.770	.769	.847	.848	1.025	.742	.753
14	.594	.731	.751	.734	.994		
15	.770	.803	.815	.836	1.094	.644	.719
16	.724	.762	.802	.789	1.318		
17	.816	1.017	.926	.871	1.157	.680	.686
18	.869	.946	1.089	1.107	1.601		
19	1.622	.945	.869	.899	1.222	.715	.739
20	.829	1.008	.870	.852	.666		
Mean	.724	.739	.754	.761	1.022	.582	.620

TABLE 9 Values of Accuracy Measure 2

Set 1 N = 100 p = 12 m = 4

Method / Population	Maximum Likelihood	Thomson's Method	Principal F. (S. M. C.)	Weighted P. F. (S. M. C.)	Centroid
Population	.000	.000	.059	.077	.123
1	.174	.153	.248	.257	.559
2	.205	.260	.261	.272	.321
3	.258	.198	.242	.261	.367
4	.263	.240	.257	.275	.258
5	.152	.190	.260	226	.999
6	.438	.465	.403	385	.628
7	.233	.232	.278	.293	.426
8	.262	287	.318	.329	.498
9	.307	.478	.429	400	.502
10	.192	.248	.211	.206	.302
11	.264	.206	.258	.264	.342
12	.276	.297	.365	.362	.530
13	.335	.331	.353	.352	.395
14	.242	.426	.444	398	.538
15	.303	.345	.377	.377	.564
16	.321	.369	.370	.360	.707
17	.300	.610	.620	.508	.729
18	.524	.646	.757	.729	1.244
19	.869	.776	.698	.733	.747
20	.388	.480	.417	.401	.308
Average	.315	.362	.378	.374	.547

and weighted principal factor solution were reversed. The difference in either mean accuracy measure between these two solutions which use the same initial approximations for communalities, was smaller than differences between mean accuracy measures of any other two solutions.

The differences in accuracy between the maximum likelihood solution, and Thomson's solution or the two solutions using S. M. C. s as approximations for communalities, are small when compared with inaccuracy in the estimated factor matrices which is due to sampling fluctuation. The differences between the maximum likelihood solution and the two approximate solutions using S. M. C. s for communalities is far more marked when the population correlation matrix is factored than when sample correlation matrices are factored. The centroid solution with highest correlations as initial approximations for communalities is noticeably less accurate than the other solutions. This is probably due more to the poor approximations for communalities than to the method of factoring, as it has been found (6) that differences in approximations for communalities have a greater effect on differences between factor matrices than differences in methods of factoring.

In order to check the computing procedures for obtaining Thomson's solution and the maximum likelihood solution, the sum of squares of non diagonal elements of the residual matrix (R - FF') and the determinant of the partial correlation matrix $D_{1/\sqrt{u}}(R - FF')\,D_{1/\sqrt{u}}$ were calculated for each of the five solutions obtained from each sample. As expected, for every sample Thomson's solution yielded the smallest sum of squares of residuals while the maximum likelihood solution yielded the partial correlation matrix

with /....

with the largest determinant. In several samples the partial correlation matrices for Thomson's solution and for the centroid solution were not positive semidefinite.

The last two columns of Table 8 show the values of the first measure of accuracy of the maximum likelihood solution and of the weighted principal factor solution for the 10 sample correlation matrices which were factored including all 16 variables. The factor matrices obtained were of order (16 x 4), but only the first 12 rows were used when calculating the accuracy index in order to make possible a comparison with the factor matrices obtained when only 12 variables were used. The approximations to the population communalities provided by the population S.M.C.s are improved if the number of variables is increased and the number of factors is held constant (14). As a result, the weighted principal factor method when applied to the population correlation matrix, provided a better approximation when all 16 variables were included than when 12 variables were used. The inclusion of the four extra variables resulted in an improvement in the accuracy of the weighted principal factor estimates for each of the 10 samples, and an improvement in accuracy of maximum likelihood estimates for all samples except the first. The average improvement in accuracy was slightly greater for the maximum likelihood estimates than for the weighted principal factor estimates. Therefore, the increase in number of variables resulted in a greater difference in accuracy between the maximum likelihood estimates and the weighted principal factor estimates. When 16 variables were used, the maximum likelihood solution was the more accurate solution for every one of the ten samples, whereas when 12 variables were

used /....

used the weighted principal factor solution was superior to the maximum likelihood solution in three out of the ten samples.

The values of the two measures of accuracy of the maximum likelihood estimates and weighted principal factor estimates obtained from the sample correlation matrices in Set 3 (n = 1500, p = 12) are shown in Table 10. The increase in sample size resulted in a marked increase in the accuracy of both solutions, the improvement being greater for the maximum likelihood solution. The average difference in accuracy between the two solutions is far greater for the samples of size 1500 than for the samples of size 100.

4.2 Estimates of individual factor loadings

The maximum likelihood factor matrices for the samples in Set 1 (N = 100, p = 12) are given in Appendix B.

Differences between factor loadings obtained by applying different methods to the same sample correlation matrix were fairly small. The magnitude of differences between factor loadings obtained by different methods is illustrated in Table 11 which shows the different estimates of each factor loading for a typical sample (No. 8 in Set 1).

The largest observed value, smallest observed value, difference between mean of estimate and population parameter, and standard deviation are given in Tables 12, 13, 14 and 15 respectively, for each factor loading. These estimates of range, bias and standard deviation of estimated factor loadings are based on small numbers of observations (20 for Set 1, 10 for Sets 2 and 3) and the effect of random error may be marked. However, a rough impression may be gained of the distribution of estimated factor

loadings /...

TABLE 10 Values of Accuracy Measures

Set 3 N = 1500 p = 12

	Index 1		Index 2	
	Maximum Likelihood	Weighted P. F. (S. M. C.)	Maximum Likelihood	Weighted [illegible]
1	.030	.194	.013	[illegible]
2	.033	.217	.022	[illegible]
3	.050	.236	.021	[illegible]
4	.031	.239	.017	[illegible]
5	.042	.192	.014	[illegible]
6	.034	.227	.015	.090
7	.057	.233	.019	.104
8	.031	.216	.019	.093
9	.037	.228	.016	.104
10	.073	.285	.024	.095
Average	.042	.227	.018	.094

TABLE 11 Factor Loadings for Sample 8

N = 100, p = 12, m = 4

Factor 1.

	Population Loadings	Maximum Likelihood	Thomson	Principal F.(S. M. C.)	Weighted P. F.(S. M. C.)	Centroid
1	.00	-.03	-.02	-.03	-.05	-.02
2	.00	-.13	-.13	-.10	-.10	-.04
3	.76	.82	.83	.75	.74	.76
4	.00	.17	.18	.16	.17	.11
5	.78	.75	.71	.73	.75	.74
6	.71	.65	.70	.68	.68	.73
7	.00	-.04	-.03	-.03	-.01	-.02
8	.40	.43	.43	.45	.46	.46
9	.00	.07	.05	.04	.05	.01
10	.00	.06	.07	.08	.08	.07
11	.00	-.04	-.02	-.03	-.02	-.07
12	.00	.07	.11	.12	.11	.19

Factor 2

	Population Loadings	Maximum Likelihood	Thomson	Principal F.(S. M. C.)	Weighted P. F.(S. M. C.)	Centroid
1	.80	.90	.91	.84	.83	.74
2	.70	.77	.83	.77	.76	.81
3	.00	.11	.13	.13	.13	.16
4	.00	.05	.15	.13	.15	.05
5	.00	.00	.00	.00	.00	.00
6	.00	.16	.18	.18	.18	.17
7	.00	-.05	.00	.01	-.03	.16
8	.00	-.03	-.01	-.03	-.03	-.04
9	.45	.51	.47	.51	.52	.58
10	.00	.19	.24	.24	.23	.23
11	.00	.15	.21	.20	.18	.18
12	.32	.37	.38	.39	.41	.33

TABLE 11 (cont.) Factor Loadings for Sample 8

N = 100, p = 12, m = 4

Factor 3.

	Population Loadings	Maximum Likelihood	Thomson	Principal F.(S.M.C.)	Weighted P.F.(S.M.C.)	Centroid
1	.51	.32	.24	.26	.24	.37
2	.00	-.10	-.23	-.19	-.19	-.09
3	.00	-.12	-.17	-.17	-.18	-.05
4	.84	.93	.81	.66	.67	.62
5	.00	.00	.00	.00	.00	.00
6	.00	-.08	-.14	-.13	-.13	-.08
7	.00	-.09	-.12	-.16	-.19	-.18
8	.00	-.01	.00	.00	-.02	.10
9	.00	.00	.00	.00	.00	.00
10	.39	.27	.28	.31	.30	.38
11	.35	.39	.40	.44	.45	.50
12	.00	.13	.12	.15	.13	.28

Factor 4.

	Population Loadings	Maximum Likelihood	Thomson	Principal F.(S.M.C.)	Weighted P.F.(S.M.C.)	Centroid
1	.00	-.08	-.17	-.10	-.03	-.27
2	.60	.43	.38	.33	.34	.18
3	.48	.50	.50	.42	.38	.42
4	.00	.00	.00	.00	.00	.00
5	.00	.00	.00	.00	.00	.00
6	.00	.04	-.05	-.02	-.01	-.05
7	.64	.65	.57	.58	.60	.48
8	.38	.29	.27	.30	.31	.40
9	.00	.00	.00	.00	.00	.00
10	.00	.22	.20	.22	.26	.13
11	.00	.04	.08	.11	.12	.17
12	.00	.04	-.01	-.00	.02	-.21

TABLE 12 Largest Estimates of Factor Loadings.

N = Sample size p = No. of variables

Factor 1

		N = 100 p = 12					N = 100 p = 16		N = 1500 p = 12	
	Population Loadings	Maximum Likelihood	Thomson	Principal F.(S. M. C.)	Weighted P. F.(S. M. C.)	Centroid	Maximum Likelihood	Weighted P. F.(S. M. C.)	Maximum Likelihood	Weighted P. F.(S. M. C.)
1	.00	.24	.26	.25	.25	.22	.27	.27	.07	.04
2	.00	.20	.28	.29	.29	.31	.24	.27	.02	.06
3	.76	.85	.86	.83	.82	.82	.86	.84	.78	.73
4	.00	.32	.32	.32	.32	.33	.31	.31	.09	.11
5	.78	1.00	.91	.84	.84	.85	.87	.85	.80	.76
6	.71	.81	.81	.79	.80	.82	.80	.79	.74	.73
7	.00	.18	.20	.20	.20	.23	.19	.18	.03	.07
8	.40	.58	.57	.58	.59	.61	.58	.58	.43	.45
9	.00	.19	.20	.19	.18	.18	.16	.17	.05	.04
10	.00	.28	.28	.29	.29	.28	.29	.29	.04	.05
11	.00	.16	.19	.19	.18	.20	.14	.15	.05	.05
12	.00	.11	.11	.12	.11	.19	.10	.10	.04	.03
		20 samples					10 samples		10 samples	

TABLE 12 (cont.) Largest Estimates of Factor Loadings.

N = Sample size p = No. of variables

Factor 2

		N = 100 p = 12					N = 100 p = 16		N = 1500 p = 12	
	Population Loadings	Maximum Likelihood	Thomson	Principal F.(S.M.C.)	Weighted P.F.(S.M.C.)	Centroid	Maximum Likelihood	Weighted P.F.(S.M.C.)	Maximum Likelihood	Weighted P.F.(S.M.C.)
1	.80	.97	.95	.84	.83	.79	.95	.90	.83	.71
2	.70	.88	.91	.86	.83	.81	.79	.73	.73	.67
3	.00	.24	.25	.25	.22	.26	.21	.20	.04	.07
4	.00	.39	.39	.42	.43	.50	.54	.54	.04	.11
5	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00
6	.00	.16	.18	.18	.18	.20	.03	.04	.03	.05
7	.00	.43	.50	.52	.50	.60	.15	.17	.07	.06
8	.00	.32	.30	.34	.31	.35	.19	.20	.07	.05
9	.45	.63	.68	.64	.64	.69	.60	.62	.50	.53
10	.00	.27	.30	.31	.28	.35	.23	.23	.04	.03
11	.00	.26	.22	.23	.22	.49	.21	.23	.02	.01
12	.32	.69	.51	.52	.51	.63	.44	.46	.36	.40
		20 samples					10 samples		10 samples	

TABLE 12 (cont.) Largest Estimates of Factor Loadings.

N = Sample size p = No. of variables

Factor 3.

		N = 100 p = 12					N = 100 p = 16		N = 1500 p = 12	
	Population Loadings	Maximum Likelihood	Thomson	Principal F.(S.M.C.)	Weighted P.F.(S.M.C.)	Centroid	Maximum Likelihood	Weighted P.F.(S.M.C.)	Maximum Likelihood	Weighted P.F.(S.M.C.)
1	.51	.83	.83	.64	.62	.71	.83	.72	.59	.52
2	.00	.19	.18	.20	.18	.29	.27	.24	.07	.07
3	.00	.13	.12	.10	.09	.14	.14	.13	.04	.00
4	.84	.99	.99	.77	.78	.73	.86	.75	.86	.69
5	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00
6	.00	.23	.24	.20	.21	.24	.18	.15	.03	.04
7	.00	.17	.15	.14	.14	.22	.15	.15	.03	.00
8	.00	.20	.19	.22	.21	.13	.11	.12	.03	.01
9	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00
10	.39	.54	.54	.59	.61	.66	.54	.58	.41	.44
11	.35	.53	.62	.56	.55	.68	.43	.46	.40	.44
12	.00	.26	.21	.22	.24	.33	.19	.18	.03	.05
		20 samples					10 samples		10 samples	

TABLE 12 (cont.) Largest Estimates of Factor Loadings.

N = Sample size p = No. of variables

Factor 4.

		N = 100 p = 12					N = 100 p = 16		N = 1500 p = 12	
	Population Loadings	Maximum Likelihood	Thomson	Principal F.(S. M. C.)	Weighted P. F.(S. M. C.)	Centroid	Maximum Likelihood	Weighted P. F.(S. M. C.)	Maximum Likelihood	Weighted P. F.(S. M. C.)
1	.00	.31	.39	.34	.36	.44	.24	.25	.04	.14
2	.60	.82	.82	.75	.74	.74	.75	.72	.66	.58
3	.48	.59	.60	.56	.55	.57	.58	.54	.53	.43
4	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00
5	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00
6	.00	.20	.28	.27	.23	.29	.14	.13	.03	.01
7	.64	.74	.81	.71	.73	.73	.71	.70	.68	.63
8	.38	.52	.50	.50	.49	.53	.53	.53	.42	.42
9	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00
10	.00	.23	.33	.38	.33	.33	.16	.18	.06	.08
11	.00	.25	.24	.22	.23	.22	.20	.23	.05	.06
12	.00	.25	.31	.31	.31	.34	.24	.25	.04	.04
		20 samples					10 samples		10 samples	

TABLE 13 Smallest Estimates of Factor Loadings

Factor 1

		N = 100 p = 12					N = 100 p = 16		N = 1500 p = 12	
	Population Loadings	Maximum Likelihood	Thomson	Principal F.(S. M. C.)	Weighted P. F.(S. M. C.)	Centroid	Maximum Likelihood	Weighted P. F.(S. M. C.)	Maximum Likelihood	Weighted P. F.(S. M. C.)
1	.00	-.19	-.18	-.19	-.19	-.23	-.16	-.17	-.03	-.06
2	.00	-.16	-.16	-.16	-.15	-.16	-.15	-.13	-.04	.00
3	.76	.57	.56	.56	.56	.52	.60	.58	.73	.69
4	.00	-.19	-.19	-.14	-.13	-.13	-.17	-.14	-.08	-.06
5	.78	.68	.64	.68	.69	.70	.70	.70	.76	.72
6	.71	.47	.51	.47	.48	.46	.62	.64	.67	.67
7	.00	-.11	-.15	-.14	-.13	-.14	-.06	-.05	-.08	-.04
8	.40	.27	.28	.29	.29	.27	.29	.29	.36	.40
9	.00	-.20	-.22	-.22	-.22	-.25	-.15	-.16	-.05	-.06
10	.00	-.29	-.26	-.26	-.27	-.23	-.29	-.27	-.08	-.07
11	.00	-.14	-.16	-.16	-.15	-.19	-.12	-.13	-.05	-.04
12	.00	-.12	-.10	-.10	-.11	-.12	-.09	-.09	-.03	-.04
		20 samples					[illegible] samples		10 samples	

TABLE 13 (cont.) Smallest Estimates of Factor Loadings

Factor 2.

	Population Loadings	N = 100 p = 12					N = 100 p = 16		N = 1500 p = 12	
		Maximum Likelihood	Thomson	Principal F.(S.M.C.)	Weighted P.F.(S.M.C.)	Centroid	Maximum Likelihood	Weighted P.F.(S.M.C.)	Maximum Likelihood	Weighted P.F.(S.M.C.)
1	.80	.53	.50	.50	.51	.43	.52	.49	.75	.65
2	.70	.43	.40	.36	.37	.34	.46	.45	.63	.57
3	.00	-.23	-.22	-.19	-.19	-.23	-.23	-.21	-.05	-.02
4	.00	-.28	-.29	-.25	-.26	-.22	-.28	-.26	-.04	.01
5	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00
6	.00	-.18	-.17	-.17	-.16	-.17	-.17	-.15	-.01	.00
7	.00	-.40	-.38	-.33	-.33	-.28	-.32	-.34	-.05	-.04
8	.00	-.24	-.26	-.24	-.22	-.23	-.19	-.18	-.05	-.05
9	.45	.36	.36	.40	.40	.42	.38	.40	.41	.45
10	.00	-.32	-.28	-.27	-.29	-.25	-.32	-.29	-.07	-.07
11	.00	-.20	-.26	-.23	-.21	-.28	-.15	-.12	-.05	-.06
12	.32	.12	.08	.12	.13	.12	.18	.18	.27	.30
		20 samples					10 samples		10 samples	

TABLE 13 (cont.) Smallest Estimates of Factor Loadings

Factor 3

		N = 100 p = 12					N = 100 p = 16		N = 1500 p = 12	
	Population Loadings	Maximum Likelihood	Thomson	Principal F.(S.M.C.)	Weighted P.F.(S.M.C.)	Centroid	Maximum Likelihood	Weighted P.F.(S.M.C.)	Maximum Likelihood	Weighted P.F.(S.M.C.)
1	.51	.12	.16	.19	.19	.03	.05	.05	.45	.41
2	.00	-.32	-.23	-.21	-.20	-.50	-.57	-.50	-.08	-.05
3	.00	-.33	-.40	-.37	-.36	-.44	-.25	-.27	-.03	-.09
4	.84	.48	.47	.49	.50	.42	.54	.51	.80	.67
5	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00
6	.00	-.20	-.22	-.22	-.22	-.27	-.14	-.15	-.06	-.06
7	.00	-.47	-.37	-.37	-.41	-.39	-.55	-.54	-.04	-.10
8	.00	-.34	-.40	-.41	-.43	-.52	-.35	-.37	-.03	-.06
9	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00
10	.39	.19	.18	.19	.20	.19	.21	.22	.34	.37
11	.35	.19	.22	.25	.24	.07	.22	.19	.29	.33
12	.00	-.20	-.19	-.20	-.20	-.18	-.16	-.18	-.03	-.04
		20 samples					10 samples		10 samples	

TABLE 13 (cont.) Smallest Estimates of Factor Loadings

Factor 4

		N = 100 p = 12					N = 100 p = 16		N = 1500 p = 12	
	Population Loadings	Maximum Likelihood	Thomson	Principal F.(S. M. C.)	Weighted P. F.(S. M. C.)	Centroid	Maximum Likelihood	Weighted P. F.(S. M. C.)	Maximum Likelihood	Weighted P. F.(S. M. C.)
1	.00	-.29	-.38	-.32	-.22	-.34	-.15	-.08	-.05	.06
2	.60	.27	.15	.05	.13	-.13	.35	.41	.54	.48
3	.48	.11	.14	.14	.11	.19	.22	.21	.45	.39
4	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00
5	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00
6	.00	-.32	-.37	-.35	-.34	-.40	-.25	-.23	-.04	-.04
7	.64	.49	.38	.33	.42	.26	.37	.42	.61	.59
8	.38	.01	-.10	-.15	-.10	-.17	.16	.15	.34	.34
9	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00
10	.00	-.19	-.19	-.22	-.21	-.26	-.16	-.17	-.05	-.04
11	.00	-.26	-.27	-.25	-.26	-.30	-.14	-.15	-.03	-.02
12	.00	-.68	-.37	-.44	-.39	-.54	-.23	-.24	-.05	-.07
		20 samples					10 samples		10 samples	

TABLE 14. Differences between means of estimated factor loadings and corresponding population factor loadings

Factor 1

		N = 100 p = 12					N = 100 p = 16		N = 1500 p = 12	
	Population Loadings	Maximum Likelihood	Thomson	Principal F.(S.M.C.)	Weighted P.F.(S.M.C.)	Centroid	Maximum Likelihood	Weighted P.F.(S.M.C.)	Maximum Likelihood	Weighted P.F.(S.M.C.)
1	.00	.01	.00	-.01	-.01	-.02	.02	.01	.00	-.03
2	.00	.01	.01	.03	.04	.04	.03	.04	.00	-.03
3	.76	.00	-.01	-.03	-.04	-.05	-.01	-.04	-.01	-.05
4	.00	.04	.05	.05	.05	.05	.06	.06	.00	.01
5	.78	.02	.01	-.01	-.01	.00	-.02	.01	.00	-.03
6	.71	-.02	-.01	-.02	-.02	.01	.00	.01	.00	-.01
7	.00	.04	.03	.05	.06	.05	.03	.04	-.01	.03
8	.40	.01	.01	.03	.03	.04	.02	.03	-.01	.02
9	.00	-.01	-.01	-.01	-.01	-.01	.02	.02	.00	-.01
10	.00	.01	.02	.02	.02	.03	.02	.02	-.01	.00
11	.00	.03	.03	.04	.04	.03	-.01	.00	-.01	.00
12	.00	-.01	-.01	-.01	-.01	.01	-.03	-.03	.00	.00
		20 samples					[illegible]ples		10 samples	

TABLE 14 (cont.) Differences between means of estimated factor loadings and corresponding population factor loadings

Factor 2

		N = 100 p = 12					N = 100 p = 16		N = 1500 p = 12	
	Population Loadings	Maximum Likelihood	Thomson	Principal F.(S. M. C.)	Weighted P. F.(S. M. C.)	Centroid	Maximum Likelihood	Weighted P. F.(S. M. C.)	Maximum Likelihood	Weighted P. F.(S. M. C.)
1	.80	-.03	-.06	-.11	-.12	-.16	-.03	-.07	-.01	-.12
2	.70	-.02	-.05	-.08	-.08	-.11	-.03	-.07	.00	-.07
3	.00	-.02	-.01	.01	.01	.01	.00	.01	.00	.04
4	.00	.03	.05	.07	.07	.08	.03	.04	-.01	.06
5	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00
6	.00	-.03	-.04	-.04	-.03	-.02	-.04	-.04	.01	.01
7	.00	-.01	-.01	.01	.00	.03	-.02	-.02	-.01	.00
8	.00	.02	.03	.04	.03	.04	.02	.02	-.01	-.01
9	.45	.04	.07	.08	.08	.13	.04	.07	.01	.05
10	.00	.04	.04	.05	.04	.04	.03	.04	.00	-.01
11	.00	.03	.03	.03	.03	.04	.04	.05	-.01	-.02
12	.32	.01	.02	.03	.04	.06	-.01	.02	.00	.03
		20 samples					10 samples		10 samples	

TABLE 14 (cont.) Differences between means of estimated factor loadings and corresponding population factor loadings

Factor 3.

		N = 100 p = 12					N = 100 p = 16		N = 1500 p = 12	
	Population Loadings	Maximum Likelihood	Thomson	Principal F. (S. M. C.)	Weighted P. F. (S. M. C.)	Centroid	Maximum Likelihood	Weighted P. F. (S. M. C.)	Maximum Likelihood	Weighted P. F. (S. M. C.)
1	.51	-.05	-.05	-.08	-.08	-.08	-.07	-.11	.01	-.04
2	.00	-.04	-.05	-.04	-.03	-.04	-.03	-.02	.01	.03
3	.00	-.05	-.07	-.08	-.08	-.08	-.01	-.03	.00	-.04
4	.84	-.04	-.07	-.18	-.17	-.21	-.10	-.15	-.01	-.16
5	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00
6	.00	-.01	-.02	-.02	-.01	-.01	-.02	-.03	-.01	-.01
7	.00	-.01	-.03	-.04	-.05	-.07	-.05	-.07	.00	-.05
8	.00	-.07	-.08	-.09	-.10	-.13	-.07	-.08	.00	-.02
9	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00
10	.39	-.02	-.01	.01	.02	.06	-.02	.00	-.01	.02
11	.35	.01	.02	.05	.05	.08	-.04	-.02	.00	.03
12	.00	-.01	-.02	-.02	-.01	.00	.03	.02	.01	.01
		20 samples					10 samples		10 samples	

TABLE 14 (cont.) Differences between means of estimated factor loadings and corresponding population factor loadings

Factor 4.

		N = 100 p = 12					N = 100 p = 16		N = 1500 p = 12	
	Population Loadings	Maximum Likelihood	Thomson	Principal F.(S.M.C.)	Weighted P.F.(S.M.C.)	Centroid	Maximum Likelihood	Weighted P.F.(S.M.C.)	Maximum Likelihood	Weighted P.F.(S.M.C.)
1	.00	.01	.04	.06	.09	.08	.01	.05	.00	.10
2	.60	-.01	-.03	-.09	-.09	-.14	-.03	-.06	-.01	-.08
3	.48	-.03	-.04	-.08	-.09	-.10	-.02	-.05	.01	-.08
4	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00
5	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00
6	.00	-.01	-.02	-.01	-.02	-.05	-.01	-.02	.00	-.02
7.	.64	-.02	-.02	-.05	-.03	-.06	-.03	-.03	.00	-.03
8	.38	-.05	-.06	-.06	-.05	-.03	-.04	-.02	.01	.00
9	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00
10	.00	.00	.03	.04	.04	.05	-.01	-.01	.00	.01
11	.00	.00	.02	.02	.03	.00	-.01	.00	.01	.03
12	.00	-.08	-.06	-.07	-.07	-.07	-.02	-.01	.00	-.01
		20 samples					10 samples		10 samples	

TABLE 15   Standard Deviations of Estimates of Factor Loadings

Factor 1.

		N = 100 p = 12					N = 100 p = 16		N = 1500 p = 12	
Variable	Population Loadings	Maximum Likelihood	Thomson	Principal F. (S. M. C.)	Weighted P. F. (S. M. C.)	Centroid	Maximum Likelihood	Weighted P. F. (S. M. C.)	Maximum Likelihood	Weighted P. F. (S. M. C.)
1	.00	.11	.11	.11	.11	.11	.14	.13	.03	.03
2	.00	.11	.12	.12	.12	.12	.13	.13	.02	.02
3	.76	.08	.07	.06	.05	.07	.07	.07	.02	.01
4	.00	.11	.11	.11	.11	.11	.14	.13	.05	.05
5	.78	.08	.07	.04	.04	.04	.06	.05	.01	.01
6	.71	.09	.08	.07	.07	.08	.06	.05	.03	.02
7	.00	.09	.09	.09	.09	.09	.07	.07	.03	.03
8	.40	.09	.09	.09	.09	.10	.10	.10	.02	.02
9	.00	.11	.11	.11	.11	.13	.11	.11	.03	.03
10	.00	.15	.15	.15	.15	.14	.16	.16	.04	.04
11	.00	.10	.10	.10	.10	.11	.09	.09	.03	.03
12	.00	.08	.08	.08	.08	.09	.06	.06	.02	.02
		20 samples					10 samples		10 samples	

TABLE 15 (cont.) Standard Deviations of Estimates of Factor Loadings

Factor 2.

Variable	Population Loadings	N = 100 p = 12 Maximum Likelihood	Thomson	Principal F. (S. M. C.)	Weighted P. F. (S. M. C.)	Centroid	N = 100 p = 16 Maximum Likelihood	Weighted P. F. (S. M. C.)	N = 1500 p = 12 Maximum Likelihood	Weighted P. F. (S. M. C.)
1	.80	.14	.13	.11	.11	.11	.15	.14	.03	.02
2	.70	.12	.14	.13	.12	.13	.11	.10	.03	.03
3	.00	.15	.15	.14	.14	.13	.15	.15	.03	.03
4	.00	.20	.20	.18	.18	.20	.27	.25	.04	.03
5	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00
6	.00	.08	.09	.08	.08	.10	.07	.06	.02	.02
7	.00	.20	.21	.20	.19	.21	.16	.17	.04	.04
8	.00	.15	.16	.16	.15	.17	.12	.12	.03	.03
9	.45	.07	.09	.07	.07	.07	.08	.08	.03	.02
10	.00	.19	.18	.18	.18	.17	.19	.18	.04	.04
11	.00	.13	.15	.14	.14	.17	.12	.13	.02	.02
12	.32	.12	.11	.10	.10	.13	.08	.09	.03	.03
		20 samples					10 samples		10 samples	

TABLE 15 (cont.) Standard Deviations of Estimates of Factor Loadings

Factor 3.

		N = 100 p = 12					N = 100 p = 16		N = 1500 p = 12	
Variable	Population Loadings	Maximum Likelihood	Thomson	Principal F. (S. M. C.)	Weighted P. F. (S. M. C.)	Centroid	Maximum Likelihood	Weighted P. F. (S. M. C.)	Maximum Likelihood	Weighted P. F. (S. M. C.)
1	.51	.18	.18	.14	.14	.16	.26	.23	.04	.03
2	.00	.13	.13	.13	.12	.17	.23	.21	.05	.04
3	.00	.11	.13	.12	.12	.16	.13	.13	.03	.03
4	.84	.14	.13	.08	.07	.08	.10	.07	.02	.01
5	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00
6	.00	.11	.12	.11	.11	.14	.09	.08	.03	.03
7	.00	.14	.12	.12	.13	.16	.19	.19	.02	.03
8	.00	.14	.15	.15	.15	.18	.14	.14	.02	.02
9	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00
10	.39	.10	.10	.10	.10	.11	.12	.12	.02	.03
11	.35	.09	.10	.09	.09	.12	.07	.08	.03	.03
12	.00	.12	.12	.12	.12	.16	.12	.12	.02	.03
		20 samples					10 samples		10 samples	

TABLE 15 (cont.) Standard Deviations of Estimates of Factor Loadings

Factor 4.

		N = 100 p = 12					N = 100 p = 16		N = 1500 p = 12	
Variable	Population Loadings	Maximum Likelihood	Thomson	Principal F.(S.M.C.)	Weighted P.F.(S.M.C.)	Centroid	Maximum Likelihood	Weighted P.F.(S.M.C.)	Maximum Likelihood	Weighted P.F.(S.M.C.)
1	.00	.20	.23	.19	.17	.21	.15	.12	.03	.03
2	.60	.16	.18	.17	.15	.19	.11	.10	.04	.03
3	.48	.12	.11	.10	.10	.11	.12	.11	.03	.01
4	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00
5	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00
6	.00	.15	.15	.14	.13	.18	.12	.12	.03	.02
7	.64	.08	.10	.08	.07	.11	.10	.08	.02	.02
8	.38	.14	.14	.15	.14	.18	.12	.12	.03	.02
9	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00
10	.00	.12	.14	.15	.14	.17	.09	.09	.04	.04
11	.00	.12	.12	.12	.12	.15	.11	.11	.03	.03
12	.00	.21	.20	.20	.20	.26	.16	.17	.03	.04
		20 samples					10 samples		10 samples	

TABLE 16. Convergence of Communality Estimates to Values greater than or equal to 1

N = 100

	Sample Number	1	2	5	6	9	10	12	13	17	19	20
p = 12	Maximum Likelihood	1	1	1	1	2	2	1	2	4	12	3
	Thomson's Solution	1		1	1	2			2	4		3
p = 16	Maximum Likelihood								1			

The number of the variable with communality estimate of 1 is given in each cell of the table.

loadings under the identification conditions used.

Differences in distributions of estimates obtained by different methods are small. The most noticeable differences are in the means of the estimates. The principal factor, weighted principal factor, and centroid methods yielded approximations to the factor loadings when applied to the population correlation matrix. As a result, the means of sample factor loadings obtained by these approximate methods differ appreciably from the population factor loadings. On the other hand, the maximum likelihood method and Thomson's method which gave exact results for the population correlation matrix, yielded sample factor loadings with means which were close to the population factor loadings.

The differences between standard deviations of estimates obtained by different methods are slight. For the samples of size 100, the differences in means between the maximum likelihood estimates and the approximate estimates are unimportant when compared with the variances of the estimates. However, for the samples of size 1500, when the variances are very much smaller, the differences in means of the maximum likelihood estimates and the approximate weighted principal factor estimates are relatively large compared to the variances.

4.3 Communality estimates

In seven of the 20 samples in Set 1 (N = 100, p = 12), Thomson's method tended towards a solution with a communality greater than one (Heywood case). Whenever this occurred, the factor matrix obtained on the iteration preceding the occurrence of the Heywood case was taken as the final solution. In each of these seven samples as well as in four additional samples the maximum determinant procedure tended towards a solution with a communality of unity (See Table 16). Communality estimates of unity occurred in some of the samples which yielded the most accurate estimates of the factor matrix. Although, in the majority of cases the communality estimate of unity occurred for the first variable which had the highest population communality of .9001, communality estimates of unity were also obtained for variables 2, 3, 4 and 12 which had population communalities of .8500, .8080, .7056, and .1024 respectively.

In the 10 samples where the maximum likelihood estimates of communalities were obtained using both 12 and 16 variables, communality estimates of unity occurred in six cases when 12 variables were used. When 16 variables were used, only one sample produced a communality estimate of unity. This sample had yielded a communality estimate of unity for a different variable when the correlation matrix of order 12 was factored. No communality estimate of unity occurred for the samples of size 1500.

Maximum likelihood communality estimates of unity and Thomson communality estimates greater than unity can therefore easily occur even when the population factor structure is reasonably well determined (4 non zero loadings on each factor, 12 variables, 3 factors), the number of cases is fairly large (100) and all population communalities are well under one. Although a population communality greater than or equal to unity is not possible,

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the occurrence of a Heywood case in a sample does not necessarily imply that the factor analysis model does not fit the population from which the sample is drawn, but may be due to sampling fluctuation in the estimate. The probability of the occurrence of a communality estimate of unity is reduced when the number of variables is increased while holding constant the number of factors or when sample size is increased.

The maximum likelihood solution for sample 19 in Set 1 was very poor. Using the weighted principal factor matrix as initial approximation, the maximum determinant computing procedure yielded a communality estimate of 1 (to 3 decimal places) for the twelfth variable which has a population communality of .1024. In addition both measures of accuracy of the factor matrix indicated a very poor result. Convergence was slow and a more accurate result would have been obtained in this particular case if a larger tolerance limit had been used, since the initial approximation was a more accurate estimate than the final solution. The analysis was repeated using the population factor matrix as initial approximation and more accurate estimates of communality were obtained, an estimate of one occurring for the first variable (See Table 17). However, it was found that the solution obtained using the weighted principal factor matrix as initial approximation provided a partial correlation matrix with a larger determinant than the solution obtained when the population factor matrix was used as initial approximation. The first solution was therefore taken to be the maximum likelihood solution (in spite of its inaccuracy).

Table /....

TABLE 17 Converged communalities for sample 19 using two different initial approximations for the factor matrix. Maximum determinant computing procedure (Tolerance limit = .0001)

N = 100 p = 12

Initial Approximation	Variable 1	2	3	4	5	6	7	8	9	10	11	12	Log determinant of Partial Corr. Matrix
Weighted P.F. (S.M.C.)	.95	.90	.77	.42	.60	.59	.46	.43	.17	.09	.12	1.00	-.2479
Pop. Factor Matrix	1.00	.85	.77	.40	.60	.59	.51	.44	.27	.08	.14	.37	-.2529
Population Communalities	.90	.85	.81	.71	.61	.50	.41	.30	.20	.15	.12	.10	

It thus appears that for some samples the determinant of the partial correlation matrix can have two or more relative maxima.

In accordance with the theoretical results (10, 14), the population S.M.C.s were smaller than the population communalities, and differences between population S.M.C.s and population communalities were larger when 12 variables were used than when 16 variables were used. Although population S.M.C.'s are lower bounds for population communalities (14), this does not necessarily apply to sample S.M.C.s and communalities, particularly when sample size is small. Corresponding to most samples of size 100, there were several sample S.M.C.s which were larger than the corresponding maximum likelihood and Thomson communalities. This occurred less frequently with samples of size 1500. Maximum likelihood estimates of S.M.C.'s have a positive bias which increases as the magnitude of the population S.M.C. decreases. Differences between means of estimates of the S.M.C.s and the expected values of these estimates (23 p. 341) were calculated and found to be small (see Table 18). The maximum likelihood estimates of communalities, which are squared multiple correlations between observed variables and the factor variables, showed a similar tendency towards positive bias (see Table 19). Bias in estimates of the smaller communalities was less than the bias in estimates of the corresponding S.M.C.s. Therefore, for samples of size 100, the means of estimates of the smaller communalities were less than the means of the corresponding S.M.C. estimates. (Table 18)

Differences between means of estimates and population communalities, largest observed values, smallest observed values and standard deviations of communality estimates are shown in Table 19.

TABLE 18. Means of estimates of communality and S. M. C.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Population Communalities	.900	.850	.808	.706	.608	.504	.410	.304	.203	.152	.123	.102	.49	.49	.49	.49
Population S. M. C. s (p = 12)	.618	.607	.599	.378	.466	.399	.282	.262	.172	.117	.095	.087				
Mean Communality Thomson (N=100, p=12)	.868	.837	.818	.664	.627	.540	.459	.355	.288	.229	.195	.186				
Mean Communality Max. L. (N=100, p=12)	.901	.872	.826	.709	.647	.518	.448	.350	.259	.214	.180	.191				
Mean Communality M. L. (N=100, p=12) excl. Sample 19	.899	.871	.829	.724	.650	.514	.448	.346	.264	.220	.183	.149				
Mean S. M. C. (N=100, p=12)	.661	.653	.658	.428	.531	.462	.371	.353	.271	.228	.207	.187				
Mean S. M. C. - Expected value	+.005	+.007	+.018	-.015	+.010	.000	+.012	+.013	+.010	+.015	+.014	.000				
Mean Comm. M. L. (N=1500, p=12)	.899	.845	.809	.693	.608	.504	.410	.296	.214	.146	.124	.104				
Mean S. M. C. (N=1500, p=12)	.618	.605	.598	.376	.467	.400	.301	.257	.182	.115	.098	.092				
Mean S. M. C. - Expected Value	-.002	-.005	-.004	-.006	-.002	-.004	.013	-.010	+.005	-.008	-.003	-.002				
Population S. M. C. s (p=16)	.694	.671	.642	.493	.496	.420	.316	.268	.178	.125	.100	.090	.409	.424	.387	.371
Mean Comm. M. L. (N=100, p=16)	.905	.854	.826	.636	.646	.539	.445	.352	.262	.214	.135	.144	.470	.584	.464	.500
Mean S. M. C. (N=1500, p=16)	.740	.722	.708	.509	.577	.512	.407	.399	.309	.242	.232	.219	.461	.554	.449	.459
Mean S. M. C. - Expected Value	.003	.006	.015	-.056	+.009	.008	-.009	.024	.009	-.013	-.003	-.008	-.033	.047	-.027	-.004

TABLE 19. Difference between mean of estimate and population parameter, range, and standard deviation, for estimates of communalities

			1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Population Communalities			.900	.850	.808	.706	.608	.504	.410	.304	.203	.152	.123	.102	.490	.490	.490	.490
Maximum Likelihood	N = 100 p = 12 20 samples	Mean - Pop. Comm.	.001	.022	.018	.003	.039	.014	.039	.046	.057	.061	.058	.089				
		Highest Value	1.000	1.000	1.000	1.000	.996	.685	.671	.550	.411	.360	.343	1.000				
		Lowest Value	.734	.637	.645	.307	.466	.339	.269	.150	.135	.064	.051	.057				
		Standard Deviation	.094	.090	.081	.187	.136	.096	.114	.096	.076	.078	.078	.196				
	N = 100 p = 16 10 samples	Mean - Pop. Comm.	.005	.004	.018	-.069	.037	.035	.035	.048	.059	.062	.012	.042	-.020	.094	-.026	.010
		Highest Value	1.000	.913	.898	.774	.762	.652	.592	.519	.382	.356	.235	.220	.585	.770	.573	.633
		Lowest Value	.806	.740	.672	.499	.486	.410	.327	.227	.144	.083	.072	.068	.366	.419	.359	.351
		Standard Deviation	.060	.056	.067	.101	.090	.090	.087	.094	.084	.094	.049	.047				
	N = 1500 p = 12 10 samples	Mean - Pop. Comm	-.001	-.005	.001	-.013	.001	.001	.000	-.008	.011	-.006	.002	.002				
		Highest Value	.939	.931	.841	.738	.638	.552	.461	.340	.246	.168	.161	.135				
		Lowest Value	.845	.797	.775	.647	.576	.454	.374	.279	.168	.121	.088	.076				
		Standard Deviation	.033	.046	.022	.030	.018	.037	.027	.019	.023	.015	.022	.021				
Thomson	N = 100 p = 12 10 samples	Mean - Pop. Comm.	-.032	-.013	.010	-.042	.019	.035	.050	.050	.085	.077	.072	.083				
		Highest Value	.999	.993	.993	.999	.821	.677	.776	.562	.474	.387	.518	.345				
		Lowest Value	.599	.631	.648	.319	.407	.387	.193	.182	.147	.070	.066	.061				
		Standard Deviation	.117	.090	.086	.174	.103	.096	.127	.098	.095	.080	.096	.078				

4.4 Computing times

Examination of Table 20 will reveal the relative computing times taken by different methods of factor analysis on the I.B.M. 7094 computer. Average times for the maximum determinant method and Thomson's method for Set 1, were calculated over the nine samples where neither method was terminated prematurely due to convergence of communality estimates to values greater than or equal to one. Computing times for these two methods do not include time required for obtaining the initial approximations.

Although the maximum determinant method required more iterations than Thomson's method before convergence within the specified limit of .0001 was attained, each iteration was very much faster so that the overall computing time was shorter.

Increasing the number of variables from 12 to 16 resulted in a marked decrease in the average number of iterations required for the maximum determinant procedure to converge within the limit of .0001. Although time per iteration was increased, overall computing time was reduced. Increasing sample size from 100 to 1500 while retaining 12 variables resulted in a smaller, but still substantial, reduction in average number of iterations and average computing time.

4.5 Procedures for estimating number of factors

Examination of Table 21 will enable a comparison to be made of the results obtained with the various procedures for estimating number of factors.

In order to examine the efficacy of a sequence of likelihood ratio

tests /....

TABLE 20. Average computing times

		Average Time (Minutes)	Average Number of Iterations**
	Maximum determinant*	.53	82
N = 100	Thomson	.94	55
p = 12	Principal factor (S. M. C.)	.04	
20 samples	Weighted principal factor (S. M. C.)	.04	
	Centroid	< .01	
N = 100 p = 16 10 samples	Maximum Likelihood	.21	22
	Weighted P. F. (S. M. C.)	.11	
N = 1500 p = 12 10 samples	Maximum Likelihood	.26	41
	Weighted P. F. (S. M. C.)	.04	

* Average times for the maximum determinant method and Thomson's method for samples of size 100 were calculated over the nine samples for which neither method converged to communalities greater than or equal to 1.

** "Iteration" refers to the set of computational steps required to obtain new estimates of all the elements of the factor matrix.

TABLE 21. Frequency distributions for number of factors estimated

Correct number of factors = 4

	No. of factors estimated	Sequence of likelihood ratio tests. 5% Sig. level	Sequence of likelihood ratio tests. 1% Sig. level	Sequence of Joreskog tests 5% Sig. level	No. of roots of Correlation matrix ≥ 1	No. of roots of Correlation matrix with S. M. C. s in diagonal ≥ 0	Saunders' procedure
Set 1	3	5	11	1			
N = 100	4	15	9	11	16		
p = 12	5			4	4		8
20 samples	6			3		6	12
	≥ 7			1		14	
Set 2	3	1	1				
N = 100	4	9	9	10	4		
p = 16	5				6		
10 samples							
Set 3	4	10	10		10		
N = 1500	≥ 7			10			
p = 12							
10 samples							

tests for estimating number of factors, the likelihood ratio test statistic was calculated for three and for four factors for each of the samples in Sets 1 and 2. From an examination of approximations to the likelihood ratio statistic for two factors obtained from the weighted principal factor solution, it was obvious that the likelihood ratio statistic for two factors would be very highly significant in every sample. Similarly, examination of approximations revealed that all likelihood ratio test statistics for three factors obtained from samples of size 1500 would be very highly significant. For these samples the tests for four factors only were computed. No likelihood ratio test statistic for four factors was significant at the 5% level for any of the samples used.

The tendency of the sequence of likelihood ratio tests to give an underestimate of the number of factors was apparent in the samples of size 100 when 12 variables were used. There was a marked drop in this tendency when 16 variables were used. The five samples in Set 1 for which the likelihood ratio test statistic was not significant at the 5% level were all included in the 10 samples which were re-analysed with the four additional variables. Using 16 variables, the likelihood ratio statistic for three factors from only one of these samples was non-significant. The means of the likelihood ratio statistic for three factors, degrees of freedom, 5% significance levels, and the values of the logs of the determinants of the population partial correlation matrices when three factors were extracted from 12 and from 16 variables are given in Table 22. Increasing number of variables from 12 to 16, thereby increasing degrees of freedom for the

TABLE 22. Likelihood Ratio Statistic for 3 factors

	12 Variables	16 Variables
Mean of likelihood ratio statistic for 3 factors (N = 100)	55.8	126.5
Degrees of freedom for χ^2 distribution (null hypothesis)	33	75
5% significance level	47.4	96.2
Log of determinant of population partial correlation matrix (3 factors obtained by maximum determinant method)	-.263	-.782

chi square distribution, resulted in a marked increase in the distance between the distribution of the likelihood ratio statistic under the alternate hypothesis and the chi square distribution of the statistic under the null hypothesis.

The power of the test for number of factors was also increased when sample size was increased. The sequence of likelihood ratio tests would have given a correct estimate of number of factors for every sample of size 1500.

Frequency distributions, means, and variances of the likelihood ratio statistics for four factors are given in Table 23. Differences between the observed frequency distributions of the likelihood ratio test statistic for four factors and the corresponding chi square frequency distributions were small for all three sets of observations. However, for all three sets of observations, the variance of the values of the likelihood ratio test statistic was smaller than the theoretical variance of the corresponding chi square distribution. This discrepancy was more marked for the samples of size 100 than for the samples of size 1500 and was largest when 16 variables were used. Because of the small numbers of observations a firm conclusion cannot be reached.

The probabilities of the test statistics for four factors are given in the last columns of Tables 6 and 7 for Sets 1 and 3.

Artificial experiments on the likelihood ratio test statistic for number of factors have been carried out by Henrysson (17) using 12 samples of size 200 from a population with 9 variables and 1 factor, and Lawley and Swanson (27), using 8 samples of size 50 from a population with 7 variables and 2 factors. Henrysson used Lawley's (24) approximation to the statistic while Lawley and Swanson used the exact form of the statistic. The variance of the values of the test statistic obtained in Henrysson's experiment was smaller than the theoretical variance of the corresponding chi square distribution. On the other hand, in Lawley and Swanson's experiment the variance of the values of the statistic was larger than the theoretical variance of the corresponding chi square distribution.

The likelihood ratio significance test is applicable only to maximum likelihood estimates of factor loadings. Replacement of the maximum likelihood factor matrix by any other factor matrix will inflate the magnitude of the test statistic. This is illustrated in Table 24 where the values of the test statistic for maximum likelihood, Thomson and Centroid estimates is shown. When Thomson or Centroid estimates were used, in several samples from Set 1, the determinant of the partial correlation matrix was negative so that the test statistic did not exist.

Column 3 of Table 24 shows the values of Lawley's approximation to the likelihood ratio statistic for four factors for samples from Set 1. This approximation was very close to the true value of the likelihood ratio statistic as the correct number of factors was taken and the elements of the partial correlation matrix were small. The approximation would be poorer if the number of factors was underestimated and the elements of the partial correlation matrix were large.

The decision procedure based on a sequence of Joreskog tests gave poor results when 12 variables were used, particularly for samples of

size /....

TABLE 23. Distribution of likelihood ratio test statistic for 4 factors

		Probability (Asympotic χ^2 distribution)					No. of Samples	Mean of Statistic	Mean of χ^2 distribution	Variance of Statistic	Variance of χ^2 distribution
		0< .2	.2 < .4	.4 < .6	.6< .8	.8 < 1.0					
N = 100 p = 12 m = 4	Observed Frequency Expected Frequency	3 5	3 5	4 5	7 5	3 5	20	24.69	24	30.84	48
N = 1500 p = 12 m = 4	Observed Frequency Expected Frequency	3 2	1 2	3 2	3 2	0 2	10	21.97	24	45.12	48
N = 100 p = 16 m = 4	Observed Frequency Expected Frequency	1 2	2 2	3 2	3 2	1 2	10	61.65	62	41.84	124

TABLE 24. Likelihood ratio statistic for four factors

N = 100 p = 12

	Max. Likelihood		Thomson	Centroid
Sample No.	L. Ratio Statistic	Lawley Approx.	L. Ratio Statistic	L. Ratio Statistic
1	15.81	15.77	*	*
2	24.39	22.36	54.20	99.04
3	18.87	18.57	36.10	126.96
4	19.95	18.41	28.01	81.74
5	23.21	22.90	258.45	91.26
6	19.13	18.50	*	88.65
7	16.86	16.38	26.76	70.23
8	27.30	25.19	*	104.25
9	27.70	28.17	*	84.82
10	26.73	26.64	37.54	68.67
11	17.65	18.23	25.67	64.80
12	26.37	23.12	64.01	96.38
13	25.00	23.44	144.26	63.40
14	35.66	34.25	50.21	142.75
15	33.90	30.23	62.85	89.97
16	28.53	29.46	49.55	*
17	26.08	28.76	*	92.10
18	32.24	31.11	46.38	*
19	22.69	21.38	31.75	*
20	25.66	23.67	*	152.04

* Partial correlation matrix not positive semi-definite.

size 1500. Differences between observed means and variances and the means and variances of the corresponding chi square distributions were large (see Table 25). Results for 16 variables and samples of size 100 were good. The Joreskog test is based on an assumption which holds approximately for the population correlation matrix used in this study, the closeness of the approximation being improved as number of varialbes is increased from 12 to 16 while the number of factors is held constant (20). The effect of the approximate fit of the model on the distribution of the test statistic becomes larger as sample size is increased. Although good results were obtained for 16 variables and sample size of 100, poorer results would have been obtained had sample size been increased while retaining 16 variables.

For the particular population matrix chosen for this study, the number of latent roots greater than or equal to unity was equal to the number of factors both when 12 variables and when 16 variables were used. The application of this criterion for the number of factors to sample correlation matrices tended to yield an overestimate for samples of size 100. This tendency was far more marked when 16 variables were used, than when 12 variables were used. The correct number of factors was given for every sample of size 1500.

Latent roots for sample correlation matrices in Set 1 are shown in Table 26. Population latent roots and the differences between means of sample latent roots and the population roots are given in Table 27. There was a tendency for the larger sample latent roots to overestimate and the smaller sample latent roots to underestimate the corresponding population

latent /....

TABLE 25. Mean and Variance of Joreskog statistic

	Mean of Statistic	Mean of χ^2 Distribution	Variance of Statistic	Variance of χ^2 Distribution
N = 100 p = 12 m = 4	48.03	35	61.88	70
N = 1500 p = 12 m = 4	339.60	35	285.28	70
N = 100 p = 16 m = 4	83.30	77	65.81	154

TABLE 26. Latent roots of sample correlation matrices (1 in Diagonal)

N = 100 p = 12

Sample No. \ Root No.	1	2	3	4	5	6	7	8	9	10	11	12
1	2.603	2.279	1.411	1.111	1.025	.855	.685	.622	.524	.378	.303	.203
2	2.749	1.984	1.679	1.283	.898	.806	.704	.669	.411	.398	.285	.134
3	2.804	2.060	1.803	1.250	.923	.753	.619	.517	.420	.382	.315	.154
4	2.668	1.954	1.713	1.420	.918	.831	.667	.548	.439	.417	.254	.172
5	2.959	2.480	1.296	1.037	.958	.823	.610	.539	.500	.405	.270	.125
6	2.651	2.128	1.469	1.174	.980	.861	.753	.612	.500	.414	.294	.164
7	3.162	2.225	1.609	1.199	.875	.702	.622	.504	.421	.275	.231	.175
8	2.748	2.227	1.440	1.172	1.024	.826	.746	.610	.418	.405	.230	.153
9	2.755	2.005	1.602	1.218	1.069	.783	.724	.669	.415	.318	.268	.174
10	2.623	1.903	1.725	1.327	.925	.812	.710	.635	.487	.363	.294	.198
11	2.928	1.938	1.414	1.202	.941	.857	.722	.631	.526	.420	.288	.133
12	2.790	2.455	1.741	1.149	.883	.764	.621	.478	.416	.280	.273	.150
13	2.644	2.033	1.791	1.337	.797	.784	.722	.627	.411	.387	.272	.196
14	2.827	2.556	1.498	1.193	.977	.736	.576	.498	.395	.352	.273	.119
15	2.983	2.405	1.375	1.034	.964	.824	.640	.586	.470	.341	.214	.163
16	2.759	1.856	1.458	1.387	1.030	.856	.741	.629	.482	.422	.234	.145
17	3.148	2.023	1.512	1.227	.906	.761	.664	.636	.419	.362	.188	.154
18	2.701	2.129	1.651	1.210	.971	.795	.670	.549	.438	.415	.261	.209
19	2.838	2.208	1.724	1.113	.863	.791	.682	.538	.466	.377	.242	.158
20	2.703	2.097	1.703	1.453	.911	.671	.634	.568	.556	.399	.203	.101
Mean	2.802	2.147	1.581	1.225	.942	.795	.676	.583	.456	.375	.260	.159
Population	2.569	2.107	1.542	1.195	.867	.864	.687	.653	.554	.450	.324	.187

TABLE 27. Differences between means of latent roots of sample correlation matrices and corresponding latent roots of the population correlation matrix.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Population Latent Roots p = 12	2.569	2.107	1.542	1.195	.867	.864	.687	.653	.554	.450	.324	.187				
Mean - Population Roots N = 100	.233	.040	.039	.030	.075	-.069	-.011	-.070	-.098	-.075	-.064	-.028				
Mean - Population Roots N = 1500	-.012	-.005	-.025	.023	.037	-.018	-.003	.000	-.003	-.001	.001	.001				
Population Latent Roots p = 16	3.083	2.631	2.028	1.640	.874	.865	.764	.714	.663	.553	.503	.435	.419	.369	.284	.173
Mean - Population Roots N = 100	.410	.021	-.004	-.107	.146	.037	.017	-.002	-.058	-.032	-.062	-.056	-.095	-.095	-.081	-.036

latent roots.

The number of latent roots greater than or equal to zero of the population correlation matrix with diagonal elements replaced by S. M. C. s was equal to the number of factors. This criterion for number of factors greatly overestimated the number of factors when applied to sample correlation matrices from Set 1. Whereas, only the first four population latent roots were greater than zero, the means of the first seven sample latent roots were greater than zero. (See Table 28)

Saunders' procedure for estimating number of factors was not followed in that, at each iteration of Thomson's method, the number of positive latent roots greater in magnitude than the absolute value of the smallest negative latent root was not taken as the number of factors for the following iteration. However, the number of positive latent roots greater in magnitude than the absolute value of the smallest negative latent root after Thomson's method has converged for four factors, will give an indication of the results of Saunders' procedure. This criterion gave an overestimate of the number of factors for every sample in Set 1, and the mean of the fifth sample latent root was greater than the absolute value of the mean of the smallest sample latent root. (See Table 28)

4.6 Conclusions

The results of this study demonstrate the superiority of maximum likelihood estimates of factor loadings over the other estimates considered. Although considerable computation is involved in the maximum determinant computing procedure, more computation is required for Thomson's method

which /....

TABLE 28. Means of latent roots of sample correlation matrices with diagonal elements replaced by S.M.C.s and by Thomson estimates of communalities

p = 12 N = 100 20 samples

		1	2	3	4	5	6	7	8	9	10	11	12
S.M.C.s in Diagonal	Population	2.04	1.57	.87	.48	-.02	-.03	-.06	-.08	-.12	-.13	-.17	-.25
	Mean	2.33	1.65	.98	.58	.22	.10	.01	-.06	-.13	-.18	-.22	-.27
Thomson Communalities in Diagonal	Population	2.20	1.78	1.07	.61	.00	.00	.00	.00	.00	.00	.00	.00
	Mean	2.45	1.79	1.14	.69	.24	.13	.07	.02	-.03	-.08	-.13	-.21

which generally provides slightly less accurate estimates. There is little justification for use of Thomson's method in preference to the maximum determinant method.

The estimates of factor loading obtained using S.M.C.s as approximations for communalities in conjunction with the principal factor method or weighted principal factor method are slightly inferior to maximum likelihood estimates, but far less computation is involved. If sample size is small, differences in accuracy between the maximum likelihood estimates and the approximations are small compared to the effect of sampling fluctuation on the estimates. These differences become larger in relation to the sampling fluctuation as sample size is increased. When sample size is small the gain in accuracy obtained by using the maximum determinant method instead of an approximate method does not warrant the additional computing time.

A maximum likelihood communality estimate of unity, or a Thomson communality estimate greater than unity can occur due to random fluctuation in the estimates. Therefore the occurrence of a sample communality greater than or equal to unity does not necessarily imply that the factor analysis model with the specified number of factors does not fit the population from which the sample is drawn.

It was found that increasing the ratio of number of variables to number of factors, or, equivalently the degree of overdetermination of the population factor structure as determined by the Lederman inequality (28), and increasing sample size, both have the following effects:-

a) Accuracy of estimates of factor loadings is increased, the increase being greater for maximum likelihood estimates than for the estimates obtained by approximate methods.

b) The probability of the occurrence of a maximum likelihood communality estimate of unity is reduced.

c) The number of iterations required for convergence of the maximum determinant computing procedure is reduced.

No method for estimating the number of factors proved completely satisfactory, but the decision procedure based on a sequence of likelihood ratio tests and the criterion of number of latent roots greater than unity of the sample correlation matrix gave results which were preferable to the results of the other methods considered. The sequence of likelihood ratio tests has a tendency to underestimate the number of factors. This tendency is reduced as sample size is increased and as the degree of overdetermination of the population factor matrix is increased. The number of latent roots greater than or equal to unity of the sample correlation matrix tends to give an overestimate of a lower bound to the number of factors. As sample size is increased the probability that this criterion will give an overestimate of the lower bound to the number of factors is reduced. A disadvantage of the criterion is that the probability of an overestimate being obtained appears to increase as the degree of over-determination of the population factor matrix is increased.

REFERENCES

1. Appel, Klaus, "Solution of eigenvalue problems with approximately known eigenvectors," Communications A.C.M., 5, 381, (1962).

2. Anderson, T.W., An introduction to multivariate statistical analysis, John Wiley and Sons, Inc., New York, (1958).

3. Anderson, T.W., and Rubin, H., "Statistical inference in factor analysis," Proc. Third Berkely Symposium, 5, 111-150, (1956).

4. Bargmann, Rolf E., A study of independence and dependence in multivariate normal analysis, Univ. North Carolina, Inst. Statist., Mimeograph Series No. 186, (1957).

5. Bargmann, Rolf E., Representative ordering and selection of variables, Virginia Polytechnic Inst., Blacksburg Va., (1962).

6. Bechtoldt, Harold P., "An empirical study of the factor analysis stability hypothesis," Psychometrika, 26, 405-432, (1961).

7. Bartlett, M.S., "Tests of significance in factor analysis," Brit. Jour. Psych. (Stat. Sec.), 3, 77-85, (1950).

8. Bartlett, M.S. "A note on the multiplying factors for various approximations," J. Roy. Stat. Soc., B, 16, 296-298, (1954).

9. Box, G.E.P., and Muller, M.E., "A note on the genpration of random normal deviates," Annals Math. Stat., 29, 610-611, (1958).

10. Dwyer, Paul S., "The contribution of an orthogonal multiple factor solution to multiple correlation," Psychometrika, 4, 163-171, (1939).

11. Dwyer, Paul S., "The square root method and its use in correlation and regression," J. A.S.A., 40, 493-503, (1945).

12. Greenstadt, John, "The determination of the characteristic roots of a matrix by the Jacobi method," Chapt. 7, Mathematical methods for digital computers, ed. Anthony Ralston, and Herbert, S. Wilf., Wiley, New York, (1960).

13. Guttman, Louis, "Some necessary conditions for common factor analysis," Psychometrika, 19, 149-161, (1954).

14. Guttman, Louis, "Best possible systematic estimates of communalities," Psychometrika, 21, 273-285, (1956).

15. Harman, Harry H., Modern factor analysis, Univ. Chicago Press, (1960).

16. Harris, Chester W., "Some Rao-Guttman relationships," Psychometrika, 27, 247-263, (1962).

17. Henrysson, Sten, "The significance of factor Loadings - Lawley's test examined by artificial samples," Brit. Jour. Psych. (Stat. Sec.), 3, 159-165, (1950).

18. Hotelling, Harold, "Analysis of a complex of statistical variables into principal components," Jour. Ed. Psych., 24, 417-441, 498-520, (1933).

19. Howe, W. G. Some contributions to factor analysis. Report No. ORNL-1919, Oak Ridge National Laboratory, Oak Ridge, Tenn., (1955).

20. Jöreskog, K. G., "On the statistical treatment of residuals in factor analysis," Psychometrika, 27, 335-354, (1962).

21. Kaiser, Henry F., "The Application of electronic computers to factor analysis," Ed. Psych. Meas., 20, 141-151, (1960).

22. Kaiser, Henry F., and Dickman, Kern, "Sample and population score matrices and sample correlation matrices from an arbitrary population correlation matrix," Psychometrika, 27, 179-182, (1962).

23. Kendall, Maurice G., and Stuart, Alan, The Advanced Theory of statistics. Volume 2. Charles Griffin and Co., London, (1961).

24. Lawley, D. N., "The estimation of factor loadings by the method of maximum likelihood," Proc. Roy. Soc. Edin., A, 60, 64-82, (1940).

25. Lawley, D. N., "The application of the maximum likelihood method to factor analysis," Brit. Jour. Psych. 33, 172-175, (1943).

26. Lawley, D. N., "A modified method of estimation in factor analysis and some large sample results," Uppsala symposium on psychological factor analysis, Nordisk Psykologi's Mon. Ser. No. 3, Uppsala, p. 35, (1953).

27. Lawley, D. N., and Swanson, Z., "Tests of significance in a factor analysis of artificial data," Brit. Jour. Stat. Psych., 7, 75-79, (1954).

28. Lederman, Walter, "On the rank of the reduced correlational matrix in multiple factor analysis," Psychometrika 2, 85-93, (1937).

29. Lord, Frederick M., "A study of speed factors in tests and academic grades," Psychometrika, 21, 31-50, (1956).

30. Maxwell, A. E., A comparative study of some factor analytic techniques. Unpublished doctoral dissertation, Univ. Edinburgh, (1957).

31. McNemar, Quinn, "On the sampling error of factor loadings," Psychometrika, 6, 141-152, (1941).

32. Mosier, Charles I., "Influence of chance error on simple structure," Psychometrika, 4, 33-44, (1939).

33. Mosier, Charles I., "Determining a simple structure when loadings for certain tests are known," Psychometrika, 4, 149-162, (1939).

34. Muller, Mervin E., "A comparison of methods for generating normal deviates on digital computers," Jour. A.C.M., 6, 376-383, (1959).

35. Olkin, I., and Roy, S. N., "On multivariate distribution theory," Ann. Math. Stat., 25, 329-339, (1954).

36. Rao, C. R., "Estimation and tests of significance in factor analysis," Psychometrika, 20, 93-111, (1955).

37. Roy, S. N., Some aspects of multivariate analysis, Wiley, New York, (1957).

38. Saunders, D. R., "Further implications of Mundy-Castle's correlations between EEG and Wechsler-Bellevue variables," Jour. Nat. Inst. Pers. Research, 8, 91-101, (1960).

39. Sokal, Robert R., "A comparison of five tests for completeness of factor extraction", Transactions Kansas Acad. Science, 62, 141-152, (1959).

40. Taussky, Olga, and Todd, John, "Generation and testing of pseudo random numbers," Symposium on Monte Carlo Methods, ed. Herbert Meyer, Wiley, New York, (1956).

41. Teichroew, "Computation of an empirical sampling distribution for the W classification statistic," Studies in item analysis and prediction, Chapt. 16, ed. Herbert Solomon, Stanford Univ. Press. (1961).

42. Thurstone, L. L., Multiple factor analysis, Univ. Chicago Press, Chicago, (1947).

43. Thomson, G. H., "Hotelling's method modified to give Spearman's g," Jour. Ed. Psych., 25, 366-374, (1934).

44. Whittle, P. "On principal components and least square methods of factor analysis," Skand. Atuar., 35, 223-239, (1952).

45. Wold, H., "Some artificial experiments in factor analysis," Uppsala Symposium on psychological factor analysis," Nordisk Psykologi's Mon. Ser. No. 3, Uppsala, (1953).

46. Wrigley, Charles, "The distinction between common and specific variance in factor theory," Brit. Jour. Stat. Psych., 10, 81-98, (1957).

47. Wrigley, Charles, "The effect upon the communalities of changing the estimate of the number of factors," Brit. Jour. Stat. Psych., 12, 34-54, (1959).

48. Wrigley, Charles, and Neuhaus, Jack O., "The use of an electronic computer in principal axes factor analysis," Jour. Ed. Psych., 46, 31-41, (1955).

APPENDIX A.

SAMPLE CORRELATION MATRICES

Set 1. (N = 100, p = 12)

SAMPLE CORRELATION MATRIX. NO. 1. (N = 100).

	1	2	3	4	5	6	7	8
1	1.000000	0.599259	0.070566	0.442227	-0.094372	-0.164388	0.026418	0.129250
2	0.599259	1.000000	0.278814	0.133155	-0.087676	-0.223377	0.326067	0.193140
3	0.070566	0.278814	1.000000	0.054313	0.523453	0.470449	0.305431	0.441121
4	0.442227	0.133155	0.054313	1.000000	-0.018996	-0.126591	0.117062	0.092903
5	-0.094372	-0.087676	0.523453	-0.018996	1.000000	0.498141	-0.014959	0.237843
6	-0.164388	-0.223377	0.470449	-0.126591	0.498141	1.000000	-0.056849	0.233383
7	0.026418	0.326067	0.305431	0.117062	-0.014959	-0.056849	1.000000	0.223520
8	0.129250	0.193140	0.441121	0.092903	0.237843	0.233383	0.223520	1.000000
9	0.375853	0.302709	-0.033006	0.112192	-0.084602	-0.126625	0.020591	0.088413
10	0.384486	0.069131	0.017848	0.467470	-0.095147	0.017326	0.121737	-0.008745
11	0.248924	0.094283	0.046543	0.301235	0.011731	-0.010654	-0.035238	-0.018329
12	0.296267	0.278623	0.049639	-0.001616	-0.055223	0.020692	0.058087	0.056732

	9	10	11	12
1	0.375853	0.384486	0.248924	0.296267
2	0.302709	0.069131	0.094283	0.278623
3	-0.033006	0.017848	0.046543	0.049639
4	0.112192	0.467470	0.301235	-0.001616
5	-0.084602	-0.095147	0.011731	-0.055223
6	-0.126625	0.017326	-0.010654	0.020692
7	0.020591	0.121737	-0.035238	0.058087
8	0.088413	-0.008745	-0.018329	0.056732
9	1.000000	0.037892	0.103520	-0.003675
10	0.037892	1.000000	0.120913	0.031727
11	0.103520	0.120913	1.000000	0.020158
12	[illegible]	0.031727	0.020158	1.000000

SAMPLE CORRELATION MATRIX. NO. 2. (N = 100).

	1	2	3	4	5	6	7	8
1	1.000000	0.656792	-0.005522	0.405633	0.141410	0.053998	0.045225	-0.071035
2	0.656792	1.000000	0.257295	0.098659	0.075834	0.070609	0.364115	0.153497
3	-0.005522	0.257295	1.000000	0.060757	0.619657	0.553855	0.349159	0.510058
4	0.405633	0.098659	0.060757	1.000000	0.012984	-0.022536	0.038222	0.027794
5	0.141410	0.075834	0.619657	0.012984	1.000000	0.533649	0.012116	0.295335
6	0.053998	0.070609	0.553855	-0.022536	0.533649	1.000000	0.025830	0.371711
7	0.045225	0.364115	0.349159	0.038222	0.012116	0.025830	1.000000	0.290305
8	-0.071035	0.153497	0.510058	0.027794	0.295335	0.371711	0.290305	1.000000
9	0.292887	0.292752	0.019400	0.010471	0.087284	0.045661	0.054279	0.068030
10	0.184143	-0.111704	-0.016652	0.424219	0.136218	0.069244	-0.014819	-0.145201
11	0.155746	0.012570	0.109643	0.266652	0.041792	-0.067928	0.167207	-0.016135
12	0.179933	0.224280	0.045347	-0.144559	-0.003732	0.201647	-0.012587	0.010739

	9	10	11	12
1	0.292887	0.184143	0.155746	0.179933
2	0.292752	-0.111704	0.012570	0.224280
3	0.019400	-0.016652	0.109643	0.045347
4	0.010471	0.424219	0.266652	-0.144559
5	0.087284	0.136218	0.041792	-0.003732
6	0.045661	0.069244	-0.067928	0.201647
7	0.054279	-0.014819	0.167207	-0.012587
8	0.068030	-0.145201	-0.016135	0.010739
9	1.000000	-0.087800	0.052145	0.214607
10	-0.087800	1.000000	0.148127	-0.133091
11	0.052145	0.148127	1.000000	-0.043156
12	0.214607	-0.133091	-0.043156	1.000000

SAMPLE CORRELATION MATRIX. NO. 3. (N = 100).

	1	2	3	4	5	6	7	8
1	1.000000	0.537371	-0.137999	0.424387	-0.135472	-0.114405	-0.129632	-0.080905
2	0.537371	1.000000	0.232402	-0.032405	-0.086397	-0.162987	0.246552	0.191935
3	-0.137999	0.232402	1.000000	0.002616	0.542187	0.548864	0.322417	0.550573
4	0.424387	-0.032405	0.002616	1.000000	0.032137	0.046869	-0.010671	-0.136372
5	-0.135472	-0.086397	0.542187	0.032137	1.000000	0.549685	-0.023154	0.443566
6	-0.114405	-0.162987	0.548864	0.046869	0.549685	1.000000	0.029111	0.438510
7	-0.129632	0.246552	0.322417	-0.010671	-0.023154	0.029111	1.000000	0.250328
8	-0.080905	0.191935	0.550573	-0.136372	0.443566	0.438510	0.250328	1.000000
9	0.395175	0.450429	-0.039445	-0.132167	-0.159630	-0.148593	-0.032200	0.010898
10	0.257019	-0.039523	-0.142676	0.447294	-0.024149	-0.046616	-0.153327	-0.011595
11	0.276019	-0.031867	-0.135849	0.294844	-0.177573	-0.099594	-0.109412	-0.160909
12	0.279932	0.138838	-0.015746	-0.047674	-0.002505	0.067998	-0.125696	-0.096371

	9	10	11	12
1	0.395175	0.257019	0.276019	0.279932
2	0.450429	-0.039523	-0.031867	0.138838
3	-0.039445	-0.142676	-0.135849	-0.015746
4	-0.132167	0.447294	0.294844	-0.047674
5	-0.159630	-0.024149	-0.177573	-0.002505
6	-0.148593	-0.046616	-0.099594	0.067998
7	-0.032200	-0.153327	-0.109412	-0.125696
8	0.010898	-0.011595	-0.160909	-0.096371
9	1.000000	-0.087025	-0.021551	0.143061
10	-0.087025	1.000000	0.181243	-0.098303
11	-0.021551	0.181243	1.000000	0.061940
12	0.143061	-0.098303	0.061940	1.000000

SAMPLE CORRELATION MATRIX. NO. 4. (N = 100).

	1	2	3	4	5	6	7	8
1	1.000000	0.488899	-0.241365	0.451942	-0.057124	0.021747	-0.118595	-0.103871
2	0.488899	1.000000	0.162064	-0.044583	-0.084848	-0.137264	0.411928	0.121159
3	-0.241365	0.162064	1.000000	-0.210959	0.595723	0.433854	0.345335	0.504435
4	0.451942	-0.044583	-0.210959	1.000000	0.004875	0.162584	-0.107569	-0.108292
5	-0.057124	-0.084848	0.595723	0.004875	1.000000	0.477138	-0.079777	0.374062
6	0.021747	-0.137264	0.433854	0.162584	0.477138	1.000000	0.047482	0.217364
7	-0.118595	0.411928	0.345335	-0.107569	-0.079777	0.047482	1.000000	0.257077
8	-0.103871	0.121159	0.504435	-0.108292	0.374062	0.217364	0.257077	1.000000
9	0.339332	0.233212	-0.106703	0.090106	-0.020446	-0.007070	-0.227634	-0.091873
10	0.320327	-0.021832	-0.163888	0.400240	-0.027140	0.025106	-0.138262	-0.122835
11	0.261965	-0.035891	-0.226739	0.392235	-0.040934	0.054401	-0.060292	-0.156882
12	0.198476	0.098292	0.023942	-0.104620	0.094179	-0.010681	-0.107581	-0.029239

	9	10	11	12
1	0.339332	0.320327	0.261965	0.198476
2	0.233212	-0.021832	-0.035891	0.098292
3	-0.106703	-0.163888	-0.226739	0.023942
4	0.090106	0.400240	0.392235	-0.104620
5	-0.020446	-0.027140	-0.040934	0.094179
6	-0.007070	0.025106	0.054401	-0.010681
7	-0.227634	-0.138262	-0.060292	-0.107581
8	-0.091873	-0.122835	-0.156882	-0.029239
9	1.000000	-0.104389	-0.045674	0.205926
10	-0.104389	1.000000	0.119394	-0.093182
11	-0.045674	0.119394	1.000000	-0.008869
12	0.205926	-0.093182	-0.008869	1.000000

SAMPLE CORRELATION MATRIX. NO. 5. (N = 100).

	1	2	3	4	5	6	7	8
1	1.000000	0.499503	-0.065575	0.553975	-0.020830	-0.101568	-0.009851	-0.024513
2	0.499503	1.000000	0.418562	0.106516	0.058426	0.139970	0.486778	0.355887
3	-0.065575	0.418562	1.000000	0.067474	0.536252	0.580315	0.526954	0.554024
4	0.553975	0.106516	0.067474	1.000000	0.065453	-0.048008	0.039739	0.062268
5	-0.020830	0.058426	0.536252	0.065453	1.000000	0.519170	0.142256	0.305515
6	-0.101568	0.139970	0.580315	-0.048008	0.519170	1.000000	0.142910	0.388577
7	-0.009851	0.486778	0.526954	0.039739	0.142256	0.142910	1.000000	0.377642
8	-0.024513	0.355887	0.554024	0.062268	0.305515	0.388577	0.377642	1.000000
9	0.382234	0.211968	-0.045416	0.053759	-0.140237	-0.114725	-0.059958	0.001685
10	0.410964	0.122036	0.023396	0.433060	0.022855	-0.071709	0.003336	-0.067523
11	0.367206	0.108049	0.109253	0.296728	-0.016653	0.019567	0.083452	0.118484
12	0.289780	0.153863	-0.043676	0.137008	0.127541	-0.059506	0.013096	-0.126977

	9	10	11	12
1	0.382234	0.410964	0.367206	0.289780
2	0.211968	0.122036	0.108049	0.153863
3	-0.045416	0.023396	0.109253	-0.043676
4	0.053759	0.433060	0.296728	0.137008
5	-0.140237	0.022855	-0.016653	0.127541
6	-0.114725	-0.071709	0.019567	-0.059506
7	-0.059958	0.003336	0.083452	0.013096
8	0.001685	-0.067523	0.118484	-0.126977
9	1.000000	0.136014	0.099952	0.032635
10	0.136014	1.000000	0.161210	0.179758
11	0.099952	0.161210	1.000000	0.072750
12	0.032635	0.179758	0.072750	1.000000

SAMPLE CORRELATION MATRIX. NO. 6. (N = 100).

	1	2	3	4	5	6	7	8
1	1.000000	0.518114	0.008311	0.251928	-0.086911	-0.015098	-0.007365	-0.107656
2	0.518114	1.000000	0.415455	-0.182796	0.034161	0.082160	0.366695	0.176983
3	0.008311	0.415455	1.000000	-0.013653	0.602587	0.474146	0.362336	0.476787
4	0.251928	-0.182796	-0.013653	1.000000	-0.007363	-0.061273	-0.079916	-0.023813
5	-0.086911	0.034161	0.602587	-0.007363	1.000000	0.551377	-0.011850	0.289163
6	-0.015098	0.082160	0.474146	-0.061273	0.551377	1.000000	0.066776	0.325513
7	-0.007365	0.366695	0.362336	-0.079916	-0.011850	0.066776	1.000000	0.276997
8	-0.107656	0.176983	0.476787	-0.023813	0.289163	0.325513	0.276997	1.000000
9	0.320344	0.443806	0.053715	-0.197304	-0.061263	-0.084624	0.050931	-0.204366
10	0.139842	-0.098702	-0.076578	0.126410	-0.048921	-0.055017	-0.185428	-0.083763
11	0.191607	-0.133745	-0.068020	0.196600	-0.094097	0.055613	-0.015982	-0.118352
12	0.352584	0.222569	-0.004765	0.039018	-0.104313	0.021200	0.081992	0.003488

	9	10	11	12
1	0.320344	0.139842	0.191607	0.352584
2	0.443806	-0.098702	-0.133745	0.222569
3	0.053715	-0.076578	-0.068020	-0.004765
4	-0.197304	0.126410	0.196600	0.039018
5	-0.061263	-0.048921	-0.094097	-0.104313
6	-0.084624	-0.055017	0.055613	0.021200
7	0.050931	-0.185428	-0.015982	0.081992
8	-0.204366	-0.083763	-0.118352	0.003488
9	1.000000	0.049104	-0.051464	0.232871
10	0.049104	1.000000	-0.010961	-0.022914
11	-0.051464	-0.010961	1.000000	0.015853
12	0.232871	-0.022914	0.015853	1.000000

SAMPLE CORRELATION MATRIX. NO. 7. (N = 100).

	1	2	3	4	5	6	7	8
1	1.000000	0.558752	-0.007881	0.401575	0.065413	0.015028	-0.118757	0.122164
2	0.558752	1.000000	0.178247	0.017525	0.030793	-0.058497	0.325291	0.283004
3	-0.007881	0.178247	1.000000	0.168282	0.719114	0.654888	0.320685	0.646510
4	0.401575	0.017525	0.168282	1.000000	0.269901	0.229930	0.002987	0.125611
5	0.065413	0.030793	0.719114	0.269901	1.000000	0.695845	0.089648	0.502701
6	0.015028	-0.058497	0.654888	0.229930	0.695845	1.000000	-0.023734	0.375007
7	-0.118757	0.325291	0.320685	0.002987	0.089648	-0.023734	1.000000	0.362623
8	0.122164	0.283004	0.646510	0.125611	0.502701	0.375007	0.362623	1.000000
9	0.386154	0.231678	-0.043455	0.090633	-0.006385	-0.108949	-0.121617	-0.032410
10	0.293474	0.044750	0.118451	0.412147	0.186344	0.124665	0.029966	0.154327
11	0.213810	0.011492	0.012719	0.368751	0.139178	0.148272	-0.135417	-0.042340
12	0.314907	0.076487	-0.141347	-0.032398	-0.087455	-0.063224	-0.241070	-0.229457

	9	10	11	12
1	0.386154	0.293474	0.213810	0.314907
2	0.231678	0.044750	0.011492	0.076487
3	-0.043455	0.118451	0.012719	-0.141347
4	0.090633	0.412147	0.368751	-0.032398
5	-0.006385	0.186344	0.139178	-0.087455
6	-0.108949	0.124665	0.148272	-0.063224
7	-0.121617	0.029966	-0.135417	-0.241070
8	-0.032410	0.154327	-0.042340	-0.229457
9	1.000000	0.105874	0.165026	0.291999
10	0.105874	1.000000	0.129204	-0.008954
11	0.165026	0.129204	1.000000	-0.011321
12	0.291999	-0.008954	-0.011321	1.000000

SAMPLE CORRELATION MATRIX. NO. 8. (N = 100).

	1	2	3	4	5	6	7	8
1	1.000000	0.634574	0.004385	0.342231	-0.021709	0.089696	-0.129314	-0.068032
2	0.634574	1.000000	0.205818	-0.077689	-0.106711	0.088087	0.257928	0.058827
3	0.004385	0.205818	1.000000	0.033063	0.615217	0.581139	0.300878	0.496324
4	0.342231	-0.077689	0.033063	1.000000	0.128819	0.047461	-0.096329	0.062276
5	-0.021709	-0.106711	0.615217	0.128819	1.000000	0.497488	-0.014958	0.253486
6	0.089696	0.088087	0.581139	0.047461	0.497488	1.000000	-0.031416	0.345551
7	-0.129314	0.257928	0.300878	-0.096329	-0.014958	-0.031416	1.000000	0.094627
8	-0.068032	0.058827	0.496324	0.062276	0.253486	0.345551	0.094627	1.000000
9	0.469981	0.362637	0.123936	0.030545	0.042990	0.156705	0.014624	-0.041222
10	0.228711	0.227325	0.145290	0.275079	0.084571	-0.035258	0.027640	0.139844
11	0.259406	0.061822	-0.049047	0.363150	-0.048130	-0.003839	0.061296	0.037395
12	0.367365	0.324326	0.098040	0.155255	0.105436	0.169110	-0.047791	-0.003272

	9	10	11	12
1	0.469981	0.228711	0.259406	0.367365
2	0.362637	0.227325	0.061822	0.324326
3	0.123936	0.145290	-0.049047	0.098040
4	0.030545	0.275079	0.363150	0.155255
5	0.042990	0.084571	-0.048130	0.105436
6	0.156705	-0.035258	-0.003839	0.169110
7	0.014624	0.027640	0.061296	-0.047791
8	-0.041222	0.139844	0.037395	-0.003272
9	1.000000	0.115742	0.222239	0.041011
10	0.115742	1.000000	0.148040	0.212858
11	0.222239	0.148040	1.000000	0.126628
12	0.041011	0.212858	0.126628	1.000000

SAMPLE CORRELATION MATRIX. NO. 9. (N = 100).

	1	2	3	4	5	6	7	8
1	1.000000	0.506228	0.062749	0.355124	0.093822	0.008477	-0.066047	-0.089322
2	0.506228	1.000000	0.319983	-0.099089	0.116895	0.085757	0.381574	0.211295
3	0.062749	0.319983	1.000000	0.167496	0.685468	0.601421	0.307123	0.389354
4	0.355124	-0.099089	0.167496	1.000000	0.142245	0.054083	0.026009	-0.176977
5	0.093822	0.116895	0.685468	0.142245	1.000000	0.639483	0.087740	0.235648
6	0.008477	0.085757	0.601421	0.054083	0.639483	1.000000	0.030900	0.246221
7	-0.066047	0.381574	0.307123	0.026009	0.087740	0.030900	1.000000	0.070596
8	-0.089322	0.211295	0.389354	-0.176977	0.235648	0.246221	0.070596	1.000000
9	0.336317	0.349353	-0.069055	-0.124802	-0.021131	0.094837	-0.034762	-0.042373
10	0.211246	-0.010217	0.141800	0.402142	0.132363	0.107613	0.043006	-0.058973
11	0.220998	0.115052	0.103730	0.100521	0.119264	0.014970	0.131812	-0.140114
12	0.217157	0.260079	0.019948	0.081699	-0.032010	-0.135078	0.127665	-0.000711

	9	10	11	12
1	0.336317	0.211246	0.220998	0.217157
2	0.349353	-0.010217	0.115052	0.260079
3	-0.069055	0.141800	0.103730	0.019948
4	-0.124802	0.402142	0.100521	0.081699
5	-0.021131	0.132363	0.119264	-0.032010
6	0.094837	0.107613	0.014970	-0.135078
7	-0.034762	0.043006	0.131812	0.127665
8	-0.042373	-0.058973	-0.140114	-0.000711
9	1.000000	0.138220	0.195143	0.057806
10	0.138220	1.000000	0.240002	0.092068
11	0.195143	0.240002	1.000000	-0.056596
12	0.057806	0.092068	-0.056596	1.000000

SAMPLE CORRELATION MATRIX. NO. 10. (N = 100).

	1	2	3	4	5	6	7	8
1	1.000000	0.408599	-0.012991	0.333962	0.046844	0.050900	-0.002507	0.031782
2	0.408599	1.000000	0.061227	-0.210232	-0.126840	-0.272285	0.317227	0.018347
3	-0.012991	0.061227	1.000000	0.017418	0.658968	0.527965	0.254253	0.348304
4	0.333962	-0.210232	0.017418	1.000000	0.051757	0.080637	-0.031896	0.023745
5	0.046844	-0.126840	0.658968	0.051757	1.000000	0.530471	0.008729	0.322297
6	0.050900	-0.272285	0.527965	0.080637	0.530471	1.000000	0.032434	0.153600
7	-0.002507	0.317227	0.254253	-0.031896	0.008729	0.032434	1.000000	0.251828
8	0.031782	0.018347	0.348304	0.023745	0.322297	0.153600	0.251828	1.000000
9	0.341310	0.361109	-0.199899	-0.144102	-0.116197	-0.107385	-0.239364	-0.036295
10	0.178537	-0.094986	0.126926	0.365593	0.028619	0.111236	0.161297	0.088021
11	0.193258	-0.040772	0.110896	0.337221	0.063770	0.102342	0.026927	0.148791
12	0.232290	0.232466	-0.137862	-0.248044	-0.089030	-0.042649	0.084669	-0.019719

	9	10	11	12
1	0.341310	0.178537	0.193258	0.232290
2	0.361109	-0.094986	-0.040772	0.232466
3	-0.199899	0.126926	0.110896	-0.137862
4	-0.144102	0.365593	0.337221	-0.248044
5	-0.116197	0.028619	0.063770	-0.089030
6	-0.107385	0.111236	0.102342	-0.042649
7	-0.239364	0.161297	0.026927	0.084669
8	-0.036295	0.088021	0.148791	-0.019719
9	1.000000	-0.116477	-0.006578	0.244565
10	-0.116477	1.000000	0.201489	-0.058076
11	-0.006578	0.201489	1.000000	-0.117700
12	0.244565	-0.058076	-0.117700	1.000000

SAMPLE CORRELATION MATRIX.   NO. 11.   (N = 100).

	1	2	3	4	5	6	7	8
1	1.000000	0.583850	0.053972	0.295420	0.244044	-0.003435	0.153577	0.061003
2	0.583850	1.000000	0.429304	-0.091210	0.204399	0.142822	0.443806	0.329162
3	0.053972	0.429304	1.000000	-0.026783	0.643206	0.524655	0.303910	0.438671
4	0.295420	-0.091210	-0.026783	1.000000	0.088407	0.024781	-0.008914	0.135428
5	0.244044	0.204399	0.643206	0.088407	1.000000	0.544825	-0.054915	0.290338
6	-0.003435	0.142822	0.524655	0.024781	0.544825	1.000000	-0.055803	0.257042
7	0.153577	0.443806	0.303910	-0.008914	-0.054915	-0.055803	1.000000	0.262174
8	0.061003	0.329162	0.438671	0.135428	0.290338	0.257042	0.262174	1.000000
9	0.391705	0.311893	0.002375	0.067595	0.136305	0.020062	0.027579	0.070846
10	0.232243	0.016110	-0.090221	0.295749	-0.082249	-0.054858	0.000729	-0.047953
11	0.249870	0.134044	0.014753	0.259347	0.115958	-0.136042	0.110707	0.177741
12	0.185341	0.196583	0.053856	0.047369	0.107551	0.004407	0.103615	0.104261

	9	10	11	12
1	0.391705	0.232243	0.249870	0.185341
2	0.311893	0.016110	0.134044	0.196583
3	0.002375	-0.090221	0.014753	0.053856
4	0.067595	0.295749	0.259347	0.047369
5	0.136305	-0.082249	0.115958	0.107551
6	0.020062	-0.054858	-0.136042	0.004407
7	0.027579	0.000729	0.110707	0.103615
8	0.070846	-0.047953	0.177741	0.104261
9	1.000000	0.057278	0.056141	0.159307
10	0.057278	1.000000	0.141866	-0.026125
11	0.056141	0.141866	1.000000	0.034437
12	0.159307	-0.026125	0.034437	1.000000

SAMPLE CORRELATION MATRIX. NO. 12. (N = 100).

	1	2	3	4	5	6	7	8
1	1.000000	0.538986	-0.135719	0.534935	-0.103496	-0.156658	0.058701	-0.010474
2	0.538986	1.000000	0.230392	0.007070	-0.074259	-0.064498	0.388333	0.309330
3	-0.135719	0.230392	1.000000	-0.121617	0.634967	0.575198	0.293069	0.552992
4	0.534935	0.007070	-0.121617	1.000000	-0.083425	-0.060713	0.024408	-0.105576
5	-0.103496	-0.074259	0.634967	-0.083425	1.000000	0.633261	0.022789	0.354156
6	-0.156658	-0.064498	0.575198	-0.060713	0.633261	1.000000	-0.080448	0.343263
7	0.058701	0.388333	0.293069	0.024408	0.022789	-0.080448	1.000000	0.318034
8	-0.010474	0.309330	0.552992	-0.105576	0.354156	0.343263	0.318034	1.000000
9	0.461943	0.447229	-0.073804	0.067400	-0.129948	-0.263908	0.130457	-0.001080
10	0.344364	-0.024506	-0.060911	0.359368	0.128253	0.002723	0.049058	-0.031364
11	0.316135	0.034743	0.073762	0.456547	0.118540	0.170712	0.118138	0.023761
12	0.177892	0.032632	-0.176931	-0.033314	-0.053240	-0.177490	-0.217325	-0.060203

	9	10	11	12
1	0.461943	0.344364	0.316135	0.177892
2	0.447229	-0.024506	0.034743	0.032632
3	-0.073804	-0.060911	0.073762	-0.176931
4	0.067400	0.359368	0.456547	-0.033314
5	-0.129948	0.128253	0.118540	-0.053240
6	-0.263908	0.002723	0.170712	-0.177490
7	0.130457	0.049058	0.118138	-0.217325
8	-0.001080	-0.031364	0.023761	-0.060203
9	1.000000	0.006383	0.077961	0.071632
10	0.006383	1.000000	0.177254	0.135703
11	0.077961	0.177254	1.000000	0.038512
12	0.071632	0.135703	0.038512	1.000000

SAMPLE CORRELATION MATRIX. NO. 13. (N = 100).

	1	2	3	4	5	6	7	8
1	1.000000	0.500321	0.042289	0.422361	0.114270	0.111316	-0.110794	-0.199574
2	0.500321	1.000000	0.339270	0.001339	-0.019686	-0.000668	0.277867	0.132056
3	0.042289	0.339270	1.000000	0.023471	0.477846	0.492266	0.370528	0.430638
4	0.422361	0.001339	0.023471	1.000000	0.161066	0.135351	-0.026215	0.024418
5	0.114270	-0.019686	0.477846	0.161066	1.000000	0.608314	-0.014869	0.197136
6	0.111316	-0.000668	0.492266	0.135351	0.608314	1.000000	0.072636	0.286892
7	-0.110794	0.277867	0.370528	-0.026215	-0.014869	0.072636	1.000000	0.289442
8	-0.199574	0.132056	0.430638	0.024418	0.197136	0.286892	0.289442	1.000000
9	0.308808	0.377823	0.127944	-0.116227	0.106118	0.142443	-0.126836	-0.062050
10	0.101349	-0.113435	0.221209	0.337364	0.195822	0.254578	0.098454	0.233326
11	0.186115	-0.071406	0.009809	0.372090	-0.166724	-0.018411	-0.082669	-0.093385
12	0.212244	0.340289	0.049661	-0.053109	-0.030664	-0.109742	-0.120773	-0.157549

	9	10	11	12
1	0.308808	0.101349	0.186115	0.212244
2	0.377823	-0.113435	-0.071406	0.340289
3	0.127944	0.221209	0.009809	0.049661
4	-0.116227	0.337364	0.372090	-0.053109
5	0.106118	0.195822	-0.166724	-0.030664
6	0.142443	0.254578	-0.018411	-0.109742
7	-0.126836	0.098454	-0.082669	-0.120773
8	-0.062050	0.233326	-0.093385	-0.157549
9	1.000000	-0.013256	-0.087568	0.221592
10	-0.013256	1.000000	0.182440	-0.115845
11	-0.087568	0.182440	1.000000	-0.026377
12	0.221592	-0.115845	-0.026377	1.000000

SAMPLE CORRELATION MATRIX. NO. 14. (N = 100).

	1	2	3	4	5	6	7	8
1	1.000000	0.553998	-0.115018	0.600318	-0.040615	-0.082144	0.026799	-0.142253
2	0.553998	1.000000	0.268676	0.127015	0.015966	-0.031234	0.295944	0.197823
3	-0.115018	0.268676	1.000000	-0.028653	0.627320	0.550595	0.348867	0.604965
4	0.600318	0.127015	-0.028653	1.000000	0.012695	-0.067840	0.106483	-0.025950
5	-0.040615	0.015966	0.627320	0.012695	1.000000	0.627555	0.101455	0.487004
6	-0.082144	-0.031234	0.550595	-0.067840	0.627555	1.000000	0.086513	0.408503
7	0.026799	0.295944	0.348867	0.106483	0.101455	0.086513	1.000000	0.247854
8	-0.142253	0.197823	0.604965	-0.025950	0.487004	0.408503	0.247854	1.000000
9	0.442705	0.395435	0.015393	0.105841	0.078618	0.032064	0.021013	-0.095622
10	0.359953	0.175726	0.030819	0.271143	-0.055010	-0.181233	0.083975	0.065308
11	0.162183	0.042570	0.079404	0.487019	0.064164	0.079634	0.162227	0.091478
12	0.292347	0.224129	-0.094259	0.086154	0.035688	0.053122	-0.008648	-0.111499

	9	10	11	12
1	0.442705	0.359953	0.162183	0.292347
2	0.395435	0.175726	0.042570	0.224129
3	0.015393	0.030819	0.079404	-0.094259
4	0.105841	0.271143	0.487019	0.086154
5	0.078618	-0.055010	0.064164	0.035688
6	0.032064	-0.181233	0.079634	0.053122
7	0.021013	0.083975	0.162227	-0.008648
8	-0.095622	0.065308	0.091478	-0.111499
9	1.000000	0.155011	0.098458	0.221064
10	0.155011	1.000000	0.155868	-0.227052
11	0.098458	0.155868	1.000000	-0.085046
12	0.221064	-0.227052	-0.085046	1.000000

SAMPLE CORRELATION MATRIX. NO. 15. (N = 100).

	1	2	3	4	5	6	7	8
1	1.000000	0.606976	-0.030497	0.438004	-0.015550	0.050841	0.012046	0.000514
2	0.606976	1.000000	0.368802	0.080885	0.153662	0.191330	0.425535	0.264303
3	-0.030497	0.368802	1.000000	-0.037790	0.557939	0.599772	0.518556	0.498396
4	0.438004	0.080885	-0.037790	1.000000	-0.132775	0.065226	0.077292	-0.091390
5	-0.015550	0.153662	0.557939	-0.132775	1.000000	0.508837	0.098163	0.206893
6	0.050841	0.191330	0.599772	0.065226	0.508837	1.000000	0.262074	0.483130
7	0.012046	0.425535	0.518556	0.077292	0.098163	0.262074	1.000000	0.218364
8	0.000514	0.264303	0.498396	-0.091390	0.206893	0.483130	0.218364	1.000000
9	0.481146	0.289585	-0.062164	0.037766	0.041790	0.001317	-0.113464	0.053500
10	0.166795	-0.031192	-0.253549	0.239088	-0.150339	-0.153298	-0.012885	-0.123189
11	0.282251	0.231068	0.069011	0.358261	-0.081675	0.030713	0.006061	-0.033694
12	0.401513	0.361610	0.043990	0.081827	-0.031475	-0.049813	0.005989	0.195771

	9	10	11	12
1	0.481146	0.166795	0.282251	0.401513
2	0.289585	-0.031192	0.231068	0.361610
3	-0.062164	-0.253549	0.069011	0.043990
4	0.037766	0.239088	0.358261	0.081827
5	0.041790	-0.150339	-0.081675	-0.031475
6	0.001317	-0.153298	0.030713	-0.049813
7	-0.113464	-0.012885	0.006061	0.005989
8	0.053500	-0.123189	-0.033694	0.195771
9	1.000000	0.046753	0.000987	0.352133
10	0.046753	1.000000	0.018173	0.047910
11	0.000987	0.018173	1.000000	0.077713
12	0.352133	0.047910	0.077713	1.000000

SAMPLE CORRELATION MATRIX. NO. 16. (N = 100).

	1	2	3	4	5	6	7	8
1	1.000000	0.598489	0.118318	0.507943	0.000215	-0.014313	0.081426	0.175293
2	0.598489	1.000000	0.346692	-0.005636	0.081281	-0.118389	0.436437	0.222366
3	0.118318	0.346692	1.000000	0.074735	0.593887	0.244011	0.433199	0.502871
4	0.507943	-0.005636	0.074735	1.000000	-0.014539	-0.028424	0.028744	0.176519
5	0.000215	0.081281	0.593887	-0.014539	1.000000	0.404240	0.136405	0.238812
6	-0.014313	-0.118389	0.244011	-0.028424	0.404240	1.000000	-0.131625	-0.003601
7	0.081426	0.436437	0.433199	0.028744	0.136405	-0.131625	1.000000	0.132568
8	0.175293	0.222366	0.502871	0.176519	0.238812	-0.003601	0.132568	1.000000
9	0.302973	0.288131	-0.086031	-0.046829	0.015186	0.010248	-0.089397	0.046774
10	0.215711	0.111152	0.285957	0.354871	0.248504	-0.071661	0.254559	0.195362
11	0.155324	0.072381	-0.002694	0.206823	0.084996	0.151670	0.008535	0.109329
12	0.265959	0.227551	0.048844	0.006148	0.014810	-0.025492	-0.035815	0.032496

	9	10	11	12
1	0.302973	0.215711	0.155324	0.265959
2	0.288131	0.111152	0.072381	0.227551
3	-0.086031	0.285957	-0.002694	0.048844
4	-0.046829	0.354871	0.206823	0.006148
5	0.015186	0.248504	0.084996	0.014810
6	0.010248	-0.071661	0.151670	-0.025492
7	-0.089397	0.254559	0.008535	-0.035815
8	0.046774	0.195362	0.109329	0.032496
9	1.000000	-0.116925	-0.000587	0.214641
10	-0.116925	1.000000	0.014257	-0.028489
11	-0.000587	0.014257	1.000000	-0.134143
12	0.214641	-0.028489	-0.134143	1.000000

SAMPLE CORRELATION MATRIX. NO. 17. (N = 100).

	1	2	3	4	5	6	7	8
1	1.000000	0.741886	0.240051	0.264913	-0.019921	0.045046	0.234938	0.204310
2	0.741886	1.000000	0.408208	0.064621	-0.021737	0.103573	0.556328	0.334737
3	0.240051	0.408208	1.000000	0.153921	0.616973	0.530490	0.413038	0.615342
4	0.264913	0.064621	0.153921	1.000000	0.022269	-0.001279	-0.108948	0.111077
5	-0.019921	-0.021737	0.616973	0.022269	1.000000	0.579172	-0.023592	0.362572
6	0.045046	0.103573	0.530490	-0.001279	0.579172	1.000000	0.154028	0.268431
7	0.234938	0.556328	0.413038	-0.108948	-0.023592	0.154028	1.000000	0.286532
8	0.204310	0.334737	0.615342	0.111077	0.362572	0.268431	0.286532	1.000000
9	0.440870	0.372230	0.150368	0.085980	0.121494	-0.023918	0.118991	0.235270
10	0.175370	0.101089	0.048312	0.343632	-0.051971	-0.133085	0.013491	0.016418
11	-0.016859	-0.078713	0.012955	0.281094	0.091700	0.155730	-0.026685	-0.050373
12	0.160275	0.079486	-0.095258	-0.019674	-0.011188	-0.125436	-0.091587	0.127810

	9	10	11	12
1	0.440870	0.175370	-0.016859	0.160275
2	0.372230	0.101089	-0.078713	0.079486
3	0.150368	0.048312	0.012955	-0.095258
4	0.085980	0.343632	0.281094	-0.019674
5	0.121494	-0.051971	0.091700	-0.011188
6	-0.023918	-0.133085	0.155730	-0.125436
7	0.118991	0.013491	-0.026685	-0.091587
8	0.235270	0.016418	-0.050373	0.127810
9	1.000000	0.134095	-0.062910	0.207643
10	0.134095	1.000000	0.049189	0.019137
11	-0.062910	0.049189	1.000000	-0.120623
12	0.207643	0.019137	-0.120623	1.000000

SAMPLE CORRELATION MATRIX. NO. 18. (N = 100).

	1	2	3	4	5	6	7	8
1	1.000000	0.597364	0.054942	0.218529	0.016010	-0.015235	0.267434	0.119220
2	0.597364	1.000000	0.327628	-0.119809	0.065488	-0.099297	0.512676	0.323097
3	0.054942	0.327628	1.000000	-0.046224	0.569692	0.360974	0.268622	0.408699
4	0.218529	-0.119809	-0.046224	1.000000	-0.030612	-0.050659	0.060701	-0.190040
5	0.016010	0.065488	0.569692	-0.030612	1.000000	0.540601	-0.045337	0.295081
6	-0.015235	-0.099297	0.360974	-0.050659	0.540601	1.000000	-0.121148	0.169246
7	0.267434	0.512676	0.268622	0.060701	-0.045337	-0.121148	1.000000	0.025400
8	0.119220	0.323097	0.408699	-0.190040	0.295081	0.169246	0.025400	1.000000
9	0.383042	0.454812	0.017901	-0.037830	0.072570	-0.085593	0.184111	0.235049
10	-0.028536	-0.198455	-0.150131	0.336407	0.008185	0.025794	-0.037590	-0.235808
11	0.243984	-0.036067	-0.037585	0.254060	0.116218	0.230187	-0.032852	0.015995
12	0.447437	0.249271	-0.093410	0.089103	-0.050387	-0.137872	0.053397	0.134857

	9	10	11	12
1	0.383042	-0.028536	0.243984	0.447437
2	0.454812	-0.198455	-0.036067	0.249271
3	0.017901	-0.150131	-0.037585	-0.093410
4	-0.037830	0.336407	0.254060	0.089103
5	0.072570	0.008185	0.116218	-0.050387
6	-0.085593	0.025794	0.230187	-0.137872
7	0.184111	-0.037590	-0.032852	0.053397
8	0.235049	-0.235808	0.015995	0.134857
9	1.000000	0.028851	-0.024526	0.053814
10	0.028851	1.000000	0.079767	-0.160024
11	-0.024526	0.079767	1.000000	0.075006
12	0.053814	-0.160024	0.075006	1.000000

SAMPLE CORRELATION MATRIX. NO. 19. (N = 100).

	1	2	3	4	5	6	7	8
1	1.000000	0.569367	-0.175438	0.546348	-0.008644	-0.137947	0.020862	-0.062933
2	0.569367	1.000000	0.135917	0.056975	0.024132	-0.156273	0.515652	0.271205
3	-0.175438	0.135917	1.000000	-0.177741	0.633129	0.560223	0.247681	0.567238
4	0.546348	0.056975	-0.177741	1.000000	-0.038979	-0.122968	-0.188064	-0.163338
5	-0.008644	0.024132	0.633129	-0.038979	1.000000	0.598412	-0.006813	0.312525
6	-0.137947	-0.156273	0.560223	-0.122968	0.598412	1.000000	-0.073824	0.392675
7	0.020862	0.515652	0.247681	-0.188064	-0.006813	-0.073824	1.000000	0.248286
8	-0.062933	0.271205	0.567238	-0.163338	0.312525	0.392675	0.248286	1.000000
9	0.336598	0.263438	-0.027792	0.226649	0.060129	-0.063333	-0.016188	0.087178
10	0.225849	0.011331	-0.058622	0.236889	0.054311	0.019362	-0.103950	0.013780
11	0.147539	-0.100300	-0.237261	0.130470	0.000644	-0.074635	-0.183410	-0.247922
12	0.277109	0.049928	-0.155086	0.170261	-0.090325	-0.030307	-0.181645	-0.132823

	9	10	11	12
1	0.336598	0.225849	0.147539	0.277109
2	0.263438	0.011331	-0.100300	0.049928
3	-0.027792	-0.058622	-0.237261	-0.155086
4	0.226649	0.236889	0.130470	0.170261
5	0.060129	0.054311	0.000644	-0.090325
6	-0.063333	0.019362	-0.074635	-0.030307
7	-0.016188	-0.103950	-0.183410	-0.181645
8	0.087178	0.013780	-0.247922	-0.132823
9	1.000000	0.025925	0.002239	0.283804
10	0.025925	1.000000	0.180225	0.128017
11	0.002239	0.180225	1.000000	-0.014945
12	0.283804	0.128017	-0.014945	1.000000

SAMPLE CORRELATION MATRIX. NO. 20. (N = 100).

	1	2	3	4	5	6	7	8
1	1.000000	0.472515	-0.244452	0.385038	0.039124	-0.090868	-0.210234	-0.065602
2	0.472515	1.000000	0.196038	-0.055026	0.002902	0.048452	0.412884	0.359140
3	-0.244452	0.196038	1.000000	-0.075005	0.638312	0.644163	0.263860	0.465443
4	0.385038	-0.055026	-0.075005	1.000000	-0.024754	-0.158344	0.020193	-0.114574
5	0.039124	0.002902	0.638312	-0.024754	1.000000	0.499570	-0.103599	0.208981
6	-0.090868	0.048452	0.644163	-0.158344	0.499570	1.000000	-0.026895	0.296802
7	-0.210234	0.412884	0.263860	0.020193	-0.103599	-0.026895	1.000000	0.237230
8	-0.065602	0.359140	0.465443	-0.114574	0.208981	0.296802	0.237230	1.000000
9	0.370178	0.346767	-0.104165	0.091575	0.040360	-0.078273	-0.036174	0.140534
10	0.282640	0.161391	-0.253617	0.333660	-0.183053	-0.233849	0.134207	-0.183654
11	0.033526	-0.025791	0.156970	0.365319	0.071501	-0.090878	0.081117	-0.020704
12	0.202457	0.130204	-0.030093	-0.066375	0.034833	-0.010178	-0.087857	-0.006665

	9	10	11	12
1	0.370178	0.282640	0.033526	0.202457
2	0.346767	0.161391	-0.025791	0.130204
3	-0.104165	-0.253617	0.156970	-0.030093
4	0.091575	0.333660	0.365319	-0.066375
5	0.040360	-0.183053	0.071501	0.034833
6	-0.078273	-0.233849	-0.090878	-0.010178
7	-0.036174	0.134207	0.081117	-0.087857
8	0.140534	-0.183654	-0.020704	-0.006665
9	1.000000	0.083739	-0.159406	0.133006
10	0.083739	1.000000	0.211123	0.040103
11	-0.159406	0.211123	1.000000	-0.328354
12	0.133006	0.040103	-0.328354	1.000000

APPENDIX B.

MAXIMUM DETERMINANT FACTOR MATRICES

Set 1. (N = 100, p = 12)

MAXIMUM LIKELIHOOD FACTOR MATRIX. NO. 1. (N = 100).
ROTATED TO FORM WITH SPECIFIED ZERO ELEMENTS.

	1	2	3	4
1	-0.138270	0.942314	0.260284	-0.157380
2	-0.138799	0.718550	-0.077865	0.487753
3	0.762580	0.241809	0.067405	0.440970
4	-0.072075	0.248149	0.762385	0.
5	0.682499	0.	-0.	0.
6	0.728576	-0.078557	-0.042905	-0.136995
7	0.028585	0.085989	0.153511	0.575707
8	0.369283	0.212577	0.066170	0.236685
9	-0.146420	0.377363	-0.	0.
10	-0.035333	0.235514	0.540180	-0.108230
11	0.004966	0.178995	0.276044	-0.057919
12	-0.024258	0.343258	-0.109545	0.013032

MAXIMUM LIKELIHOOD FACTOR MATRIX. NO. 2. (N = 100).
ROTATED TO FORM WITH SPECIFIED ZERO ELEMENTS.

	1	2	3	4
1	0.175763	0.778757	0.574232	-0.180258
2	0.096496	0.783679	0.175296	0.394569
3	0.764892	-0.050258	0.000146	0.559769
4	0.069904	-0.051805	0.755246	-0.
5	0.804450	0.	-0.	-0.
6	0.689271	0.011064	-0.115626	0.051962
7	0.011318	0.114615	0.116721	0.627128
8	0.379600	-0.026094	-0.076683	0.407166
9	0.066607	0.361099	-0.	-0.
10	0.120069	-0.197456	0.493461	-0.185826
11	0.049488	-0.065040	0.375126	0.098603
12	0.095650	0.337652	-0.182466	-0.027464

MAXIMUM LIKELIHOOD FACTOR MATRIX. NO. 3. (N = 100).
ROTATED TO FORM WITH SPECIFIED ZERO ELEMENTS.

	1	2	3	4
1	-0.187767	0.670172	0.626689	-0.287550
2	-0.128033	0.789185	0.168494	0.420775
3	0.746465	0.178833	0.013025	0.434058
4	0.042543	-0.235467	0.941827	-0.
5	0.718954	0.	0.	-0.
6	0.798673	-0.008083	0.010757	-0.116772
7	0.095974	0.058362	-0.001869	0.532007
8	0.571001	0.233837	-0.105237	0.233586
9	-0.189616	0.536808	0.	-0.
10	-0.060833	-0.105927	0.450071	-0.120946
11	-0.138527	-0.010780	0.320208	-0.180822
12	0.017742	0.297292	0.021718	-0.243525

MAXIMUM LIKELIHOOD FACTOR MATRIX. NO. 4. (N = 100).
ROTATED TO FORM WITH SPECIFIED ZERO ELEMENTS.

	1	2	3	4
1	-0.065250	0.674351	0.559212	0.294773
2	-0.116181	0.428775	-0.078666	0.801820
3	0.744701	-0.228746	-0.274693	0.401847
4	0.007449	0.039059	0.773160	-0.
5	0.803037	0.	-0.	-0.
6	0.595720	-0.129071	0.228144	0.038825
7	-0.077098	-0.355430	-0.118648	0.703532
8	0.432292	-0.143748	-0.136601	0.288563
9	0.013716	0.520049	-0.	-0.
10	-0.031337	0.061305	0.485179	-0.024193
11	-0.096594	0.007389	0.467778	-0.026069
12	0.112081	0.353946	-0.095556	-0.023170

MAXIMUM LIKELIHOOD FACTOR MATRIX. NO. 5. (N = 100).
ROTATED TO FORM WITH SPECIFIED ZERO ELEMENTS.

	1	2	3	4
1	-0.028901	0.928601	0.369349	-0.005988
2	0.089881	0.596759	-0.129618	0.698206
3	0.734628	-0.051535	0.018505	0.553225
4	0.101981	0.314532	0.717035	-0.
5	0.721696	0.	0.	-0.
6	0.733659	-0.016758	-0.174146	0.080722
7	0.171182	-0.024978	0.061005	0.710242
8	0.436671	-0.001940	-0.020123	0.440657
9	-0.090750	0.408710	0.	-0.
10	0.020309	0.270013	0.435628	0.007105
11	0.103186	0.260318	0.348025	0.031310
12	0.000981	0.280499	0.079018	-0.028980

MAXIMUM LIKELIHOOD FACTOR MATRIX. NO. 6. (N = 100).
ROTATED TO FORM WITH SPECIFIED ZERO ELEMENTS.

	1	2	3	4
1	-0.091780	0.532558	0.834926	0.102213
2	0.036290	0.764335	0.067219	0.570027
3	0.635837	0.081742	-0.043580	0.582536
4	-0.010308	-0.279381	0.478797	0.
5	0.946986	0.	0.	0.
6	0.583228	-0.061321	0.062074	0.188307
7	-0.013071	0.053708	-0.117103	0.592937
8	0.308932	-0.164206	-0.053086	0.513145
9	-0.060412	0.591133	0.	0.
10	-0.037938	-0.017666	0.193614	-0.155580
11	-0.088282	-0.171362	0.331535	-0.020182
12	-0.104016	0.219703	0.258702	0.097896

MAXIMUM LIKELIHOOD FACTOR MATRIX. NO. 7. (N = 100).
ROTATED TO FORM WITH SPECIFIED ZERO ELEMENTS.

	1	2	3	4
1	0.077141	0.967778	0.168323	0.144473
2	0.038958	0.510877	-0.146005	0.593863
3	0.853317	-0.102082	-0.104624	0.297893
4	0.321696	0.246507	0.820953	-0.
5	0.848953	0.	-0.	-0.
6	0.812174	-0.021380	-0.029394	-0.154374
7	0.098776	-0.245408	0.036220	0.728950
8	0.582167	0.023727	-0.071980	0.453394
9	-0.034730	0.402964	-0.	-0.
10	0.201271	0.215990	0.362921	0.049288
11	0.143381	0.175687	0.329835	-0.151614
12	-0.098393	0.386284	-0.124045	-0.210572

MAXIMUM LIKELIHOOD FACTOR MATRIX. NO. 8. (N = 100).
ROTATED TO FORM WITH SPECIFIED ZERO ELEMENTS.

	1	2	3	4
1	-0.028819	0.903450	0.320422	-0.081711
2	-0.134258	0.773389	-0.102351	0.427018
3	0.821909	0.119272	-0.124081	0.498591
4	0.173684	0.052846	0.933617	0.
5	0.747361	0.	0.	0.
6	0.650670	0.164655	-0.084735	0.038458
7	-0.039900	-0.049743	-0.093872	0.650492
8	0.428758	-0.026936	-0.010691	0.286452
9	0.073098	0.511335	0.	0.
10	0.059054	0.189868	0.271399	0.216109
11	-0.040146	0.147723	0.390953	0.037088
12	0.070310	0.374029	0.131909	0.037969

MAXIMUM LIKELIHOOD FACTOR MATRIX. NO. 9. (N = 100).
ROTATED TO FORM WITH SPECIFIED ZERO ELEMENTS.

	1	2	3	4
1	0.105725	0.639331	0.564966	0.120558
2	0.142264	0.629281	-0.005703	0.763757
3	0.834576	-0.112015	-0.005935	0.355759
4	0.184099	-0.191796	0.805862	0.
5	0.821718	0.	0.	0.
6	0.762718	0.044596	-0.120658	-0.067464
7	0.093089	-0.149999	-0.024456	0.605630
8	0.331463	-0.003821	-0.285960	0.215915
9	0.008616	0.553197	0.	0.
10	0.182886	-0.022188	0.429919	-0.025922
11	0.115123	0.133301	0.192480	0.020883
12	-0.072546	0.123833	0.170193	0.253304

MAXIMUM LIKELIHOOD FACTOR MATRIX. NO. 10. (N = 100).
ROTATED TO FORM WITH SPECIFIED ZERO ELEMENTS.

	1	2	3	4
1	0.063002	0.589240	0.586823	0.195304
2	-0.163102	0.541344	-0.102004	0.818248
3	0.846666	-0.150192	-0.067563	0.334530
4	0.043698	-0.226069	0.791491	-0.
5	0.777779	0.	-0.	-0.
6	0.687739	-0.026855	0.073895	-0.168668
7	0.078907	-0.167935	-0.047280	0.508627
8	0.352307	-0.069849	0.022848	0.141785
9	-0.130515	0.627664	-0.	-0.
10	0.107202	-0.186437	0.421786	0.081273
11	0.115540	-0.070387	0.389553	0.072305
12	-0.076968	0.418149	-0.105185	-0.020909

MAXIMUM LIKELIHOOD FACTOR MATRIX. NO. 11. (N = 100).
ROTATED TO FORM WITH SPECIFIED ZERO ELEMENTS.

	1	2	3	4
1	0.244475	0.828962	0.213085	0.071340
2	0.204796	0.642211	-0.213754	0.652266
3	0.644357	-0.162281	-0.076688	0.591866
4	0.088649	0.111083	0.844155	0.
5	0.998203	0.	0.	0.
6	0.546075	-0.178079	-0.026880	0.204720
7	-0.055052	0.140941	-0.007751	0.566766
8	0.290578	-0.038107	0.118862	0.476889
9	0.136557	0.437825	0.	0.
10	-0.082017	0.188398	0.343429	-0.014425
11	0.115343	0.213678	0.273244	0.044249
12	0.107451	0.202497	0.005849	0.068201

MAXIMUM LIKELIHOOD FACTOR MATRIX. NO. 12. (N = 100).
ROTATED TO FORM WITH SPECIFIED ZERO ELEMENTS.

	1	2	3	4
1	-0.122314	0.818918	0.502921	-0.247128
2	-0.070382	0.808651	-0.093079	0.344084
3	0.743562	0.139548	-0.085752	0.469075
4	-0.089455	0.099694	0.879564	-0.
5	0.846140	0.	-0.	-0.
6	0.757255	-0.071157	0.018004	0.059957
7	-0.015270	0.233183	0.024406	0.592349
8	0.429535	0.237133	-0.102837	0.406335
9	-0.205488	0.533398	-0.	-0.
10	0.087403	0.127994	0.404798	-0.188696
11	0.139660	0.102619	0.513976	0.037615
12	-0.081896	0.144071	-0.053648	-0.311204

ROTATED TO FORM WITH SPECIFIED ZERO ELEMENTS.

	1	2	3	4
1	0.150223	0.527925	0.676449	-0.028391
2	-0.024533	0.739818	0.194717	0.643228
3	0.578434	0.006585	-0.043484	0.555116
4	0.149130	-0.189941	0.747212	0.
5	0.802776	0.	0.	0.
6	0.764831	-0.068444	0.035317	C.096180
7	0.003598	-0.132039	-0.058957	0.601833
8	0.266125	-0.239853	-0.109611	0.524430
9	0.191614	0.517045	0.	0.
10	0.284410	-0.320235	0.318464	C.106493
11	-0.095389	-0.196889	0.454669	-0.025777
12	-0.048069	0.429430	0.029845	0.024222

MAXIMUM LIKELIHOOD FACTOR MATRIX. NO. 14. (N = 100).

ROTATED TO FORM WITH SPECIFIED ZERO ELEMENTS.

	1	2	3	4
1	-0.046218	0.840569	0.458569	-0.059525
2	0.021165	0.719597	-0.031011	0.591290
3	0.738855	-0.044702	-0.032525	0.481410
4	0.013155	0.214501	0.916885	-C.
5	0.846546	0.	-0.	-0.
6	0.759058	-0.025529	-0.072777	-C.045004
7	0.138080	0.014404	0.111772	0.487017
8	0.553185	-0.102877	-C.010817	0.424595
9	0.060344	0.535762	-0.	-0.
10	-0.059753	0.233870	0.278168	0.103087
11	0.078864	-0.060611	0.530122	0.147191
12	0.021229	0.368239	-0.020279	-0.116826

MAXIMUM LIKELIHOOD FACTOR MATRIX. NO. 15. (N = 100).

ROTATED TO FORM WITH SPECIFIED ZERO ELEMENTS.

	1	2	3	4
1	-0.021529	0.811481	0.381414	C.240494
2	0.194758	0.498744	0.078446	0.727156
3	0.769165	-0.208892	0.130901	0.438056
4	-0.188244	0.086376	0.954679	0.
5	0.704061	0.	0.	0.
6	0.697024	-0.060713	0.211695	0.100769
7	0.176722	-0.252895	0.138363	0.692709
8	0.496223	-0.028677	0.003661	0.236359
9	0.073938	0.588836	0.	0.
10	-0.278140	0.109341	0.185832	-0.040573
11	-0.048665	0.143684	0.350780	0.149564
12	0.030231	0.431734	0.048880	0.191879

MAXIMUM LIKELIHOOD FACTOR MATRIX. NO. 16. (N = 100).
ROTATED TO FORM WITH SPECIFIED ZERO ELEMENTS.

	1	2	3	4
1	0.013187	0.611569	0.634095	0.271920
2	0.108094	0.576816	0.062428	0.752043
3	0.839820	-0.101349	0.087110	0.412308
4	-0.016860	-0.103810	0.902373	0.
5	0.714638	0.	0.	0.
6	0.469160	0.121515	-0.015187	-0.321805
7	0.163598	-0.167162	0.008451	0.676130
8	0.426145	-0.020723	0.199697	0.234993
9	-0.031479	0.504366	0.	0.
10	0.186214	-0.174040	0.383116	0.228782
11	0.026528	0.078388	0.210937	-0.014148
12	0.051701	0.333219	0.051723	0.048643

MAXIMUM LIKELIHOOD FACTOR MATRIX. NO. 17. (N = 100).
ROTATED TO FORM WITH SPECIFIED ZERO ELEMENTS.

	1	2	3	4
1	-0.014053	0.875462	0.123826	0.156428
2	-0.029432	0.753992	-0.058733	0.550142
3	0.691827	0.159752	0.112107	0.580426
4	0.024863	0.163232	0.986066	-0.
5	0.895629	0.	-0.	-0.
6	0.618456	0.004728	-0.017676	0.198934
7	-0.030106	0.173634	-0.138471	0.737498
8	0.433922	0.180220	0.071868	0.409487
9	0.113391	0.509468	-0.	-0.
10	-0.063895	0.150636	0.325161	0.026025
11	0.074179	-0.061757	0.293392	-0.038770
12	-0.014163	0.242417	-0.059732	-0.184502

MAXIMUM LIKELIHOOD FACTOR MATRIX. NO. 18. (N = 100).
ROTATED TO FORM WITH SPECIFIED ZERO ELEMENTS.

	1	2	3	4
1	0.028171	0.784109	0.422255	-0.267134
2	0.075643	0.877349	-0.046736	0.269958
3	0.707116	0.196898	-0.051789	0.391659
4	-0.032755	-0.087395	0.685335	-0.
5	0.803930	0.	0.	-0.
6	0.653994	-0.124020	0.037078	-0.149674
7	-0.034617	0.430060	0.169547	0.523464
8	0.398207	0.315047	-0.285550	0.011603
9	0.003256	0.488661	0.	-0.
10	-0.052322	-0.229554	0.402157	0.028238
11	0.162989	0.033483	0.347744	-0.260091
12	-0.086253	0.400631	0.108047	-0.286936

MAXIMUM LIKELIHOOD FACTOR MATRIX. NO. 19. (N = 100).
ROTATED TO FORM WITH SPECIFIED ZERO ELEMENTS.

	1	2	3	4
1	-0.006827	0.824795	0.423161	0.307713
2	0.014506	0.615727	-0.321881	0.642189
3	0.798864	-0.075874	-0.331724	0.109999
4	-0.019610	0.391923	0.518153	-0.
5	0.776822	0.	-0.	-0.
6	0.740399	-0.091881	-0.015026	-0.172427
7	0.023327	0.084895	-0.469485	0.485117
8	0.515589	0.017909	-0.336497	0.222787
9	0.025381	0.414612	-0.	-0.
10	0.057336	0.192411	0.215776	-0.065045
11	-0.107462	0.024062	0.333082	-0.032464
12	-0.116258	0.691833	-0.199044	-0.682715

MAXIMUM LIKELIHOOD FACTOR MATRIX. NO. 20. (N = 100).
ROTATED TO FORM WITH SPECIFIED ZERO ELEMENTS.

	1	2	3	4
1	0.052278	0.849150	0.453386	-0.246097
2	0.003949	0.786204	-0.102007	0.605649
3	0.850081	-0.149168	-0.088065	0.496936
4	-0.000974	0.032810	0.786952	-0.
5	0.750852	0.	-0.	-0.
6	0.689901	-0.024175	-0.195546	0.074133
7	-0.113622	-0.026732	-0.012056	0.715124
8	0.317890	0.125000	-0.200087	0.394878
9	-0.045127	0.440759	-0.	-0.
10	-0.294492	0.169479	0.404607	0.116078
11	0.063205	-0.160284	0.501891	0.248533
12	0.079276	0.258271	-0.124004	-0.140834

APPENDIX C.

Computing procedure for obtaining an oblique transformation matrix Λ (m x m) such that the sum of squares of differences between corresponding elements of the transformed factor matrix ($F\Lambda$) and the simple structure factor matrix Φ (p x m) is a minimum.

A transformation matrix Λ (m x m) is required such that

$$\text{trace} \quad (\Phi - F\Lambda)' \quad (\Phi - F\Lambda)$$

is a minimum and

$$\text{Diag} \quad (\Lambda' \Lambda) = I.$$

Computation is simplified if the factor matrix F is in principal axes form (which may be obtained by Thurstone's method of derived principal axes (42)). Then

$$F'F = D_\gamma$$

where D_γ is a diagonal matrix of order m.

The transformation matrix Λ satisfies the equation

$$D_\gamma \Lambda + \Lambda D_\beta = F'\Phi$$

where D_β is a diagonal matrix of order m with unknown diagonal elements.

The following computational procedure may be used to obtain Λ

1. Calculate

$$G \text{ (m x m)} = F' \Phi$$

2. One column of Λ is calculated at a time. Obtain initial approximations for the elements of the j^{th} column (j = 1 .... m) of Λ from

$$\lambda_{ij}^{(o)} = \frac{g_{ij}}{\gamma_i} \qquad (i = 1 \ldots m)$$

Find the position k of the largest element $\lambda_{kj}^{(o)}$ of this column vector.

The superscript (n) denotes the number of the iteration.

3. Compute the sum of squares of the elements of the j^{th} column of Λ

$$s^{(n)} = \sum_{i=1}^{m} (\lambda_{ij}^{(n)})^2$$

4. If $|1 - s^{(n)}|$ falls within a specified tolerance limit, iteration is terminated and step 9 is carried out. If the tolerance limit is exceeded step 5 is carried out.

5. Compute

$$x^{(n)} = \frac{\lambda_{kj}^{(n)}}{\sqrt{s^{(n)}}}$$

6. If n is greater than or equal to 2 compute

$$y^{(n)} = y^{(n-1)} + (x^{(n)} - y^{(n-1)}) \frac{1 - s^{(n-1)}}{s^{(n)} - s^{(n-1)}}$$

If n is equal to 0 or 1 let

$$y^{(n)} = x^{(n)}$$

7. Compute

$$\beta_j^{(n)} = -\gamma_j + \frac{g_{kj}}{y^{(n)}}$$

8. Compute the next approximations to the elements of the j^{th} column of Λ

$$\lambda_{ij}^{(n+1)} = \frac{g_{ij}}{\gamma_i + \beta_j^{(n)}}$$

Return to Step 3 and carry out the next iteration.

9. Normalize the vector.

$$\lambda_{ij} = \frac{\lambda_{ij}^{(n)}}{\sqrt{s^{(n)}}} \qquad (i = 1, \ldots, m)$$

Unless the m columns of Λ have been obtained, return to step 2 and repeat the procedure for the next column of Λ.

When the above procedure was applied to factor matrices of order 12 x 4 and a tolerance limit of .000001 was used, a maximum of six iterations was required to obtain a column of Λ.

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