The Shuffled Conic Power Flow Equations: An Improved Angle-Inclusive Conic Model

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***Abstract*—This paper presents the Shuffled Conic Power Flow (PF) equations which enable a novel voltage angle substitution within a second-order cone (SOC) PF model. Computational experiments compare the performance of the novel voltage angle substitution against a conventional approximation used to model voltage angles within a state-of-the-art SOC PF model. Solving the PF problem for four radial distribution networks and the optimal power flow (OPF) problem for four demand scenarios in one of these networks, we show the following. First, voltage angles were improved in around 95% of cases when solving the PF problem and around 96% of cases when solving the OPF problem. Second, there is a particular improvement in voltage angles when voltage magnitudes divert from 1 p.u., and when there is demand growth. The results are also compared to benchmark software, Matpower.**

***Index Terms*—Optimal Power flow (OPF), Power flow (PF), Second-order cone programming (SOCP), Voltage angles.**

I. Introduction

The optimal power flow (OPF) problem plays a key role in the design and operation of electric power systems. However, as the power flow equations (a key constraint within the OPF problem) are nonconvex, a range of relaxations and approximations have been proposed [1]. Within these formulations, second-order cone (SOC) power flow (PF) models have been given an increasing attention due to their computational efficiency and accuracy. However, early studies [1]–[3] propose SOC PF models where voltage angles are not explicitly modelled. In particular, [2] proposed a model for radial networks which eliminates voltage angles (though they can be recovered, as explained in [4]). Studies [1], [3] present models for radial and mesh networks which also eliminate voltage angles, with [1] proposing angle recovery algorithms.

More recent SOC PF models have turned towards directly accounting for all power flow variables including voltage angles. This allows for a representation of the complete voltage drop phasor (e.g. [5], [6]), provides the ability to directly use the SOC PF model on mesh networks, without requiring recovery algorithms (e.g. [7]), and gives the opportunity to directly obtain voltage phase solutions (e.g. [8]). Voltage angles can also be important for active reconfiguration on real-world distribution networks [9], [10]. SOC PF models in [11], [12] directly account for voltage angles, where they are incorporated using a range of methods including linear approximations and McCormick envelopes. In the SOC PF of [13], the load flow problem is solved using an iterative procedure. The SOC PF model of [8], replaces the nonconvex terms in the voltage angle equation by their convex hulls. Finally, the SOC PF models [5]–[7], [14]–[16] use a conventional approximation to incorporate voltage angles.

However, aforementioned works propose SOC PF models which either require algorithms to be solved, introduce additional constraints, or use a conventional approximation to incorporate voltage angles. In this study, we provide a novel substitution to account for voltage angles within an SOC PF model, and show that compared to a conventional approximation used within the SOC PF model of [15], voltage angles are improved in around 95% and 96% of cases when solving the PF and OPF problems, respectively. Furthermore, the voltage angle improvement is greatest in cases where voltage magnitudes divert from 1 p.u. and when active demand increases (which is important, since electricity demand is expected to almost double by 2050 [17]). Given the works above, the contributions of this paper are listed below.

1) This paper presents the Shuffled Conic Power Flow equations which enable a novel voltage angle substitution within an SOC PF model. The Shuffled Conic PF equations are a conic quadratic format of the "Shuffled PF equations", which we derive by *shuffling* the variables of the AC PF equations. Note that, even though named hereby, the Shuffled PF equations have been derived by various methods to form SOC PF models in [5]–[7], [14]–[16], which use a conventional approximation to model voltage angles (shown in Section II-A).

2) Numerical results show the accuracy of the proposed method, comparing the performance of the novel voltage angle substitution within the Shuffled Conic PF equations against a conventional approximation used to model voltage angles within the state-of-the-art SOC PF of [15]. This is shown solving the PF problem for four radial distribution networks and the OPF problem for four demand scenarios on one of these networks. Results are also compared to Matpower [18].

In this paper, Section II presents the methodology, Section III the numerical results, and Section IV the conclusions.

This letter was supported by EPSRC under Grant no. EP/T517914/1. For the purpose of open access, the author(s) has applied a Creative Commons Attribution (CC BY) license to any Accepted Manuscript version arising.

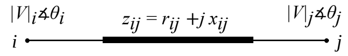

Fig. 1. Model used for the lines of the network [2].

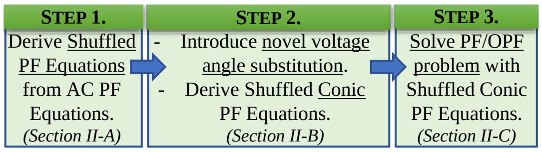

Fig. 2. Diagram of proposed methodology in Section II.

II. Methodology

Let $z_{ij} = r_{ij} + \jmath x_{ij}$ be the complex impedance of line $(i,j) \in E$, where E consists of lines (i,j) and (j,i) such that $i,j \in N$, and N represents the set of buses (Fig. 1). Then $g_{ij} - \jmath b_{ij} := 1/z_{ij}$ is the line admittance, where $g_{ij} := r_{ij}/(r_{ij}^2 + x_{ij}^2)$ and $b_{ij} := x_{ij}/(r_{ij}^2 + x_{ij}^2)$, $|V_i|$ and θ_i are the voltage magnitude and angle at bus i, respectively, and $\theta_{ij} = \theta_i - \theta_j$. The real and reactive power flows are:

$$P_{ij} = g_{ij}(|V_i|^2 - |V_i||V_j|\cos\theta_{ij}) + b_{ij}|V_i||V_j|\sin\theta_{ij} \quad \text{(1a)}$$

$$Q_{ij} = b_{ij}(|V_i|^2 - |V_i||V_j|\cos\theta_{ij}) - g_{ij}\,|V_i||V_j|\sin\theta_{ij} \quad \text{(1b)}$$

The remainder of this section presents the proposed approach, which is summarized in Fig. 2.

A. The Shuffled Power Flow Equations

This section presents the Shuffled PF equations which we derive by *shuffling* the variables in (1), and therefore are sufficient to represent the AC PF equations (1) (shown in Appendix A). The Shuffled PF Equations, still nonlinear, have also been derived by various methods such as using the voltage drop equation in [7], [14], by extending a branch flow model in [5], [6], [15], [16], while in [5], [6] it is explained that they are sufficient to represent the voltage drop phasor. The Shuffled PF equations are:

$$r_{ij}P_{ij} + x_{ij}Q_{ij} = \frac{|V_i|^2 - |V_j|^2}{2} + (r_{ij}^2 + x_{ij}^2)\frac{P_{ij}^2 + Q_{ij}^2}{2|V_i|^2} \quad \text{(2a)}$$

$$x_{ij}\,P_{ij} - r_{ij}\,Q_{ij} = |V_i|\,|V_j|\sin\theta_{ij} \quad \text{(2b)}$$

In the next section, we present a novel substitution while linearizing equations (2b), which does not use the conventional approximation $|V_i||V_j| \approx 1$ p.u. like the papers above.

B. The Shuffled Conic PF Equations

This section presents the Shuffled Conic PF equations which are a conic quadratic format of Equations (2), and enable a novel voltage angle substitution within an SOC PF model. To formulate them, we first linearize (2b) as follows.

Using Taylor series expansion we set $\sin\theta_{ij} = \theta_{ij}, \forall \theta_i \in \mathbb{R}$ (a widely used assumption). Therefore, (2b) becomes,

$$x_{ij}\,P_{ij} - r_{ij}\,Q_{ij} = |V_j|\,(|V_i|\,\theta_i) - |V_i|\,(|V_j|\,\theta_j) \quad \text{(2b}'\text{)}$$

We introduce a novel voltage angle substitution using a new variable, θ_i^V, which is defined as follows,

$$\theta_i^V = |V_i|\,\theta_i \quad (3)$$

Therefore, (2b′) becomes,

$$x_{ij}\,P_{ij} - r_{ij}\,Q_{ij} = |V_j|\theta_i^V - |V_i|\theta_j^V \quad (4)$$

At Equation (4) we set $|V_i| = |V_j| \approx 1 p.u.$, and (2b) becomes,

$$x_{ij}\,P_{ij} - r_{ij}\,Q_{ij} = \theta_i^V - \theta_j^V \quad (5)$$

We then convexify (2a) as in [1]-Part I, [3]. Therefore we set,

$$I_{ij}^{sq} = \frac{P_{ij}^2 + Q_{ij}^2}{|V_i^{\text{sq}}|}, \text{ where } |V_i^{\text{sq}}| = |V_i|^2 \quad (6)$$

and (2a) becomes,

$$r_{ij}\,P_{ij} + x_{ij}\,Q_{ij} = \frac{|V_i^{\text{sq}}| - |V_j^{\text{sq}}|}{2} + \frac{r_{ij}^2 + x_{ij}^2}{2}\,I_{ij}^{sq} \quad (7)$$

Relaxing equation (6) as in [3], the Shuffled Conic PF Equations are (5), (7), and $I_{ij}^{sq} \geq (P_{ij}^2 + Q_{ij}^2)/|V_i^{\text{sq}}|$.

C. The Power Flow and Optimal Power Flow Problems

We test the Shuffled Conic PF Equations on two problems:

1) The Power Flow problem: To solve the power flow problem, we formulate an optimization problem as in [2] in model (5), (7), (8)-(11) below. The relaxation gap of (9), which represents the total current squared, is minimized by (8). Constraints (10) form the power balance equations, where over-satisfaction of the real and reactive demand, Pd_i and Qd_i, respectively is set as in [3], in order for the equality to hold in (9) at optimal solutions (this is shown in the proof of Proposition 1 at Appendix B which is based on [1], [3]), and (11) set the voltage magnitudes and angles at the slack bus, whose values can be altered without loss of generality.

$$\min \textstyle\sum_{(i,j)\in E} I_{ij}^{sq} \quad (8)$$

subject to (5), (7), and

$$I_{ij}^{sq} \geq (P_{ij}^2 + Q_{ij}^2)/|V_i^{\text{sq}}|, \quad \forall (i,j) \in E \quad (9)$$

$$\textstyle\sum_{j:(i,j)\in E} P_{ij} \leq -Pd_i, \sum_{j:(i,j)\in E} Q_{ij} \leq -Qd_i, i = 2, \ldots N \quad (10)$$

$$|V_s^{\text{sq}}| = 1, \quad \theta_s^V = 0, \quad s: \text{slack bus} \quad (11)$$

Model (5), (7), (8)-(11) forms the Shuffled Conic PF model which is a second-order cone programming (SOCP) problem and can be solved by commercial solvers, since (9) can be written as the following second-order cone [3]:

$$I_{ij}^{sq} + |V_i^{\text{sq}}| \geq \left\| \begin{bmatrix} 2\,P_{ij} & 2\,Q_{ij} & (I_{ij}^{sq} - |V_i^{\text{sq}}|) \end{bmatrix}^T \right\|_2 \quad (12)$$

Proposition 1. *At any optimal solution of the model (5), (7), (8)-(11), constraint (9) holds as an equality.*

Proof. See Appendix B. □

2) The Optimal Power Flow problem: Extending the model above, the OPF problem is shown below.

$$\min \textstyle\sum_{i\in N} \beta_i\,Pg_i \quad (13)$$

subject to (5), (7), (9), (11), and

$$\textstyle\sum_{j:(i,j)\in E} P_{ij} \leq Pg_i - Pd_i, \sum_{j:(i,j)\in E} Q_{ij} \leq Qg_i - Qd_i, \quad \forall i \in N \quad (14)$$

$$I_{ij}^{sq} \leq \overline{I_{ij}^{sq}}, \ \forall (i,j) \in E \quad (15)$$

$$\underline{V_i^{sq}} \leq V_i^{sq} \leq \overline{V_i^{sq}}, \forall i \in N \quad (16)$$

$$\underline{Pg_i} \leq Pg_i \leq \overline{Pg_i}, \quad \underline{Qg_i} \leq Qg_i \leq \overline{Qg_i}, \quad \forall i \in N \quad (17)$$

where β_i the term of the generator cost function at bus i in [\$/MWh], Pg_i/Qg_i are the variables for the real/reactive generation at node i, and $\_/^-$ indicate the lower/upper limits of variables. The Shuffled Conic OPF model (5), (7), (9), (11), (13)-(17) is an SOCP problem, which can be solved by commercial solvers, since (9) can be written as the SOC (12). Objective function (13) minimizes the cost of generators; this implicitly minimizes the I^{sq} term because losses within the branches need to be met by the generation. Constraints (14) are power balance constraints, and (15)-(17) impose technical limits. At the optimal solution, (9) holds as an equality similarly to Proposition 1, as shown in Appendix B.

III. Numerical Results

This section compares the performance of the Shuffled Conic PF equations and the SOC PF equations of [15][1]. Section III-A shows results solving the PF problem and Section III-B solving the OPF problem. PF and OPF models are coded in GAMS 38.1.0 using MOSEK on a desktop with an Intel Core i5-6600 CPU at 3.30 GHz and 32 GB of RAM. The results are benchmarked against Matpower [18].

A. Power Flow Problem Results

This section, shows results solving the PF problem for four small to medium scale radial distribution networks with 15, 33, 69 and 85 buses [18]–[21] (also used in [2]), which we assume are single-phase or three-phase balanced. Results are shown in Fig. 3 and Table I, where voltage angle absolute errors (AE) in Fig. 3 and AE in Table I-(A, B) are calculated according to the results obtained by Matpower [18].

On the left y-axes of Fig. 3-(A), we observe that using the Shuffled Conic PF equations, voltage angle AE are significantly improved when voltage magnitudes divert from 1 p.u.,

[1]Even though [15] calculates voltage angle *difference* between buses, we show voltage angles *per bus* to enable direct comparison with our results.

TABLE I
PF Problem Results

	$\lvert N\rvert$=15	$\lvert N\rvert$=33	$\lvert N\rvert$=69	$\lvert N\rvert$=85
(A) Voltage Angle Mean Absolute Errors (compared to Matpower [18])				
Shuffled Conic PF	1.56%	5.98%	1.20%	1.28%
SOC PF of [15]	5.97%	10.16%	3.83%	10.42%
Improvement	73.87%	**41.14%**	68.67%	**87.72%**
(B) Voltage Angle Max. Absolute Errors (compared to Matpower [18])				
Shuffled Conic PF	2.94%	27.11%	2.43%	2.58%
SOC PF of [15]	8.33%	31.38%	10.78%	15.12%
Improvement	64.71%	**13.61%**	77.46%	**82.94%**
(C) Execution Time (seconds)				
Shuffled Conic PF	0.1060	0.1044	0.1170	0.1295
SOC PF of [15]	0.0915	0.0873	0.0926	0.1012

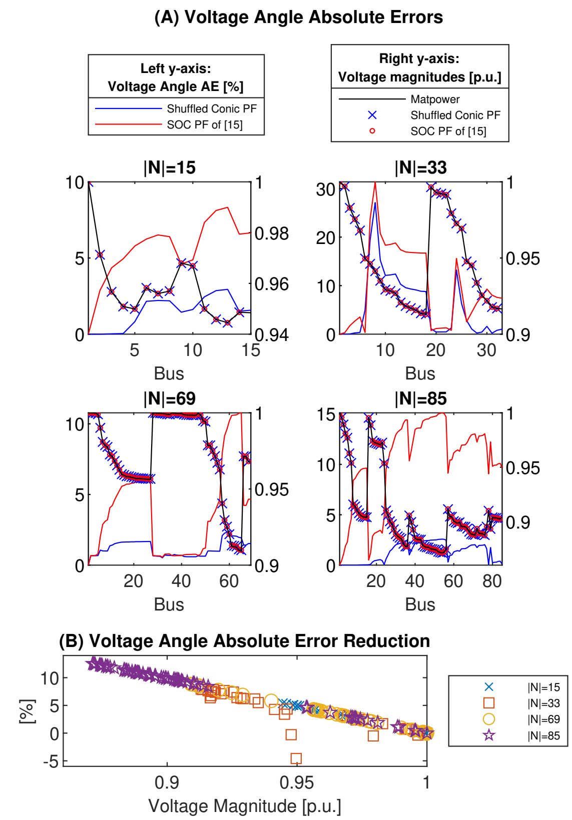

Fig. 3. PF Problem Results using four networks with $|N| = 15, 33, 69, 85$.

compared to using the SOC PF of [15]; this is particularly visible for the case where $|N|$=85. This is also shown in Fig. 3-(B), which shows the difference between the voltage angle AE of Fig. 3-(A). Additionally, Fig. 3-(B) shows that voltage angles are reduced in around 95% of cases, in almost half of the cases they are reduced by at least 5%, and in around 21% of cases by at least 10%. The right y-axes of Fig. 3-(A), show the accuracy of the proposed method in the calculation of voltage magnitudes, delivering near identical results to Matpower [18]. The method in [15] provides similar results to the Shuffled Conic PF, which is attributed to the fact that it also uses equations (7) and (9).

In Tables I-(A, B), it is shown that in all case studies, using the substitution in (3), voltage angle mean and maximum AE are reduced. In particular, mean AE are improved between 41.14% (for $|N|$=33) and 87.72% (for $|N|$=85), and maximum AE between 13.61% (for $|N|$=33) and 82.94% (for $|N|$=85). In terms of execution times, both models show a similar performance (Table I-(C)).

The improved accuracy of the proposed model in the voltage angle calculation compared to the method in [15] is attributed to the fact that in the proposed substitution only one voltage magnitude is set equal to 1 in Eq. (2b), instead of the product $|V_i||V_j|\approx 1$ as in [15]. This result indicates that the substitution could be most effective in future networks with increased loading which results in greater changes in voltage magnitude.

TABLE II
OPF PROBLEM RESULTS

	Base Demand	+5%	+10%	+20%
(A) COST [$/hour] (Objective Function, Eq. (13))				
Shuffled Conic PF	44.34	46.69	49.06	53.82
SOC PF of [15]	44.34	46.69	49.06	53.82
Matpower	44.34	46.69	49.06	53.82
(B) EXECUTION TIME (seconds)				
Shuffled Conic PF	0.1148	0.1076	0.1236	0.1208
SOC PF of [15]	0.0976	0.0975	0.0885	0.0982

B. Optimal Power Flow Problem Results

In this section, we solve the OPF problem for the the network $|N|$=33. Extracting generator and feeder data from [22], we assume that there are generators at buses 18, 22, 25, and 33. Voltage limits are set to 1p.u.±6%. Branch flow limits are not imposed for this network in Matpower [18], therefore constraints (15) are omitted from the OPF models of this section. Given our conclusion from Section III-A, and that electricity demand is expected to increase [17], we perform computational experiments for four active demand scenarios. In the first scenario, demand is the same as in Section III-A called, *Base demand* scenario, and in the other three, demand is increased by 5%, 10% and 20%. Results are shown in Fig. 4 and Table II, where voltage angle AE and mean/maximum voltage angle AE are compared to Matpower [18].

In Figs. 4-(A, B), it is shown that voltage angle AE are reduced, using the Shuffled Conic PF equations compared to the SOC PF equations of [15]; with a particular improvement when voltage magnitudes divert from 1 p.u. (as shown in Section III-A). In Fig. 4-(C), it is shown that both mean and maximum voltage angle AE using the Shuffled Conic PF are below the mean/maximum voltage angle AE using the SOC PF of [15]. In Fig. 4-(D), it is shown that, on average, using the Shuffled Conic PF equations within an OPF problem, voltage angle maximum and mean AE are improved around 50% and 70% in all active demand scenarios, compared to using the SOC PF equations of [15].

In Fig. 4-(E), it is also shown that using the proposed voltage angle substitution, voltage angle AE are reduced as demand grows. Furthermore, we calculated that using the Shuffled Conic PF equations to solve the OPF problem, voltage angle AE are reduced in around 96% of cases, in around 65% of cases they are reduced by at least 3%, and in around 17% of cases by least 5%. Finally, Table II shows that both models provide highly accurate objective functions compared to Matpower [18], and both models show a similar performance in terms of execution times.

IV. CONCLUSIONS AND FUTURE WORK

This paper presented the Shuffled Conic Power Flow equations, which enable a novel voltage angle substitution within a second-order cone power flow model and compared the performance of this substitution against a state-of-the-art SOC PF model which uses a conventional approximation to incorporate voltage angles. We showed that using the novel substitution, within a PF and an OPF problem, voltage angles were improved in around 95% and 96% of cases respectively. Also, the Shuffled Conic PF equations provided more accurate results for voltage angles in cases where voltage magnitudes diverted from 1 p.u. and with the increase of active demand. The latter is important, since electricity demand is expected to increase in the future. Future research may investigate the linearization of the Shuffled PF equations, and the applicability of this method to unbalanced and mesh power networks.

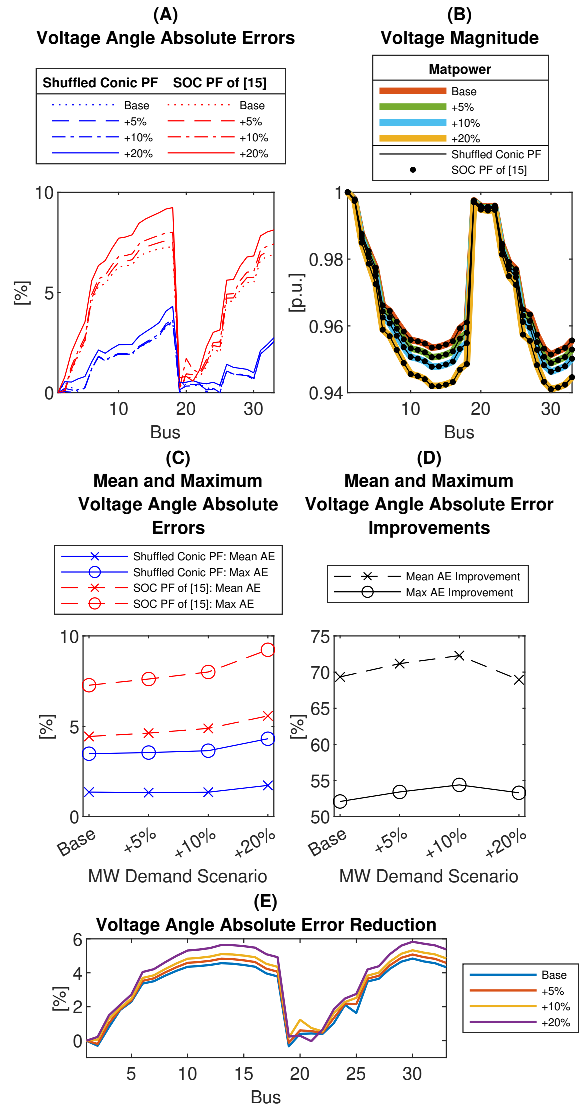

Fig. 4. OPF Problem Results using a network with $|N| = 33$.

APPENDIX A

Proposition 2. *Equations (1) can be written as the Shuffled PF Equations.*

Proof. First, re-arranging Equations (1) with respect to $|V_i|^2 - |V_i|\,|V_j|\cos\theta_{ij}$ and $|V_i||V_j|\sin\theta_{ij}$, we obtain:

$$x_{ij}P_{ij} - r_{ij}Q_{ij} = |V_i||V_j|\sin\theta_{ij} \quad (18)$$
$$r_{ij}P_{ij} + x_{ij}Q_{ij} = |V_i|^2 - |V_i||V_j|\cos\theta_{ij} \quad (19)$$

We next show that (19) can be written as (2a). First, we multiply and divide with $2\,|V_i|^2$ the right-hand side of (19) and then add and subtract $|V_i|^2|V_j|^2/2|V_i|^2$: $r_{ij}\,P_{ij} + x_{ij}\,Q_{ij} =$

$$\frac{2\,(|V_i|^2)^2}{2\,|V_i|^2} - \frac{2\,|V_i|^2|V_i|\,|V_j|\cos\theta_{ij}}{2\,|V_i|^2} + \frac{|V_i|^2|V_j|^2 - |V_i|^2|V_j|^2}{2\,|V_i|^2}.$$

Then, since $\sin^2\theta_{ij} + \cos^2\theta_{ij} = 1$, $r_{ij}\,P_{ij} + x_{ij}\,Q_{ij} =$

$$\frac{(|V_i|^2)^2 + (|V_i|^2)^2}{2\,|V_i|^2} - \frac{2\,|V_i|^2|V_i|\,|V_j|\cos\theta_{ij}}{2\,|V_i|^2}$$
$$+ \frac{(|V_i|^2|V_j|^2)\,(\sin^2\theta_{ij} + \cos^2\theta_{ij}) - |V_i|^2|V_j|^2}{2\,|V_i|^2},$$ or equivalently

$$\frac{|V_i|^2 - |V_j|^2}{2} + \frac{(|V_i||V_j|\sin\theta_{ij})^2 + (|V_i|^2 - |V_i||V_j|\cos\theta_{ij})^2}{2|V_i|^2}.$$

Given (18)-(19), we obtain $r_{ij}P_{ij} + x_{ij}Q_{ij} =$

$$\frac{|V_i|^2 - |V_j|^2}{2} + \frac{(x_{ij}P_{ij} - r_{ij}Q_{ij})^2 + (r_{ij}P_{ij} + x_{ij}Q_{ij})^2}{2|V_i|^2},$$ or

$$r_{ij}P_{ij} + x_{ij}Q_{ij} = \frac{|V_i|^2 - |V_j|^2}{2} + (r_{ij}^2 + x_{ij}^2)\frac{P_{ij}^2 + Q_{ij}^2}{2|V_i|^2} \quad (20)$$

which is Equation (2a). The proof of Equation (2b) is similar and is omitted for brevity. This completes the proof. □

Appendix B

Proof of Proposition 1. We will show that there cannot be an optimal solution with $I_{ij}^{sq} > (P_{ij}^2 + Q_{ij}^2)/|V_i^{\text{sq}}|$. Assuming the contrary, let $X^* := (P^*,\ Q^*,\ |V^{\text{sq}}|^*,\ \theta^{V^*}, I^{sq*})$ be an optimal solution such that $\exists$ a line $(k,l) \in E$: $I_{kl}^{sq*} > ({P_{kl}^*}^2 + {Q_{kl}^*}^2)/|V_k^{\text{sq}}|^*$. For $\varepsilon > 0$, consider a point $\Psi \neq X^*$:

$$|V^{\text{sq}}|^{\Psi} = |V^{\text{sq}}|^*,\ \theta^{V\,\Psi} = \theta^{V^*},\ P_{kl}^{\Psi} = P_{kl}^* - \varepsilon r_{kl}/2,$$
$$P_{-kl}^{\Psi} = P_{-kl}^*,\ Q_{kl}^{\Psi} = Q_{kl}^* - \varepsilon x_{kl}/2,\ Q_{-kl}^{\Psi} = Q_{-kl}^*,$$
$$I_{kl}^{sq\,\Psi} = I_{kl}^{sq*} - \varepsilon,\ I_{-kl}^{sq\ \Psi} = I_{-kl}^{sq\ *}$$

where a negative index means $\forall (i,j) \in E : (i,j) \neq (k,l)$.

We next show that Ψ is a feasible solution, and has a lower objective value than X^*. It can be easily shown that point Ψ satisfies constraints (5), (7) $\forall (i,j) \in E$ (including (k,l)), and (9), (10) $\forall (i,j) \in E$, apart from (k,l). For (k,l), constraint (9) becomes:

$$(I_{kl}^{sq*} - \varepsilon) \geq \{(P_{kl}^* - \varepsilon r_{kl}/2)^2 + (Q_{kl}^* - \varepsilon x_{kl}/2)^2\}/|V_k^{\text{sq}}|^*,$$ or
$$I_{kl}^{sq*}|V_k^{\text{sq}}|^* - {P_{kl}^*}^2 - {Q_{kl}^*}^2 \geq$$
$$\varepsilon(|V_k^{\text{sq}}|^* - P_{kl}^* r_{kl} - Q_{kl}^* x_{kl} + \varepsilon r_{kl}^2/4 + \varepsilon x_{kl}^2/4)$$

and since $I_{kl}^{sq*}|V_k^{\text{sq}}|^* - {P_{kl}^*}^2 - {Q_{kl}^*}^2 > 0$ by assumption, there always exists a sufficiently small $\varepsilon > 0$ which satisfies constraints (9) for line (k,l). Finally, constraints (10) are also satisfied for line (k,l) since $\varepsilon r_{kl}/2 > 0$ and $\varepsilon x_{kl}/2 > 0$, which are subtracted from the left-hand side of the constraints. However, the objective value of Ψ (which is $\sum_{(i,j)\in E} I_{ij}^{sq*} - \varepsilon$) is smaller than the objective of X (which is $\sum_{(i,j)\in E} I_{ij}^{sq*}$), which contradicts the assumption of the optimality of X. □

We note that for the Shuffled Conic OPF model, constraint (9) can be proven to hold as an equality at the optimal solution in a similar way. In particular, X^* can be extended to include Pg^* and Qg^*, and Ψ can be extended to include $Qg_k^{\Psi} = Qg_k^* - \epsilon x_{kl}/2$, $Qg_{-k}^{\Psi} = Qg_{-k}^*$, $Pg_k^{\Psi} = Pg_k^* - \epsilon r_{kl}/2$, and $Pg_{-k}^{\Psi} = Pg_{-k}^*$.

References

[1] M. Farivar and S. H. Low, "Branch Flow Model: Relaxations and Convexification—(Parts I, II)," *IEEE Trans. Power Syst.*, vol. 28, no. 3, pp. 2554–2564, 2013.

[2] R. Jabr, "Radial distribution load flow using conic programming," *IEEE Trans. Power Syst.*, vol. 21, no. 3, pp. 1458–1459, 2006.

[3] M. Farivar, C. R. Clarke, S. H. Low, and K. M. Chandy, "Inverter VAR control for distribution systems with renewables," in *Proc. IEEE SmartGridComm Conf.*, pp. 457–462, 2011.

[4] B. Kocuk, S. S. Dey, and X. A. Sun, "Strong SOCP relaxations for the optimal power flow problem," *Operations Research*, vol. 64, pp. 1177–1196, Dec 2016.

[5] Z. Yuan and M. Paolone, "Properties of convex optimal power flow model based on power loss relaxation," *Electric Power Systems Research*, vol. 186, p. 106414, 2020.

[6] Z. Yuan, "Formulations and approximations of branch flow model for general power networks," *J. Modern Power Syst. Clean Energy*, pp. 1–12, 2022.

[7] M. Baradar, M. R. Hesamzadeh, and M. Ghandhari, "Second-Order Cone Programming for Optimal Power Flow in VSC-Type AC-DC Grids," *IEEE Trans. Power Syst.*, vol. 28, no. 4, pp. 4282–4291, 2013.

[8] Z. Yuan, M. R. Hesamzadeh, and D. R. Biggar, "Distribution Locational Marginal Pricing by Convexified ACOPF and Hierarchical Dispatch," *IEEE Trans. Smart Grid*, vol. 9, no. 4, pp. 3133–3142, 2018.

[9] N. Bottrell, S. Terry, N. Ash, and L. Grella, "Active Response to Distribution Network Constraints," in *CIRED 2019*, pp. 457–462, 2011.

[10] UK Power Networks, "Active Response Project Progress Report." https://innovation.ukpowernetworks.co.uk/wp-content/uploads/2019/05/Active-Response-Project-Progress-Report-July-2018.pdf, 2018. (Accessed: 31.10.2022).

[11] Z. Yuan and M. R. Hesamzadeh, "Second-order cone AC optimal power flow: convex relaxations and feasible solutions," *J. Modern Power Syst. Clean Energy*, vol. 7, no. 2, pp. 268–280, 2019.

[12] Z. Yuan and M. R. Hesamzadeh, "Improving the Accuracy of Second-Order Cone AC Optimal Power Flow by Convex Approximations," in *2018 IEEE ISGT Asia*, pp. 172–177, 2018.

[13] R. A. Jabr, "A Conic Quadratic Format for the Load Flow Equations of Meshed Networks," *IEEE Trans. Power Syst.*, vol. 22, no. 4, pp. 2285–2286, 2007.

[14] M. Baradar and M. R. Hesamzadeh, "AC Power Flow Representation in Conic Format," *IEEE Trans. Power Syst.*, vol. 30, no. 1, pp. 546–547, 2015.

[15] M. M.-U.-T. Chowdhury and S. Kamalasadan, "A New Second-Order Cone Programming Model for Voltage Control of Power Distribution System With Inverter-Based Distributed Generation," *IEEE Trans. Ind. Appl.*, vol. 57, no. 6, pp. 6559–6567, 2021.

[16] M. M.-U.-T. Chowdhury and S. Kamalasadan, "An Angle Included Optimal Power Flow (OPF) Model for Power Distribution Network Using Second Order Cone Programming (SOCP)," in *2020 IEEE Ind. Applicat. Soc. Annu. Meeting*, pp. 1–7, 2020.

[17] IEA. International Energy Agency, "World Energy Outlook 2022," 2022.

[18] R. D. Zimmerman, C. E. Murillo-Sánchez, and R. J. Thomas, "MATPOWER: Steady-State Operations, Planning, and Analysis Tools for Power Systems Research and Education," *IEEE Trans. Power Syst.*, vol. 26, no. 1, pp. 12–19, 2011.

[19] D. Das, D. Kothari, and A. Kalam, "Simple and efficient method for load flow solution of radial distribution networks," *Int. J. Elect. Power Energy Syst.*, vol. 17, no. 5, pp. 335–346, 1995.

[20] M. Baran and F. Wu, "Network reconfiguration in distribution systems for loss reduction and load balancing," *IEEE Trans. Power Del.*, vol. 4, no. 2, pp. 1401–1407, 1989.

[21] D. Das, "Optimal placement of capacitors in radial distribution system using a Fuzzy-GA method," *Int. J. Elect. Power Energy Syst.*, vol. 30, no. 6, pp. 361–367, 2008.

[22] S. H. Dolatabadi, M. Ghorbanian, P. Siano, and N. D. Hatziargyriou, "An Enhanced IEEE 33 Bus Benchmark Test System for Distribution System Studies," *IEEE Trans. Power Syst.*, vol. 36, no. 3, pp. 2565–2572, 2021.