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ON ORDERED TOPOLOGICAL VECTOR GROUPS – NEW RESULTS

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Abstract. The theory of ordered topological vector spaces has been treated in a great number of articles and books. On the other hand, topological vector groups were introduced and studied by D. A. Raikov [On B-complete topological vector groups, Studia Math. 31 (1968), 296–305] and P. S. Kenderov [On topological vector groups, Mat. Sb. 10 (1970), 531–546]. These are vector spaces with a topology in which addition is continuous, but multiplication by scalars is continuous only if the scalar field is taken with the discrete topology. In this paper we introduce ordered topological vector groups and investigate their structure, in particular exploring them in the case when they need not be locally convex.

1. Introduction

When considering real examples of applications of vector spaces of functions and sequences that are important for mathematical analysis, the following terms are often used: inequality, positive, positive operator and positive linear form. Many topological vector spaces encountered in functional analysis have a natural ordering, hence it is of great interest to examine the relation of topology and ordering. For example, it is important to know when a positive linear form is continuous, or when a continuous linear form is equal to the difference of two positive linear forms. In an attempt to answer these and similar questions about the relationship between topology and ordering on vector space, the theory of ordered normed and, in particular, Banach spaces was born. Further, the theory of locally convex ordered spaces and topological ordered vector spaces were developed. The first results in this area of functional analysis were obtained in the years 1935–1937, when the development of the theory of topological vector spaces began. Many mathematicians deserve the merit for the development of the theory of ordered topological vector spaces, among them: F. Riesz, L. Kantorovič, G. Freudenthal, G. Birkhoff, S. Kakutani, M. G. Krein, A. Rutman, R. Nakano, J. Namioka, H. Schaefer, N. Kung, Y. Wong and others. A more detailed presentation of chronological development of this theory is given in [30], particularly in its preface. For some more recent sources see [3,7,10,26].

Topological vector groups (not to be confused with topological groups) were introduced and studied by D. A. Raikov [24] and P. S. Kenderov [14]. These are vector spaces with a topology in which addition is continuous, but multiplication by scalars is continuous only if the scalar field is taken with the discrete topology. In this paper we introduce ordered topological vector groups and investigate their

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structure, in particular exploring them in the case when they need not be locally convex.

2. Preliminaries

In this section, we first collect some basic notions and facts on ordered vector spaces that will be used in the paper. For details, we refer to [3, 7, 10, 26, 30].

Throughout the paper, $(E, \preceq)$ or (E, C) will denote a real ordered vector space with zero element θ and the positive cone $C = \{x \in E : x \succeq \theta\}$, so that for $x, y \in E$ we have $x \preceq y$ if and only if $y - x \in C$. Here, the cone C is an arbitrary non-empty convex subset of E satisfying $\lambda C \subseteq C$ whenever $\lambda \geq 0$ (i.e., the relation $\preceq$ is not, in general, supposed to be antisymmetric and the cone need not satisfy that $C \cap (-C) = \emptyset$). The cone C is said to be generating if $E = C - C$ holds.

An interval in (E, C) is every set of the form $[x, y] = \{z \in E : x \preceq z \preceq y\} = (x + C) \cap (y - C)$. A set $A \subseteq E$ is order-bounded if it is contained in an interval. We will use the following notation:

$$[A] = (A + C) \cap (A - C) = \bigcup \{[a, b] : a \preceq b,\ a, b \in A\},$$
$$S(A) = \bigcup \{[-u, u] : u \in A \cap C\}.$$

The set A is called order-convex if $A = [A]$, o-convex if it is convex and order-convex, and solid if $A = S(A)$. It is absolutely order-convex, resp. positively order-convex, if $[-x, x] \subseteq A$ whenever $x \in A \cap C$, resp. $[0, x] \subseteq A$ whenever $x \in A \cap C$.

A linear mapping f from an ordered vector space (E, C) to another one (F, K) is said to be positive if $f(C) \subseteq K$.

An ordered vector space (E, C) is called a Riesz space if the order $\preceq$ is antisymmetric and for each two elements $x, y \in E$ there exists $\sup\{x, y\}$ in E (the supremum is defined in a standard way). In a Riesz space the relation $[\theta, u] + [\theta, v] = [\theta, u + v]$ holds for all $u, v \in C$. The following notation is commonly used: $|x| = \sup\{x, -x\}$, and $|x + y| \preceq |x| + |y|$ holds for all $x, y \in E$. A subspace F of a Riesz space (E, C) is a Riesz space itself if for each $x \in F$, also $\sup\{x, \theta\} \in F$. A solid subspace of a Riesz space is said to be its l-ideal.

In a Riesz space (E, C), a set $A \subseteq E$ is solid if and only if, for all $x, y \in E$, $y \in A$ and $|x| \preceq |y|$ implies that $x \in A$. The solid kernel $\operatorname{sk} A = \{x \in E : [-|x|, |x|] \subseteq A\}$ of a set $A \subseteq E$ is the largest solid subset contained in A.

Now we introduce the basic objects of our investigation.

By a *topological vector group* (TVG, for short), following [24] and [14], we will understand a vector space E endowed with a topology $\mathcal{T}$ such that operations $(x, y) \mapsto x + y$ (from $E \times E$ to E) and, for each $\lambda \in \mathbb{R}$, $x \mapsto \lambda x$ (from E to E) are $\mathcal{T}$-continuous. We note that the difference between TVG and a topological vector space (TVS) is that in a TVS the mapping $(\lambda, x) \mapsto \lambda x$ is continuous (as a function in two variables). In a TVG this mapping is also continuous, but when the scalar field is taken with the discrete topology. A *locally convex group* (LCG) is a TVG having a zero neighborhood basis formed by absolutely convex subsets. It is known that a TVG (resp. LCG) is a TVS (resp. LCS) if and only if it has a zero neighborhood basis formed by absorbing subsets.

If (E, C) is an ordered vector space, and $(E, \mathcal{T})$ a TVG, then the triple $(E, C, \mathcal{T})$ is called an *ordered topological vector group* (OTVG, for short). In a similar way an *ordered locally convex group* (OLCG), *ordered topological vector space* (OTVS) and *ordered locally convex space* (OLCS) are defined.

3. Open decomposition property of OTVGs

The open decomposition property for OLCSs was introduced and studied in [9] and [29]. Here, we will introduce and study this property for OTVGs in more details.

Definition 3.1. An OTVG $(E, C, \mathcal{T})$ is said to have an open decomposition property if $U \cap C - U \cap C$ is a $\mathcal{T}$-neighborhood of zero for each $\mathcal{T}$-neighborhood of zero U.

Note that in the case of an OTVS, an additional condition is imposed that $E = C - C$. However, for a discrete OTVG, if C is a proper subspace of E, then $E \neq C - C$. By [4], an ordered Banach space E with a closed cone C has an open decomposition property if and only if $E = C - C$.

A subspace A of an ordered vector space (E, C) is said to be *positively generated* if $A \subseteq A \cap C - A \cap C$. The following lemma gives a characterization of OTVGs with the open decomposition property in terms of positively generated subsets.

Lemma 3.2. *For each OTVG $(E, C, \mathcal{T})$ the following conditions are equivalent:*

(a) *$(E, C, \mathcal{T})$ has the open decomposition property;*
(b) *$(E, C, \mathcal{T})$ has a zero neighborhood basis formed by positively generated subsets.*

Proof. For each $\mathcal{T}$-neighborhood of zero V there is a symmetric $\mathcal{T}$-neighborhood of zero U such that $V \supseteq U + U = U - U \supseteq U \cap C - U \cap C$. Since $U \cap C \subseteq U \cap C - U \cap C$, i.e. $U \cap C - U \cap C \subseteq (U \cap C - U \cap C) \cap C - (U \cap C - U \cap C) \cap C$ and so the group $(E, C, \mathcal{T})$ has the 0-neighborhood basis formed by positively generated subsets. This proves that (a) implies (b).

Conversely, if V is a $\mathcal{T}$-neighborhood of zero, then there is a positively generated $\mathcal{T}$-neighborhood of zero U such that $V \supseteq U$, and then $V \cap C \supseteq U \cap C$, and hence $V \cap C - V \cap C \supseteq U \cap C - U \cap C \supseteq U$. Hence, the group $(E, C, \mathcal{T})$ has the open decomposition property. $\square$

Recall (see, e.g., [2]) that a sequence $\mathcal{U} = (U_n)_{n=1}^{\infty}$ of balanced and absorbing subsets of a vector space is called a *string* if $U_{n+1} + U_{n+1} \subseteq U_n$ holds for each $n \in \mathbb{N}$. A string $\mathcal{U}$ in a TVS $(E, \mathcal{T})$ is called *$\mathcal{T}$-topological* if each U_n is a $\mathcal{T}$-neighborhood of zero. Note that if U is an absolutely convex neighborhood of zero then it generates an associated $\mathcal{T}$-topological string $\mathcal{U} = (\frac{1}{2^{n-1}} U)_{n \in \mathbb{N}}$.

Proposition 3.3. *Let $(E, C, \mathcal{T})$ be an OTVG such that the cone C is generating. Then the following conditions are equivalent:*

(a) *$(E, C, \mathcal{T})$ has the open decomposition property.*
(b) *For each $\mathcal{T}$-topological string $\mathcal{V}$, the string $\mathcal{V} \cap C - \mathcal{V} \cap C$ is $\mathcal{T}$-topological.*

Proof. It is obvious that (a) implies (b), since if $\mathcal{V} = (V_n)_{n\in\mathbb{N}}$ is a $\mathcal{T}$-topological string, then $V_n \cap C - V_n \cap C$ is a $\mathcal{T}$-neighborhood of zero, for each $n \in \mathbb{N}$, because of the open decomposition property of $(E, C, \mathcal{T})$. Conversely, if V is a $\mathcal{T}$-neighborhood of zero, then there exists a $\mathcal{T}$-topological string $\mathcal{V} = (V_n)_{n\in\mathbb{N}}$ with $V_1 = V$. But then, using (b), we get that $\mathcal{V}\cap C - \mathcal{V}\cap C = (V_n \cap C - V_n \cap C)_{n\in\mathbb{N}}$ is a $\mathcal{T}$-topological string. In particular, $V_1 \cap C - V_1 \cap C = V \cap C - V \cap C$ is a $\mathcal{T}$-neighborhood of zero. □

In each OTVG $(E, C, \mathcal{T})$, the collection of all $\mathcal{T}$-topological strings generates a topological vector space $(E, \overline{\mathcal{T}})$. It is easy to see that $\overline{\mathcal{T}}$ is the finest linear topology coarser than the given topology $\mathcal{T}$.

Proposition 3.4. *Let $E = C - C$. If an OTVG $(E, C, \mathcal{T})$ has the open decomposition property, then so does $(E, C, \overline{\mathcal{T}})$.*

Proof. Let $\mathcal{V} = (V_n)_{n\in\mathbb{N}}$ be a $\overline{\mathcal{T}}$-topological string. This means that V_n, $n \in \mathbb{N}$ are $\overline{\mathcal{T}}$-neighborhoods of zero. By the previous proposition, $\mathcal{V} \cap C - \mathcal{V} \cap C = (V_n \cap C - V_n \cap C)_{n\in\mathbb{N}}$ is a $\overline{\mathcal{T}}$-topological string. This means that $(E, C, \overline{\mathcal{T}})$ has the open decomposition property. □

Corollary 3.5. *If $E = C - C$ then (E, C, t^f) has the open decomposition property, where t^f is the finest linear topology on E.*

Remark 3.6. Similarly as in Proposition 3.4, it can be proved that open decomposition property is preserved when passing:

(1) from an OTVS $(E, C, \mathcal{T})$ to the associated OLCS $(E, \mathcal{T}^\circ)$, where the topology $\mathcal{T}^\circ$ has as the zero neighborhood basis the collection of all absolutely convex $\mathcal{T}$-neighborhoods of zero;
(2) from an OLCG $(E, C, \mathcal{T})$ to the associated OLCS $(E, C, \operatorname{loc}\mathcal{T})$, where the topology $\operatorname{loc}\mathcal{T}$ has as the zero neighborhood basis the collection of all absorbing absolutely convex $\mathcal{T}$-neighborhoods of zero.

In particular, if $E = C - C$ then (E, C, t^c) is an OLCS with the open decomposition property, where t^c is the finest locally convex topology on E.

The converses of these assertions do not hold. It is illustrated in one case by the following example.

Example 3.7. (Inspired from [23]) Consider the cone $C = \{(x, y) : x > 0, y > 0\} \cup \{(0, 0)\}$ in $\mathbb{R}^2$ and let $(\mathbb{R}^2, C, \mathcal{T})$ be the OLCG having as zero neighborhood basis the collection of sets of the form $I_\varepsilon = \{(x, 0) : |x| \le \varepsilon\}$ for $\varepsilon > 0$. Obviously, $\mathbb{R}^2 = C - C$ and $\operatorname{loc}\mathcal{T} = t^c$, hence $(\mathbb{R}^2, C, \operatorname{loc}\mathcal{T})$ has the open decomposition property. However, $(E, C, \mathcal{T})$ does not posses this property since $I_\varepsilon \cap C - I_\varepsilon \cap C = \{0\}$ for each $\varepsilon > 0$. As the OLCG $(\mathbb{R}^2, C, \mathcal{T})$ is not discrete, it does not have the open decomposition property.

If $\mathcal{U}$ is a zero neighborhood basis of an OTVG $(E, C, \mathcal{T})$ whose elements are symmetric sets U, i.e., $U = -U$ (such basis always exists, see, e.g., [23]), then from [24] we know that the elements of $\mathcal{U}$ have the following properties:

(1) $0 \in U$ for each $U \in \mathcal{U}$.
(2) For each $U \in \mathcal{U}$ and each $\lambda \neq 0$ there exists $V \in \mathcal{U}$ such that $V \subseteq \lambda U$.

(3) For each $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $V + V \subseteq U$.

The converse is also true: each filter basis $\mathcal{U}$ having properties (1), (2) and (3) generates a unique topological vector group on (E, C).

Proposition 3.8. *If $(E, C, \mathcal{T})$ is an OTVG with the zero neighborhood basis $\mathcal{U}$ then $\mathcal{U} \cap C - \mathcal{U} \cap C = \{U \cap C - U \cap C : U \in \mathcal{U}\}$ is also a zero neighborhood basis of some OTVG.*

Proof. Let us prove that $\mathcal{U} \cap C - \mathcal{U} \cap C$ is a filter basis with properties (1), (2) and (3). If $U \cap C - U \cap C$ and $V \cap C - V \cap C$ belong to $\mathcal{U} \cap C - \mathcal{U} \cap C$ for some $U, V \in \mathcal{U}$, then there exists $W \in \mathcal{U}$ such that $W \subseteq U \cap V$. Further, $W \cap C \subseteq U \cap C$ and $W \cap C \subseteq V \cap C$, hence $W \cap C - W \cap C \subseteq (U \cap C - U \cap C) \cap (V \cap C - V \cap C)$. Since $0 \in U$ for each $U \in \mathcal{U}$, then $0 \in U \cap C - U \cap C$. If $V \cap C - V \cap C \in \mathcal{U} \cap C - \mathcal{U} \cap C$ for some $V \in \mathcal{U}$ then, for each $\lambda \neq 0$, there exists $U \in \mathcal{U}$ satisfying $V \subseteq \lambda U$. It follows that $V \cap C - V \cap C \subseteq \lambda U \cap C - \lambda U \cap C = \lambda(U \cap C - U \cap C)$ (the last equality follows easily since the sets U and $U \cap C - U \cap C$ are symmetric and C is a cone).

It remains to prove the property (3). If $V \cap C - V \cap C$ belongs to $\mathcal{U} \cap C - \mathcal{U} \cap C$ for some $V \in \mathcal{U}$ then there exists $U \in \mathcal{U}$ with $U + U \subseteq V$. It further means that

$$(U \cap C - U \cap C) + (U \cap C - U \cap C) \subseteq (U + U) \cap C - (U + U) \cap C \subseteq V \cap C - V \cap C,$$

and the proposition is proved. □

The topology with the zero neighborhood basis described in the previous proposition is denoted by $\mathcal{T}_D$. It has the open decomposition property by Proposition 3.2 since its basis is formed by positively generated subsets. Also, $\mathcal{T} \leq \mathcal{T}_D$. Indeed, if V is a $\mathcal{T}$-neighborhood of zero, then there exists a symmetric $\mathcal{T}$-neighborhood of zero U such that $V \supseteq U + U = U - U \supseteq U \cap C - U \cap C$. Thus, an OTVG $(E, C, \mathcal{T})$ has the open decomposition property if and only if $\mathcal{T} = \mathcal{T}_D$.

The topology $\mathcal{T}_D$ can be characterized using the notion of positive linear mapping (see Section 2).

Proposition 3.9. *Let $(E, C, \mathcal{T})$ and $(F, K, \mathcal{P})$ be OTVGs. If a mapping f from $(E, C, \mathcal{T})$ to $(F, K, \mathcal{P})$ is positive and continuous then it is continuous from $(E, \mathcal{T}_D)$ to $(F, \mathcal{P}_D)$.*

Proof. If U is a symmetric $\mathcal{P}$-neighborhood of zero, then $f^{-1}(U \cap K - U \cap K) \supseteq f^{-1}(U) \cap C - f^{-1}(U) \cap C$. Indeed, let $x = y - z$ where $y, z \in f^{-1}(U) \cap C$. Then $f(x) = f(y - z) = f(y) - f(z) \in U \cap f(C) - U \cap f(C) \subseteq U \cap K - U \cap K$, i.e., $x \in f^{-1}(U \cap K - U \cap K)$. This proves the proposition. □

Corollary 3.10. *If an OTVG $(E, C, \mathcal{T})$ has the open decomposition property, then a linear mapping $f \colon E \to F$ is continuous from $(E, \mathcal{T})$ into $(F, \mathcal{P})$ if and only if it is continuous from $(E, \mathcal{T})$ into $(F, \mathcal{P}_D)$.*

Note that the converse of the previous corollary does not hold. It is enough to take in a vector space E the trivial cone $C = \{0\}$ and the indiscrete topology $\mathcal{T}$ and in a vector space F some topology with the open decomposition property which is not indiscrete.

The topology $\mathcal{T}_D$ is best characterized by the following corollary of Proposition 3.9.

Corollary 3.11. *The topology $\mathcal{T}_D$ is the weakest among all topologies with the open decomposition property which are finer than $\mathcal{T}$, i.e.,*

$$\mathcal{T}_D = \inf\{\mathcal{T}_\alpha : \mathcal{T}_\alpha \geq \mathcal{T} \text{ and } \mathcal{T}_\alpha \text{ has the open decomposition property}\}.$$

Remark 3.12. For an arbitrary TVG $(E, \mathcal{T})$ there is a cone C such that $(E, C, \mathcal{T})$ has the open decomposition property (it is enough to take $C = E$). A discrete TVG has the open decomposition property for each positive cone. However, an indiscrete TVG has this property if and only if $E = C - C$.

If $\{C_\alpha\}_{\alpha\in I}$ is a family of cones in a vector space E then $C = \bigcap_{\alpha\in I} C_\alpha$ is also a cone in E. It is easy to see that, in this case, if $(E, C_\alpha, \mathcal{T})$ are OTVGs and $(E, C, \mathcal{T})$ has the open decomposition property then each $(E, C_\alpha, \mathcal{T})$ has the same property. The following example shows that the converse is not true.

Example 3.13. Let $(E, C, \mathcal{T})$ be the OTVG considered in Example 3.7 which has the open decomposition property, is not discrete and for which $C \cap (-C) = \{0\}$. Since $U \cap C - U \cap C = U \cap (-C) - U \cap (-C)$ for each symmetric $\mathcal{T}$-neighborhood of zero, it follows that the cones C and $-C$ induce the same topology $\mathcal{T}_D$. Hence, $(E, C, \mathcal{T})$ has the open decomposition property if and only if $(E, -C, \mathcal{T})$ has this property. However, the OTVG $(E, \{0\}, \mathcal{T})$ does not have this property because it is not discrete.

In a similar way it can be shown that the open decomposition property is inherited from $(E, C, \mathcal{T})$ to $(E, \overline{C}, \mathcal{T})$, but not conversely.

In what follows we consider some further inheritance properties connected with the open decomposition property. We start with the following example.

Example 3.14. Let $(E, C, \mathcal{T})$ be the OTVG considered in Example 3.7. Then $(\mathbb{R}, \{0\}, \operatorname{loc}\mathcal{T})$ is an OTVG and $(\mathbb{R}, \{0\})$ is closed and finite codimensional subspace of $(\mathbb{R}^2, C)$, where $\{0\} = C \cap \mathbb{R}$. The OTVG $(\mathbb{R}, \{0\}, \operatorname{loc}\mathcal{T})$ does not have the open decomposition property because $(\operatorname{loc}\mathcal{T})_D$ is the discrete topology on $\mathbb{R}$. This means that the open decomposition property is, in general, not inherited by projective limits.

However, it can be proved that if a subspace contains the positive cone as a subset, then such subspace inherits the open decomposition property from the original space.

It is known ([30]) that the open decomposition property is inherited by inductive limits in the category of OLCSs. Similarly, for OTVSs we have

Proposition 3.15. *Let (E, C) be an ordered vector space, $(E_\alpha, C_\alpha, \mathcal{T}_\alpha)$ be OTVSs with the open decomposition property and let $f_\alpha \colon E_\alpha \to E$ be positive linear mappings. If E is the linear hull of $\bigcup_{\alpha\in I} f_\alpha(E_\alpha)$, then the inductive topology $\mathcal{T}$ on E has the open decomposition property.*

Proof. Let $\mathcal{V} = (V_n)_{n\in\mathbb{N}}$ be a $\mathcal{T}$-topological string in E. According to Proposition 3.3, it is enough to prove that $\mathcal{V} \cap C - \mathcal{V} \cap C$ is a $\mathcal{T}$-topological string. By the way neighborhoods of zero are constructed in inductive limits (see, e.g., [2]), $f_\alpha^{-1}(\mathcal{V}) = (f_\alpha^{-1}(V_n))_{n\in\mathbb{N}}$ is a $\mathcal{T}_\alpha$-topological string for each $\alpha \in I$. This further means that $f_\alpha^{-1}(\mathcal{V}) \cap C_\alpha - f_\alpha^{-1}(\mathcal{V}) \cap C_\alpha = (f_\alpha^{-1}(V_n) \cap C_\alpha - f_\alpha^{-1}(V_n) \cap C_\alpha)_{n\in\mathbb{N}}$ is a $\mathcal{T}_\alpha$-topological string, for each $\alpha \in I$. It follows from the obvious inclusion $f_\alpha^{-1}(V_n)\cap C_\alpha - f_\alpha^{-1}(V_n)\cap C_\alpha \subseteq f_\alpha^{-1}(V_n \cap C - V_n \cap C)$ that $\mathcal{V} \cap C - \mathcal{V} \cap C$ is a $\mathcal{T}$-topological string. $\square$

In particular, the open decomposition property is inherited by quotients.

It was proved in [30] that the associated topologies with the open decomposition property of the direct sum and product topologies, in the category of OLCSs, satisfy the following relations:

$$\left(\bigoplus_\alpha \mathcal{T}_\alpha\right)_D = \bigoplus_\alpha (\mathcal{T}_\alpha)_D, \tag{3.1}$$

$$\left(\prod_\alpha \mathcal{T}_\alpha\right)_D = \prod_\alpha (\mathcal{T}_\alpha)_D. \tag{3.2}$$

By [30, Theorem 3.19] it is $(\prod_\alpha V_\alpha) \cap C - (\prod_\alpha V_\alpha) \cap C = \prod_\alpha (V_\alpha \cap C - V_\alpha \cap C)$, and hence the proof of the relation (3.2) is the same in the category of OTVSs. For the direct sums we have

Proposition 3.16. *If $\{(E_\alpha, C_\alpha, \mathcal{T}_\alpha) : \alpha \in I\}$ is a family of OTVSs, and $(E, C, \bigoplus_\alpha \mathcal{T}_\alpha)$, $(C = \bigoplus_\alpha C_\alpha)$, their direct sum (in the sense of* [2]*), then the relation* (3.1) *holds.*

Proof. According to [2, §4, p. 20], the topologies $\bigoplus_\alpha (\mathcal{T}_\alpha)_D$ and $(\bigoplus_\alpha \mathcal{T}_\alpha)_D$ are generated by the strings $(V_n)_{n\in\mathbb{N}}$ and $(U_n \cap C - U_n \cap C)_{n\in\mathbb{N}}$, respectively, where $V_n = \sum_{k=1}^\infty \big\{\bigcup_\alpha \big(U^\alpha_{2^{n-1}k} \cap C_\alpha - U^\alpha_{2^{n-1}k} \cap C_\alpha\big)\big\}$ and $U_n = \sum_{k=1}^\infty \big\{\bigcup_\alpha U^\alpha_{2^{n-1}k}\big\}$. It has to be proved that $V_n = U_n \cap C - U_n \cap C$ for each n. If $x \in V_n$ then $x = \sum_{i=1}^m x_i^{\alpha_i}$, $x_i^{\alpha_i} \in U^{\alpha_i}_{2^{n-1}} \cap C_{\alpha_i} - U^{\alpha_i}_{2^{n-1}} \cap C_{\alpha_i}$, i.e., $x = \sum_{i=1}^m (u_i^{\alpha_i} - v_i^{\alpha_i}) = \sum_{i=1}^m u_i^{\alpha_i} - \sum_{i=1}^m v_i^{\alpha_i}$. This means that $x \in U_n \cap C - U_n \cap C$ because $u_i^{\alpha_i}, v_i^{\alpha_i} \in U^{\alpha_i}_{2^{n-1}} \cap C_{\alpha_i}$ and so $\sum_{i=1}^m u_i^{\alpha_i}, \sum_{i=1}^m v_i^{\alpha_i} \in U_n \cap C$.

Conversely, if $x \in U_n \cap C - U_n \cap C$ then $x = u - v$, where $u = \sum_{i=1}^m u_i^{\alpha_i}$ and $v = \sum_{i=1}^m v_i^{\alpha_i}$, $u, v \in U_n \cap C$, hence $x = \sum_{i=1}^m (u_i^{\alpha_i} - v_i^{\alpha_i}) \in V_n$. It is clear that from $u, v \in C$ it follows that $u_i^{\alpha_i}, v_i^{\alpha_i} \in C_{\alpha_i}$. $\square$

4. Order-convex OTVGs

OTVSs and OLCSs that have order-convex subsets as elements of a zero neighborhood basis were studied in [4, 15, 18, 26, 29, 30]. Particularly well-known are the results on OLCSs called locally o-convex spaces (having zero neighborhood basis formed by absolutely convex and order-convex subsets (see Section 2)) [15].

In this section, we will study OTVGs and OLCGs that have order-convex subsets as elements of a zero neighborhood basis. We will also show some results obtained while studying OTVSs and OLCSs not present in the cited works.

Definition 4.1. An OTVG $(E, C, \mathcal{T})$ is order-convex, if it has a zero neighborhood basis formed by order-convex subsets.

Since, as already mentioned, every TVG has a zero neighborhood basis formed by symmetric subsets, the following result follows.

Lemma 4.2. *An OTVG $(E, C, \mathcal{T})$ is order-convex if and only if it has a zero neighborhood basis formed by symmetric order-convex subsets.*

Corollary 4.3. *If an OLCG $(E, C, \mathcal{T})$ is order-convex, then it has a zero neighborhood basis formed by absolutely convex, order-convex subsets.*

Since a symmetric order-convex subset is absolutely order-convex, and a symmetric absolutely order-convex subset is positively order-convex (for terminology, see Section 2), then it is clear that order-convex OTVG has a zero neighborhood basis formed by symmetric absolutely order-convex subsets, resp. by symmetric positively order-convex subsets.

For bounded order-convex subsets of OTVGs the following is valid similarly as for bounded subsets of OTVSs.

Lemma 4.4. *If $(E, C, \mathcal{T})$ is an order-convex OTVG, then the order-convex hull $[A]$ of any $\mathcal{T}$-bounded subset A is a $\mathcal{T}$-bounded subset.*

If $(E, C, \mathcal{T})$ is an order-convex OTVS, then clearly any interval $[x, y]$ is $\mathcal{T}$-bounded, because $[x, y]$ is the order-convex hull of the set $\{x, y\}$. However in an order-convex OTVG $(E, C, \mathcal{T})$ this is not necessarily so, as not every $\mathcal{T}$-neighborhood of zero is necessarily absorbing. The following proposition shows when it is true.

Proposition 4.5. *An order-convex OTVG $(E, C, \mathcal{T})$ is an OTVS if and only if $[x, x]$ is a $\mathcal{T}$-bounded subset for any $x \in E$.*

Proof. If $(E, C, \mathcal{T})$ is an order-convex OTVS, then $[x, x]$ is a $\mathcal{T}$-bounded subset, because $[x, x]$ is the order-convex hull of the set $\{x\}$. Conversely, let $e \in E$ and let V be a $\mathcal{T}$-neighborhood of zero. We shall prove that V is an absorbing subset of E. Since $e \in [e, e]$, then there exists $\lambda \geq 0$ such that $e \in [e, e] \subseteq \lambda V$, and it follows that the zero-neighborhood V is absorbing. □

For order-convex OTVGs the following statement is true, similarly as for order-convex OTVSs.

Proposition 4.6. *If $(E, C, \mathcal{T})$ is an order-convex OTVG that is Hausdorff, then the cone C is antisymmetric, that is $C \cap (-C) = \{0\}$.*

Proof. Since the OTVG $(E, C, \mathcal{T})$ has a zero neighborhood basis formed by order-convex subsets, then for any $\mathcal{T}$-neighborhood of zero $U = [U]$ it follows that $\{0\} \subseteq U$, hence $[\{0\}] \subseteq U = [U]$, that is $[0, 0] = C \cap (-C) \subset \bigcap U = \{0\}$. □

The converse of the previous assertion is not true. For instance an indiscrete OTVG is order-convex for any cone $C \subseteq E$, yet it is not Hausdorff. The space itself E is an order-convex set for any cone C, because $(E + C) \cap (E - C) = E$.

As OTVG $(E, E, \mathcal{T})$ has the open-decomposition property, so OTVG $(E, \{0\}, \mathcal{T})$ is order-convex, because for any $\mathcal{T}$-neighborhood of zero U, $U = (U + \{0\}) \cap (U - \{0\}) = [U]$. If $(E, C, \mathcal{T})$ is an arbitrary OTVG, then to a zero neighborhood basis $\mathcal{U}$ formed by symmetric subsets the following basis can be associated: $[\mathcal{U}] = \{[U] : U \in \mathcal{U}\}$; in this case it can be proven that it determines a unique OTVG on E that is order-convex, in a similar way as for $\mathcal{U} \cap C - \mathcal{U} \cap C$. Because of the inclusion $U \subseteq [U]$ for any $U \in \mathcal{U}$, clearly one obtains the topology $\mathcal{T}_F \leq \mathcal{T}$. Taking the last inequality into account, it is natural to pose the question of when the topology $\mathcal{T}_F$ is indiscrete, and when it is Hausdorff.

Proposition 4.7. *For any OTVG $(E, C, \mathcal{T})$ the following statements are equivalent:*

(1) $\overline{C} = E$
(2) $\mathcal{T}_F$ *is an indiscrete topology.*

Proof. For any $\mathcal{T}$-neighborhood of zero V we have:

$$(V + \overline{C}) \cap (V - \overline{C}) = (V + E) \cap (V - E) = E \,.$$

According to Proposition 4.8 (shown below) it follows that $\mathcal{T}_F$ is an indiscrete topology, that means that (1) implies (2). If we prove that $\overline{C}^{\mathcal{T}} = \overline{C}^{\mathcal{T}_F}$ for any OTVG $(E, C, \mathcal{T})$ then it is clear that (2) implies (1). Obviously $\overline{C}^{\mathcal{T}} \subseteq \overline{C}^{\mathcal{T}_F}$. It is also true that

$$\begin{aligned}\overline{C}^{\mathcal{T}_F} &= \bigcap (C + [U]) = \bigcap (C + (U + C) \cap (U - C)) \subseteq \bigcap (C + U + C)\\ &\subseteq \bigcap (C + U) = \overline{C}^{\mathcal{T}} .\end{aligned}$$

□

In the cited papers (e.g. [15]) it is proven using seminorms, that an OLCS $(E, C, \mathcal{T})$ is order-convex if and only if the OLCS $(E, \overline{C}, \mathcal{T})$ is order-convex. We shall prove the same for OTVGs in an elementary way:

Proposition 4.8. *For any OTVG $(E, \overline{C}, \mathcal{T})$ it holds $\mathcal{T}_F = \mathcal{T}_{\overline{F}}$, where $\mathcal{T}_{\overline{F}}$ has for zero neighborhood basis the sets of the form $(U + \overline{C}) \cap (U - \overline{C})$ for any $U \in \mathcal{U}$.*

Proof. Since $(U + \overline{C}) \cap (U - \overline{C}) \supseteq (U + C) \cap (U - C)$ if follows that $\mathcal{T}_F \geq \mathcal{T}_{\overline{F}}$. Conversely, if $V \in \mathcal{U}$, then there exists $U \in \mathcal{U}$ such that $V \supseteq U + U$, that is, $V + C \supseteq U + U + C \supseteq U + \overline{C}$ and $V - C \supseteq U + U - C \supseteq U - \overline{C}$ ($\overline{C} \subseteq U + C$ for any $U \in \mathcal{U}$). It means that $(V + C) \cap (V - C) \supseteq (U + \overline{C}) \cap (U - \overline{C})$, that is $\mathcal{T}_F \leq \mathcal{T}_{\overline{F}}$. □

Corollary 4.9. *An OTVG $(E, C, \mathcal{T})$ (this also means ULCG or OTVS) is order-convex if and only if the OTVG $(E, \overline{C}, \mathcal{T})$ is order-convex.*

The following proposition shows when the topology $\mathcal{T}_F$ is Hausdorff.

Proposition 4.10. *For any OTVG $(E, C, \mathcal{T})$ the following statements are equivalent:*

(1) *The cone $\overline{C}$ is antisymmetric, that is $\overline{C}^{\mathcal{T}} \cap (-\overline{C}^{\mathcal{T}}) = \{0\}$.*
(2) *Topology $\mathcal{T}_F$ is Hausdorff.*

Proof. If $x \in \overline{\{0\}}^{\mathcal{T}_F}$, it means that $x \in (U + C) \cap (U - C)$ for any $\mathcal{T}$-neighborhood of zero U, that is $(x - U) \cap C \neq \emptyset$ and $(x - U) \cap (-C) \neq \emptyset$, therefore $x \in \overline{C}^{\mathcal{T}} \cap (-\overline{C}^{\mathcal{T}}) = \overline{C}^{\mathcal{T}_F} \cap (-\overline{C}^{\mathcal{T}_F}) = \{0\}$. It follows that $x = 0$, that is the topology $\mathcal{T}_F$ is Hausdorff, and so (1) implies (2). The converse follows from Propositions 4.6. and 4.8. □

The following theorem answers when a discrete OTVG is order-convex.

Proposition 4.11. *For any discrete OTVG $(E, C, \mathcal{T})$ the following statements are equivalent:*

(1) *OTVG $(E, C, \mathcal{T})$ is order-convex.*
(2) *The cone C is antisymmetric.*

Proof. Since $\mathcal{T}$ is discrete, $(\{0\}+C)\cap(\{0\}-C) = C\cap(-C) \subset \{0\}$. Hence, $C\cap(-C)=\{0\}$, that is the cone C is antisymmetric, so (1) implies (2). Conversely, if the cone C is antisymmetric, then clearly $[\{0\}]=\{0\}$, that is $\mathcal{T}=\mathcal{T}_F$, and the given discrete OTVG $(E,C,\mathcal{T})$ is then order-convex, so (2) implies (1). $\square$

Example 4.12. Let $(E,C,\mathcal{T})$ be the OLCG where $E=\mathbb{R}^2$, $C=\{(x,y): x\geq 0, y\geq 0\}$ and topology $\mathcal{T}$ has for a zero neighborhood basis the sets of the form $I_\varepsilon=\{(x,0): |x|\leq\varepsilon\}$, $\varepsilon>0$. Then $(E,C,\mathcal{T})$ is order-convex. Indeed, $(I_\varepsilon+C)\cap(I_\varepsilon-C) = \{(x,y): x\geq-\varepsilon, y\geq 0\}\cap\{(x,y): x\leq\varepsilon, y\leq 0\} = I_\varepsilon$. Hence, $(E,C,\mathcal{T})$ has for zero neighborhood basis absolutely convex order-convex subsets.

In Section 6 we will show that for an OLCG $(E,C,\mathcal{T})$ which is order-convex, the associated OLCS $(E,C,\operatorname{loc}\mathcal{T})$ need not be order-convex (Example 6.2).

In what follows we prove a proposition that further characterizes topology $\mathcal{T}_F$.

Proposition 4.13. *If f is a continuous positive linear map from an OTVG $(E,C,\mathcal{T})$ to another one $(F,K,\mathcal{P})$, then it is continuous when treated as mapping from the OTVG $(E,C,\mathcal{T}_F)$ into OTVG $(F,K,\mathcal{P}_F)$.*

Proof. If U is a $\mathcal{P}$-neighborhood of zero, then $f^{-1}([U]) = f^{-1}[(U+K)\cap(U-K)] = f^{-1}(U+K)\cap f^{-1}(U-K) \supseteq (f^{-1}(U)+C)\cap(f^{-1}(U)-C)$, because if $x\in f^{-1}(U)+C$, then $f(x)\in U+f(C)\subseteq U+K$. This means that f is a continuous map from OTVG $(E,C,\mathcal{T}_F)$ to OTVG $(F,K,\mathcal{P}_F)$. $\square$

Corollary 4.14. *If an OTVG $(F,K,\mathcal{P})$ is order-convex, then a positive linear map from OTVG $(E,C,\mathcal{T})$ to OTVG $(F,K,\mathcal{P})$ is continuous if and only if it is continuous from OTVG $(E,C,\mathcal{T}_F)$ to OTVG $(F,K,\mathcal{P}_F)$.*

Corollary 4.15. *The topology $\mathcal{T}_F$ is the strongest among topologies $\mathcal{T}_\alpha$ weaker than $\mathcal{T}$ and for which the OTVG $(E,C,\mathcal{T}_\alpha)$ is order-convex for any α, that is $\mathcal{T}_F = \sup_\alpha\{\mathcal{T}_\alpha : \alpha\in I, \mathcal{T}_\alpha\leq\mathcal{T}$ and OTVG $(E,C,\mathcal{T}_\alpha)$ is order-convex$\}$.*

In subsequent propositions we give some properties of order-convex OTVSs. A string $\mathcal{U}=(U_n)_{n\in\mathbb{N}}$ in an OTVS $(E,C,\mathcal{T})$ is *order-convex* if all elements U_n are order-convex. The following proposition shows that the set of order-convex topological strings in an OTVS is nonempty.

Proposition 4.16. *If $\mathcal{U}=(U_n)_{n\in\mathbb{N}}$ is a $\mathcal{T}$-topological string in an OTVS $(E,C,\mathcal{T})$, then $[\mathcal{U}]=([U_n])_{n\in\mathbb{N}}$ is a $\mathcal{T}_F$-topological (so also $\mathcal{T}$-topological) string in the OTVS $(E,C,\mathcal{T}_F)$.*

Proof. The sets $[U_n]_{n\in\mathbb{N}}$ are absorbing and balanced $\mathcal{T}_F$-neighborhoods of zero. We shall prove that $[U_{n+1}]+[U_{n+1}]\subseteq[U_n]$ for any $n\in\mathbb{N}$. Let $x=u+v$, where $u,v\in[U_{n+1}]$; then there exist $a,b,c,d\in U_{n+1}$ with $a\leq b$, $c\leq d$ such that $u\in[a,b]$ and $v\in[c,d]$. It follows that $x\in[a+c,b+d]$, $a+c\leq b+d$, $a+c\in U_{n+1}+U_{n+1}\subseteq U_n$ and $b+d\in U_{n+1}+U_{n+1}\subseteq U_n$, therefore $x\in[U_n]$. $\square$

Proposition 4.17. *If $(E,C,\mathcal{T})$ is an OTVS, then there exists a set of order-convex strings in E with the following properties:*

(i) *For any two strings $\mathcal{U}$ and $\mathcal{V}$ from this set, there exists another string $\mathcal{W}$ from the same set such that $\mathcal{W}\subseteq\mathcal{U}\cap\mathcal{V}$.*

(ii) *Members of the strings from this set form a zero $\mathcal{T}_F$-neighborhood basis.*

Proof. On the basis of Proposition 4.16, there exists in the OTVS $(E, C, \mathcal{T})$ a nonempty set of order-convex $\mathcal{T}$-topological strings. According to [2, §1, p. 6], if $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ and $\mathcal{V} = (V_n)_{n\in\mathbb{N}}$ are two strings from the given set, then there exists a $\mathcal{T}$-topological string $\mathcal{W} = (W_n)_{n\in\mathbb{N}}$ that is contained in $\mathcal{U}\cap\mathcal{V} = (U_n\cap V_n)_{n\in\mathbb{N}}$. This means that also $[W_n] \subseteq V_n \cap U_n$ for any $n \in \mathbb{N}$ because $\mathcal{U}$ and $\mathcal{V}$ are order-convex strings, that is $[\mathcal{W}] = ([W_n])_{n\in\mathbb{N}}$ is the required order-convex string in $\mathcal{U}\cap\mathcal{V}$. The part (ii) is clear, because if U is a $\mathcal{T}_F$-neighborhood of zero, then there exists a $\mathcal{T}$-neighborhood of zero V, such that $U \supseteq [V]$. But according to Proposition 4.16 the neighborhood V generates an order-convex string $[\mathcal{V}] = ([V_n])_{n\in\mathbb{N}}$, $V_1 = V$ and $U \supseteq [V_1]$. □

The following proposition is the converse of the previous one (it follows directly from [2, §1, p. 7]).

Proposition 4.18. *If a set of order-convex strings in an OTVS $(E, C, \mathcal{T})$ satisfies part (i) of Proposition 4.17 then this set of strings generates a linear order-convex topology. The zero neighborhood basis of this topology is formed by elements of strings of the given set.*

Since the finest linear topology t^f on any vector space E is Hausdorff, then according to Proposition 4.6 the OTVS (E, C, t^f) is not necessarily order-convex. Indeed, it is enough to consider a non antisymmetric cone. According to Propositions 4.18 and 4.13, it follows that t^f_F is the finest linear order-convex topology. From Proposition 4.10 it is Hausdorff if and only if $\overline{C}^{t^f_F}$ is an antisymmetric cone.

The following proposition characterizes the finest linear order-convex topology.

Proposition 4.19. *For any OTVS $(E, C, \mathcal{T})$ that is order-convex, the following statements are equivalent:*

(1) *$\mathcal{T}$ is the finest linear order-convex topology.*
(2) *Any positive linear map from the OTVS $(E, C, \mathcal{T})$ to an arbitrary OTVS $(F, K, \mathcal{P})$ that is order-convex, is continuous.*

Proof. If V is a balanced order-convex $\mathcal{P}$-neighborhood of zero, then it generates a $\mathcal{P}$-topological string $\mathcal{V} = (V_n)_{n\in\mathbb{N}}$, $V_1 = V$. Since $\mathcal{T}$ is the finest linear order-convex topology on (E, C), then $f^{-1}(\mathcal{V}) = (f^{-1}(V_n))_{n\in\mathbb{N}}$ is a $\mathcal{T}$-topological string, because it is obviously order-convex. This implies that f is a continuous map because $f^{-1}(V) = f^{-1}(V_n) \supseteq f^{-1}(V_2)$, so (1) implies (2). Conversely, if f is a continuous linear positive map from an order-convex OTVS $(E, C, \mathcal{T})$ to an arbitrary order-convex OTVS $(F, K, \mathcal{P})$, then it follows that $\mathcal{T} = t^f_F$, if we consider (E, C, t^f_F) instead of $(F, K, \mathcal{P})$ and instead of f we consider the identity map in E. □

In the next propositions we will investigate hereditary properties of order-convex OTVGs and OTVSs. It is clear that the OTVG $(E, C, \mathcal{T})$ is order-convex if and only if OLCS $(E, C, \mathcal{T})$ is order-convex. For OTVGs the following proposition can be easily proved.

Proposition 4.20. *If $(E, C, \mathcal{T})$ is an order-convex OTVG, and $H \subseteq E$, then the OTVG $(H, C \cap H, \mathcal{T}|H)$ is order-convex.*

It is an open question whether $(\mathcal{T}_F|H) = (\mathcal{T}|H)_F$ holds true, which would be a more general result than Proposition 4.20. From Corollary 4.15, it follows that $(\mathcal{T}_F|H) \leq (\mathcal{T}|H)_F$.

The category of order-convex OTVSs is invariant with respect to the projective limits, because a finite number of intersections of order-convex subsets is order-convex itself. In the case of an inductive limit this is not true in general. It is proved in [26] that the quotient space of an order-convex OTVS is not necessarily order-convex. In [26], it is also shown that an arbitrary product and direct sum of order-convex OLCSs is an order-convex OLCSs. More general results for OLCSs are proved in [30, Theorems 5.20 and 5.21]: $\prod_\alpha \mathcal{T}_{\alpha F} = \prod_\alpha (\mathcal{T}_\alpha)_F$ and $\bigoplus_\alpha \mathcal{T}_{\alpha F} = (\bigoplus_\alpha \mathcal{T}_\alpha)_F$.

We will show that the last statement is also true for OTVSs (the proof for the products is the same as in [30]). The proof for the direct sum must be different, because in the general case the inductive limit and direct sum in the TVS category differ from those in the category of LCSs (see [2] for details).

Proposition 4.21. *If $\{(E_\alpha, C_\alpha, \mathcal{T}_\alpha) : \alpha \in I\}$ is a family of OTVSs and $(E, C, \bigoplus_\alpha \mathcal{T}_\alpha)$ their direct sum in the TVP category (see [2, §4, p. 21]) then $\bigoplus_\alpha \mathcal{T}_{\alpha F} = (\bigoplus_\alpha \mathcal{T}_\alpha)_F$.*

Proof. Following [2, §4, p. 20], the topologies $\bigoplus_\alpha \mathcal{T}_{\alpha F}$ and $(\bigoplus_\alpha \mathcal{T}_\alpha)_F$ are generated by the strings $(U_n)_{n\in\mathbb{N}}$ and $([V_n])_{n\in\mathbb{N}}$, where $V_n = \sum_{k=1}^{\infty} \left\{\cup_\alpha U^\alpha_{2^{n-1}k}\right\}$ and $U_n = \sum_{k=1}^{\infty} \left\{\cup_\alpha \left[U^\alpha_{2^{n-1}k}\right]\right\}$. We will prove that $[V_n] = U_n$, for any $n \in \mathbb{N}$. If $x \in [V_n]$, then there exist $a, b \in V_n$, $a \leq b$, such that $a = \sum_{i=1}^{m} x^{\alpha_i}_{2^{n-1}}$, $b = \sum_{i=1}^{m} y^{\alpha_i}_{2^{n-1}}$ and also $\sum_{i=1}^{m} x^{\alpha_i}_{2^{n-1}} \leq x \leq \sum_{i=1}^{m} y^{\alpha_i}_{2^{n-1}}$. Then it follows that $\prod_{\alpha_i}(a) \leq \prod_{\alpha_i}(x) \leq \prod_{\alpha_i}(b)$, that is $x^{\alpha_i}_{2^{n-1}} \leq \prod_{\alpha_i}(x) \leq y^{\alpha_i}_{2^{n-1}}$. Since $x^{\alpha_i}_{2^{n-1}}, y^{\alpha_i}_{2^{n-1}} \in U^{\alpha_i}_{2^{n-1}}$ and $x = \sum_{i=1}^{m} \prod_{\alpha_i}(x)$, then $x \in U_n$ because $\prod_{\alpha_i}(x) = x_{\alpha_i} \in \left[U^{\alpha_i}_{2^{n-1}}\right]$.

Conversely, if $x \in U_n$, then $x = \sum_{i=1}^{m} x^{\alpha_i}_{2^{n-1}}$, where $x^{\alpha_i}_{2^{n-1}} \in \left[U^{\alpha_i}_{2^{n-1}}\right]$. This means that for any α_i there exist $v^{\alpha_i}_{2^{n-1}}, u^{\alpha_i}_{2^{n-1}} \in U^{\alpha_i}_{2^{n-1}}$ and $v^{\alpha_i}_{2^{n-1}} \leq u^{\alpha_i}_{2^{n-1}}$; then it follows that $a = \sum_{i=1}^{m} v^{\alpha_i}_{2^{n-1}} \leq x \leq \sum_{i=1}^{m} u^{\alpha_i}_{2^{n-1}} = b$. Clearly $a \leq b$, $a, b \in V_n$, and it follows that $x \in [V_n]$. □

A discrete LCG $\mathcal{V}(E, E^*)$ is order-convex if and only if, the positive cone C is antisymmetric (Proposition 4.11). $\mathcal{V}(E, E^*)$ is the finest LCG and $\mathcal{V}_F(E, E^*)$ is the finest order-convex LCG. Since $\overline{C}^{\mathcal{V}(E,E^*)} = C$, then according to Proposition 4.10, if OLCG $(E, C, \mathcal{V}(E, E^*))$ is not order-convex, then the topology $\mathcal{V}_F(E, E^*)$ is not Hausdorff. It follows that OLCG $(E, C, \mathcal{V}(E, E^*))$ is Hausdorff if and only if it is discrete. The finest order-convex LCG on (E, C) has the following subspace for a basis in the neighborhood of zero: $[\{0\}] = C \cap (-C)$.

The following proposition characterizes the finest order-convex LCG.

Proposition 4.22. *For any order-convex OLCG $(E, C, \mathcal{T})$ the following statements are equivalent:*

(1) *$\mathcal{T}$ is the finest order-convex LCG.*
(2) *Any positive linear map from OLCG $(E, C, \mathcal{T})$ to an arbitrary OLCG $(F, K, \mathcal{P})$ that is order-convex is continuous.*

Proof. If V is a balanced, convex and order-convex $\mathcal{P}$-neighborhood of zero, then it is clear that $f^{-1}(V) = f^{-1}((V + K) \cap (V - K)) \supseteq (f^{-1}(V) + C) \cap (f^{-1}(V) - C) \supseteq$

$C \cap (-C)$, that is $f^{-1}(V)$ is a $\mathcal{T} = \mathcal{V}_F(E, E^*)$-neighborhood of zero. Therefore, (1) implies (2). The converse is obtained considering that $(F, K, \mathcal{P}) = (E, C, \mathcal{V}_F(E, E^*))$ and taking f to be the identity map on E. □

5. Solid OTVGs

Definition 5.1. An OTVG $(E, C, \mathcal{T})$ is solid, if it has a zero neighborhood basis formed by solid subsets.

Since a solid subset is balanced, then it is clear that a solid OTVG $(E, C, \mathcal{T})$ has a zero neighborhood basis formed by balanced subsets, therefore of solid and symmetric subsets.

Example 5.2. By [14, Remark, p. 533], there exists a Hausdorff TVG-topology $\mathcal{T}$ on $\mathbb{R}$, strictly weaker than the standard Euclidean topology t. As shown in [23, Example B, p. 43], there is no $\mathcal{T}$-neighborhood of zero which is balanced and different from $\mathbb{R}$.

Indeed, if $U \neq \mathbb{R}$ were a balanced $\mathcal{T}$-neighborhood of zero, then it would be of the form $U = (-\lambda, \lambda)$ or $U = [-\lambda, \lambda]$ for some $\lambda > 0$. If V is an arbitrary t-neighborhood of zero, then for some $\varepsilon > 0$ it would be $V \supseteq [-\varepsilon, \varepsilon] = \frac{\varepsilon}{\lambda}[-\lambda, \lambda] \supseteq \frac{\varepsilon}{\lambda}U$, where $\frac{\varepsilon}{\lambda}U$ is a $\mathcal{T}$-neighborhood of zero, so V would also be a $\mathcal{T}$-neighborhood of zero, which would imply that $t \leq \mathcal{T}$, which is a contradiction.

It follows that the OTVG $(E, C, \mathcal{T})$ is not solid for any of possible cones, otherwise it would have a zero neighborhood basis formed by balanced subsets.

If $\mathcal{U}$ is a zero neighborhood basis of a topology $\mathcal{T}$ formed by symmetric subsets, then $S(\mathcal{U}) = \{S(U) : U \in \mathcal{U}\}$ (see Section 2) is a zero neighborhood basis of the topology $\mathcal{T}_S$ of a solid OTVG. Since the sets A and $S(A)$ are incomparable in general, it follows that the topologies $\mathcal{T}$ and $\mathcal{T}_S$ are incomparable. The following proposition is proved similarly as the one for OLCS in [30].

Proposition 5.3. *If $(E, C, \mathcal{T})$ is an OTVG, then the following statements are true:*

(i) $\mathcal{T}_F \leq \mathcal{T}_S \leq \mathcal{T}_D$;
(ii) *OTVG $(E, C, \mathcal{T}_S)$ is order-convex with the open decomposition property;*
(iii) $\mathcal{T}_S = (\mathcal{T}_D)_F = (\mathcal{T}_F)_D$.

Corollary 5.4. (i) *If an OTVG $(E, C, \mathcal{T}_S)$ is order-convex with the open decomposition property then it is solid.*
(ii) *A discrete OTVG is solid with respect to some cone if and only if it is order-convex, that is if and only if the cone is antisymmetric.*
(iii) *An indiscrete OTVG is solid with respect to some cone C if and only if it has the open decomposition property, that is if and only if $E = C - C$.*
(iv) *If f is a positive continuous linear map from OTVG $(E, C, \mathcal{T})$ to OTVG $(F, K, \mathcal{P})$, then f is continuous from OTVG $(E, C, \mathcal{T}_S)$ to OTVG $(F, K, \mathcal{P}_S)$.*
(v) *An OTVG $(E, C, \mathcal{T})$ is solid if and only if the OTVG $(E, -C, \mathcal{T})$ is solid.*

The following proposition shows that the set of solid strings (i.e., strings whose terms are solid sets) in an OTVG $(E, C, \mathcal{T})$ is nonempty.

Proposition 5.5. *Let $E = C - C$. If $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ is a $\mathcal{T}$-topological string in OTVS $(E, C, \mathcal{T})$, then $S(\mathcal{U}) = (S(U_n))_{n\in\mathbb{N}}$ is a $\mathcal{T}_S$-topological string in the OTVS $(E, C, \mathcal{T}_S)$.*

Proof. Since $S(U_n) = \bigcup\{[-x, x] : x \in U_n \cup C\}$, that is a solid subset of E, then it is clearly balanced. Since, for each $c \in C$, $S(\{c\}) = [-c, c]$, $S(U_n)$ is an absorbing subset in E, because $E = C - C$. If $x = a + b$, where $a, b \in S(U_{n+1})$, then there exist $u, v \in U_{n+1} \cap C$ such that $a \in [-u, u]$ and $b \in [-v, v]$, that is, $a + b \in [-u, u] + [-v, v] \subseteq [-(u + v), u + v]$. Since $u + v \in U_{n+1} + U_{n+1} \subseteq U_n$, it follows that $x = a + b \in S(U_n)$. Therefore $S(U_{n+1}) + S(U_{n+1}) \subseteq S(U_n)$. □

Corollary 5.6. *If $(E, C, \mathcal{T})$ is an OTVS then there exists a set of solid strings in E with the following properties:*

(i) *For any two strings $\mathcal{U}$ and $\mathcal{V}$ in this set, there exists a string $\mathcal{W}$ such that $\mathcal{W} \subseteq \mathcal{U} \cap \mathcal{V}$.*
(ii) *Elements of strings of this set form a zero neighborhood basis for the topology $\mathcal{T}_S$.*

Corollary 5.7. *If $\mathcal{F}$ is a set of strings in an OTVS $(E, C, \mathcal{T})$ that generates topology $\mathcal{T}$, then $S(\mathcal{F})$ is a set of strings in E, that generates topology $\mathcal{T}_S$.*

According to Proposition 5.3(iii), the topology $\mathcal{T}_S$ can be expressed by the topologies $\mathcal{T}_D$ and $\mathcal{T}_F$; hence, the following proposition is a direct consequence of Propositions 3.16 and 4.21.

Proposition 5.8. *If $\{(E_\alpha, C_\alpha, \mathcal{T}_\alpha) : \alpha \in I\}$ is a family of OTVSs, $(E, C_1, \prod_\alpha \mathcal{T}_\alpha)$ is their product, and $(E, C_2, \bigoplus_\alpha \mathcal{T}_\alpha)$ their direct sum, then $\prod_\alpha \mathcal{T}_{\alpha S} = (\prod_\alpha \mathcal{T}_\alpha)_S$ and $\bigoplus_\alpha \mathcal{T}_{\alpha S} = (\bigoplus_\alpha \mathcal{T}_\alpha)_S$.*

In [30], [26] and [18] the authors studied the so-called *order-bounded* locally convex topology on an arbitrary vector space (E, C), denoted as $\mathcal{T}_b$. It has for zero neighborhood basis the set of all absolutely convex subsets of E that absorb order-bounded subsets (such subsets are called *order-bornivorous*).

Topology $\mathcal{T}_b$ is not necessarily Hausdorff for an arbitrary cone C. In particular if we take $C = E$, then $\mathcal{T}_b$ is an indiscrete locally convex topology, so it is not Hausdorff. Then order-bounded subsets are in fact all subsets of E, therefore also the space E itself. It is obvious, because for any cone $C \subseteq E$: $[x, y] = (x + C) \cap (y - C)$.

In [30, p. 75], it was shown that locally convex order-bounded topology $\mathcal{T}_b$ is the finest locally solid topology on (E, C), if $E = C - C$ and if for any two elements $u, v \in C$, $[\theta, u] + [\theta, v] = [\theta, u + v]$ holds (note that the last condition is satisfied, e.g., in Riesz spaces – see the next section).

For linear order-bounded topology (here denoted as $\mathcal{T}^b$) we will show the following proposition.

Proposition 5.9. *If (E, C) is an ordered vector space such that the cone C is generating and that for $u, v \in C$, $[\theta, u] + [\theta, v] = [\theta, u + v]$, then $\mathcal{T}^b$ is the finest linear solid topology on (E, C).*

Proof. First we will show that the topology $\mathcal{T}^b$ is solid, namely if V is a $\mathcal{T}^b$-neighborhood of zero, we will show that there exists a solid $\mathcal{T}^b$-neighborhood of zero U such that $V \supseteq U$. It is clear that the neighborhood V generates a $\mathcal{T}^b$-topological string $\mathcal{V} = (V_n)_{n\in\mathbb{N}}$, $V = V_1$ (not necessarily unique). We shall now define a sequence of subsets $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ in the following way:

$$U_n = \bigcup \{[-x, x] : x \in V_{n+1} \text{ and } [\theta, x] \subseteq V_{n+1}\}.$$

All subsets U_n are clearly solid, being unions of solid subsets. To prove that $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ is a solid string, one has to prove that U_n are absorbing subsets, and that $U_{n+1} + U_{n+1} \subseteq U_n$.

In order to prove that U_n are absorbing we will prove that $U_n \cap C = \{x \in C : [\theta, x] \subseteq V_{n+1}\}$. Indeed, if $a \in U_n \cap C$, then $a \in U_n$ and $a \in C$, that is there exists $x \in C$ such that $[\theta, x] \subseteq V_{n+1}$ and $a \in [-x, x]$. Since $a \in C$, then $a \in [-x, x] \cap C$, or $[\theta, a] \subseteq [\theta, x] \subseteq V_{n+1}$, that is, $a \in \{x \in C : [\theta, x] \subseteq V_{n+1}\}$. The converse statement is obvious because $[\theta, x] \subseteq [-x, x]$ for any $x \in C$.

Since U_n are solid subsets, they are absorbing if and only if they absorb positive elements. Therefore, if some $c \in C$ is not an element of the set kU_n for any natural number k, then $\frac{1}{k}c \notin \{x \in C : [\theta, x] \subset V_{n+1}\}$, that is, $[\theta, c] \nsubseteq kV_{n+1}$, and the latter is contrary to the hypothesis that $\mathcal{V} = (V_n)_{n\in\mathbb{N}}$ is a $\mathcal{T}^b$-topological string.

We will now show that $U_{n+1} + U_{n+1} \subseteq U_n$ for any $n \in \mathbb{N}$. If $a = u + v \in U_{n+1} + U_{n+1}$, then there exist $x, y \in V_{n+2}$, $[\theta, x] \subseteq V_{n+2}$, $[\theta, y] \subseteq V_{n+2}$ and $u \in [-x, x]$, $v \in [-y, y]$. Also, $a = u + v \in [-(x + y), x + y]$, $x + y \in V_{n+2} + V_{n+2} \subseteq V_{n+1}$ and $[\theta, x + y] \subseteq V_{n+2} + V_{n+2} \subseteq V_{n+1}$. This means that $a \in U_{n+1}$.

Now we will prove that $U_n \subseteq V_n$ for any $n \in \mathbb{N}$. Indeed, if $u \in U_n$, then there exists $x \in V_{n+1}$, $[\theta, x] \subseteq V_{n+1}$ and $u \in [-x, x]$, then it follows that $\frac{u+x}{2} \in [\theta, x]$ and $\frac{x-u}{2} \in [\theta, x]$, that is $\frac{u+x}{2} \in V_{n+1}$ and $\frac{x-u}{2} \in V_{n+1}$. Now, $u = \frac{x+u}{2} - \frac{x-u}{2} \in V_{n+1} - V_{n+1} = V_{n+1} + V_{n+1} \subseteq V_n$. Therefore, the sequence $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ is a solid string, which means that it is also $\mathcal{T}^b$-topological and $V = V_1 \supseteq U_1 = U$.

We conclude that $\mathcal{T}^b$ is a linear topology with a zero neighborhood basis formed by solid subsets. Since order-bounded subsets are bounded for any linear solid topology, and that $\mathcal{T}^b$ is the finest linear topology for which order-bounded subsets are topologically bounded, then $\mathcal{T}^b$ is the finest linear solid topology. □

Corollary 5.10. *If (E, C) is an ordered vector space as in the previous proposition, then $t^f_F = \mathcal{T}^b$ (t^f is the finest topology on E).*

In the following propositions we will state some properties of linear order-bounded topology $\mathcal{T}^b$ without particular assumptions for the cone C.

Proposition 5.11. *If $\mathcal{T}^b$ is a linear order-bounded topology on ordered vector space (E, C), then $(\mathcal{T}^b)^\circ = \mathcal{T}_b$ ($(\mathcal{T}^b)^\circ$ is a locally convex topology on (E, C), which has a zero neighborhood basis formed by absolutely convex $\mathcal{T}^b$-neighborhoods of zero).*

Proof. If U is a $(\mathcal{T}^b)^\circ$-neighborhood of zero, then there exists an absolutely convex $\mathcal{T}^b$-neighborhood of zero V, such that $U \supseteq V$ and V absorbs order-bounded subsets. From the definition of topology $\mathcal{T}_b$, it follows that V is a $\mathcal{T}_b$-neighborhood of zero, which means that U is also a $\mathcal{T}_b$-neighborhood of zero, i.e., $(\mathcal{T}_b)^\circ \leq \mathcal{T}_b$.

Conversely if W is an absolutely convex $\mathcal{T}_b$-neighborhood of zero, then the natural string $\left(\frac{1}{2^{n-1}}W\right)_{n\in\mathbb{N}}$ is order-bornivorous, that is $\mathcal{T}^b$-topological. Therefore, W is a $\mathcal{T}_b$-neighborhood of zero, and since it is absolutely convex, it is also a $(\mathcal{T}^b)^\circ$-neighborhood of zero. We conclude that $\mathcal{T}_b \leq (\mathcal{T}^b)^0$. □

In [30, p. 68], a necessary and sufficient condition is given for an absolutely convex subset V to be order-bornivorous in the ordered vector space (E, C). The sequence $\{x_n\} \subseteq E$ is an *ordered Mackey null sequence* if there exist a real decreasing sequence $\{\varepsilon_n\}$ tending to 0 and an order-bounded subset $A \subseteq E$ such that $x_n \in \varepsilon_n A$, for any $n \in \mathbb{N}$. The following proposition gives a characterization of order-bornivorous strings in an ordered vector space (E, C).

Proposition 5.12. *A string $\mathcal{V} = (V_n)_{n\in\mathbb{N}}$ in the ordered vector space (E, C) is order-bornivorous if, and only if, every V_n absorbs order-Mackey null sequences.*

Proof. The necessity of condition is evident, because for any ordered Mackey null sequence $\{x_k\}$ there exists a real sequence $\{\varepsilon_k\}$ and an order-bounded subset A, such that $x_k \in \varepsilon_k A$ for any $k \in \mathbb{N}$. Since any V_n absorbs A, there exists $\lambda > 0$ such that $x_k \in \varepsilon_k \lambda V_n \subseteq \lambda' V_n$, where $\lambda\varepsilon_k \leq \lambda'$. On the other hand, if there exists an order-bounded subset A and a member V_{n_0} of the string $\mathcal{V}$ such that $A \nsubseteq k^2 V_{n_0}$, then there exists a sequence $x_k \in A$ such that the ordered Mackey null sequence $\left\{\frac{1}{k}x_k\right\}$ is not absorbed by V_{n_0}, what contradicts the hypothesis on the string $\mathcal{V} = (V_n)_{n\in\mathbb{N}}$. □

A linear map f from an ordered vector space (E, C) to a TVS $(F, \mathcal{P})$ is *order-bounded*, if it maps order-bounded subsets of the space (E, C) into $\mathcal{P}$-bounded subsets of TVS $(F, \mathcal{P})$, [30, p. 69]. The following proposition and its corollary characterize linear order-bounded topology $\mathcal{T}^b$ on the ordered vector space (E, C).

Proposition 5.13. *Let $(E, C, \mathcal{T})$ be an OTVS and $\mathcal{T}^b$ a linear order-bounded topology on (E, C). Then the following statements are equivalent:*

(a) $\mathcal{T} \geq \mathcal{T}^b$;
(b) *Any string $\mathcal{V} = (V_n)_{n\in\mathbb{N}}$ whose members absorb ordered Mackey null sequences is $\mathcal{T}$-topological;*
(c) *Any order-bounded linear map from $(E, C, \mathcal{T})$ to an arbitrary TVS $(F, \mathcal{P})$ is continuous.*

Proof. (a) implies (b) because if $\mathcal{V} = (V_n)_{n\in\mathbb{N}}$ is a string with above mentioned property, then according to Proposition 5.12 it is order-bornivorous, therefore a $\mathcal{T}^b$-topological string, hence, $\mathcal{T}$-topological, because $\mathcal{T} \geq \mathcal{T}^b$.

(b) implies (c): if U is a balanced $\mathcal{P}$-neighborhood of zero, then it generates a $\mathcal{P}$-topological string $(U_n)_{n\in\mathbb{N}}$, such that $f^{-1}((U_n)_{n\in\mathbb{N}}) = (f^{-1}(U_n))_{n\in\mathbb{N}}$ is an order-bornivorous string in (E, C). According to Proposition 5.12, the members $f^{-1}(U_n)$ absorb ordered Mackey null sequences, so because of (b) the string $(f^{-1}(U_n))_{n\in\mathbb{N}}$ is $\mathcal{T}$-topological. It means that $f^{-1}(U) = f^{-1}(U_1)$ is a $\mathcal{T}$-neighborhood of zero, that is f is a $\mathcal{T}$-$\mathcal{P}$ continuous map.

(c) implies (a): The identity map from $(E, \mathcal{T})$ to $(E, \mathcal{T}^b)$ is obviously order-bounded and because of (c) it is continuous, therefore $\mathcal{T} \geq \mathcal{T}^b$. □

Corollary 5.14. *If $(E, C, \mathcal{T})$ is an OTVS, then the following statements are equivalent:*

(a) *$\mathcal{T} = \mathcal{T}^b$;*
(b) *Any string $\mathcal{V} = (V_n)_{n\in\mathbb{N}}$ whose members absorb ordered Mackey null sequences in (E, C) is $\mathcal{T}$-topological and any order-bounded subset is $\mathcal{T}$-bounded;*
(c) *Any order-bounded linear map from $(E, C, \mathcal{T})$ to an arbitrary TVS $(F, \mathcal{P})$ is continuous and any order-bounded subset of (E, C) is $\mathcal{T}$-bounded.*

Corollary 5.15. *Linear order-bounded topology $\mathcal{T}^b$ is the finest linear topology on (E, C) for which ordered Mackey null sequences are topological null sequences.*

In any OTVS $(E, C, \mathcal{T})$ the following two propositions hold true.

Proposition 5.16. *For any linear topology $\mathcal{T}$ on an ordered vector space (E, C), $\mathcal{T}_F \leq \mathcal{T}^b$ holds.*

Proof. First we shall prove that $\mathcal{T}^b$ is the finest linear topology on (E, C) for which order-bounded subsets are topologically bounded, that is, $\mathcal{T}^b$-bounded. If $\mathcal{T}'$ is such a topology and V one of its neighborhoods of zero, then the generated string $\mathcal{V} = (V_n)_{n\in\mathbb{N}}$ is obviously order-bornivorous, and this means that $\mathcal{V}$ is a $\mathcal{T}^b$-topological string. It follows that $V = V_1$ is a $\mathcal{T}^b$-neighborhood of zero and $\mathcal{T}' \leq \mathcal{T}^b$. According to Lemma 4.4, order-bounded subsets are $\mathcal{T}_F$-bounded so that $\mathcal{T}_F \leq \mathcal{T}^b$. □

Proposition 5.17. *If f is a positive linear mapping from an ordered vector space (E, C) into another one (F, K), then f is continuous from $(E, C, \mathcal{T}^b)$ to $(F, K, \mathcal{P}^b)$.*

Proposition 5.18. *For any ordered vector space (E, C) the following conditions are equivalent:*
(a) *The cone C is generating;*
(b) *The space $(E, C, \mathcal{T}^b)$ has the open decomposition property.*

Proof. (b) implies (a), because, for each $\mathcal{T}^b$-neighborhood of zero U, $U \cap C - U \cap C$ is also a $\mathcal{T}^b$-neighborhood of zero, i.e., for each $x \in E$, there exists $\lambda > 0$ such that $x \in \lambda(U \cap C - U \cap C) = \lambda U \cap C - \lambda U \cap C$, i.e., $x \in C - C$.

Let us prove that (a) implies (b). Suppose that for each $\mathcal{T}^b$-topological string $\mathcal{V} = (V_n)_{n\in\mathbb{N}}$, the string $\mathcal{V} \cap C - \mathcal{V} \cap C$ is $\mathcal{T}^b$-topological. It is enough to prove that, for each $n \in \mathbb{N}$, $V_n \cap C - V_n \cap C$ absorbs order-bounded subsets. If this is not the case, then there exist $x \in C$ and $n_0 \in \mathbb{N}$ such that $[-x, x] \nsubseteq k^2(V_{n_0} \cap C - V_{n_0} \cap C)$ for any $k \in \mathbb{N}$. Since $E = C - C$, it follows that $x_k = a_k - b_k$, where $a_k, b_k \in C$ and $\frac{1}{k}a_k - \frac{1}{k}b_k \notin k(V_{n_0} \cap C) - k(V_{n_0} \cap C)$, which is impossible since $\{\frac{1}{k}a_k\}$ and $\{\frac{1}{k}b_k\}$ are positive order-Mackey zero-sequences which are, by Proposition 5.12 absorbed by each $V_n \cap C$. □

We note that the corresponding result in locally convex case was proved in [30], but using duality theory.

Corollary 5.19. *If (E, C) is an ordered vector space such that $E = C - C$, then the linear order-bounded topology $\mathcal{T}^b$ is solid if and only if it is order-convex.*

6. Some results on Riesz TVGs and Riesz TVSs

Among ordered vector spaces, Riesz spaces (see the definition in Section 1) have much richer structure. Here we introduce and study some properties of Riesz TVGs.

Definition 6.1. A Riesz space is called a Riesz TVG (RTVG, for short) if it is endowed with a topology of vector group having a zero neighborhood basis formed by solid subsets. Similarly, a Riesz locally convex group (RLCG) is defined.

If (E, C) is a Riesz space and $(E, \mathcal{T})$ is a TVG, then $(E, C, \mathcal{T})$ is an RTVG if and only if any of the following conditions is satisfied:

(1) The mapping $(x, y) \mapsto \sup\{x, y\}$ is uniformly continuous on $E \times E$.
(2) The mapping $x \mapsto \sup\{x, \theta\}$ is uniformly continuous on E.
(3) $(E, C, \mathcal{T})$ is an order-convex OTVG with the open decomposition property.
(4) $(E, C, \mathcal{T})$ is an order-convex OTVG and the mapping $x \mapsto \sup\{x, \theta\}$ is continuous at θ.
(5) If $\{x_\alpha : \alpha \in D\}$ and $\{y_\alpha : \alpha \in D\}$ are nets in E, $|x_\alpha| \preceq |y_\alpha|$ for each $\alpha \in D$ and $\{y_\alpha\}$ $\mathcal{T}$-converges to zero, than $\{x_\alpha\}$ $\mathcal{T}$-converges to zero.

It was proved in [30, p. 137] that if $(E, C, \mathcal{T})$ is an RTVS then (E, C) is an Archimedean ordered vector space (i.e., $nx \preceq y$ for some $y \in E$ and all $n \in \mathbb{N}$ implies that $x \preceq \theta$). The following example shows that this may not be true for RTVGs.

Example 6.2. Let (E, C) be a Riesz space which is not Archimedean (see [30, p. 120]), and let (E, C, d) be an RTVG (d – the discrete topology). By Corollary 3.11 and Proposition 4.11, (E, C, d) is an order-convex UTVG with the open decomposition property, hence, by Proposition 5.3 it is solid, i.e., (E, C, d) is an RTVG since (E, C) is a Riesz space. According to [30, Proposition 11.2], on the given Riesz space (E, C) there is no vector topology $\mathcal{T}$ such that $(E, C, \mathcal{T})$ is an RTVS.

It follows also that if $(E, C, \mathcal{T})$ is an RLCG, then the associated OLCS $(E, C, \operatorname{loc}\mathcal{T})$ need not be an RLCS (in other words $(\operatorname{loc}\mathcal{T})_S \neq \operatorname{loc}\mathcal{T}$). Also, linear order-bounded topology $\mathcal{T}^b$ need not be order-convex.

Lemma 6.3. *A subset U of E is a $\mathcal{T}$-neighborhood of zero in RTVS $(E, C, \mathcal{T})$ if and only if $\operatorname{sk} U$ is a $\mathcal{T}$-neighborhood of zero.*

Proof. The condition is trivially sufficient since $\operatorname{sk} U \subseteq U$. Conversely, if U is a $\mathcal{T}$-neighborhood of zero, then there exists a solid neighborhood of zero V, such that $U \supseteq V$. By the definition of the solid kernel, it follows that $U \supseteq \operatorname{sk} U \supseteq V$, i.e., $\operatorname{sk} U$ is a $\mathcal{T}$-neighborhood of zero. □

Corollary 6.4. *In an RTVS $(E, C, \mathcal{T})$, for each $\mathcal{T}$-topological string $\mathcal{U}$ there exists a solid $\mathcal{T}$-topological string $\mathcal{V}$ such that $\mathcal{U} \supseteq \mathcal{V}$.*

The next proposition is concerned with the extension of a topological string from an l-ideal to a topological string in the whole space.

Proposition 6.5. *Let $(E, C, \mathcal{T})$ be an RTVS and let F be its l-ideal (with the induced topology). If $\mathcal{V} = (V_n)_{n \in \mathbb{N}}$ is a solid topological string in $(F, C \cap F, \mathcal{T}|F)$, then there exists a solid $\mathcal{T}$-topological string $\mathcal{U} = (U_n)_{n \in \mathbb{N}}$ in E such that $\mathcal{U} \cap F = \mathcal{V}$.*

Proof. Define, for each $n \in \mathbb{N}$, subsets U_n of E by

$$U_n = \{x \in E : y \in V_n \text{ whenever } y \in F \text{ and } \theta \preceq y \preceq |x|\}.$$

In order to prove that the sequence $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ is a $\mathcal{T}$-topological string satisfying $\mathcal{U} \cap F = \mathcal{V}$ we have to show that, for each $n \in \mathbb{N}$: 1° U_n is a solid subset of E; 2° $U_{n+1} + U_{n+1} \subseteq U_n$; 3° $U_n \cap F = V_n$; 4° U_n is a $\mathcal{T}$-neighborhood of zero.

1° Let $b \in U_n$ and $|a| \preceq |b|$. By the definition of the set U_n, if $y \in F$ and $\theta \preceq y \preceq |a|$, this means that $\theta \preceq y \preceq |a| \preceq |b|$, i.e., $y \in V_n$ because $b \in U_n$.

2° Let $x = a+b$, where $a, b \in U_{n+1}$. If $y \in F$ and $\theta \preceq y \preceq |x|$ then $\theta \preceq y \preceq |a+b| \preceq |a| + |b|$ and, since $[\theta, |a| + |b|] = [\theta, |a|] + [\theta, |b|]$ holds, it follows that $y = y_1 + y_2$ where $y_1 \in [\theta, |a|]$ and $y_2 \in [\theta, |b|]$. Hence, $y = y_1 + y_2 \in V_{n+1} + V_{n+1} \subseteq V_n$, i.e., $x = a + b \in U_n$.

3° Let $a \in V_n$ and let $y \in F$ such that $\theta \preceq y \preceq |a|$. Then $|y| \preceq |a|$, hence, $y \in V_n$ since V_n is a solid subset of F. Conversely, if $a \in U_n \cap F$, then clearly $a \in V_n$ by the definition of the set U_n and since U_n and V_n are solid sets.

4° Suppose, to the contrary, that U_n is not a $\mathcal{T}$-neighborhood of zero. Then for each solid $\mathcal{T}$-neighborhood of zero U there exists $x_U \in F$ with $x_U \notin U_n$. This means that for each U there exists $y_U \in F$ with $\theta \preceq y_U \preceq |x_U|$ and $y_U \notin V_n$. Since the net $\{x_U, \mathcal{U}, \supseteq\}$ tends to zero in the space $(E, C, \mathcal{T})$, by the property (5) given after Definition 6.1, the net $\{y_U, \mathcal{U}, \supseteq\}$ tends to zero in $(E, C, \mathcal{T})$, hence also in $(F, C \cap F, \mathcal{T}|F)$. But this is impossible since $y_U \notin V_n$. Thus, $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ is a $\mathcal{T}$-topological string. □

Following [2], a string in an OTVG $(E, C, \mathcal{T})$ will be called:

- *bornivorous* if all of its terms are bornivorous (i.e., absorb $\mathcal{T}$-bounded sets);
- *order-bornivorous* if all of its terms are order-bornivorous (i.e., absorb order-bounded sets);
- *$\mathcal{T}$-closed* if all of its terms are $\mathcal{T}$-closed.
- *locally topological* if each of its terms intersect every $\mathcal{T}$-bounded subset of E by a neighborhood of zero in the induced topology.

The following proposition produce similar information as in Proposition 6.5, but for different kinds of strings.

Proposition 6.6. *(i) Let $(E, C, \mathcal{T})$ be an RTVS and F its l-ideal endowed by the induced topology. If $\mathcal{V} = (V_n)_{n\in\mathbb{N}}$ is a bornivorous solid string in $(F, C \cap F, \mathcal{T}|F)$, then there exists a bornivorous solid string $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ in E such that $\mathcal{U} \cap F = \mathcal{V}$.*

(ii) If the string $\mathcal{V}$ is $\mathcal{T}|F$-closed, then $\mathcal{U}$ is $\mathcal{T}$-closed.

(iii) If the string $\mathcal{V}$ is locally topological, then $\mathcal{U}$ is locally topological

Proof. (i) Define the sets $U_n \subseteq E$ in the same way as in the proof of Proposition 6.5. It has to be proved that each U_n is a bornivorous subset in the space $(E, C, \mathcal{T})$. Suppose the contrary – then there exists a solid bounded set $A \subseteq E$ such that $A \nsubseteq kU_{n_0}$ for some n_0 and each $k \in \mathbb{N}$. Hence, there is a sequence $(x_k)_{k\in\mathbb{N}}$ in A such that $\frac{1}{k}a_k \notin U_{n_0}$ for each k. By the definition of subset U_n, there exists a sequence (b_k) in F such that $\theta \preceq b_k \preceq |\frac{1}{k}a_k|$ and $b_k \notin V_{n_0}$. Hence, $\theta \preceq kb_k \preceq |a_k|$, implying that the sequence (kb_k) is contained in $A \cap F$, because A is a solid set and F is a subspace. Since $A \cap F$ is a bounded subset of F, there exists $\lambda > 0$ such that (kb_k) is contained in λV_{n_0}, hence in kV_{n_0} for $k \in \mathbb{N}$ large enough. But this is impossible because $b_k \notin V_{n_0}$ for each $k \in \mathbb{N}$. This contradiction finishes the first part of proof.

(ii) Let us prove now that $\overline{U}_n = U_n$ for each $n \in \mathbb{N}$. If $x \in \overline{U}_n$ then there exists a net $\{x_\tau : \tau \in D\}$ in U_n $\mathcal{T}$-converging to x. Let $y \in F$ and $\theta \preceq y \preceq |x|$. Then $\theta \preceq \inf\{y, |x_\tau|\} \preceq |x_\tau| \in U_n$ and $\inf\{y, |x_\tau|\} \in F$. By the definition of U_n, it follows that $\inf\{y, |x_\tau|\} \in V_n$. Using condition (2) formulated after Definition 6.1, we conclude that $\inf\{y, |x_\tau|\}$ $\mathcal{T}$-converges to $\inf\{y, |x|\} = y \in \overline{V_n} = V_n$. This means that $x \in U_n$ and the proof is over.

(iii) Suppose, to the contrary, that for each solid $\mathcal{T}$-neighborhood of zero U of the space $(E, C, \mathcal{T})$ there exists a solid $\mathcal{T}$-bounded subset A such that $A \cap U \nsubseteq A \cap U_{n_0}$ for some $n_0 \in \mathbb{N}$. This means that there exists a net $\{x_U : U \in \mathcal{U}\}$ in $A \cap U$ with $x_U \notin A \cap U_{n_0}$, hence, for each U, $x_U \notin U_{n_0}$. Thus, by the definition of subset U_{n_0}, there exists a net $\{y_U : U \in \mathcal{U}\}$ such that $y_U \in F$ with $\theta \preceq y_U \preceq |x_U|$ and $y_U \notin V_{n_0}$. Since U is a solid subset, then $y_U \in A \cap U \cap F$, and the sequence $\{y_U : U \in \mathcal{U}\}$ is a bounded subset in the induced topology of F. On the other hand, the string $\mathcal{V}$ is bornivorous, hence, there exists $\lambda > 1$ such that $\frac{1}{\lambda} y_U \in V_{n_0}$, for each U. Since $\frac{1}{\lambda} y_U \in F$, $\theta \preceq \frac{1}{\lambda} y_U \preceq y_U \preceq |x_U|$ and $\frac{1}{\lambda} y_U \in V_{n_0}$ and so $x_U \in U_{n_0}$, contrary to the supposition. □

The previous results will be used to obtain hereditary properties of some special classes of RTVSs. First, we give characterization of these classes, not given in [16,17] (which are, to the best of our knowledge, the only papers in which such classes of RTVSs were investigated without using convexity conditions).

Definition 6.7. An RTVS $(E, C, \mathcal{T})$ is bornological (barreled, quasi-barreled, locally topological) if $(E, \mathcal{T})$ is bornological (barreled, quasi-barreled, locally topological) in the category of TVSs, i.e. (see [2]) if any bornivorous ($\mathcal{T}$-closed, bornivorous $\mathcal{T}$-closed, locally topological) string in it is $\mathcal{T}$-topological.

Proposition 6.8. *For an RTVS $(E, C, \mathcal{T})$ the following conditions are equivalent:*

(a) *$(E, C, \mathcal{T})$ is a bornological RTVS.*
(b) *Each positive and bounded linear mapping from E into an arbitrary RTVS $(F, K, \mathcal{P})$ is continuous;*
(c) *Each positive and bounded linear mapping from E into an arbitrary Fréchet RTVS $(F, K, \mathcal{P})$ is continuous;*
(d) *Each solid bornivorous string in $(E, C, \mathcal{T})$ is $\mathcal{T}$-topological.*

Proof. The only nontrivial part is that (d) implies (a). Let $\mathcal{V} = (V_n)_{\mathbb{N}}$ be a bornivorous string in $(E, C, \mathcal{T})$. Since each V_n absorbs solid bounded subsets of E, then $\operatorname{sk} V_n \neq \emptyset$ and $\operatorname{sk} V_n$ is a bornivorous subset of E. It is easy to show that $(\operatorname{sk} V_n)_{n\in\mathbb{N}}$ is a string in $(E, C, \mathcal{T})$. Now (d) implies that $(\operatorname{sk} V_n)_{n\in\mathbb{N}}$ is $\mathcal{T}$-topological, and so is $\mathcal{V}$, which completes the proof. □

Similarly, the following propositions can be proved.

Proposition 6.9. *For an RTVS $(E, C, \mathcal{T})$ the following conditions are equivalent:*

(a) *$(E, C, \mathcal{T})$ is a quasi-barelled RTVS.*
(b) *Each solid and closed bornivorous string in $(E, C, \mathcal{T})$ is $\mathcal{T}$-topological.*

Proposition 6.10. *For an RTVS $(E, C, \mathcal{T})$ the following conditions are equivalent:*

(a) *$(E, C, \mathcal{T})$ is a locally topological RTVS.*

(b) *Each positive and locally continuous linear mapping from E into an arbitrary RTVS $(F, K, \mathcal{P})$ is continuous;*
(c) *Each positive and locally continuous linear mapping from E into an arbitrary Fréchet RTVS $(F, K, \mathcal{P})$ is continuous;*
(d) *Each solid locally topological string in $(E, C, \mathcal{T})$ is $\mathcal{T}$-topological.*

Here, a mapping is called locally continuous if its restriction to each bounded subset is continuous.

Combining Propositions 6.6–6.10 we obtain the following result.

Proposition 6.11. *Every l-ideal of a bornological (resp. quasi-barreled, locally topological) RTVS is bornological (resp. quasi-barreled, locally topological) in the induced topology.*

Remark 6.12. The previous result was, in another way, in the cases of bornological and quasi-barreled RTVSs, obtained by Keim [17]. The respective result in the category of RLCSs was proved earlier by Kawai in [13].

These results differ from the known situation in the category of topological vector spaces (or locally convex spaces) where, in order that a subspace of a bornological (resp. quasi-barreled) TVS (or LCS) be bornological (resp. quasi-barreled) one needs additional condition, e.g., that the subspace is of finite codimension (see [1] for the TVS case and [27,28] for the LCS case). The respective result for locally topological TVSs is unknown.

The conclusion similar to the given in Proposition 6.11 for barreled RTVSs does not hold. Namely, if $\mathcal{V} = (V_n)_{n\in\mathbb{N}}$ is a $\mathcal{T}$-closed string in an OTVS $(E, C, \mathcal{T})$, then the sequence $(\operatorname{sk} V_n)_{n\in\mathbb{N}}$ need not be a string. Indeed, if this were the case, then the finest linear topology t^f on the ordered vector space (E, C) from Example 6.2 would be solid, which is not the case.

In the category of OTVSs, in a way more natural than the bornological and quasi-barreled spaces are the classes introduced in the following way.

Definition 6.13. An RTVS $(E, C, \mathcal{T})$ is said to be order-bornological (resp. order-quasi-barreled) if each order-bornivorous string (resp. order-bornivorous and $\mathcal{T}$-closed string) in it is $\mathcal{T}$-topological. Here, a string is called order-bornivorous if all of it terms absorb all order-bounded subsets of E.

A characterization of order-quasi-barreled RTVSs is given in the following proposition.

Proposition 6.14. *An RTVS $(E, C, \mathcal{T})$ is order-quasi-barreled if and only if each solid, $\mathcal{T}$-closed and order-bornivorous string in it is $\mathcal{T}$-topological.*

Proof. The condition is obviously necessary. In order to prove sufficiency, let $\mathcal{V} = (V_n)_{n\in\mathbb{N}}$ be a $\mathcal{T}$-closed and order-bornivorous string. Since $(E, C, \mathcal{T})$ is an RTVS, it follows that the set $\operatorname{sk} V_n$ is also order-bornivorous for each $n \in \mathbb{N}$, i.e., $(\operatorname{sk} V_n)_{n\in\mathbb{N}}$ is a solid, $\mathcal{T}$-closed and order-bornivorous string, hence it is $\mathcal{T}$-topological. By Lemma 6.3, the same is true for $\mathcal{V}$. Thus, $(E, C, \mathcal{T})$ is order-quasi-barreled. □

Both mentioned classes of RTVSs are stable with respect to quotients by arbitrary l-ideals, but not with respect to passing to l-ideals. In order to obtain a positive result in this direction, we introduce the following notions.

An RTVS $(E, C, \mathcal{T})$ is said to be *σ-order-complete* if each majorated increasing sequence in it has a supremum. Its l-ideal F is said to be *σ-normal* if the supremum of each sequence in F with the mentioned property belongs to F.

Proposition 6.15. *If $(F, C \cap F, \mathcal{T}|F)$ is a σ-normal ideal in a σ-order-complete RTVS $(E, C, \mathcal{T})$ which is order-bornological (resp. order-quasi-barreled) then $(F, C \cap F, \mathcal{T}|F)$ is order-bornological (resp. order-quasi-barreled) itself.*

Proof. We present the proof in the order-quasi-barreled case; the proof in order-bornological case is similar.

Having the previous proposition in mind, it suffices to prove that each solid, $\mathcal{T}|F$-closed and order-bornivorous string $\mathcal{V} = (V_n)_{n\in\mathbb{N}}$ in F can be extended to a string of the same kind in the given space $(E, C, \mathcal{T})$. Let $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ be the string constructed as in Proposition 6.5. Taking into account Proposition 6.6, it is enough to prove that each U_n is an order-bornivorous subset of E, moreover, it is enough to prove that it abosrbs all positive elements in E. Suppose that this is not the case; then there exists n_0 such that for some $c \in C$ and each $k \in \mathbb{N}$, $\frac{1}{k}c \notin U_{n_0}$ holds. By the definition of the set U_n, for each positive integer k there exists $y_k \in F$ such that $\theta \preceq y_k \preceq \frac{1}{k}c$ and $y_k \notin V_{n_0}$. Since the space $(E, C, \mathcal{T})$ is σ-ordered-complete, then the set $\{ky_k : k \in \mathbb{N}\}$ has the supremum $y \in E$ which actually belongs to F since it is a σ-normal l-ideal. This means that $\{ky_k : k \in \mathbb{N}\} \subseteq [0, y] \subseteq F$, but it is not absorbed by V_{n_0}, which is not possible. □

7. Conclusion

The theory of topological vector, in particular locally convex, spaces has become a part of general language of functional analysis and its applications. Theory of ordered topological vector spaces, as a part of it, is very important, as it can be applied in various analytical problems where order between elements of a space plays significant role. In these structures, linear operations (addition and multiplication by scalars) are supposed to be continuous (the latter as a function in two variables).

It was rather long time ago when an idea appeared to consider the structure where multiplication by scalars is treated as continuous mapping just as a function of one variable ($x \mapsto \lambda x$), for each fixed scalar λ, which is equivalent with considering the scalar field taken with discrete topology. Although some interesting results were obtained, there was no further detailed investigation of such structures, called topological vector groups.

We have tried in this article to fill the mentioned gap, by deriving several properties of ordered topological vector groups, in particular Riesz topological vector groups as one of the most interesting examples. Several results have been obtained, some of them similar to the ones in ordered topological vector spaces, but some that are significantly different. Various examples illustrate these differences.

Obviously, these investigations are just a beginning and there are several other parts of the theory of (ordered) topological vector spaces for which the respective results for (ordered) topological vector groups need to be investigated and obtained in, possibly, modified versions. Some of them are the following:

- Define countably barreled, countably quasi-barreled and (DF) RTVSs (analogously to respective classes of TVSs from [2]) and investigate their properties.
- Investigate which part of duality theory of (ordered) LCSs can be transfered to (ordered) LCGs.
- Explore the structure obtained when bornological vector spaces of [6] are endowed with partial order.
- Try to apply Burkholder's theorem on the convergence of martingales within solid locally convex groups (see [5]).
- Consider the details of relationship between the design concepts and topology within OTVG (see [8]).

Some other results were obtained, e.g., in the papers [11, 12, 19–22, 25].

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