ON THE APPEARANCE OF ORIENTED TREES IN TOURNAMENTS

by

ALISTAIR BENFORD

A thesis submitted to
The University of Birmingham
for the degree of
DOCTOR OF PHILOSOPHY

School of Mathematics
College of Engineering and Physical Sciences
The University of Birmingham
February 2023

Abstract

We consider how large a tournament must be in order to guarantee the appearance of a given oriented tree. Sumner's universal tournament conjecture states that every $(2n-2)$-vertex tournament should contain a copy of every n-vertex oriented tree. However, it is known that improvements can be made over Sumner's conjecture in some cases by considering the number of leaves or maximum degree of an oriented tree. To this end, we establish the following results.

(1) There exists $C > 0$ such that any $(n + Ck)$-vertex tournament contains a copy of every n-vertex oriented tree with k leaves.

(2) For each k, there exists $n_0 \in \mathbb{N}$, such that, whenever $n \geqslant n_0$, any $(n + k - 2)$-vertex tournament contains a copy of every n-vertex oriented tree with at most k leaves.

(3) For every $\alpha > 0$, there exists $n_0 \in \mathbb{N}$ such that, whenever $n \geqslant n_0$, any $((1+\alpha)n+k)$-vertex tournament contains a copy of every n-vertex oriented tree with k leaves.

(4) For every $\alpha > 0$, there exists $c > 0$ and $n_0 \in \mathbb{N}$ such that, whenever $n \geqslant n_0$, any $(1 + \alpha)n$-vertex tournament contains a copy of any n-vertex oriented tree with maximum degree $\Delta(T) \leqslant cn$.

(5) For all countably-infinite oriented graphs H, either (i) there is a countably-infinite tournament not containing H, or (ii) every countably-infinite tournament contains a *spanning* copy of H.

(1) improves the previously best known bound of $n + O(k^2)$. (2) confirms a conjecture of Dross and Havet. (3) provides an asymptotic form of a conjecture of Havet and Thomassé. (4) improves a result of Mycroft and Naia which applies to trees with polylogarithmic maximum degree. (5) extends the problem to the infinite setting, where we also consider sufficient conditions for the appearance of oriented graphs satisfying (i).

Coauthors

The material in Chapter 2 is joint work with Richard Montgomery and based on [4].

The material in Chapter 3 is joint work with Richard Montgomery and based on [5].

The material in Chapter 4 is joint work with Louis DeBiasio and based on parts of [3].

Dedicated to my parents, Penny and Steve,

from whom I have learned the most.

ACKNOWLEDGEMENTS

First and foremost, I would like to thank Richard Montgomery, for the dedication, kindness, knowledge, and expertise he has offered to me as my supervisor. I have learned so much from him, and I consider myself privileged to have been his student.

I would like to thank Andrew Treglown for the advice and supervision he has given to me in the later stages of my PhD. Many thanks to Louis DeBiasio for his insights and mentorship during the preparation of a highly rewarding paper. I also give thanks to Candida Bowtell for her collaboration and friendship.

My time at the University of Birmingham has been enriched by the many wonderful people at the School of Mathematics, and in particular all those in the combinatorics group. I'm also very grateful for the mutual support of my fellow PhD students, and especially those who filled The Dungeon with so much liveliness and humour.

Keeping loneliness at bay during covid lockdowns would not have been possible without a rigorous schedule of regular calls and online activities, and so I would like to thank friends both old and new at Monday coffees, Friday(ish) quizzes, Sunday brunches, as well as all who kept me company on other days with calls, catch-ups, gaming sessions, and meetings of the Election Prediction Committee. I would especially like to thank Eleanor, who is always there to listen.

I can't overstate my gratitude for the continued love and support of my family, who give me so much joy and happiness on trips home. Finally, my sincerest thanks to Allison, who has been there for me since the beginning of this journey; I am very fortunate to have your care.

CONTENTS

CHAPTER 1

INTRODUCTION AND PRELIMINARY MATERIAL

A fundamental class of combinatorial problems ask how large a certain discrete structure needs to be, in order to guarantee it contains some specified substructure. For example, due to Ramsey's theorem, it has long been known that, given any finite graph H, if the edges of a sufficiently large complete graph are coloured red and blue in any fashion then it must contain a monochromatic copy of H. How large such a complete graph must be for different graphs H is the central consideration of Ramsey Theory, alongside generalisations using more colours and other discrete structures, such as hypergraphs, hypercubes, integers, and infinite graphs.

This thesis studies the natural analogue of these questions where we orient edges instead of colouring them. That is, given an oriented graph H, how large does a complete graph need to be before orienting its edges in any fashion (giving a *tournament*) guarantees a copy of H within these edges? Unlike for colourings, it is not true here that any oriented graph H is guaranteed to appear in a sufficiently large tournament, for if H contains a directed cycle then H is not contained in any *transitive* tournament (that is, a tournament G for which there exists a labelling $V(G) = \{v_1, \ldots, v_m\}$ such that $v_i \to v_j$ whenever $i < j$). On the other hand, an n-vertex oriented graph is acyclic if and only if it is a subgraph of the transitive tournament on n vertices, and it is well-known (see [12]) that every tournament on at least 2^{n-1} vertices contains a transitive tournament of order n.

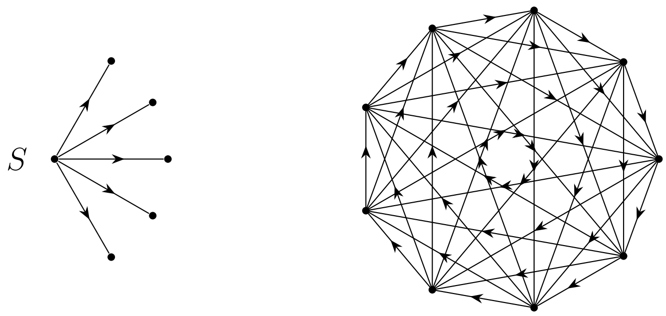

Figure 1.1: The oriented tree S and a regular tournament on $2n-3$ vertices containing no copy of S.

Thus, if we define the *unavoidability* of an oriented graph H to be the smallest m such that every m-vertex tournament contains a copy of H, then every acyclic oriented graph has a well-defined unavoidability.

Motivated by the requirement for H to be acyclic, the unavoidability of oriented graphs has been most extensively studied in the case of oriented trees. The aim of this thesis is to present several new results in this setting. So suppose T is an n-vertex oriented tree. In some cases, determining the unavoidability of T is straightforward. For example, if P is an n-vertex directed path (that is, a path with all edges oriented forward) then it is easy to see that any n-vertex tournament contains a copy of P. Indeed, if G is an n-vertex tournament, we can label the vertices $V(G) = \{v_1, \dots, v_n\}$ to maximise the number of pairs $i < j$ with $v_i \to v_j$. We would then find that $v_i \to v_{i+1}$ for every $i < n$, else we could swap v_i and v_{i+1}, a contradiction to the maximisation. Therefore, P is contained in any n-vertex tournament, and so has unavoidability n. For another example, consider an n-vertex star S (see Figure 1.1). In a regular tournament (i.e., a tournament where every vertex has the same in- and out-degree) on $2n-3$ vertices, every vertex has out-degree $n-2$, and hence such a tournament contains no copy of S. On the other hand, by considering the average out-degree, any tournament on $2n-2$ vertices contains a vertex with out-degree at least $n-1$. Thus, S has unavoidability $2n-2$.

Determining the unavoidability for trees in general is usually not as easy as these cases. However, in 1971 Sumner conjectured that every $(2n-2)$-vertex tournament should contain a copy of every n-vertex oriented tree (see, e.g., [26]). The first major step towards Sumner's conjecture was taken by Häggkvist and Thomason [14] in 1991, who showed that $O(n)$ vertices in a tournament are sufficient to find a copy of any n-vertex oriented tree. The constant implicit in this result has been improved in the intervening years by Havet [15], Havet and Thomassé [17], El Sahili [11], and Dross and Havet [10]. In particular, the result of Dross and Havet that any $\lceil \frac{21}{8}n - \frac{47}{16} \rceil$-vertex tournament contains a copy of every n-vertex oriented tree remains the best current bound applicable for all n. Significantly, however, Sumner's conjecture has been proved exactly for all sufficiently large n, by Kühn, Mycroft and Osthus [22], so that the conjecture remains open for only finitely many oriented trees.

Meanwhile, there has also been an extensive amount of investigation into other questions on the unvaoidability of oriented trees. We have already seen that Sumner's conjecture is far from tight in the case of a directed path, which is contained in any tournament on the same number of vertices. One line of investigation is to ask which other n-vertex trees are *unavoidable*, in the sense that they are guaranteed to appear in any tournament of size n. Thomason [29] showed in 1986 that the behaviour for oriented paths holds more generally, proving that there is some n_0 such that, whenever $n \geqslant n_0$, any n-vertex tournament contains a copy of every n-vertex oriented path, confirming a conjecture of Rosenfeld [28]. In 2000, Havet and Thomassé [18] showed that the optimal value of n_0 is 8, with the only paths that are not unavoidable being the antidirected paths of lengths 3, 5, and 7, which have unavoidability $n+1$, rather than n. A claws (i.e., a collection of directed paths which meet only at their common start vertex) with maximum degree at most $19n/50$ is known to be unavoidable due to Lu, Wang and Wong [24]. By defining a large class of oriented trees that are unavoidable, Mycroft and Naia [25] proved in 2018 that if T is selected uniformly at random from the set of all labelled oriented trees on n vertices, then asymptotically almost surely T is unavoidable.

Another approach to the unavoidability problem is to consider whether we can impose stronger bounds than Sumner's conjecture if we fix some structural property of the tree.

For example, Häggkvist and Thomason [14] showed in 1991 that the number of additional vertices required in the tournament can be bounded as a function of the number of leaves in the tree. That is, for each k, there is some smallest $g(k)$ such that every $(n + g(k))$-vertex tournament contains a copy of every n-vertex tree with k leaves. We note that, because every $(n+1)$-vertex tournament contains a copy of every n-vertex oriented path, $g(2) = 1$, and also that the example of an n-vertex star implies $g(k) \geqslant k - 1$. Motivated in part by these observations, Havet and Thomassé [16] generalised Sumner's conjecture by suggesting that $g(k) = k - 1$, as follows.

Conjecture 1.1. *Every $(n + k - 1)$-vertex tournament contains a copy of every n-vertex oriented tree with k leaves.*

While the upper bound shown by Häggkvist and Thomason on $g(k)$ was exponential in k^3, it was recently improved to $144k^2 - 280k + 124$ by Dross and Havet [10]. In the same paper, Dross and Havet also provided further evidence for Conjecture 1.1, by proving that every $(n + k - 1)$-vertex tournament contains a copy of every n-vertex k-leaf *arborescence* (that is, a tree with all paths branching outwards, or all paths branching inwards, from some designated root vertex).

The first three results of this thesis make further progress towards Conjecture 1.1, which we state now. The following theorem provides the first linear bound on $g(k)$.

Theorem 1.2. *There is some $C > 0$ such that every $(n+Ck)$-vertex tournament contains a copy of every n-vertex oriented tree with k leaves.*

If true, Conjecture 1.1 would be tight whenever $k = n - 1$ (i.e., whenever it is covered by Sumner's conjecture), but for general n and k, we only have examples showing that the tournament may need to have at least $n + k - 2$ vertices (as described below). From the result of Havet and Thomassé [18] on oriented paths we know that $n + k - 2$ is best possible if $k = 2$ and $n \geqslant 8$, while Ceroi and Havet [6] proved that $n + k - 2$ is also best possible if $k = 3$ and $n \geqslant 5$. Dross and Havet [10] conjectured that, for each k,

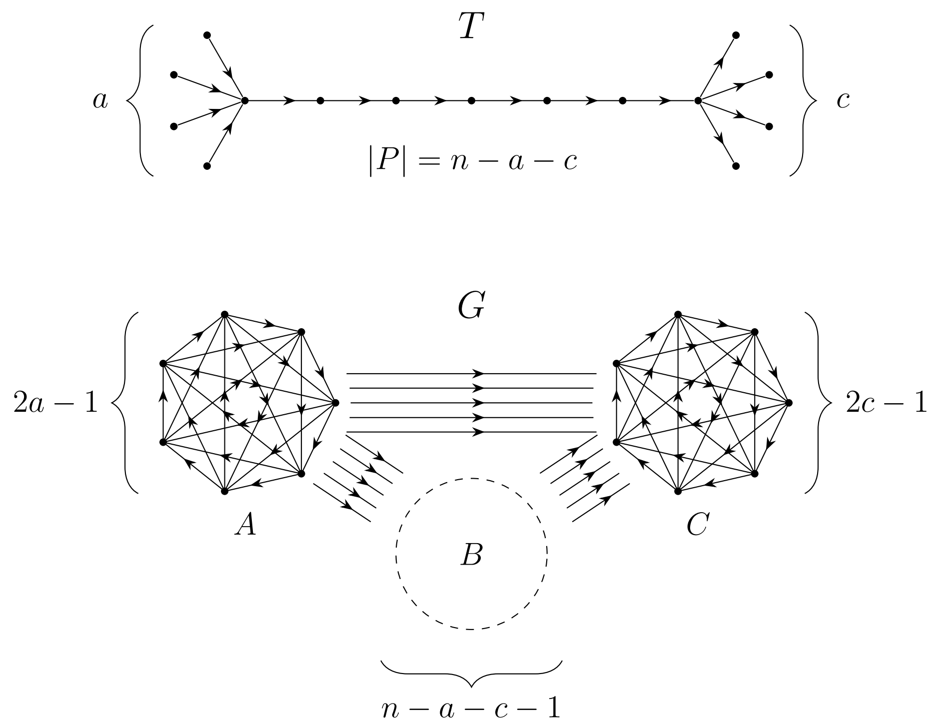

Figure 1.2: The oriented tree T and a tournament G containing no copy of T.

if n is sufficiently large then $n+k-2$ is best possible. Our second result confirms this conjecture, as follows.

Theorem 1.3. *For each k, there is some n_0 such that, for each $n \geqslant n_0$, every $(n+k-2)$-vertex tournament contains a copy of every n-vertex oriented tree with k leaves.*

The following 'double-star' example of Allen and Cooley (see [21]), illustrated in Figure 1.2, shows that Theorem 1.3 is tight. Given $a, c, n \in \mathbb{N}$ with $n > a+c$, form a tree T by taking a directed path P with $n-a-c$ vertices and attaching a in-leaves to the first vertex of P and c out-leaves to the last vertex of P. The resulting oriented tree T has n vertices and $a+c$ leaves. Construct the following $(n+a+c-3)$-vertex tournament G. Let $V(G) = A \cup B \cup C$, where A, B and C are disjoint with $|A| = 2a-1$, $|B| = n-a-c-1$, and $|C| = 2c-1$. Orient the edges of G so that $G[A]$ and $G[C]$ are regular tournaments, $G[B]$ is an arbitrary tournament, and all other edges are directed from A to B, from B to C, or from A to C. As every vertex in A has $a-1$ in-neighbours, if G contains a copy of T then the first vertex of P must be copied to $B \cup C$. Similarly, as every vertex in C has

$c-1$ out-neighbours, any copy of T in G must have the last vertex of P copied to $A \cup B$. But then every vertex of P must be copied into B, a contradiction as $|B| = n-a-c-1$. Thus, taking $a, c \in \mathbb{N}$ such that $a+c=k$, we have that the n-vertex tree T with k leaves does not appear in the $(n+k-3)$-vertex tournament G.

Theorems 1.2 and 1.3 are of course only useful if the oriented tree has a small number of leaves, with the largest gap between the results and Conjecture 1.1 occuring whenever $k = \Omega(n)$. Accordingly, we also aim to prove an upper bound on the unavoidability of oriented trees with many leaves. As a stepping stone towards proving Sumner's conjecture for large n [22], Kühn, Mycroft and Osthus first proved an asymptotic form of the conjecture, showing that $2(1+o(1))n$ vertices are enough to guarantee the appearance of an n-vertex oriented tree [21]. To address the case where $k = \Omega(n)$, we prove a similar asymptotic form of Conjecture 1.1, as follows.

Theorem 1.4. *Let $\alpha > 0$. There exists $n_0 \in \mathbb{N}$ such that for any $n \geqslant n_0$, if G is a $((1+\alpha)n+k)$-vertex tournament and T is an n-vertex oriented tree with k leaves, then G contains a copy of T.*

In the work of Kühn, Mycroft and Osthus on Sumner's conjecture, moving from the asymptotic form to an exact version for large n was achieved by noting that significantly fewer than $2n-2$ vertices are required in most cases, and sharpening the bound of $2(1+\alpha)n$ is thus most involved for certain 'star-like' classes of trees. It is natural to ask whether it is also possible to use Theorem 1.4 as a stepping stone towards proving Conjecture 1.1 for large n. In fact, the bound on unavoidability obtained in the proof of Theorem 1.4 is not strictly $((1+\alpha)n+k)$, but rather $(1+\gamma+\alpha)n$, where $\gamma \in [0,1]$ is a parameter depending on T satisfying $\gamma \lesssim k/n$. In many cases, we find γ is significantly less than k/n, and so fewer than $n+k-1$ vertices are required. In the case where $\gamma \approx k/n$ and $k = \Omega(n)$, the resulting tree has many of the 'star-like' properties that could allow for careful analysis to remove the αn error term, similar to the work of Kühn, Mycroft and Osthus. However, a new difficulty arises when $k = o(n)$, where, unlike for Sumner's conjecture, Conjecture 1.1 allows very little extra space in the tournament relative to the size of the tree being embedded. If k is extremely small then Theorem 1.3 can apply instead; however, this still leaves a significant range (cases such as $k = \Theta(\log n)$ or $k = \Theta(\sqrt{n})$) where neither approach proves fruitful and new ideas will be needed to make further progress towards Conjecture 1.1.

We turn now to consider whether stronger bounds than Sumner's conjecture are also possible if, instead of restricting the number of leaves in the oriented tree, we restrict the maximum degree. Indeed, it is already known that only a few additional vertices are required in a tournament to guarantee the appearance of an oriented tree with low maximum degree, however many leaves it has. Specifically, Kühn, Mycroft and Osthus [21] proved in 2010 that, if Δ is a fixed constant, then every $(1 + o(1))n$-vertex tournament contains a copy of every n-vertex oriented tree with maximum degree $\Delta(T) \leqslant \Delta$. Mycroft and Naia [25] later showed that the same conclusion holds even if the bound on $\Delta(T)$ is relaxed to one polylogarithmic in n. We will relax the bound on $\Delta(T)$ much further still, showing that a degree bound linear in n is sufficient, as follows.

Theorem 1.5. *Let $\alpha > 0$. There exists $c > 0$ and $n_0 \in \mathbb{N}$ such that for any $n \geqslant n_0$, if G is a $(1+\alpha)n$-vertex tournament and T is an n-vertex oriented tree with $\Delta(T) \leqslant cn$, then G contains a copy of T.*

Despite these results, it appears that bounding unavoidability based on maximum degree is more difficult than for the number of leaves. Indeed, the following question of Kühn, Mycroft and Osthus [21] remains open.

Question 1.6. *Does there exist a function h such that any $(n+h(\Delta))$-vertex tournament contains a copy of every oriented tree with maximum degree at most Δ?*

Mycroft and Naia [25] further asked whether $h(\Delta) = 2\Delta - 4$ is sufficient as long as n is much larger than Δ, noting that the earlier 'double-star' example (with $a = c = \Delta - 1$) demonstrates this would be tight for each Δ. On this question, perhaps the least is known for aborescences. The balanced binary arborescence B_d is the arborescence on $2^{d+1} - 1$ vertices in which every leaf is of distance d from the root, and every non-leaf vertex has

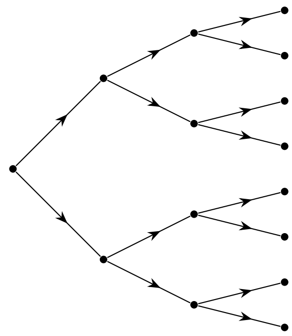

Figure 1.3: The binary arborescence B_3.

exactly 2 out-neighbours. It is not known whether there exists an absolute constant C such that any tournament on $|B_d| + C$ vertices contains a copy of B_d, and so this is a critical case for further study in relation to Question 1.6.

We next consider how the notion of unavoidability may be extended to infinite oriented graphs. We say that a countably-infinite oriented graph H is *unavoidable* if H is a subgraph of every countably-infinite tournament (note that this differs from the definition of an unavoidable oriented graph in the finite setting). Unlike the finite setting, it is not true that a countably-infinite oriented graph is unavoidable if and only if it is acyclic. For example, if K is the tournament on $\mathbb{N}$ with $E(K) = \{(i, j) : i < j\}$, then K does not have a copy of any oriented graph containing either a vertex of infinite in-degree or an infinite backward directed path. Similarly, the reversal of K does not have a copy of any oriented graph containing either a vertex of infinite out-degree or an infinite forward directed path. Therefore, for a countably-infinite oriented graph to be unavoidable, it must at least be acyclic, locally-finite (i.e., every vertex is incident with only finitely many edges), and have no infinite directed paths.

So our first goal is to characterise which countably-infinite oriented graphs are unavoidable. Having done that, the next goal is to get quantitative results for unavoidable countably-infinite oriented graphs along the lines of Sumner's conjecture and unavoidability results for finite oriented trees. For instance, motivated by recent Ramsey-type results regarding monochromatic subgraphs in edge-colourings of $K_{\mathbb{N}}$ [9, 8, 7, 23, 2], it would be natural try to prove that there exists $d > 0$ such that for every countably-infinite unavoidable oriented tree T and every tournament K on $\mathbb{N}$, there is an embedding $\phi : T \to K$ such that $\phi(V(T)) \subseteq \mathbb{N}$ has upper density at least d. So it is perhaps surprising that we prove the following result which both characterises unavoidable oriented graphs and proves that all such countably-infinite unavoidable oriented graphs are unavoidable in a very strong sense (in a way which makes the quantitative question mentioned above irrelevant).

Theorem 1.7. *Let H be a countably-infinite oriented graph. The following are equivalent:*

(i) *H is acyclic, locally-finite, and has no infinite directed paths.*

(ii) *H is contained in every countably-infinite tournament.*

(iii) *H is a spanning subgraph of every countably-infinite tournament.*

Despite this characterisation, there are still many interesting questions that can be asked in the infinite setting. A natural one is to consider how Theorem 1.7 may generalise to uncountable cardinals if the conditions of (i) are modified accordingly. Indeed, a version of Theorem 1.7 does still hold for oriented graphs of cardinality $\aleph_1$ which are acyclic, locally-*countable*, and have no infinite directed paths (for a discussion of this generalisation to $\aleph_1$, and the potential for generalisations to even larger cardinals, we direct the reader to the results of DeBiasio and Larson in Section 3 of [3]). Another question is, if we drop one or more of the conditions of (i), can we still guarantee a copy of H in a tournament on $\mathbb{N}$ if we additionally ask for the tournament to have some prescribed density of forward edges? In Section 4.3 we will state the latter question precisely, and prove a quantitive answer in the case where H is the infinite forward directed path.

In the remainder of this chapter, we give notation, definitions, and preliminary results that will be useful throughout this thesis. The proofs of Theorems 1.2 and 1.3 both use techniques that rely heavily on median orders, and are presented in Chapter 2. Theorems 1.4 and 1.5 are proven under a common framework, using regularity and random homomorphisms to reduce these theorems to critical cases amenable to a more direct study, and so these results are proven together in Chapter 3. Finally, in Chapter 4 we will prove Theorem 1.7 and consider the forward density version of the infinite case described above. Before all proofs, we provide further discussion of the results and relevant techniques, as well as proof sketches.

1.1 Notation

A *digraph* G consists of a vertex set $V(G)$ and an edge set $E(G)$, where each edge $e \in E(G)$ is an ordered pair (u, v) of vertices, which we write as uv. We write $|G| = |V(G)|$ for the order of G, and refer to edges of the form vv as *looped edges*. If $uv \in E(G)$, then we say that u *dominates* v (written $u \to_G v$), that v is an *out-neighbour* of u, and that u is an *in-neighbour* of v. Given $v \in V(G)$, the *out-neighbourhood* of v, written $N^+_G(v)$, is the set of out-neighbours of v in $V(G)$, and the *in-neighbourhood* of v, written $N^-_G(v)$ is the set of in-neighbours of v in $V(G)$. Throughout, we use $+$ and $-$ interchangeably with 'out' and 'in' respectively. For $X, Y \subseteq V(G)$ and $\diamond \in \{+, -\}$, we write $N^\diamond_G(X) = (\cup_{v \in X} N^\diamond_G(v)) \setminus X$ and $N^\diamond_G(X, Y) = N^\diamond_G(X) \cap Y$. For each $\diamond \in \{+, -\}$, the $\diamond$-*degree* of v in G is $d^\diamond_G(v) = |N^\diamond_G(v)|$, and for $X, Y \subseteq V(G)$ we also write $d^\diamond_G(X, Y) = |N^\diamond_G(X, Y)|$. For a vertex v, we also define its neighbourhood to be $N_G(v) = N^+_G(v) \cup N^-_G(v)$ and its degree to be $d_G(v) = |N_G(v)|$, and similarly define $N_G(X) = N^+_G(X) \cup N^-_G(X)$ for a set $X \subseteq V(G)$. We denote by $G[X]$ the induced sub-digraph of G with vertex set X and let $G - X = G[V(G) \setminus X]$. Subscripts are omitted wherever they are clear from context, as are rounding signs wherever they are not crucial.

If G, H are digraphs, a *homomorphism* ϕ from H to G is a function $\phi : V(H) \to V(G)$ such that $\phi(u)\phi(v) \in E(G)$ whenever $uv \in E(H)$. We sometimes write $\phi : H \to G$ to denote a homomorphism from H to G. We say G *contains* H if there is an injective homomorphism ϕ from H to G, and refer to such a ϕ as an *embedding* of H into G.

An *oriented graph* is a digraph with at most one edge between any pair of vertices. The *underlying graph* of an oriented graph G is the (non-oriented) graph H with $V(H) = V(G)$

and $E(H) = \{\{u, v\} : uv \in E(G)\}$. A *tournament* G is a digraph whose underlying graph is a complete graph, so for each $u, v \in V(G)$ with $u \neq v$, exactly one of uv or vu is in $E(G)$. An *oriented tree* (respectively, *oriented path*) is a digraph whose underlying graph is a tree (respectively, path). The *maximum degree* of an oriented tree T is the maximum degree of its underlying tree, and denoted $\Delta(T)$. A *directed path* from v_0 to v_ℓ is a path of the form $v_0 \to v_1 \to \ldots \to v_\ell$, and we refer to $\{v_1, \ldots, v_{\ell-1}\}$ as the *internal vertices* of P. The *length* of a path P is $|P| - 1$, and denoted $\ell(P)$. We say a subpath P of a forest T is a *bare path* if all of the internal vertices v of P have $d_T(v) = 2$, and we denote by $T - P$ the digraph formed from T by removing all the edges and internal vertices of P. A *directed cycle* on n vertices is the oriented graph $\vec{C}_n$ with $V(\vec{C}_n) = \{x_1, \ldots, x_n\}$ and $E(\vec{C}_n) = \{x_1x_2, \ldots, x_{n-1}x_n, x_nx_1\}$. Say that an oriented graph is *acyclic* if it contains no directed cycles.

Having proved, for example, a result holds for $\diamond = +$, we will occasionally deduce the same result for $\diamond = -$ by *directional duality.* That is, reversing all the relevant orientations and applying the result with $\diamond = +$ implies, after reversing the edges again, the result with $\diamond = -$. Where the symbol $\pm$ appears in a formula, we mean the formula holds for both $+$ and $-$ in place of $\pm$. Given $n \in \mathbb{N}$, we use $[n]$ to denote the set $\{1, \ldots, n\}$. For a set X and a function $f : X \to \mathbb{R}$, if $A \subseteq X$ we will often write $f(A)$ to mean $\sum_{x \in A} f(x)$ and $f(x_1, \ldots, x_k)$ to mean $f(\{x_1, \ldots, x_k\})$. For an event E_n depending on the parameter n, we will say that E_n holds *with high probability* if $\mathbb{P}(E_n) \to 1$ as $n \to \infty$. We also use standard hierarchy notation. That is, for $a, b \in (0, 1]$, we write $a \ll b$ to mean that there is a non-decreasing function $f : (0, 1] \to (0, 1]$ such that the subsequent statement holds whenever $a \leqslant f(b)$.

1.2 Embedding results for oriented trees

We will often embed small parts of a tree into a subset of a tournament with many spare vertices. To do this we could use any result embedding an n-vertex tree into a tournament with $O(n)$ vertices, but for convenience we will use the following result of El Sahili [11].

Theorem 1.8 ([11, Corollary 2])**.** *For each $n \geqslant 2$, every $(3n-3)$-vertex tournament contains a copy of every n-vertex oriented tree.*

The following corollary shows how Theorem 1.8 can be used to extend a partial copy of a tree to a full copy, provided each vertex in the partial copy has sufficient out- and in-degree to the remaining vertices in the tournament.

Corollary 1.9. *Let G be a tournament with disjoint subsets $U, V \subseteq V(G)$. Let T be a tree, and suppose $T' \subseteq T$ is a subtree such that there is a copy S' of T' in $G[V]$. If $d^{\pm}_G(v, U) \geqslant 3|V(T) \setminus V(T')|$ for every $v \in V$, then S' can be extended to a copy of T in G, with $T - V(T')$ copied to U.*

Proof. Label the components of $T - V(T')$ as $T_1, \ldots, T_r$, and take the largest $s \leqslant r$ such S' can be extended to a copy S of $T[V(T') \cup (\cup_{i\in[s]} V(T_i))]$. Suppose that $s < r$. Then, if $\diamond \in \{+, -\}$ is such that T_{s+1} is attached to T' by a $\diamond$-neighbour, and $v \in V(S')$ is the copy of the attachment point, then

$$d^{\diamond}_G(v, U \setminus V(S)) \geqslant 3|V(T) \setminus V(T')| - |T_1| - \ldots - |T_s| \geqslant 3|T_{s+1}|,$$

and so, by Theorem 1.8, $N^{\diamond}_G(v, U \setminus V(S))$ contains a copy of T_{s+1}, contradicting the maximality of s. Thus, S is a copy of T in G. □

While Theorem 1.8 would suffice to prove all of our main results, Theorem 1.2 can be proven with a significantly lower value of C by instead using Corollary 1.11, which we derive from the following theorem of Dross and Havet [10].

Theorem 1.10. *For each $n \geqslant 2$, every $\lceil \frac{3}{2}n + \frac{3}{2}k - 2 \rceil$-vertex tournament contains a copy of every n-vertex oriented tree with k leaves.*

Corollary 1.11. *Let $n, r, k \geqslant 1$, and suppose G is a tournament with at least $\frac{3}{2}n+\frac{3}{2}k-2r$ vertices and T is an oriented forest with n vertices, r components and, in total, k leaves and isolated vertices. Then, G contains a copy of T.*

Proof. Label the components of T as $T_1, \ldots, T_r$, and say, for each $i \in [r]$, that T_i has n_i vertices and, in total, k_i isolated vertices and leaves. Note that $n_i + 3k_i \geqslant 4$ for each $i \in [r]$. Take the largest $s \leqslant r$ for which there are vertex-disjoint subgraphs $S_i \subseteq G$, $i \in [s]$ such that, for each $i \in [s]$, S_i is a copy of T_i. Suppose $s < r$, for otherwise we have already found a copy of T in G, and note that

$$\left| G - \bigcup_{i\in[s]} V(S_i) \right| \geqslant |G| - n + n_{s+1}$$

$$\geqslant \sum_{i\in[r]\setminus\{s+1\}} \frac{n_i + 3k_i - 4}{2} + \frac{3n_{s+1}}{2} + \frac{3k_{s+1}}{2} - 2 \geqslant \frac{3n_{s+1}}{2} + \frac{3k_{s+1}}{2} - 2.$$

Therefore, by Theorem 1.10, $G - \cup_{i\in[s]} V(S_i)$ contains a copy of T_{s+1}, a contradiction. □

1.3 Embedding results for oriented paths

There will be many places in our proofs where it will be useful to embed a subpath of a tree into a tournament after the endpoints of the path have already had their embedding fixed (or, if there is a limited number of candidates for the embedding of the endpoints of the path). This embedding is most difficult if the path in question is a directed path, and indeed, embedding such paths will form much of the technical work of Chapter 2. On the other hand, for paths which instead have at least one change of direction, we can rely on some established results quoted here.

To discuss the changes of direction in a path and recall these results, we use the terminology of blocks. A *block* of an oriented path P is a maximal directed subpath. When we introduce an oriented path we assume it has an associated overall direction, and thus a first and last vertex as well as a first block and a last block. When the path is a directed path we will always assume the associated direction is the natural one, i.e., the one in which the first vertex has no in-neighbours.

We will often embed a path P into a tournament G while furthermore requiring the first vertex of P to be embedded into a fixed set of two vertices of G (and sometimes also requiring the last vertex of P to be embedded into another fixed set of two vertices of G). The following two results of Thomason show this is possible, provided that any restricted endvertex of P is next to a block of length 1, and also that the G has one more vertex than P (or two more vertices than P, if both endvertices are restricted).

Theorem 1.12. *Let P be an oriented path of order n with first block of length 1. Let G be a tournament of order $n+1$ and X be a subset of $V(G)$ of order at least 2. Then, there is a copy of P in G with first vertex in X.*

Theorem 1.13. *Let P be a non-directed oriented path of order n with first and last block of length 1. Let G be a tournament of order $n+2$ and X and Y be disjoint subsets of $V(G)$ of order at least 2. If P does not consist of three blocks with length one, then there is a copy of P in G with first vertex in X and last vertex in Y.*

Theorem 1.12 is a special case of [29, Theorem 1], and Theorem 1.13 is a special case of [29, Theorem 5]. In Section 3.6.1 we will consider a class of tournaments in which oriented paths can be robustly embedded. For this, we need a result more general, but less tight, than those stated above. The following corollary is a direct consequence of Thomason's results.

Corollary 1.14. *Let P be a non-directed path of order n with at least two blocks. Let G be a tournament, and let X and Y be subsets of $V(G)$ of order at least $n+2$. Then, there is a copy of P in G with first vertex in X and last vertex in Y.*

Proof. If P consists of just two blocks, then the result follows from [29, Theorem 3]. If P consists of exactly three blocks, and the middle block has length 1, then the result follows from [29, Theorem 4]. In all other cases, the result follows from [29, Theorem 5]. □

1.4 Properties of trees

Here we collect a number of elementary properties of oriented trees for use later. Our first proposition considers the number of maximal bare paths in a (non-oriented) tree with k leaves, as follows.

Proposition 1.15. *An n-vertex tree T with $k \geqslant 2$ leaves has at most $2k-3$ maximal bare paths, one of which must have length at least $(n-1)/(2k-3)$, and at most $2k-2$ vertices whose degree is not 2.*

Proof. For the appropriate r, let $P_1, \dots, P_r$ be the maximal bare paths in T, and label vertices such that, for each $i \in [r]$, P_i is an x_i, y_i-path. Note that the tree T' formed from T by replacing each path P_i, $i \in [r]$, by a single undirected edge has r edges, $r+1$ vertices, k leaves and no degree 2 vertices. Therefore,

$$2(|T'|-1) = 2e(T') = \sum_{v \in V(T')} d_{T'}(v) \geqslant k + 2(|T'|-k) + |\{v : d_{T'}(v) \geqslant 3\}|,$$

and thus $|\{v : d_{T'}(v) \geqslant 3\}| \leqslant k-2$. As $|\{v : d_T(v) \geqslant 3\}| = |\{v : d_{T'}(v) \geqslant 3\}|$, T has at most $2k-2$ vertices whose degree is not 2. Furthermore, $|T'| = r+1 \leqslant k + (k-2)$, so that $r \leqslant 2k-3$. Finally, as $\sum_{i\in[r]} \ell(P_i) = e(T) = n-1$, one of the paths P_i, $i \in [r]$, has length at least $(n-1)/(2k-3)$. □

In the main embedding for both Theorem 1.2 and Theorem 1.3, we will embed collections of small subtrees with directed paths between them. The next two propositions (appropriately applied to an auxiliary oriented tree with vertices representing subtrees and edges representing paths) will give us an order in which these trees and paths will be embedded along a median order of the tournament. We use Proposition 1.16 for Theorem 1.2, and Proposition 1.17 for Theorem 1.3.

Proposition 1.16. *Every oriented tree T with $n \geqslant 1$ vertices has a vertex partition $V(T) = V_1 \cup \ldots \cup V_s$ of non-empty sets, for some $s \in [n]$, such that, for each edge $e \in E(T)$, for some $i \in [s-1]$, e is an edge directed from V_i to V_{i+1}.*

Proof. Noting that the statement is trivially true if $|T| \leqslant 2$, we prove this by induction on $|T|$. Suppose then it is true for all oriented trees with fewer than $n \geqslant 3$ vertices. We may assume, by directional duality, that T has an out-leaf. Let T' be formed from T by removing such an out-leaf, t say, and let $s \in [n-1]$ be such that there is a vertex partition $V(T') = V_1 \cup \ldots \cup V_s$ of non-empty sets, such that, for each edge $e \in E(T')$, for some $i \in [s-1]$, e is an edge directed from V_i to V_{i+1}. Let $V_{s+1} = \emptyset$. Let j be such that the in-neighbour of t in T is in V_j, and add t to V_{j+1}. Taking the non-empty sets from $V_1, \ldots, V_{s+1}$ completes the proof of the inductive step, and hence the proposition. □

Proposition 1.17. *Every n-vertex oriented tree T has labellings $V(T) = \{t_1, \ldots, t_n\}$ and $E(T) = \{e_1, \ldots, e_{n-1}\}$, such that, for every $j \in [n-1]$, there is some $i_1, i_2 \in [n]$ with $i_1 \leqslant j < i_2$ and $e_j = t_{i_1}t_{i_2}$.*

Proof. We proceed by induction on n, noting the proposition is trivial for $n = 1$. For $n > 1$, we may assume, by directional duality, that T has an out-leaf. Let t_n be this out-leaf, and e_{n-1} its adjacent edge. By the inductive hypothesis, there are labellings $V(T-t_n) = \{t_1, \ldots, t_{n-1}\}$ and $E(T-t_n) = \{e_1, \ldots, e_{n-2}\}$, such that, for every $j \in [n-2]$, $e_j = t_{i_1}t_{i_2}$ for some $i_1 \leqslant j < i_2$. Taking $V(T) = \{t_1, \ldots, t_n\}$ and $E(T) = \{e_1, \ldots, e_{n-1}\}$ completes the proof. □

1.5 Probabilistic results

Parts of our embeddings will be random, or use some reserved random set. To analyse these parts, we will use the following probabilistic bounds. The first is a Chernoff bound [13, Corollary 2.21], and the second is Hoeffding's inequality [13, Corollary 2.28].

Lemma 1.18. *If X is a binomial variable with standard parameters n and p, denoted $X = \mathrm{Bin}(n, p)$, and ε satisfies $0 < \varepsilon \leqslant 3/2$, then*

$$\mathbb{P}(|X - \mathbb{E}X| \geqslant \varepsilon \mathbb{E}X) \leqslant 2 \exp\left(-\varepsilon^2 \mathbb{E}X/3\right).$$

Theorem 1.19. *Let $X_1, \ldots, X_n$ be independent random variables with X_i bounded by the interval $[a_i, b_i]$ for $i \in [n]$. Let $X = \sum_{i\in[n]} X_i$. Then, for any $t > 0$, we have*

$$\mathbb{P}(|X - \mathbb{E}X| \geqslant t) \leqslant 2\exp\left(-\frac{2t^2}{\sum_{i\in[n]}(b_i - a_i)^2}\right).$$

It will often be convenient for most of the vertices to have large in- and out-degree into a reserved random set, for which we use the following result.

Proposition 1.20. *Fix $p > 0$. Let G be a tournament with $n \leqslant |G| \leqslant 3n$. Let $U \subseteq V(G)$ be a random subset, with elements from $V(G)$ chosen independently at random with probability p. Let V' be the set of vertices $v \in V(G) \setminus U$ for which $d^{\pm}(v, U) \geqslant p^2 n$. Then, with high probability, $pn/2 \leqslant |U| \leqslant 4pn$, and $|V(G) \setminus V'| \leqslant 12pn$.*

Proof. By Lemma 1.18 and the fact that $pn \leqslant \mathbb{E}|U| \leqslant 3pn$, we have $pn/2 \leqslant |U| \leqslant 4pn$ with high probability. If $v \in V(G)$ is such that $d^{\pm}_G(v) \geqslant 2pn$, then, by setting $\varepsilon = 1/2$ in Lemma 1.18, the probability that $d^{\pm}(v, U) \geqslant p^2 n$ fails for v is at most $4\exp(-p^2 n/6)$. Any set of $4pn+1$ vertices in G contains a vertex with out-degree at least $2pn$ and a vertex with in-degree at least $2pn$. So at most $4pn$ vertices v of G have $d^+_G(v) < 2pn$ and at most $4pn$ vertices of G have $d^-_G(v) < 2pn$. Therefore, the probability that $|V(G)\setminus V'| \leqslant |U| + 8pn$ fails is at most $12n\exp(-p^2 n/6)$. So U satisfies both $pn/2 \leqslant |U| \leqslant 4pn$ and $|V(G)\setminus V'| \leqslant 12pn$ with high probability. □

CHAPTER 2

EMBEDDING ORIENTED TREES USING MEDIAN ORDERS

In this chapter we present proofs of the following results, which we recall from the introduction.

Theorem 1.2. *There is some $C > 0$ such that every $(n+Ck)$-vertex tournament contains a copy of every n-vertex oriented tree with k leaves.*

Theorem 1.3. *For each k, there is some n_0 such that, for each $n \geqslant n_0$, every $(n+k-2)$-vertex tournament contains a copy of every n-vertex oriented tree with k leaves.*

To prove Theorems 1.2 and 1.3, we use median orders, a technique first used to embed trees in tournaments by Havet and Thomassé [17]. In particular, we exploit the property that pairs of vertices in a median order can be robustly connected by directed paths with length 3 travelling in the direction of the order (see Lemma 2.4), using this repeatedly in our embeddings. We have not optimised the value of C reachable with our methods as this will not reach a plausibly optimal bound, but we show that Theorem 1.2 holds for some $C < 500$. We do not calculate an explicit function $n_0(k)$ for Theorem 1.3, but our methods show that we may take $n_0(k) = k^{O(k)}$. However, it seems likely some function $n_0(k)$ satisfying Theorem 1.3 with $n_0(k) = O(k)$ exists.

We next recall median orders and their basic properties, before proving Lemma 2.4, a crucial tool in the proofs of this chapter. In Section 2.2 we prove Theorem 1.2, and in Section 2.3 we prove Theorem 1.3, sketching the proofs beforehand in each case.

2.1 Median orders

Given a tournament G, an ordering $\sigma = v_1, \ldots, v_n$ of $V(G)$ is a *median order* if it maximises the number of pairs $i < j$ with $v_i v_j \in E(G)$. The following lemma gives two simple fundamental properties of median orders (see, e.g., [10, Lemma 9]).

Lemma 2.1. *Let G be a tournament and $v_1, \ldots, v_n$ a median order of G. Then, for any two indices i, j with $1 \leqslant i < j \leqslant n$, the following properties hold.*

(i) *$v_i, v_{i+1}, \ldots, v_j$ is a median order of the induced subtournament $G[\{v_i, v_{i+1}, \ldots, v_j\}]$.*

(ii) *v_i dominates at least half of the vertices $v_{i+1}, v_{i+2}, \ldots, v_j$, and v_j is dominated by at least half of the vertices $v_i, v_{i+1}, \ldots, v_{j-1}$. In particular, each vertex v_i, $1 \leqslant i < n$, dominates its successor v_{i+1}.*

Median orders contain short directed paths from any vertex to any vertex later in the order, as follows (in combination with Lemma 2.1 (i)).

Corollary 2.2. *Let $n \geqslant 2$. If $v_1, \ldots, v_n$ is a median order of the n-vertex tournament G, then G contains a directed path from v_1 to v_n with length at most 2.*

Proof. Suppose $v_1 v_n \notin E(G)$, for otherwise such a path exists, and let $V = \{v_2, \ldots, v_{n-1}\}$. Then, by Lemma 2.1 (ii), $|N^+(v_1, V)| = |N^+(v_1)| \geqslant \frac{n-1}{2} > |V|/2$. Similarly, $|N^-(v_n, V)| > |V|/2$. Therefore, there is some $w \in V$ such that $v_1 w v_n$ is a directed path. □

Median orders have been used particularly effectively to embed arborescences in tournaments. An *out-arborescence* (respectively, *in-arborescence*) is an oriented tree T with a root vertex $t \in V(T)$ such that, for every $v \in V(T)$, the path between t and v in T is directed from t to v (respectively, from v to t). Dross and Havet [10] used median orders to prove that any $(n + k - 1)$-vertex tournament contains a copy of any n-vertex arborescence with k leaves. We will use their result in the following slightly stronger form (see [10, Theorem 12]).

Theorem 2.3. *Let A be an n-vertex out-arborescence with $k \geqslant 1$ out-leaves and root r. Let G be a tournament on $n+k-1$ vertices and let $\sigma = v_1, \dots, v_{n+k-1}$ be a median order of G. Then, there is an embedding ϕ of A in G such that $\phi(r) = v_1$.*

In both the proofs of Theorem 1.2 and 1.3, we will take a median order, $\sigma = v_1, \dots, v_m$ say, of an m-vertex tournament, G say, and carefully partition this order into intervals before embedding different parts of the tree into each interval. Having found parts of a tree in distinct intervals, we will often wish to join two of them with a directed path. The following lemma shows that this is possible across a median order, even in cases where the interval in between the vertices to be joined contains some forbidden vertices.

Lemma 2.4. *Suppose G is an n-vertex tournament with a median order $\sigma = v_1, \dots, v_n$. Then, for any set $A \subseteq V(G) \setminus \{v_1, v_n\}$ with $|A| \leqslant (n-8)/6$, there is a directed v_1, v_n-path in $G - A$ with length 3.*

Proof. If there are some distinct $x, y \in (N^+_G(v_1) \cap N^-_G(v_n)) \setminus A$, then assume, by relabelling if necessary, that $xy \in E(G)$ and observe that v_1xyv_n is a path with length 3 in $G - A$, as required. Therefore, suppose that $|(N^+_G(v_1) \cap N^-_G(v_n)) \setminus A| \leqslant 1$.

By Lemma 2.1 (ii), we have $|N^+_G(v_1) \setminus \{v_n\}|, |N^-_G(v_n) \setminus \{v_1\}| \geqslant (n-2)/2$. Let $B_1 = N^+_G(v_1) \setminus (A \cup N^-_G(v_n) \cup \{v_n\})$ and $B_2 = N^-_G(v_n) \setminus (A \cup \{v_1\})$. Note that $|B_1| \geqslant n/2 - 2 - |A| > 0$ and $|B_2| \geqslant n/2 - 1 - |A|$. Let $B_0 = V(G) \setminus (B_1 \cup B_2 \cup \{v_1, v_n\})$, so that

$$|B_0| = n - 2 - |B_1| - |B_2| \leqslant n - 2 - (n/2 - 2 - |A|) - (n/2 - 1 - |A|) = 2|A| + 1. \quad (2.1)$$

Colour vertices in B_0, B_1 and B_2 respectively green, red and blue. If any blue vertex, x say, has a red in-neighbour, y say, then v_1yxv_n is a path with length 3 in $G - A$, as required. Therefore, suppose that every in-neighbour of each blue vertex is a green vertex or a blue vertex, for otherwise we have the desired path.

Let j be the largest integer such that v_j is blue. Let $A_1 = A \cap \{v_2, \dots, v_{j-1}\}$ and $A_2 = A \cap \{v_{j+1}, \dots, v_{n-1}\}$, so that $|A_1| + |A_2| = |A|$. For the appropriate r, let $I_1, \dots, I_r$ be the maximal intervals of $v_2, \dots, v_{j-1}$ consisting of only red and green vertices. Observe that, for each $i \in [r]$, the vertex after I_i in σ is blue, and has at least $|I_i|/2$ in-neighbours in I_i by Lemma 2.1 (ii), all of which must be green. Thus, every interval I_i, $i \in [r]$, contains at least as many green vertices as red vertices.

As every red or green vertex before v_j in σ is in some interval I_i, $i \in [r]$, we have that there are at least as many green vertices as there are red vertices in $\{v_2, \dots, v_{j-1}\}$. As $|N_G^+(v_1) \cap \{v_2, \dots, v_j\}| \geqslant (j-1)/2$ by Lemma 2.1 (ii), at least $(j-1)/2 - |A_1| - 1$ of the vertices in $\{v_2, \dots, v_{j-1}\}$ are red. Therefore, there are at least $(j-1)/2 - |A_1| - 1$ green vertices in $\{v_2, \dots, v_{j-1}\}$. By (2.1) and the definition of A_2, we have that there at most $2|A|+1-|A_2|$ green vertices in $\{v_2, \dots, v_{j-1}\}$. Thus, $2|A|+1-|A_2| \geqslant (j-1)/2-|A_1|-1$. Rearranging, and using that $|A_1| + |A_2| = |A|$, we get $3|A| \geqslant 2|A_2| + j/2 - 5/2$.

Now, by Lemma 2.1 (ii), $|N_G^-(v_n) \cap (\{v_{j+1}, \dots, v_{n-1}\})| \geqslant (n-1-j)/2$, so, as v_j is the last blue vertex in σ, there are at least $(n-1-j)/2$ vertices in A_2. Thus, $3|A| \geqslant 2|A_2| + j/2 - 5/2 \geqslant (n-j) + j/2 - 7/2 = n - j/2 - 7/2$. As $j \leqslant n-1$, we have $3|A| \geqslant (n-6)/2$, contradicting that $|A| \leqslant (n-8)/6$. □

2.2 Proof of Theorem 1.2

In Section 2.2.1, we use the results quoted in Section 1.3 to show that it is enough to prove Theorem 1.2 in the case where all bare paths of T are directed. That is, we reduce the proof to showing the following result.

Theorem 2.5. *There is some $C > 0$ such that each $(n+Ck)$-vertex tournament contains a copy of every n-vertex oriented tree with k leaves in which every bare path is a directed path.*

To prove Theorem 2.5, we first remove $O(k)$ long directed paths from T to leave a forest with size linear in k. The components of this forest we embed into carefully chosen intervals of a median order with $O(k)$ spare vertices in total, using Corollary 1.11. It remains then to embed the long directed paths, where we only have a constant number of spare vertices per path. This we do with Lemma 2.6 in Section 2.2.2. A simple modification of Dross and Havet's procedure for embedding arborescences into median orders (which they used to prove Theorem 2.3) allows directed paths from specified first vertices to be embedded efficiently into a median order. To embed such paths with both endvertices specified, we adapt this procedure, using it to embed most of the directed paths, but, as soon as all but three edges of any path are embedded, using Lemma 2.4 to connect the path to its desired last vertex. This allows us to find a set of directed paths while having only constantly many spare vertices per path (see Lemma 2.6), which we use to prove Theorem 2.5 in Section 2.2.3.

2.2.1 Reduction to trees with only directed bare paths

To prove Theorem 1.2 from Theorem 2.5, we take a tree T, remove most of the middle section of the maximal bare paths with at least 6 blocks, and duplicate each new leaf created by this removal. (Here, a *duplicated vertex* is a new vertex with exactly the same in- and out-neighbourbood as the matching original vertex.) Calling the resulting forest T', if we have an embedding of T' then the duplication of a leaf gives us two options to embed the original vertex from T. This will allow us to use the results in Section 1.3 to embed the deleted path given enough other vertices in the tournament (with no further restriction on these other vertices).

Not every maximal bare path in T' will be directed, but each such path will have at most 5 blocks. Adding a dummy leaf at any vertex in two blocks will give a forest T'' containing T' whose maximal bare paths are all directed, allowing us to apply Theorem 2.5 to each component. Importantly, T', and hence T'', will still have $O(k)$ leaves.

Proof of Theorem 1.2 from Theorem 2.5. Using Theorem 2.5, let $C \geqslant 8$ be large enough that, for every $\bar{n}$ and $\bar{k}$, every $(\bar{n}+(C-8)\bar{k})$-vertex tournament contains a copy of every $\bar{n}$-vertex oriented tree with (at most) $9\bar{k}$ leaves in which every bare path is a directed path. Let G be an $(n+Ck)$-vertex tournament, and let T be an n-vertex oriented tree with k leaves.

For the appropriate r, let $P_1, \dots, P_r$ be the maximal bare paths in T, and label vertices such that, for each $i \in [r]$, P_i is an x_i, y_i-path. By Proposition 1.15, we have $r \leqslant 2k-3$. Let $I \subseteq [r]$ be the set of $i \in [r]$ such that P_i has at least 6 blocks.

For each $i \in I$, let $P_i^{(1)}$ and $P_i^{(2)}$ be the first two blocks of P_i from x_i, and let $P_i^{(3)}$ and $P_i^{(4)}$ be the first two blocks of P_i from y_i. Let $e_i^{(1)}$ be the furthest edge of $P_i^{(2)}$ from x_i on P_i, and let $e_i^{(2)}$ be the furthest edge of $P_i^{(4)}$ from y_i on P_i. Let $Q_i = (P_i - \sum_{j=1}^4 P_i^{(j)}) + e_i^{(1)} + e_i^{(2)}$.

Note that, for each $i \in I$, the first and last block of Q_i have length 1, its endvertices have degree 2 in T, and it has at least 4 blocks (and thus length at least 4). Label vertices so that, for each $i \in I$, Q_i is a u_i, v_i-path. Let T' be the forest formed from T by, for each $i \in I$, deleting the edges of Q_i and creating two new vertices, u_i' and v_i', so that u_i' is a duplicate of u_i and v_i' is a duplicate of v_i. Note that u_i, u_i', v_i and v_i' are all leaves of T'.

Let B be the set of vertices with degree 2 in T' with either no in-neighbour or no out-neighbour, so that they lie in the intersection of two (consecutive) blocks. Observe that each such vertex must lie on some path P_i, $i \in [r] \setminus I$, or on $P_i^{(1)} \cap P_i^{(2)}$ or $P_i^{(3)} \cap P_i^{(4)}$ for some $i \in I$. Therefore, $|B| \leqslant 4(r - |I|) + 2|I|$. Now, form T'' from T' by taking each $v \in B$ and adding a new out-neighbour as a leaf, calling the new vertex u_v. We note here that all bare paths of T'' are directed paths.

Note that, if $\bar{T}$ is a component of T', and q is the number of paths Q_i adjacent to $\bar{T}$ that are deleted when forming T' from T, then $\bar{T}$ has at most $k - q + 2q \leqslant k + |I|$ leaves. Furthermore, T' has in total $n + 2|I| - \sum_{i \in I}(|Q_i| - 2) \leqslant n + 2|I| - 3|I| = n - |I|$ vertices. Therefore, as $r \leqslant 2k-3$, each component of T'' has at most $k + |I| + |B| \leqslant 9k$ leaves and T'' in total has at most $n - |I| + |B| \leqslant n + 8k$ vertices. Iteratively and vertex-disjointly, embed as many different components from T'' into G as possible. If a component of T'', say a tree $\bar{T}$ with $\bar{n}$ vertices and $\bar{k}$ leaves, is left unembedded then there are at least

$$|G| - (|T''| - |\bar{T}|) \geqslant (n + Ck) - (n + 8k) + \bar{n} \geqslant \bar{n} + (C - 8)k$$

vertices not used in the embedding, and $\bar{k} \leqslant 9k$. Thus, by the choice of C, we can embed

$\bar{T}$ using the unused vertices in G, a contradiction. Thus, G contains a copy of T'', S'' say.

For each $v \in B$, delete the copy of u_v from S'', and let the resulting copy of T' be S'. Note that, as $C \geqslant 8$ and $|I| \leqslant 2k-3$,

$$|V(G) \setminus V(S')| = n + Ck - |T'| = n + Ck - \Big(n + 2|I| - \sum_{i \in I}(|Q_i| - 2)\Big) \geqslant \sum_{i \in I}(|Q_i| - 2),$$

and take vertex disjoint sets A_i, $i \in I$, in $V(G) \setminus V(S')$ with $|A_i| = |Q_i| - 2$ for each $i \in I$.

For each $i \in I$, let $\bar{u}_i, \bar{u}'_i, \bar{v}_i, \bar{v}'_i$ be the copy of u_i, u'_i, v_i, v'_i respectively in S'. Using Theorem 1.13, for each $i \in I$, find a copy of Q_i, say R_i, in $G[A_i \cup \{\bar{u}_i, \bar{u}'_i, \bar{v}_i, \bar{v}'_i\}]$ starting at $\bar{u}_i$ or $\bar{u}'_i$ and ending at $\bar{v}_i$ or $\bar{v}'_i$. Take then S', and, for each $i \in I$, delete from T' any vertices in $\{\bar{u}_i, \bar{u}'_i, \bar{v}_i, \bar{v}'_i\}$ which are not an endvertex of R_i and add the path R_i. Note that this gives a copy of T. □

2.2.2 Joining vertex pairs with directed paths disjointly

We now connect multiple pairs of vertices with directed paths, where the start vertex for each path lies in a set B_1, and the end vertex lies in another set B_2, and the vertices of B_1 come before the vertices of B_2 in a median order. With Lemma 2.4 we can find such paths; the challenge here is to find these paths when they collectively must use almost all of the intermediate vertices in the median order. To do this, we find most of the paths using a procedure of Dross and Havet [10] for embedding arborescences, modifying it with Lemma 2.4 to attach each path to the correct end vertex when most of the path has been found.

Lemma 2.6. *Let G be an $(m_0 + m_1 + m_2)$-vertex tournament, and suppose $\sigma = v_1, \dots, v_{m_0+m_1+m_2}$ is a median order of G. Let $B_1 \subseteq V(G)$ be the first m_1 vertices of G according to σ, let $B_2 \subseteq V(G)$ be the last m_2 vertices of G according to σ, and let $B_0 = V(G) \setminus (B_1 \cup B_2)$. Let $(x_1, \dots, x_r) \in B_1^r$ and $(y_1, \dots, y_r) \in B_2^r$. For each $i \in [r]$, let*

$\ell_i \geqslant 5$. *Suppose finally that*

$$m_0 \geqslant m_1 + m_2 + \sum_{i\in[r]} \ell_i + 22r - 15. \tag{2.2}$$

Then, there are internally vertex-disjoint directed paths $P_1, \ldots, P_r$ in G such that, for each $i \in [r]$, P_i is a directed x_i, y_i-path with length ℓ_i and internal vertices in B_0.

Proof. Let B_1' be the first $(m_1 + 2r - 2)$ vertices of B_0 according to σ, and let B_2' be the last $(m_2 + 2r - 2)$ vertices of B_0 according to σ. Choose a set $X' = \{x_1', \ldots, x_r'\} \subseteq B_1'$ of distinct vertices such that $x_i' \in N^+(x_i)$ for each $i \in [r]$. This is possible as, if for $i \in [r]$ we have chosen $x_1', \ldots, x_{i-1}'$, letting $U_i = \{w \in B_1 : x_i <_\sigma w \leqslant_\sigma v_{m_1}\}$, then Lemma 2.1 (ii) gives

$$\begin{aligned}|N^+(x_i, B_1') \setminus \{x_1', \ldots, x_{i-1}'\}| &= |N^+(x_i, U_i \cup B_1') \setminus (U_i \cup \{x_1', \ldots, x_{i-1}'\})| \\ &\geqslant \frac{|U_i| + |B_1'|}{2} - |U_i| - |\{x_1', \ldots, x_{i-1}'\}| = \frac{|B_1'| - |U_i|}{2} - (i-1) \\ &\geqslant \frac{(m_1 + 2r - 2) - (m_1 - 1)}{2} - (r-1) > 0.\end{aligned}$$

Similarly, choose a set $Y' = \{y_1', \ldots, y_r'\} \subseteq B_2'$ of distinct vertices such that $y_i' \in N^-(y_i)$ for each $i \in [r]$.

Let A be a digraph formed by taking the disjoint union of directed paths Q_i, $i \in [r]$, where Q_i has length $\ell_i - 5$ for each $i \in [r]$. For $i \in [r]$, let b_i be the first vertex and c_i be the last vertex of Q_i. Note that A has $\sum_{i\in[r]}(\ell_i - 4)$ vertices.

Let $n_1 = m_0 - m_2 - 20r + 13$. We now give a procedure which produces a partial embedding ϕ of A into $G[\{v_{m_1+1}, \ldots, v_{m_1+n_1}\}]$. Throughout, if a vertex v_j of G is the image of a vertex of A, we say that it is *hit* and denote its pre-image by $a_j \in V(A)$. The sets W_j record vertices of G already used for the last two internal vertices of the paths $P_1, \ldots, P_r$ found by stage j.

- Initially, set $W_{m_1+1} = \emptyset$ and $\phi(b_i) = x_i'$ for each $i \in [r]$ (so that $x_1', \ldots, x_r'$ are hit).

- For $j = m_1 + 1$ to $m_1 + n_1$ in turn, do the following.

(a) If v_j is hit and $a_j = c_i$ for some $i \in [r]$, then, if possible, let $w_{i,1}, w_{i,2} \in \{v_{j+1}, \dots, v_{m_1+m_0}\} \setminus (W_j \cup Y')$ be such that $w_{i,1}$ and $w_{i,2}$ are not yet hit, and $v_j \to w_{i,1} \to w_{i,2} \to y'_i$ in G. Set $W_{j+1} = W_j \cup \{w_{i,1}, w_{i,2}\}$. If it is not possible to find such a $w_{i,1}$ and $w_{i,2}$, then simply set $W_{j+1} = W_j$.

(b) If v_j is hit and $a_j \notin \{c_1, \dots, c_r\}$, then extend ϕ if possible by assigning the first not-yet-hit out-neighbour of v_j in $\{v_{j+1}, \dots, v_{m_1+n_1}\} \setminus W_j$ to the out-neighbour of a_j in A. Set $W_{j+1} = W_j$.

(c) If $v_j \in W_j$, then set $W_{j+1} = W_j$.

(d) If $v_j \notin W_j$ and v_j is not hit, then say that v_j is *failed*. Set $W_{j+1} = W_j$.

Note that, for each $m_1+1 \leqslant j \leqslant m_1+n_1$, the vertices in W_j are never hit, so that this procedure is well-defined. We first show that the paths with length 3 in (a) are always found, as follows.

Claim 2.7. *For each $m_1+1 \leqslant j \leqslant m_1+n_1$, if v_j is hit and $a_j = c_i$ for some $i \in [r]$, then the procedure finds vertices $w_{i,1}$ and $w_{i,2}$ as described in (a).*

Proof of Claim 2.7. Suppose j satisfies $m_1+1 \leqslant j \leqslant m_1+n_1$, v_j is hit and $a_j = c_i$ for some $i \in [r]$, so that, at stage j, we carry out (a). Let s denote the number of times (a) was carried out before stage j. As W_j contains only vertices found in these previous instances of (a), we have $|W_j| \leqslant 2s$.

At stage j, each path Q_i has at most one vertex embedded by ϕ to $\{v_j, v_{j+1}, \dots, v_{m_1+n_1}\}$. Moreover, if a path Q_i has a vertex embedded by ϕ to $\{v_{j+1}, \dots, v_{m_1+n_1}\}$, then (a) has not been carried out for that c_i. Thus, at most $r-1-s$ vertices in $\{v_{j+1}, \dots, v_{m_1+n_1}\}$ have been hit. Let W' be the union of W_j, $Y' \setminus \{y'_i\}$, and the hit vertices in $\{v_{j+1}, \dots, v_{m_1+n_1}\}$. Thus, as $s \leqslant r-1$,

$$|W'| \leqslant 2s + (r-1) + (r-1-s) \leqslant 3(r-1). \tag{2.3}$$

Let j' be such that $v_{j'} = y'_i$, and note that, as $y'_i \in B'_2$, $j' \geqslant m_1 + m_0 - m_2 - 2r + 3$, so that, as $n_1 = m_0 - m_2 - 20r + 13$, we have

$$j' - j + 1 \geqslant m_1 + m_0 - m_2 - 2r + 4 - m_1 - n_1 = 18(r-1) + 9 \geqslant 6|W'| + 9. \qquad (2.4)$$

Therefore, by Lemma 2.4, vertices $w_{i,1}$ and $w_{i,2}$ exist in $\{v_j, v_{j+1}, \ldots, v_{j'}\} \setminus (W_j \cup Y')$ which have not yet been hit so that $v_j \to w_{i,1} \to w_{i,2} \to v_{j'} = y'_i$ in G. ⊡

If the procedure finds a full embedding of A into $G[\{v_{m_1+1}, v_{m_1+2}, \ldots, v_{m_1+n_1}\}]$, then observe that, for each $i \in [r]$, the image of Q_i and the path $\phi(c_i) \to w_{i,1} \to w_{i,2} \to y'_i$ together give a path, P'_i say, with length $\ell_i - 2$ which is directed from $\phi(b_i) = x'_i$ to y'_i. Furthermore, the paths P'_i, $i \in [r]$, are vertex-disjoint with vertices in B_0. Taking P_i to be the path $x_i P'_i y_i$ for each $i \in [r]$ gives the desired result.

All that remains to show is that the procedure produces a full embedding ϕ of A. Let $W = W_{m_1+n_1+1}$ and note that $|W| \leqslant 2r$. Assume for a contradiction that the procedure does not yield an embedding of A into G. Then the set, F say, of failed vertices in $\{v_{m_1+1}, \ldots, v_{m_1+n_1}\}$ has $|F| > n_1 - |A| - |W|$. Let $U \subseteq V(A)$ be the set of embedded vertices at the end of the procedure. Let L be the set of vertices of A which are the last embedded vertex on some path Q_i. Note we have $|L| = r$.

Say a vertex $a \in V(A)$ is *active at stage* j if $\phi(a) \in \{v_{m_1+1}, \ldots, v_{j-1}\}$ and a has an out-neighbour b that is not embedded in $\{v_{m_1+2}, \ldots, v_j\}$ (i.e., either b is not embedded or $\phi(b) \in \{v_{j+1}, \ldots, v_{m_1+n_1}\}$). Now, if $v_j \in F$ comes before some vertex in $X' = \{x'_1, \ldots, x'_r\} \subseteq B'_1$, then it is possible there will be no active vertex at stage j. However, because we have assumed that the procedure does not yield an embedding of A into G, if $v_j \in F$ and $j \geqslant 2m_1 + 2r - 1$, then there must be some active vertex at stage j, for otherwise all the vertices of A would be embedded in $\{v_{m_1+1}, \ldots, v_{j-1}\}$.

Let $\bar{F} = \{v_j \in F : j \geqslant 2m_1 + 2r - 1\}$, so that, for each $v_j \in \bar{F}$ we can define r_j to be the largest index such that a_{r_j} is active for j. Note, by the definition of an active vertex, $r_j < j$. Furthermore, as $|F| > n_1 - |A| - |W|$, $B'_1 = \{v_{m_1+1}, \ldots, v_{2m_1+2r-2}\}$ contains at least r vertices in the embedding (those in X'), and $|A| = \sum_{i\in[r]}(\ell_i - 4)$, we have

$$|\bar{F}| > n_1 - |A| - |W| - (m_1 + 2r - 2 - r) \geqslant m_0 - m_2 - 20r + 13 - \sum_{i\in[r]} \ell_i + r - m_1 + 2 \overset{(2.2)}{\geqslant} 3r. \quad (2.5)$$

For each $v_j \in \bar{F}$, set $I_j = \{v_i : r_j < i \leqslant j\}$. We now bound from above the number of vertices of $\bar{F}$ in I_j, as follows.

Claim 2.8. *If $v_j \in \bar{F}$, then $|I_j \cap F| \leqslant |I_j \cap \phi(L)| + |I_j \cap W|$.*

Proof of Claim 2.8. Let $J = (I_j \cap N^+(v_{r_j})) \setminus W$. As the out-neighbour of a_{r_j} is never embedded in I_j, all the vertices in J must be hit by the start of stage r_j. Thus, as $F \cap W = \emptyset$, we have $I_j \cap F \subseteq I_j \cap N^-(v_{r_j})$, so that

$$|I_j \cap F| \leqslant |I_j \cap N^-(v_{r_j})|. \quad (2.6)$$

Now, let A_{r_j} and A_{j-1} be the sub-digraphs of $G[v_{m_1+1}, \dots, v_j]$ which are the image of the partial embedding ϕ at the end of stage r_j and stage $j-1$, respectively, restricted to the vertex set $\{v_{m_1+1}, \dots, v_j\}$. Observe the following.

- Each vertex of J is the last vertex of a path of A_{r_j}, as it is hit by the end of stage r_j and occurs later in σ than r_j.
- Any vertex in I_j which is the last vertex of some path of A_{j-1} must be the image of some c_i, for otherwise it is active for j, contradicting the definition of r_j. Thus, because L is the set of vertices of A which are the last embedded vertex on some path Q_i, such a vertex is in $I_j \cap \phi(L)$.
- As $r_j \leqslant j-1$, $A_{r_j} \subseteq A_{j-1}$, and $V(A_{j-1}) \setminus V(A_{r_j}) \subseteq I_j$, so A_{j-1} must have at least as many paths terminating in I_j as A_{r_j} does.

Combining these three observations we have $|J| \leqslant |I_j \cap \phi(L)|$, and hence

$$|I_j \cap N^+(v_{r_j})| \leqslant |I_j \cap \phi(L)| + |I_j \cap W|. \quad (2.7)$$

Now, by Lemma 2.1 (ii), $|I_j \cap N^-(v_{r_j})| \leqslant |I_j \cap N^+(v_{r_j})|$. Together with (2.6) and (2.7), this proves the claim. $\boxdot$

Let M be the set of indices j such that $v_j \in \bar{F}$, and I_j is maximal for inclusion among the sets I_i, with $v_i \in \bar{F}$. We will show that the sets I_j, $j \in M$ are disjoint. If $i, j \in M$ with $i < j$ and $I_i \cap I_j \neq \emptyset$, then we have $r_j < i$. Observe that, as a_{r_j} is active for j and $\phi(a_{r_j}) \in \{v_0, \ldots, v_{i-1}\}$, a_{r_j} is also active for i, and hence $r_i \geqslant r_j$. Thus, $I_i \subseteq I_j$ and, as $i < j$, $I_i \neq I_j$, and hence I_i is not maximal for inclusion among the sets $I_{i'}$, with $v_{i'} \in \bar{F}$, a contradiction.

Since $v_j \in I_j$ for all $v_j \in \bar{F}$, we have $\bar{F} \subseteq \cup_{j\in M} I_j$. As the sets I_j, $j \in M$, are pairwise disjoint, $|\bar{F}| \leqslant \sum_{j\in M} |I_j \cap F|$. By Claim 2.8, we therefore obtain

$$|\bar{F}| \leqslant \sum_{j\in M} |I_j \cap F| \leqslant \sum_{j\in M} (|I_j \cap \phi(L)| + |I_j \cap W|) \leqslant |\phi(L)| + |W| \leqslant 3r,$$

contradicting (2.5). This completes the proof of the lemma. $\square$

2.2.3 Proof of Theorem 2.5

Given Lemma 2.6 it is now straight-forward to prove Theorem 2.5. Given an n-vertex oriented tree T with k leaves whose maximal bare paths are directed, we label such paths with length at least 5 as $P_1, \ldots, P_r$, for the appropriate r (which, by Proposition 1.15, satisfies $r = O(k)$). We can then consider T to be formed of small vertex-disjoint subtrees $T_1, \ldots, T_{r+1}$ connected by the paths $P_1, \ldots, P_r$. We use Proposition 1.16 to group these subtrees into classes, with the classes ordered so that each path P_i goes from some class to the next class. Given then a tournament G with $n + 50k$ vertices, we divide a median order into intervals, with one interval for each class of subtrees and one for the set of paths between each pair of consecutive classes (see (2.9)). Then, we then use Corollary 1.11 to embed the subtrees T_i into their interval in the median order before using Lemma 2.6 to embed the paths P_i with interior vertices in their interval in the median order.

Proof of Theorem 2.5. We will prove this with $C = 50$, so let $\bar{n} = n + 50k$. Let T be an n-vertex oriented tree with k leaves in which every bare path is a directed path, and let G be a $\bar{n}$-vertex tournament. Let B be the set of vertices of T which do not have degree 2, so that, by Proposition 1.15, $|B| \leqslant 2k - 2$. Remove all maximal bare paths of length at least 5 from T. Let r be the number of removed paths, noting that, by Proposition 1.15, $r \leqslant 2k - 3$, and label these paths as $P_1, \dots, P_r$ (where we recall $\ell(P_i)$ denotes the length of P_i). Say the remaining forest F has component trees $T_1, \dots, T_{r+1}$, and, for each $i \in [r + 1]$, let k_i be the number of leaves of T_i if $|T_i| \geqslant 2$, and let $k_i = 1$ if $|T_i| = 1$. Note that F is a union of $(|B| - 1 - r)$ maximal bare paths of T with length at most 4 between vertices in B, resulting in a forest with $r + 1$ components. Thus, we have that $|F| \leqslant |B| + 3(|B| - 1 - r) \leqslant 8k - 3r - 11$. Observing that every leaf or isolated vertex of F is in B, we have $\sum_{i \in [r+1]} k_i \leqslant |B| \leqslant 2k - 2$. We also note that

$$|F| = \sum_{i \in [r+1]} |T_i| \quad \text{and} \quad \sum_{i \in [r]} \ell(P_i) = |T| - |F| + r = n - \sum_{i \in [r+1]} |T_i| + r. \tag{2.8}$$

Let S be the oriented tree on vertex set $[r + 1]$ with $ij \in E(S)$ whenever there is a directed path from T_i to T_j in T. By applying Proposition 1.16 to S, let $s \leqslant r + 1$ be such that there is a partition $I_1, \dots, I_s$ of $[r + 1]$ into non-empty sets such that, for each distinct $i, j \in [r+1]$, and $i' \in [s]$, if $i \in I_{i'}$ and there is a directed path from T_i to T_j in T, then $i' < s$ and $j \in I_{i'+1}$. For each $i \in [s - 1]$, let J_i be the set of indices $j \in [r]$ such that P_j is directed from $T_{i'}$ to $T_{j'}$ for some $i' \in I_i$ and $j' \in I_{i+1}$, and note that $\cup_{i \in [s-1]} J_i = [r]$.

Let $\sigma = v_1, \dots, v_{\bar{n}}$ be a median order of G. In this median order take consecutive intervals

$$V_1, U_1, V_2, U_2, V_3, \dots, V_{s-1}, U_{s-1}, V_s, \tag{2.9}$$

appearing in that order, such that, for each $j \in [s]$,

$$|V_j| = \left\lceil \frac{3}{2} \sum_{i \in I_j} (|T_i| + k_i) \right\rceil - 2|I_j| \leqslant \frac{3}{2} \sum_{i \in I_j} (|T_i| + k_i) + \frac{1}{2} - 2|I_j|, \tag{2.10}$$

and, for each $j \in [s-1]$,

$$|U_j| = |V_j| + |V_{j+1}| + \sum_{i \in J_j} \ell(P_i) + 22|J_j| - 15. \tag{2.11}$$

Note that this is possible, as

$$\begin{aligned}
\sum_{j=1}^{s} |V_j| + \sum_{j=1}^{s-1} |U_j| &\overset{(2.11)}{\leqslant} 3\sum_{j=1}^{s} |V_j| + \sum_{j \in [r]} \ell(P_j) + 22 \sum_{j \in [s-1]} |J_j| - 15(s-1) \\
&\overset{(2.10)}{\leqslant} \frac{9}{2} \sum_{i \in [r+1]} (|T_i| + k_i) + \frac{3}{2}s - 6\sum_{j=1}^{s} |I_j| + \sum_{j \in [r]} \ell(P_j) + 22r - 15(s-1) \\
&\overset{(2.8)}{\leqslant} n + r + \frac{7}{2}|F| + \frac{9}{2} \sum_{i \in [r+1]} k_i - 6(r+1) + 22r \\
&\leqslant n + \frac{7}{2}(8k - 3r - 11) + \frac{9}{2}(2k-2) + 17r - 6 \\
&\leqslant n + 37k + \frac{13}{2}r \leqslant n + 50k,
\end{aligned}$$

where we have used that $r \leqslant 2k - 3$. By Corollary 1.11 and (2.10), a copy of $\cup_{i \in I_j} T_i$ exists in $G[V_j]$ for each $j \in [s]$. By Lemma 2.6 and (2.11), for each $j \in [s-1]$, the $|J_j|$ paths P_i, $i \in J_j$, between $\cup_{i \in I_j} T_i$ and $\cup_{i \in I_{j+1}} T_i$ can then be embedded in the intervals V_j, U_j, V_{j+1} with the appropriate first and last vertex in V_j and V_{j+1}, respectively, and internal vertices in U_j. This completes the embedding of T, and hence the proof of the theorem. □

2.3 Proof of Theorem 1.3

We now turn to Theorem 1.3, where we aim to embed to embed an n-vertex oriented tree T with $k \ll n$ leaves in an arbitrary $(n + k - 2)$-vertex tournament. This is not too difficult if T contains a long path P which has first and last block of length 1 and whose endvertices have degree 2 in T. Indeed, similar to the previous reduction in Section 2.2.1, each component of $T - P$ can be embedded separately using Theorem 1.2 (with duplicated vertices for the endpoints of P), and then an appropriate result from Section 1.3 can be used to extend the embedding to a full copy of T. Therefore, the most difficult cases for consideration will be when the bare paths of T have few changes of direction.

Thus, as an illustrative case, let us first sketch Theorem 1.3 for trees consisting of a directed path between two arborescences, as follows. Suppose we have a directed path P, an in-arborescence S with root the first vertex of P, and an out-arborescence S' with root the last vertex of P, and suppose that $S \cup P \cup S'$ is an oriented tree with n vertices. Say S has k in-leaves and S' has k' out-leaves, and the tournament G has $m := n+k+k'-2$ vertices and a median order $v_1, \ldots, v_m$. Using Lemma 2.1 (i) and Theorem 2.3 (via directional duality), we can embed S into $G[\{v_1, \ldots, v_{|S|+k-1}\}]$ with the root vertex embedded to $v_{|S|+k-1}$. Similarly, we can embed S' into $G[\{v_{m-|S'|-k'+2}, \ldots, v_m\}]$ with the root vertex of S' embedded to $v_{m-|S'|-k'+2}$. Finally, by Lemma 2.1 (ii), we have $v_{|S|+k-1} \to v_{|S|+k} \to \ldots \to v_{m-|S'|-k'+2}$, so we can use this path to embed the $n - |S| - |S'| + 2 = m - |S| - |S'| - k - k' + 4$ vertices of P and complete an embedding of T into G.

Essentially, all our embeddings will look like this, where P will be a very long path, but with some additional subtrees and paths found within the interval we use to embed P. For example, suppose now the tree T also has a subtree F which shares one vertex, t say, with S, where t only has out-neighbours in F. If P is a long path (compared to $|F|, |S|, |S'|$) then we can embed $T = F \cup S \cup P \cup S'$ into a tournament G with $m := |T| + k + k' - 2$ vertices as follows. Carry out the above embedding of S and S' into the start and end respectively of a median order $v_1, \ldots, v_m$ of G and note that the path $Q := v_{|S|+k-1} \to v_{|S|+k} \to \ldots \to v_{m-|S'|-k'+2}$ has $|F| - 1 + |P|$ vertices. If s is the embedding of $t \in V(S)$, then by Lemma 2.1 (ii) and as $|Q| \geqslant |P| - 1 \gg |F|, |S|$, s will have many out-neighbours in this path, enough that we can easily embed $F - t$ among the out-neighbours of s in Q (using, in particular, Corollary 1.11). However, we wish to do this so that there is a directed path between $v_{|S|+k-1}$ and $v_{m-|S'|-k'+2}$ covering exactly the $|Q| - (|F| - 1) = |P|$ vertices of $V(Q)$ which are not used to embed $F - t$.

To do this, before embedding F, we first find a short directed $v_{|S|+k-1}, v_{m-|S'|-k'+2}$-path R with vertices in $V(Q)$ so that every vertex in $V(Q)$ has at least one out-neighbour on R occurring after some in-neighbour on R. Obtaining this property is straightforward, by ensuring R covers a randomly selected subset of $V(Q)$. The path R will be short enough that we can embed $F - t$ in the out-neighbours of s in $V(Q)$ while avoiding $V(R)$. Once $F - t$ has been embedded, we slot the remaining vertices in $V(Q)$ into R one by one. This will be easy as, even after some vertices have been slotted into the R, every vertex v in $V(Q)$ will still have an out-neighbour on R occurring after some in-neighbour on R, allowing us to find consecutive vertices u_1, u_2 on R such that $u_1 \to v \to u_2$, thus enabling the insertion of v also (see Claim 2.13). Note that, in the language of absorption (as codified by Rödl, Ruciński and Szemerédi [27]), R is a path which can absorb any set of vertices from the interval of the median order between its first and last vertex.

More generally, we can embed small trees attached with an out-edge from $S \cup P \cup S'$, as long as the attachment point is not too late in P, and also not in S', by embedding such small trees within the interval for the path P. Similarly, we can embed small trees attached with an in-edge from $S \cup P \cup S'$, as long as the attachment point is not too early in P, and also not in S. We can also use Lemma 2.4 to add short paths between vertices in the interval from P that are not too close together. We therefore decompose any n-vertex tree T with k leaves by finding a digraph D which can be built in this way and which contains T.

Roughly speaking, we call the digraph D a *good decomposition for* T if it contains T and can be built from some $S \cup P \cup S'$ as described above by adding digraphs in these ways; this is defined precisely in Section 2.3.1. In Section 2.3.2, we show that there is a good decomposition for any tree without a subpath that we could otherwise deal with using Section 1.3 as before. Then, in Section 2.3.3, we show it is possible to embed any good decomposition of any n-vertex tree with k leaves into an $(n + k - 2)$-vertex tournament. In fact, only leaves inside S and S' will contribute to the number of vertices needed to embed a good decomposition, implying fewer than $n + k - 2$ vertices are needed in many cases. Finally, in Section 2.3.4, we put this together to prove Theorem 1.3.

2.3.1 (r,m)-good decompositions

We now define a good decomposition precisely, using the following definition of a path partition.

Definition 2.9. *Say a sequence of paths $P_1 \dots P_\ell$ is a* path partition *of a path P if $P = \cup_{i\in[\ell]} P_i$ and, for each $i \in [\ell-1]$, the end vertex of P_i is the start vertex of P_{i+1}, and all the paths are otherwise pairwise vertex disjoint.*

Roughly speaking, as depicted in Figure 2.1, an (r,m)-good decomposition for a tree T is a digraph D with $T \subseteq D$, such that D can be constructed by taking a long directed path P from the root of an in-arborescence S_1 to the root of an out-arborescence S_{r+1}, attaching small forests F_i to a limited number of well-separated subpaths S_i of P, and, finally, attaching short directed paths Q_i between some of these well-separated subpaths and forests. More precisely, we define an (r,m)-good decomposition as follows.

Definition 2.10. *Say that a digraph D is an* (r,m)-good decomposition *for an n-vertex oriented tree T if $V(D) = V(T)$, and, for some distinct $x, y \in V(D)$, there is a directed x,y-path P with path partition*

$$P = P_1 S_2 P_2 S_3 \dots P_{r-1} S_r P_r, \tag{2.12}$$

an in-arborescence S_1 with root x, an out-arborescence S_{r+1} with root y, and

- *forests F_i^+, F_i^-, $i \in [r+1]$, and*
- *for some $0 \leqslant \ell \leqslant 2r$, vertices s_i, t_i and directed s_i,t_i-paths Q_i, $i \in [\ell]$,*

such that, letting $F_i = F_i^- \cup F_i^+$ for each $i \in [r+1]$, the following hold.

A1 *$T \subseteq S_1 \cup P \cup S_{r+1} \cup (\cup_{i\in[r+1]} F_i) \cup (\cup_{i\in[\ell]} Q_i) = D$.*

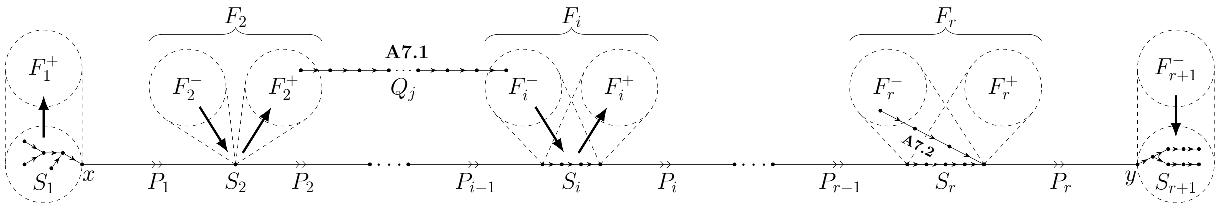

Figure 2.1: An (r, m)-good decomposition.

A2 *The following sets, over $i \in [r+1]$ and $j \in [\ell]$, form a partition of $V(T) = V(D)$:*

$$V(P),\ V(F_i^+) \setminus V(S_i),\ V(F_i^-) \setminus V(S_i),\ V(S_1) \setminus \{x\},\ V(S_{r+1}) \setminus \{y\},\ V(Q_j) \setminus \{s_j, t_j\}.$$

A3 *For each $i \in [r]$, P_i has length at least $2000m$.*

A4 *For each $i \in [r+1]$ and $\diamond \in \{+, -\}$, $V(S_i) \subseteq V(F_i^\diamond)$, $|F_i^\diamond| \leqslant m$, and $F_i^\diamond$ is a forest in which each component has exactly one vertex in S_i, which furthermore has only $\diamond$-neighbours in $F_i^\diamond$.*

A5 *$E(F_1^-) = E(F_{r+1}^+) = \emptyset$ and $|S_1|, |S_{r+1}| \geqslant 2$.*

A6 *The total number of in-leaves of S_1 and out-leaves of S_{r+1} is at most the number of leaves of T.*

A7 *For each $i \in [\ell]$, one of the following holds.*

A7.1 *For some $1 \leqslant j < j' \leqslant r+1$, Q_i is a directed path from F_j to $F_{j'}$ with length $3(j'-j)+1$.*

A7.2 *For some $2 \leqslant j \leqslant r$, Q_i is a directed path with length 3 from $V(F_j^-) \setminus V(S_j)$ to the last vertex of S_j.*

It should be noted that, in our proof, D will be almost identical to T, with only a few possible additional edges to ensure P is indeed a directed path from S_1 to S_2. Strictly speaking, these edges are included only for the sake of illustration, as the presence of P helps indicate the link between the embeddings in the proof and the embeddings in the examples sketched earlier. For the embeddings in the proof, all extra edges will only ever be embedded to consecutive vertices in a median order, and so they present no additional difficulty.

2.3.2 Finding a good decomposition

As noted previously, the most difficult cases for Theorem 1.3 occur when an oriented tree mostly consists of directed bare paths. We will handle such cases by embedding a corresponding good decomposition. To find a good decomposition, we first arrange these directed bare paths in order of decreasing length. Identifying a point where the length of these paths drops significantly (perhaps including all the paths), we show that removing these long paths creates a forest in which each component is much smaller than each of the removed paths. Next, we order these paths and components using Proposition 1.17. Taking (essentially) the removed paths as the paths P_i, carefully chosen directed subpaths S_i of the components of the forest (see **B1**–**B4** later) and some dummy edges if necessary will form the path in (2.12). After the careful selection in **B1**–**B4**, we will be able to divide naturally the rest of the tree into the other sets in the decomposition.

Lemma 2.11. *Let $1/n \ll \mu \ll 1/k$. Let T be an n-vertex oriented tree with $k \geqslant 2$ leaves. Suppose T contains no bare path of length at least μn which has first and last block of length 1 and whose endvertices have degree 2 in T. Then, for some $r \leqslant 10k$ and $m \geqslant \mu n$, T has an (r, m)-good decomposition.*

Proof. We will construct an (r, m)-good decomposition using the notation in Definition 2.10, and confirm that each of **A1**–**A7** hold.

Let p be the number of maximal bare paths of T, and let them be $T'_1, \dots, T'_p$. By Proposition 1.15, we have $p \leqslant 2k - 3$. Observe that each T'_i has fewer than μn edges that are not contained in the first two blocks or the last two blocks, for otherwise, taking the last edge of the second block, and the first edge of the penultimate block, and all the edges between them on T'_i, gives a bare path with length at least μn with first and last block of length 1 whose endvertices have degree 2 in T. Let q be the number of maximal directed bare paths of T with length at least μn, and let them be $T_1, \dots, T_q$ with length $\ell_1, \dots, \ell_q$ respectively, so that $\ell_1 \geqslant \ell_2 \geqslant \dots \geqslant \ell_q$. By the above observation, we find $q \leqslant 4p \leqslant 8k - 12$, and $|T - T_1 - \dots - T_q| \leqslant (2k-3)(4\mu n + \mu n) \leqslant 10k\mu n$. Furthermore, as $\mu \ll 1/k$, we must have that $q \geqslant 1$ and $\ell_1 \geqslant n/2q \geqslant n/20k$.

Now, let $r \in [q-1]$ be the smallest integer such that $\ell_r > 10^6 k \ell_{r+1}$, if it exists, and $r = q$ otherwise. Let $m = \ell_r/2500$. Note that, as $\ell_1 \geqslant n/20k$ and $\mu \ll 1/k$,

$$m \geqslant \frac{\ell_1}{2500 \cdot (10^6 k)^{r-1}} \geqslant \frac{n/20k}{2500 \cdot (10^6 k)^{8k-12}} \geqslant \mu n. \tag{2.13}$$

Note that $r \leqslant q \leqslant 10k$ and $m \geqslant \mu n$, as required. As $T - T_1 - \dots - T_r$ is the union of $T - T_1 - \dots - T_q$ and at most $8k - 12$ paths of length at most $\ell_r/10^6 k$, we have $|T - T_1 - \dots - T_r| \leqslant 10k\mu n + m/4 \leqslant m/2$. Note that $T - T_1 - \dots - T_r$ has $r+1$ components. Say these are $R_1, \dots, R_{r+1}$, and note that $|R_i| \leqslant |T - T_1 - \dots - T_r| \leqslant m/2$ for each $i \in [r+1]$.

Using Proposition 1.17, relabel the components $\{R_1, \dots, R_{r+1}\}$ and paths $\{T_1, \dots, T_r\}$, and define functions $i^-, i^+ : [r] \to [r+1]$, so that, for every $j \in [r]$, T_j is a directed path from $R_{i^-(j)}$ to $R_{i^+(j)}$, and $i^-(j) \leqslant j < i^+(j)$.

For each $j \in [r]$, label vertices so that T_j is an x'_j, y'_j-path directed from $x'_j \in V(R_{i^-(j)})$ to $y'_j \in V(R_{i^+(j)})$. Let $I \subseteq \{2, \dots, r\}$ be the set of i with $y'_{i-1} \in V(R_i)$, $x'_i \in V(R_i)$, and such that the path between y'_{i-1} and x'_i in R_i is not directed from y'_{i-1} to x'_i. For each $j \in [r]$, let Q_j^+ be the path consisting of the last $3(i^+(j) - j - 1) + 1 \geqslant 1$ edges of T_j. For each $j \in [r] \setminus I$, let Q_j^- be the path consisting of the first $3(j - i^-(j)) + 1 \geqslant 1$ edges of T_j. For each $j \in I$, let Q_j^- be the path consisting of the first 3 edges of T_j. Note that the lengths of the paths Q_j^+, Q_j^- are always much smaller than the length of the path T_j.

For each $i \in [r]$, let P_i be such that $T_i = Q_i^- P_i Q_i^+$ is a path partition. Label vertices so that P_i is an x_i, y_i-path directed from x_i to y_i. Note that each path P_i is T_i with up to $3r + 1$ edges removed from each end. As the original length of such a path was at least

$\ell_r = 2500m$, and we have $1/n \ll \mu \ll 1/r$, we have by (2.13) that **A3** holds.

Let $x = x_1$ and note that $Q_1^- = x_1'x$. Let $S_1 \subseteq R_1 + x_1'x$ be the maximal in-arborescence in $R_1 + x_1'x$ with root x. Note we have that $|S_1| \geqslant 2$. Let $y = y_r$ and note that $Q_r^+ = yy_r'$. Let S_{r+1} be the maximal out-arborescence in $R_{r+1} + yy_r'$ with root y. Note we have $|S_{r+1}| \geqslant 2$.

If k_0 is the number of in-leaves of S_1, then as its root x is an out-leaf, S_1 has $k_0 + 1$ leaves. Similarly, if k_1 is the number of out-leaves of S_{r+1}, then S_{r+1} has $k_1 + 1$ leaves. Now, take the path, S say, between S_1 and S_{r+1} in T and note that the tree $S_1 \cup S \cup S_{r+1}$ has $(k_0 + 1) + (k_1 + 1) - 2 = k_0 + k_1$ leaves. Noting that T has at least as many leaves as $S_1 \cup S \cup S_{r+1} \subseteq T$ completes the proof that **A6** holds.

Now, for each $i \in \{1, r+1\}$ and each $\diamond \in \{+, -\}$, let $F_i^\diamond \subseteq S_i \cup R_i$ be the digraph formed from the union of the paths in $(S_i \cup R_i) - E(S_i)$ from $V(S_i)$ which start with a $\diamond$-edge, and let $F_i = F_i^+ \cup F_i^- = (S_i \cup R_i) - E(S_i)$. Note that, by the maximality of S_1 as an in-arborescence and the maximality of S_{r+1} as an out-arborescence, we have that $E(F_1^-) = E(F_{r+1}^+) = \emptyset$, completing the proof that **A5** holds. For each $i \in \{1, r+1\}$, $|F_i| \leqslant |R_i| + 1 \leqslant m/2 + 1 \leqslant m$, so **A4** holds as well for $i \in \{1, r+1\}$.

We now construct y_{i-1}, x_i-paths S_i, for each $2 \leqslant i \leqslant r$. For each such i, we consider $Q_{i-1}^+ \cup R_i \cup Q_i^-$, and add up to two edges (according to the cases below) before finding a directed path S_i through the resulting digraph. We next divide into cases **B1**–**B4** according to whether $y_{i-1}' \in V(R_i)$ (i.e., if $i^+(i-1) = i$) and whether $x_i' \in V(R_i)$ (i.e., if $i^-(i) = i$). These cases are depicted in Figure 2.2. Note that, if $y_{i-1}' \in V(R_i)$ then Q_{i-1}^+ consists of only the edge $y_{i-1}y_{i-1}'$, and if $x_i' \in V(R_i)$ with $i \notin I$, then Q_i^- consists of only the edge $x_i'x_i$. Precisely, for each $2 \leqslant i \leqslant r$, we do the following.

B1 If y_{i-1}' and x_i' are both in $V(R_i)$, then do the following.

- **B1.1** If the y_{i-1}', x_i'-path in the tree R_i is a directed path from y_{i-1}' to x_i', then let S_i be the directed path from y_{i-1} to x_i in $R_i + y_{i-1}y_{i-1}' + x_i'x_i$.
- **B1.2** If the y_{i-1}', x_i'-path in the tree R_i is not a directed path from y_{i-1}' to x_i' (i.e., if

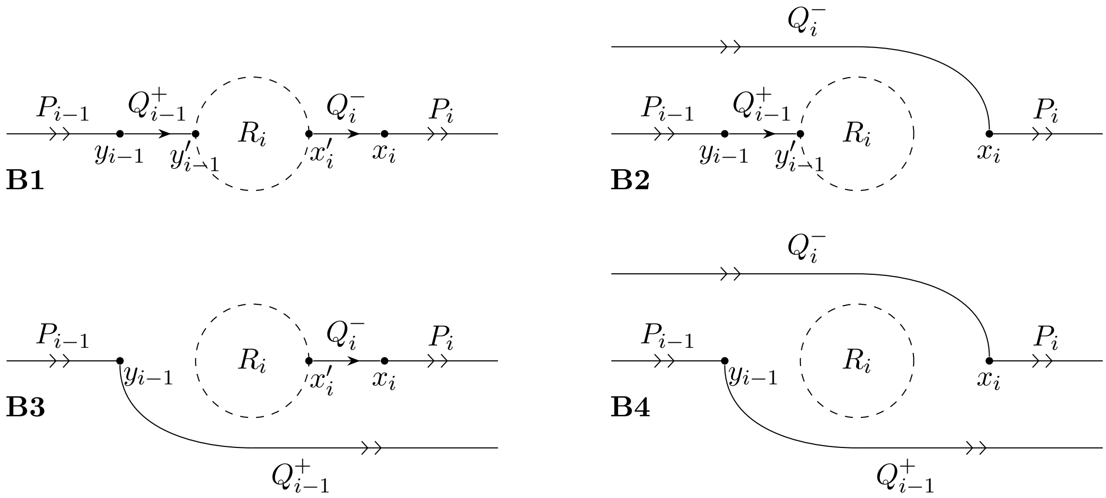

Figure 2.2: Cases **B1-B4**.

$i \in I$), then let S'_i be the maximal directed subpath from y'_{i-1} that it contains. Let S_i be the path consisting of the edge $y_{i-1}y'_{i-1}$, followed by S'_i, followed by a new edge from the endvertex of S'_i to x_i.

B2 If $y'_{i-1} \in V(R_i)$ and $x'_i \notin V(R_i)$, then let S_i be the path $y_{i-1}y'_{i-1}x_i$.

B3 If $y'_{i-1} \notin V(R_i)$ and $x'_i \in V(R_i)$, then let S_i be the path $y_{i-1}x'_ix_i$.

B4 If $y'_{i-1}, x'_i \notin V(R_i)$, then let $z \in V(R_i)$ be arbitrary, and let S_i be the path $y_{i-1}zx_i$.

Now, for each $2 \leqslant i \leqslant r$, we choose F^+_i, F^-_i and $F_i = F^+_i \cup F^-_i$. To do so, for each $2 \leqslant i \leqslant r$ and each $\diamond \in \{+, -\}$, let $F^\diamond_i \subseteq S_i \cup R_i$ be the digraph formed from the union of the paths in $(S_i \cup R_i) - E(S_i)$ from $V(S_i)$ which start with a $\diamond$-edge, and let $F_i = F^+_i \cup F^-_i = (S_i \cup R_i) - E(S_i)$. Note that F^+_i and F^-_i could consist of a single vertex. For each $2 \leqslant i \leqslant r$, $|F_i| = |R_i| + 2 \leqslant m/2 + 2 \leqslant m$. We now have that **A4** holds for each $i \in [r + 1]$, as required.

Let ℓ be the number of paths $Q^\diamond_i$, $i \in [r]$, $\diamond \in \{+, -\}$ with length greater than 1, so that $0 \leqslant \ell \leqslant 2r$. Relabel these paths arbitrarily as $Q_i, i \in [\ell]$. Note that, as we created no new vertices, we have that $V(D) \subseteq V(T)$ (with equality once we confirm $T \subseteq D$ below). It is left then to prove that **A1**, **A2**, and **A7** hold and check the properties at the start of Definition 2.10.

Note that, for each $2 \leqslant i \leqslant r$, S_i was a directed y_{i-1}, x_i-path. Therefore, as $x = x_1$ and $y = y_r$,

$$P := P_1 S_2 P_2 S_2 \dots P_{r-1} S_r P_r \tag{2.14}$$

is a path partition of the directed x, y-path P. Furthermore, we have that S_1 is an in-arborescence with root x and that S_{r+1} is an out-arborescence with root y.

Now, by construction, $T \subseteq P \cup S_1 \cup S_{r+1} \cup (\cup_{i\in[r+1]} F_i) \cup (\cup_{i\in[r],\diamond\in\{+,-\}} Q_i^\diamond) = D$. Whenever Q_i^+ has length 1 and $i < r$, we have that $i^+(i) = i+1$, so S_{i+1} is chosen in **B1.1**, **B1.2**, or **B2**, and hence $Q_i^+ = y_{i-1}y'_{i-1} \subseteq S_{i+1}$. Note that Q_r^+ has length 1, and $Q_r^+ = yy'_r$ is in S_{r+1}. Whenever Q_i^- has length 1 and $i > 1$, we must have that $i \notin I$ and $i^-(i) = i$, and therefore S_i is chosen in **B1.1** or **B3**, so that $Q_i^- = x'_i x_i \subseteq S_i$. Note that Q_1^- has length 1, and $Q_1^- = x'_1 x$ is in S_1. Therefore, $P \cup (\cup_{i\in[r],\diamond\in\{+,-\}} Q_i^\diamond) = P \cup (\cup_{i\in[\ell]} Q_i) + x'_1 x + yy'_r$, and so $T \subseteq P \cup S_1 \cup S_{r+1} \cup (\cup_{i\in[r+1]} F_i) \cup (\cup_{i\in[\ell]} Q_i) = D$ and **A1** holds.

Furthermore, note that $V(R_i)$, $i \in [r+1]$, and $V(T_i) \setminus \{x'_i, y'_i\}$, $i \in [r]$, form a partition of $V(T)$. For each $i \in [r]$, $V(Q_i^-) \setminus \{x_i, x'_i\}$, $V(P_i)$ and $V(Q_i^+) \setminus \{y_i, y'_i\}$ form a partition of $V(T_i) \setminus \{x'_i, y'_i\}$. For each $2 \leqslant i \leqslant r$, by the choice of F_i^+ and F_i^-, $V(F_i^+) \setminus V(S_i)$, $V(F_i^-) \setminus V(S_i)$ and $V(S_i) \setminus \{y_{i-1}, x_i\}$ form a partition of R_i, while $V(F_1^-) \setminus V(S_1) = \emptyset$, $V(F_1^+) \setminus V(S_1)$ and $V(S_1) \setminus \{x_1\}$ partition $V(R_1) \setminus \{x_1\}$, and $V(F_{r+1}^-) \setminus V(S_{r+1})$, $V(F_{r+1}^+) \setminus V(S_{r+1}) = \emptyset$ and $V(S_{r+1}) \setminus \{y_r\}$ partition $V(R_{r+1}) \setminus \{y_r\}$. As $V(P) = (\cup_{i\in[r]} V(P_i)) \cup (\cup_{2\leqslant i\leqslant r}(V(S_i) \setminus \{y_{i-1}, x_i\}))$, the sets listed in **A2** form a partition of $V(T)$.

Therefore, we need only show that, for each path $i \in [\ell]$, either **A7.1** or **A7.2** holds. If $Q_i = Q_j^+$ for some $j \in [r]$, then Q_i is a directed y_j, y'_j-path of length $3(i^+(j) - (j+1)) + 1 > 1$, so that $i^+(j) > j + 1$. As $y_j \in V(S_{j+1}) \subseteq V(F_{j+1})$ and $y'_j \in V(R_{i^+(j)}) \subseteq V(F_{i^+(j)})$, **A7.1** holds for Q_i. If $Q_i = Q_j^-$ for some $j \in [r] \setminus I$, then Q_i is a directed x'_j, x_j-path of length $3(j - i^-(j)) + 1 > 1$, so that $i^-(j) < j$. As $x'_j \in V(R_{i^-(j)}) \subseteq V(F_{i^-(j)})$, and $x_j \in V(S_j) \subseteq V(F_j)$, **A7.1** holds for Q_i. Finally, if $Q_i = Q_j^-$ for some $j \in I$, then S_j was chosen in **B1.2**. From the choice of the relevant maximal directed path S'_j, the first vertex x'_j of Q_i is in $V(F_j^-) \setminus V(S_j)$ and the last vertex x_j of Q_i is also the last vertex of S_j, and therefore **A7.2** holds. □

2.3.3 Embedding a good decomposition

We now show that it is possible to embed an (r, m)-good decomposition D of a n-vertex tree T with k leaves into an $(n+k-2)$-vertex tournament G, when $1/n \ll 1/r, 1/k, m/n$. For our sketch we will use the notation of Definition 2.10. We take a median order of G and find within it consecutive disjoint intervals $V_1, U_1, V_2, U_2, \ldots, V_r, U_r, V_{r+1}$ with carefully chosen sizes. We will embed S_1 into $G[V_1]$ while embedding its root to the last vertex of V_1 under σ, using Theorem 2.3, and similarly embed S_{r+1} into V_{r+1} so that its root is embedded to the first vertex of V_{r+1} under σ. For each $i \in \{2, \ldots, r\}$, we will have $|V_i| = |S_i|$ and embed the directed path S_i into $G[V_i]$ using the ordering provided by σ.

As described at the start of this section, for each $i \in [r]$, we then find a short path R_i from the last vertex of V_i under σ to the first vertex of V_{i+1} under σ which can 'absorb' any subset of vertices from U_i (see Claim 2.13). We then embed the forests F_i^+, F_i^-, $i \in [r+1]$ and directed paths Q_i, $i \in [\ell]$, into $\cup_{i\in[r]}(U_i \setminus V(R_i))$, before incorporating the right number of vertices into each path R_i. More specifically, as depicted in Figure 2.3, for each $i \in [r]$, we will divide U_i into six parts, $U_{i,1}, \ldots, U_{i,6}$, again with carefully chosen sizes. The sets $U_{i,1}$ and $U_{i,6}$ will be small and covered by R_i (aiding the desired 'absorption' property of R_i). We will embed $V(F_i^+) \setminus V(S_i)$ into $U_{i,2} \setminus V(R_i)$, using **A4** and that typical vertices in V_i (the image of S_i) have plenty of out-neighbours in $U_{i,2}$ (see Claim 2.14) and $V(R_i)$ is small. Similarly, we will embed $V(F_{i+1}^-) \setminus V(S_{i+1})$ into $U_{i,4} \setminus V(R_i)$ (see also Claim 2.14). We will embed paths Q_j satisfying **A7.2** using the appropriate set $U_{i,5}$ (see Claim 2.15). We will then embed paths Q_j satisfying **A7.1** using different sets $U_{i,3}$ (see Claim 2.16). As we chose the size of the sets U_i, $i \in [r]$, carefully, for each $i \in [r]$, we will then have the correct number of vertices unused in U_i to absorb into R_i and complete the embedding of P_i, and hence also the embedding of $T \subseteq D$.

Lemma 2.12. *Let $1/n \ll \mu, 1/r, 1/k$ and $m \geqslant \mu n$. Every tournament with $n+k-2$ vertices contains a copy of every n-vertex oriented tree with k leaves which has an (r,m)-good decomposition.*

Proof. Note that we can additionally assume that $\mu \ll 1/r, 1/k$. Let G be an $(n+k-2)$-vertex tournament and suppose that the n-vertex tree T with k leaves has an (r,m)-good decomposition D using the notation in Definition 2.10. Let k_0 be the number of in-leaves of S_1 and let k_1 be the number of out-leaves of S_{r+1}. By **A5**, we have $k_0, k_1 \geqslant 1$ and by **A6** we have $k_0 + k_1 \leqslant k$.

Let $I_1 \subseteq [\ell]$ be the set of $i \in [\ell]$ satisfying **A7.1**. Let $I_2 = [\ell] \setminus I_1$, so that, by **A7**, each $i \in I_2$ satisfies **A7.2**. For each $i \in I_1$, using **A7.1**, let $q_i, r_i \in [r+1]$ with $q_i < r_i$ be such that Q_i is a directed path from F_{q_i} to F_{r_i} with length $3(r_i - q_i) + 1$. For each $i \in [r]$, let a_i be the number of $j \in I_1$ for which $q_j \leqslant i < r_j$. For each $i \in I_2$, using **A7.2**, let $2 \leqslant s_i \leqslant r$ be such that Q_i is a directed path from $V(F^-_{s_i}) \setminus V(S_{s_i})$ to the last vertex of S_{s_i}. For each $i \in [r]$, let b_i be the number of $j \in I_2$ with $s_j = i+1$ (and note that we always have $b_r = 0$).

Let $\sigma = v_1, \ldots, v_{n+k-2}$ be a median order of G. Take in $v_1, \ldots, v_{n+k-2}$ consecutive disjoint intervals

$$V_1, U_1, V_2, U_2, V_3, \ldots, V_r, U_r, V_{r+1}$$

such that $|V_1| = |S_1| + k_0 - 1$, $|V_{r+1}| = |S_{r+1}| + k_1 - 1$, and, for each $2 \leqslant i \leqslant r$, $|V_i| = |S_i|$, and, for each $i \in [r]$,

$$|U_i| = |P_i| - 2 + |V(F^+_i) \setminus V(S_i)| + |V(F^-_{i+1}) \setminus V(S_{i+1})| + 3a_i + 2b_i \tag{2.15}$$

$$\geqslant |P_i| - 2 \overset{\textbf{A3}}{\geqslant} 2000m - 1. \tag{2.16}$$

Note that this is possible, as, by **A4** and **A5**, $|F^-_1| = |S_1|$ and $|F^+_{r+1}| = |S_{r+1}|$, so that, using **A4**, we have

$$\begin{aligned}
\sum_{i=2}^{r}|V_i|+\sum_{i=1}^{r}|U_i| &= \sum_{i=2}^{r}|S_i|+\sum_{i=1}^{r}(|P_i|-2+|F_i^+|+|F_{i+1}^-|-|S_i|-|S_{i+1}|+3a_i+2b_i)\\
&= \sum_{i=2}^{r}|S_i|+\sum_{i=1}^{r}(|P_i|-2)+\sum_{i=1}^{r+1}(|F_i^+|+|F_i^-|-2|S_i|)+\sum_{i\in[r]}(3a_i+2b_i)\\
&\overset{(2.12)}{=} |P|-2+\sum_{i=1}^{r+1}(|F_i^+|+|F_i^-|-2|S_i|)+3\sum_{i\in I_1}(r_i-q_i)+2|I_2|\\
&\overset{\textbf{A7.1,A7.2}}{=} |P|-2+\sum_{i=1}^{r+1}|(V(F_i^+)\cup V(F_i^-))\setminus V(S_i)|+\sum_{i\in[\ell]}(|Q_i|-2)\\
&\overset{\textbf{A2}}{=} n-|S_1|-|S_{r+1}|,
\end{aligned}$$

and hence,

$$\sum_{i=1}^{r+1}|V_i|+\sum_{i=1}^{r}|U_i| = n+k_0+k_1-2 \overset{\textbf{A6}}{\leqslant} n+k-2.$$

Next, for each $i\in[r]$, partition U_i as intervals $U_{i,1},\dots,U_{i,6}$ in that order such that

$$|U_{i,1}|=m,\ |U_{i,2}|=10m,\ |U_{i,4}|=110m,\ |U_{i,5}|=100m,\ |U_{i,6}|=m \tag{2.17}$$

$$\text{and } |U_{i,3}|=|U_i|-222m \overset{(2.15)}{\geqslant} 1700m. \tag{2.18}$$

Note also, by **A4**, that, for each $i\in\{2,\dots,r\}$,

$$|V_i|=|S_i|\leqslant|F_i|\leqslant m. \tag{2.19}$$

For each $i\in[r]$, let U_i' be a subset of U_i where each vertex is included independently at random with probability $\mu/20$. By Lemma 2.1 (ii) $v_1v_2\dots v_{n+k-2}$ forms a directed path in that order, so there is a directed path from the last vertex of V_i under σ to the first vertex of V_{i+1} under σ, whose vertex set covers $U_{i,1}\cup U_i'\cup U_{i,6}$ and whose vertex order is a suborder of σ. Let R_i be a shortest such path. We now prove that, with positive probability, the 'absorption property' we need for R_i holds, as well as a bound on $|R_i|$.

Claim 2.13. *With positive probability, for each $i \in [r]$, $|V(R_i) \setminus (U_{i,1} \cup U_{i,6})| \leqslant m$, so that $|R_i| \leqslant 3m$, and, for any $U \subseteq U_i \cup V(R_i)$ with $V(R_i) \subseteq U$, there is a directed path with the same start vertex and end vertex as R_i but with vertex set U.*

Proof of Claim 2.13. Let $p = \mu/20$ and $i \in [r]$. Note that, by Lemma 1.18, as $|U_i| \leqslant n$ and $1/n \ll \mu, 1/r$, we have, with probability at least $1 - 1/3r$ that $|U_i'| \leqslant 2pn$. For each $v \in U_i \setminus (U_{i,1} \cup U_{i,6})$, let $\mathbf{E}_v$ be the following event.

$\mathbf{E}_v$: There are $u \in N^-(v) \cap U_i'$ and $u' \in N^+(v) \cap U_i'$ with $u <_\sigma v <_\sigma u'$.

Now, by Lemma 2.1 (ii), for each $v \in U_i \setminus (U_{i,1} \cup U_{i,6})$, we have

$$|\{u \in N^-(v) \cap U_i : u <_\sigma v\}| \geqslant \frac{|\{u \in U_i : u <_\sigma v\}|}{2} \geqslant \frac{|U_{i,1}|}{2} \overset{(2.17)}{=} \frac{m}{2},$$

and

$$|\{u \in N^+(v) \cap U_i : u >_\sigma v\}| \geqslant \frac{|\{u \in U_i : u >_\sigma v\}|}{2} \geqslant \frac{|U_{i,6}|}{2} \overset{(2.17)}{=} \frac{m}{2},$$

so that $\mathbb{P}(\mathbf{E}_v \text{ does not hold}) \leqslant 2(1-p)^{m/2} \leqslant 2\exp(-pm/2) \leqslant 2\exp(-\mu^2 n/40)$. Therefore, as $1/n \ll \mu, 1/r$, a union bound implies that, with probability at least $1 - 1/3r$, $\mathbf{E}_v$ holds for each $v \in U_i \setminus (U_{i,1} \cup U_{i,6})$. Thus, with probability at least $1/3$, we have, for each $i \in [r]$, that $\mathbf{E}_v$ holds for each $v \in U_i \setminus (U_{i,1} \cup U_{i,6})$, and $|U_i'| \leqslant 2pn$. Assuming these events occur, we now prove that the property in the claim holds for each $i \in [r]$.

By Corollary 2.2 and the minimality of R_i, any two vertices in $U_{i,1} \cup U_i' \cup U_{i,6}$, with no vertices between them on R_i from $U_{i,1} \cup U_i' \cup U_{i,6}$ have at most 1 vertex between them on R_i. As the vertices from $U_{i,1} \cup U_{i,6}$ form two intervals on R_i, just after the first vertex and just before the last vertex of R_i respectively, $|V(R_i) \setminus (U_{i,1} \cup U_{i,6})| \leqslant 2 + 2|U_i'| + 1 \leqslant 4pn + 3 \leqslant m$.

Now, take any set $U \subseteq U_i \cup V(R_i)$ with $V(R_i) \subseteq U$. Let R_U be a directed path with the same endvertices as R_i which contains every vertex of R_i in order according to σ and for which $V(R_U) \subseteq U$, and which, under these conditions, has the maximum possible length. Note that this exists as R_i itself satisfies these conditions. Suppose, for contradiction, that there is some $v \in U \setminus V(R_U)$. Note that $v \in U_i \setminus (U_{i,1} \cup U_{i,6})$.

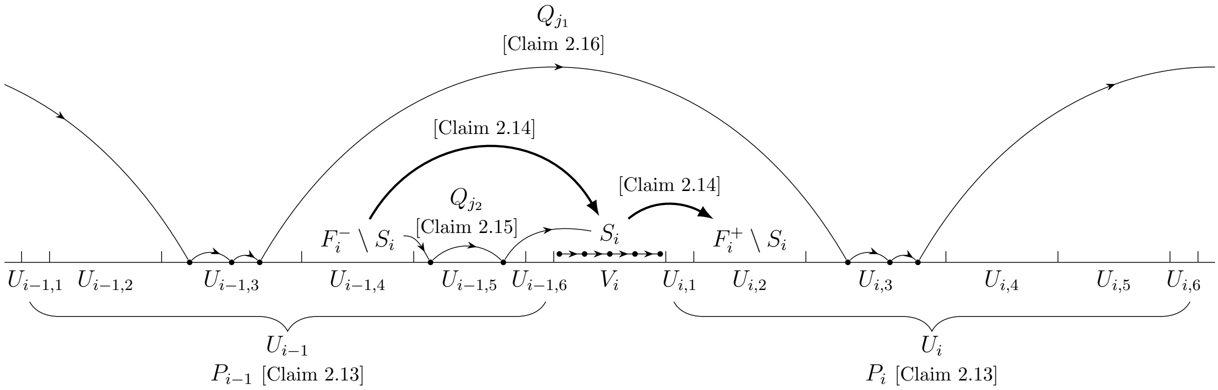

Figure 2.3: Embedding an (r, m)-good decomposition (as depicted in Figure 2.1) into a median order, with the claims used to embed each part.

Let ℓ be the length of R_U and label vertices so that $R_U = u_0u_1 \dots u_\ell$. As $\mathbf{E}_v$ holds and $U_i' \subseteq V(R_i) \subseteq V(R_U)$, we can take $j = \min\{j' \in \{0, 1, \dots, \ell\} : u_{j'} \in N^-(v)\}$ and find that $u_j <_\sigma v$. Let $j'' \in \{0, 1, \dots, \ell\}$ be the smallest $j'' > j$ such that $u_{j''} \in N^+(v)$, noting that this is well-defined also by $\mathbf{E}_v$.

Observe that $u_{j''-1} \notin N^+(v)$, so that, as G is a tournament, $u_{j''-1} \in N^-(v)$ and therefore

$$u_0u_1 \dots u_{j''-1}vu_{j''} \dots u_\ell,$$

is a directed path with the same endvertices as R_U (and hence R_i) which contains every vertex of R_i in order according to σ. As this path has vertex set $\{v\} \cup V(R_U) \subseteq U$ and $v \notin V(R_U)$, this path contradicts the maximality of R_U. Thus, $V(R_U) = U$, so that R_U is a directed path with the same endvertices as R_i and with vertex set U, as required. $\boxdot$

Assume then, that the property in Claim 2.13 holds. We now show three further claims, before embedding T. This embedding, annotated with which part of the embedding is done with each claim, is depicted in Figure 2.3. For each $i \in [r+1]$, we will use the following claim to embed the vertices in $V(F_i^+) \setminus V(S_i)$ to $U_{i,2}$ (if $i \neq r+1$) and embed the vertices in $V(F_i^-) \setminus V(S_i)$ to $U_{i-1,4}$ (if $i \neq 1$) so that they attach appropriately to an embedding of S_i into the vertex set V_i.

Claim 2.14. *For each $i \in [r]$ and $v \in V_i$, we have $|N^+(v, U_{i,2}) \setminus V(R_i)| \geqslant 3m$, and, for* *each $i \in [r]$ and $v \in V_{i+1}$, we have $|N^-(v, U_{i,4}) \setminus V(R_i)| \geqslant 3m$.*

Proof of Claim 2.14. Let $i \in [r]$ and $v \in V_i$, and take $V_{i,v} = \{u \in V_i : u >_\sigma v\}$. By Lemma 2.1 (ii), we have that

$$|N^+(v, U_{i,2})| \geqslant |N^+(v, V_{i,v} \cup U_{i,1} \cup U_{i,2})| - |V_{i,v} \cup U_{i,1}| \geqslant \frac{|V_{i,v} \cup U_{i,1} \cup U_{i,2}|}{2} - |V_{i,v} \cup U_{i,1}|$$
$$= \frac{|U_{i,2}| - |V_{i,v} \cup U_{i,1}|}{2} \geqslant \frac{|U_{i,2}| - |V_i \cup U_{i,1}|}{2} \overset{(2.17),(2.19)}{\geqslant} \frac{10m - m - m}{2} = 4m.$$

Therefore, by the property from Claim 2.13, $|N^+(v, U_{i,2}) \setminus V(R_i)| \geqslant |N^+(v, U_{i,2})| - (|R_i| - |U_{i,1}| - |U_{i,6}|) \geqslant 3m$.

Let then $i \in [r]$ and $v \in V_{i+1}$ and let $V'_{i+1,v} = \{u \in V_{i+1} : u <_\sigma v\}$. By Lemma 2.1 (ii), we have similarly that

$$|N^-(v, U_{i,4})| \geqslant |N^-(v, V'_{i+1,v} \cup U_{i,4} \cup U_{i,5} \cup U_{i,6})| - |V'_{i+1,v} \cup U_{i,5} \cup U_{i,6}|$$
$$\geqslant \frac{|U_{i,4}| - |V'_{i+1,v} \cup U_{i,5} \cup U_{i,6}|}{2} \overset{(2.17),(2.19)}{\geqslant} \frac{110m - 100m - m - m}{2} = 4m.$$

Therefore, by the property from Claim 2.13 again, $|N^-(v, U_{i,4}) \setminus V(R_i)| \geqslant |N^-(v, U_{i,4})| - (|R_i| - |U_{i,1}| - |U_{i,6}|) \geqslant 3m$. ⊡

Recall that, for each $i \in I_2$, Q_i is a directed path of length 3 from $V(F^-_{s_i}) \setminus V(S_{s_i})$ to the last vertex of S_{s_i}. The following claim will be used to embed such a path when its first and last vertex have already been embedded into $U_{s_i-1,4}$ and V_{s_i} respectively.

Claim 2.15. *For each $2 \leqslant j \leqslant r$, $v \in U_{j-1,4}$, $w \in V_j$ and $U \subseteq U_{j-1,4} \cup U_{j-1,5}$ with $|U| \leqslant 2m$, there is a directed v,w-path in G with length 3 and internal vertices in $(U_{j-1,4} \cup U_{j-1,5}) \setminus (U \cup V(R_{j-1}))$.*

Proof of Claim 2.15. Let $A_{j,v,w,U} = \{u \in U \cup V(R_{j-1}) \cup V_j : v <_\sigma u <_\sigma w\}$, and note that, by (2.19) and the choice of R_i according to Claim 2.13, $|A_{j,v,w,U}| \leqslant 6m$. The number of vertices between v and w in σ is at least $|U_{j-1,5}| + |U_{j-1,6}| = 101m > 6|A_{j,v,w,U}| + 8$. Therefore, by Lemma 2.4, there is a directed v,w-path in G with length 3 and internal vertices in $\{u \notin A_{j,v,w,U} : v <_\sigma u <_\sigma w\}$. Because $U_{j-1,6} \subseteq V(R_{j-1})$, we have $\{u \notin A_{j,v,w,U} : v <_\sigma u <_\sigma w\} \subseteq (U_{j-1,4} \cup U_{j-1,5}) \setminus (U \cup V(R_{j-1}))$, and so the claim holds. $\boxdot$

For each $i \in [6]$, let $U_{0,i} = U_{r+1,i} = \emptyset$, and note that, by **A4** and **A6**, $|V_1|, |V_{r+1}| \leqslant m + k \leqslant 2m$. For each $i \in [r+1]$, let $\bar{V}_i = U_{i-1,4} \cup U_{i-1,5} \cup U_{i-1,6} \cup V_i \cup U_{i,1} \cup U_{i,2}$, and note that, by (2.17) and (2.19), $|\bar{V}_i| \leqslant 225m$. Note that $\bar{V}_1 U_{1,3} \bar{V}_2 U_{2,3} \ldots \bar{V}_r U_{r,3} \bar{V}_{r+1}$ are consecutive intervals in σ.

Recall that, for each $i \in I_1$, Q_i is a directed path from F_{q_i} to F_{r_i} with length $3(r_i - q_i) + 1$. The following claim will be used to embed such a path when its first and last vertex have already been embedded into $\bar{V}_{q_i}$ and $\bar{V}_{r_i}$ respectively.

Claim 2.16. *For each $1 \leqslant i < j \leqslant r+1$, $v \in \bar{V}_i$, $w \in \bar{V}_j$ and $U \subseteq V(G)$ with $|U| \leqslant m$, there is a directed v,w-path in G with length $3(j-i)+1$ and exactly 3 vertices in each set $U_{i',3} \setminus (U \cup V(R_{i'}))$, $i \leqslant i' < j$.*

Proof of Claim 2.16. First we will choose vertices $u_{i'}$, $i \leqslant i' < j$ between $u_{i-1} := v$ and w in the median order, with $u_{j-1}w \in E(G)$ before carefully applying Lemma 2.4 to each consecutive pair of vertices in $v, u_i, u_{i+1}, \ldots, u_{j-1}$ to get, together with $u_{j-1}w$, a v,w-path with length $3(j-i)+1$.

To do this, first, for each i', $i \leqslant i' \leqslant j-2$, let $u_{i'}$ be the last vertex in $U_{i',3} \setminus (U \cup V(R_{i'}))$ under σ. Let $U'_{j-1,3}$ be the set of the last $250m$ vertices of $U_{j-1,3}$ under σ, and let $\bar{V}_{j,w} = \{w' \in \bar{V}_j : w' <_\sigma w\}$, so that $|\bar{V}_{j,w}| \leqslant |\bar{V}_j| \leqslant 225m$. Note that, by Lemma 2.1 (ii), we have

$$\begin{aligned} |N^-(w, U'_{j-1,3}) \setminus (U \cup V(R_{j-1}))| &\geqslant |N^-(w, \bar{V}_{j,w} \cup U'_{j-1,3})| - |\bar{V}_{j,w}| - |U \cup V(R_{j-1})| \\ &\geqslant \frac{|U'_{j-1,3}| - |\bar{V}_{j,w}|}{2} - |U \cup V(R_{j-1})| \geqslant \frac{250m - 225m}{2} - 4m > 0. \end{aligned}$$

Let u_{j-1} then be the last vertex of $N^-(w, U_{j-1,3}) \setminus (U \cup V(R_{j-1}))$ under σ, noting that there are fewer than $250m$ vertices in $U_{j-1,3}$ after u_{j-1} under σ. Let $u_{i-1} = v$.

For each $i \leqslant i' < j$, we will show there exists a directed $u_{i'-1}, u_{i'}$-path $T_{i'}$ with length

3 and internal vertices in $U_{i',3} \setminus (U \cup V(R_{i'}))$. Noting that $T_iT_{i+1} \dots T_{j-1}w$ is a directed path with length $3(j-i)+1$ and exactly three vertices in each set $U_{i',3} \setminus (U \cup V(R_{i'}))$, $i \leqslant i' < j$, will then complete the proof of the claim.

Let then $i \leqslant i' < j$ and let $A_{i'} = \{u \in U_{i'-1,3} \cup \bar{V}_{i'} \cup ((U \cup V(R_{i'})) \cap U_{i',3}) : u_{i'-1} <_\sigma u <_\sigma u_{i'}\}$. Note that, for each $i \leqslant i' < j$, by the choice of $u_{i'}$ there are at most $|U \cup V(R_{i'-1})| \leqslant 4m$ vertices after $u_{i'-1}$ in $U_{i'-1,3}$ under σ, so $|A_{i'}| \leqslant 4m + 225m + |U \cup V(R_{i'})| \leqslant 233m$. In addition, recall that there are fewer than $250m$ vertices in $U_{j-1,3}$ after u_{j-1} under σ. Therefore, by (2.18), for each $i \leqslant i' < j$, there are at least $1700m - 250m > 6|A_{i'}| + 8$ vertices in $U_{i',3}$ before $u_{i'}$ under σ. So, by Lemma 2.4, there is a directed $u_{i'-1}, u_{i'}$-path $T_{i'}$ with length 3 and internal vertices in $\{u \notin A_{i'} : u_{i'-1} <_\sigma u <_\sigma u_{i'}\} \subseteq U_{i',3} \setminus (U \cup V(R_{i'}))$, as required. ⊡

We are now ready to embed the (r, m)-good decomposition D into G, as follows. Begin with the empty embedding $\phi : \emptyset \to V(G)$. For each $2 \leqslant i \leqslant r$, recalling that $|V_i| = |S_i|$, extend ϕ to embed the directed path S_i onto the vertices in V_i in the order given by σ. Note that the vertices of each interval V_i form a directed path in this order by Lemma 2.1 (ii).

Let x' be the last vertex of V_1 under σ, and let y' be the first vertex of V_{r+1} under σ. Recall that P, as defined in (2.12), is a directed x, y-path, S_1 is an in-arboresence with k_0 in-leaves and root x, and S_{r+1} is an out-arboresence with k_1 out-leaves and root y. Therefore, as $|V_1| = |S_1| + k_0 - 1$ and $|V_{r+1}| = |S_{r+1}| + k_1 - 1$, by Theorem 2.3 (applied twice, once with directional duality) we can extend ϕ to embed S_1 into V_1 such that $\phi(x) = x'$ and embed S_{r+1} into V_{r+1} such that $\phi(y) = y'$.

Now, for each $i \in [r+1]$ and $v \in V(S_i)$, let F_v^- be the component of F_i^- containing v and let F_v^+ be the component of F_i^+ containing v. For each vertex $v \in V(S_i)$ in increasing order of $\phi(v)$ under σ, greedily and disjointly extend ϕ to embed $F_v^- - v$ into $N^-(\phi(v), U_{i-1,4}) \setminus V(R_{i-1})$ and $F_v^+ - v$ into $N^+(\phi(v), U_{i,2}) \setminus V(R_i)$. Note this is possible for each $v \in V(S_i)$ as, by **A5**, if $|E(F_v^-)| > 0$, then $i \geqslant 2$ and thus, by Claim 2.14,

$$|N^-(\phi(v), U_{i-1,4}) \setminus (V(R_{i-1}) \cup (\cup_{u \in V(S_i):\phi(u)<_\sigma \phi(v)} \phi(F_u^-)))| \geqslant 3m - (|F_i^-| - |V(F_v^-) \setminus \{v\}|) \overset{\mathbf{A4}}{\geqslant} 3|V(F_v^-) \setminus \{v\}|,$$

so that a copy of $F_v^- - v$ in $N^-(\phi(v), U_{i-1,4}) \setminus (V(R_{i-1}) \cup (\cup_{u \in V(S_i):\phi(u)<_\sigma \phi(v)} \phi(F_u^-)))$ exists by Theorem 1.8. Similarly, for each $v \in V(S_i)$, this is also possible for $F_v^+ - v$.

For each $i \in [\ell]$, say that Q_i is a directed path from x_i to y_i. For each $i \in [\ell]$ in turn, extend ϕ to cover $V(Q_i) \setminus \{x_i, y_i\}$, by doing the following.

- If $i \in I_1$, recall that q_i, r_i are such that Q_i is a directed path from F_{q_i} to F_{r_i} with length $3(r_i - q_i) + 1$, where $q_i < r_i$, and note that $\phi(x_i) \in \phi(V(F_{q_i})) \subseteq \bar{V}_{q_i}$ and $\phi(y_i) \in \phi(V(F_{r_i})) \subseteq \bar{V}_{r_i}$. Embed Q_i as a directed $\phi(x_i), \phi(y_i)$-path with length $3(r_i - q_i) + 1$ and exactly three vertices in $U_{i',3} \setminus (V(R_{i'}) \cup (\cup_{j \in [i-1]} \phi(V(Q_j))))$, for each $q_i \leqslant i' < r_i$. Note that this is possible, by Claim 2.16, as when we look for such a path we have $| \cup_{j \in [i-1]} \phi(V(Q_i))| \leqslant \ell \cdot (3r + 2) \leqslant m$ as $\ell \leqslant 2r$, $1/n \ll \mu \ll 1/r$ and $m \geqslant \mu n$.

- If $i \in I_2$, recall that $2 \leqslant s_i \leqslant r$ is such that Q_i is a directed path with length 3 from $V(F_{s_i}^-) \setminus V(S_{s_i})$ to the last vertex of S_{s_i}, and note that $\phi(x_i) \in \phi(V(F_{s_i}^-) \setminus V(S_{s_i})) \subseteq U_{s_i-1,4}$ and $\phi(y_i) \in \phi(V(S_{s_i})) \subseteq V_{s_i}$. Embed Q_i as a directed path with length 3 from $\phi(x_i)$ to $\phi(y_i)$ with interior vertices in $(U_{s_i-1,4} \cup U_{s_i-1,5}) \setminus (\phi(V(F_{s_i}^-)) \cup (\cup_{j \in [i-1]} \phi(V(Q_j))) \cup V(R_{s_i-1}))$. Note that this possible, by Claim 2.15, as when we look for such a path we have, by **A4**, $|\phi(V(F_{s_i}^-))| + | \cup_{j \in [i-1]} \phi(V(Q_j))| \leqslant m + \ell \cdot (3r + 2) \leqslant 2m$.

Finally, we extend ϕ to cover the internal vertices of P_i, for each $i \in [r]$. For each $i \in [r]$, let $U_i'' = \big(V(R_i) \cup U_i\big) \setminus \phi\big(V(F_i^+) \cup V(F_{i+1}^-) \cup (\cup_{j \in [\ell]} V(Q_j))\big)$. Note that $V(R_i) \cup U_i$

contains exactly the vertices in U_i and the endvertices of R_i. Therefore,

$$\begin{aligned}|U_i''| &= |U_i| + 2 - (|F_i^+| - |S_i|) - (|F_{i+1}^-| - |S_{i+1}|) - |\cup_{j\in[\ell]} (V(Q_j) \cap U_i)| \\ &\overset{(2.15)}{=} (|P_i| + 3a_i + 2b_i) - 3|\{j \in I_1 : q_j \leqslant i < r_j\}|) - 2|\{j \in I_2 : s_j = i + 1\}| \\ &= (|P_i| + 3a_i + 2b_i) - 3a_i - 2b_i = |P_i|.\end{aligned}$$

By Claim 2.13, for each $i \in [r]$, there is a directed path between the endvertices of R_i with vertex set U_i''. Using these paths, for each $i \in [r]$, extend the embedding ϕ to cover P_i, for each $i \in [r]$. This completes the embedding ϕ of $D = P \cup S_1 \cup S_{r+1} \cup (\cup_{i\in[r+1]} F_i) \cup (\cup_{i\in[\ell]} Q_i)$, and hence, by **A1**, G contains a copy of T. □

2.3.4 Proof of Theorem 1.3

Given Lemmas 2.11 and 2.12, it is now straight-forward to prove Theorem 1.3.

Proof of Theorem 1.3. Note that, due to the result of Thomason [29] quoted in the introduction, we may assume that $k \geqslant 3$. Let n_0 and μ be such that $1/n_0 \ll \mu \ll 1/k$. Let T be a tree with $n \geqslant n_0$ vertices and k leaves, and let G be a tournament with $n + k - 2$ vertices.

If there are no vertices x and y with degree 2 in T and a bare x, y-path P with length at least μn with first and last block of length 1, then, by Lemma 2.11, T has an (r, m)-good decomposition for some $m \geqslant \mu n$ and $r \leqslant 10k$. In this case, by Lemma 2.12, G contains a copy of T. Thus, we can assume that T contains vertices x and y with degree 2 in T and a bare x, y-path P with length at least μn with first and last block of length 1.

Suppose first, that $k = 3$. Note that in this case P must lie in a maximal bare path of T with one endvertex that is a leaf. Say this leaf is z, and assume, by relabelling x and y if necessary, that the path, Q say, from x to z in T contains y (and hence P). Let $T' = T - (V(Q) \setminus \{x\})$. Noting that x is a leaf of T', duplicate x to get the tree T'' with the new leaf x'. Note that T'' has 4 leaves and $|T| - |Q| + 2 \leqslant n - \mu n + 1$ vertices. Therefore, by

Theorem 1.2, as $1/n \ll \mu, 1/k$, G contains a copy of T'', S'' say. Let s and s' be the copy of x and x' in S'' respectively. Note that $|G-(V(S'')\setminus\{s,s'\})| = n+1-(n-|Q|) = |Q|+1$. By Theorem 1.12, there is a copy, Q' say, of Q with x embedded to $\{s,s'\}$. Then $S''\cup Q'$ gives a copy of T.

Therefore, we have that $k \geqslant 4$. In this case, let $T' = T - (V(P)\setminus\{x,y\})$. Noting that x and y are leaves of T', create T'' by duplicating x and y to get the new vertices x' and y' respectively, and adding the edge xy. Note that T'' has $k+2$ leaves and $|T|-|P|+4 \leqslant n-\mu n+3$ vertices. Therefore, by Theorem 1.2, as $1/n \ll \mu, 1/k$, G contains a copy of T'', S'' say. Let s, s', t and t' be the copy of x, x', y and y' in S'' respectively. Note that $|G-(V(S'')\setminus\{s,s',t,t'\})| = n+k-2-(n-|P|) = |P|+k-2 \geqslant |P|+2$. By Theorem 1.13, there is a copy, P' say, of P with x embedded to $\{s,s'\}$ and y embedded to $\{t,t'\}$. Observing that $S''\cup P'$ contains a copy T completes the proof that G contains a copy of T in this case. □

CHAPTER 3

EMBEDDING ORIENTED TREES USING THE REGULARITY LEMMA

In this chapter we present proofs of the following results, which we recall from the introduction.

Theorem 1.4. *Let $\alpha > 0$. There exists $n_0 \in \mathbb{N}$ such that for any $n \geqslant n_0$, if G is a $((1+\alpha)n+k)$-vertex tournament and T is an n-vertex oriented tree with k leaves, then G contains a copy of T.*

Theorem 1.5. *Let $\alpha > 0$. There exists $c > 0$ and $n_0 \in \mathbb{N}$ such that for any $n \geqslant n_0$, if G is a $(1+\alpha)n$-vertex tournament and T is an n-vertex oriented tree with $\Delta(T) \leqslant cn$, then G contains a copy of T.*

Both proofs make use of a reduction of the theorems to critical cases which can be embedded using the regularity lemma (introduced properly in Section 3.3). As there is significant overlap in this reduction for the two theorems, many parts of this chapter will apply to both results. Though our methods apply to all the trees covered by these theorems, Theorem 1.2 already implies both theorems for those trees with $o(n)$ leaves. Thus, the critical case for consideration are those trees with $\Omega(n)$ leaves. The most difficult cases arise when these leaves are all close to each other within the tree, connected via some smaller core tree (see Figure 3.1). The challenge is then to be able to distribute these leaves around the tournament despite their location being quite tightly restricted

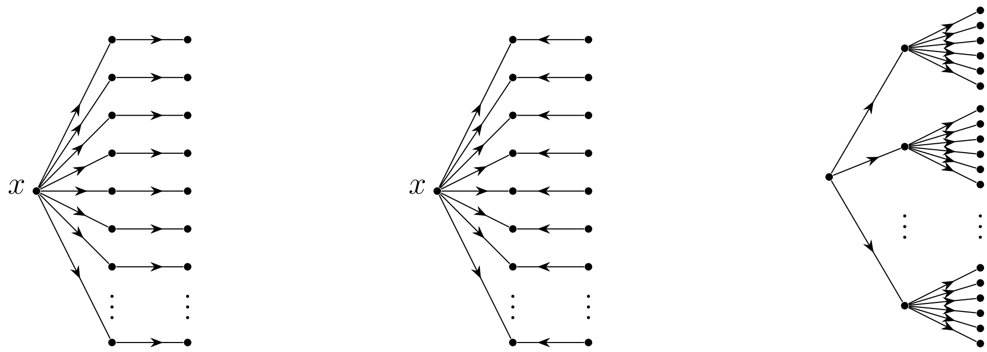

Figure 3.1: Examples of oriented trees with many leaves close to one another. While the first two trees have $\Delta(T) \approx n/2$, the third tree may be realised with $\Delta(T) \leqslant cn$ for a small constant c, making it a case of particular interest for Theorem 1.5.

by the location of the vertices in the core tree. For Theorem 1.5, the maximum degree condition will imply the core tree cannot be too small, and we exploit this to distribute the vertices of the core tree around the tournament. Here, the key novelty in our methods is the identification of the small core tree in the most challenging cases, and its embedding around the tournament.

For Theorem 1.4, we will be able to contract this small core tree in the most challenging cases to a single vertex without increasing the number of leaves. As the core tree is small, if we can find a copy of this contracted tree then we will be able to recover the original tree using suitable regularity techniques. The critical case will then be trees which have one very high degree vertex, whose removal results in components of at most constant size. Further simplification will allow us to assume that this high degree vertex has either in-degree or out-degree 0. This simplification focuses in on the hardest cases in our proof. To embed a tree T with one high out-degree vertex x with in-degree 0 into a tournament G, a natural approach is to place x at the vertex of G with highest out-degree, maximising the attachment possibilities for the components of $T - \{x\}$. Often, this is a good strategy (indeed, this approach will always succeed for the first tree depicted in Figure 3.1), but when many vertices of T are reached from x by travelling along a path beginning with a forward edge followed by a backward edge (such as for the second tree depicted in

Figure 3.1), this may fail. The key to our proof is to use the failure in these cases to identify structural properties of the tournament, and thus a better location for the high out-degree vertex. This is the most significant novelty in our proof of Theorem 1.4, and enables the most difficult trees to be found in tournaments.

3.1 Outline of proof of Theorem 1.4 and Theorem 1.5

We will now sketch the proofs for both Theorem 1.4 and Theorem 1.5 together in more detail. We have already discussed the need to take particular care with trees which contain some small core subtree that restricts the distribution of the other vertices in the tree around the tournament. Therefore, we will identify a small core in any tree T, from which T can be recovered by first appending a collection of constant-sized trees, then connecting components by constant-length paths, and then iteratively attaching a small number of additional leaves. This decomposition is independent of the directions of the edges of T, and so we state it for non-oriented trees. More precisely, given any tree T, we find $T_0 \subseteq T_1 \subseteq T_2 \subseteq T_3 \subseteq T_4 = T$ (shown in Figure 3.2), such that

i) T_0 is small,

ii) T_1 is formed by adding constant-sized trees, each attached with an edge to some vertex of T_0,

iii) T_2 is formed by adding (unattached) constant-sized trees to T_1,

iv) T_3 is formed by adding long but constant-sized paths connecting the components of T_2, and

v) T_4 is formed by attaching constant-sized trees to T_3, so that few vertices are added in total.

Having found such a decomposition, we need a strategy for embedding these pieces. We note first that the vertices added in iii) and v) above pose little trouble given the spare vertices in our tournament. Indeed, within, say, any $\alpha n/2$ vertices in a tournament, any oriented tree with up to $\alpha n/6$ vertices can be found using known results (see Theorem 1.8).

This allows the new constant-sized trees in iii) to be found greedily. For v), we note that setting aside a small random subset of the spare vertices preserves some in- and out-neighbours for almost all the remaining vertices in the tournament (see Proposition 1.20). Carrying out the embedding for i)–iv) within the tournament induced on these *good* remaining vertices, will then allow us to extend the embedding greedily to cover the final vertices in v) (see Corollary 1.9).

Thus, our focus is on how to embed the vertices in T_0 so that this can be extended to an embedding of T_1, and how to embed the paths at iv). In certain tournaments the paths at iv) can also be embedded straightforwardly by reserving a random set of vertices for this purpose. Where this is not possible, by removing a small set of vertices from the tournament, we will be able to partition the vertices into a sequence of linearly-sized sets, with all edges between the sets directed forward along the sequence. This partition allows us to divide the tree naturally into pieces, which can then be found separately along the sequence of sets. We note that this is a streamlined version of a decomposition due to Kühn, Mycroft and Osthus [21, 22].

Let us assume then that the tournament is sufficiently well connected that the paths at iv) can be embedded within a reserved random set of vertices. We need then to embed T_0 so that the vertices of $T_1 - V(T_0)$ can be distributed throughout the tournament. To do this we will use the regularity lemma for digraphs, so that we may assign vertices to clusters before embedding them. The challenge is to identify some good clusters for T_0, for which we can assign the vertices in $V(T_1) \setminus V(T_0)$ across the other regularity clusters. The whole of T_1 can then be embedded using relatively standard regularity techniques, in combination with the result that any oriented tree with ℓ vertices can be found in tournaments with only $O(\ell)$ vertices.

Embedding the core T_0 of the tree and extending it to cover T_1 is the only part where the proofs of Theorem 1.4 and Theorem 1.5 differ. For Theorem 1.4, the core tree can always be embedded within a single regularity cluster, which will allow us to reduce the problem to embedding trees where the core is a single vertex, x say (which may have very high degree), and further reduction will allow us to assume that all components of $T - \{x\}$ are attached to x by out-edges. Similar to the discussion in the introduction, here it would be natural to try embedding x to a cluster with as many out-edges as possible in a suitable 'reduced digraph' (see Section 3.3). This may fail, but we try this anyway, essentially embedding as much of T_1 as possible. If the embedding fails, it will be due to certain structural properties of the tournament which will allow us to move the embedding of x, along with some of the embedded vertices, to complete the embedding. This part of the proof, with its division into a number of detailed subcases, is the most technical aspect of our proof, but solves the key problem and allows the proof of Theorem 1.4.

Fortunately, embedding T_0 and extending this to cover T_1 is less involved for Theorem 1.5. The maximum degree condition in this case ensures that T_0 necessarily contains at least a large constant number of vertices (for example, for the third tree shown in Figure 3.1, we may identify T_0 with the star consisting of all non-leaf vertices). Thus, it is possible to distribute the vertices of T_0 across several regularity clusters if required for the even distribution of $V(T_1) \setminus V(T_0)$ throughout the tournament. For this, we identify a particular caterpillar-like structure which spans most of the clusters in the regularity graph (see Section 3.5.1).

Each aspect of the proof is discussed in more detail before it is carried out. In Section 3.2, we define precisely our tree decomposition and show that such a decomposition can always be found. We then state and discuss the regularity lemma is Section 3.3. In Section 3.4, we embed the core tree T_0 and extend the embedding to cover T_1 for Theorem 1.4 (see Theorem 3.9), deferring the most technical parts to Section 3.7 where we prove a key intermediate result, Theorem 3.13. In Section 3.5, we embed the core tree T_0 and extend the embedding to cover T_1 for Theorem 1.5 (see Theorem 3.18). These embeddings of T_0 extended to T_1 allow us then to prove both Theorem 1.4 and 1.5 in Section 3.6. We then finish with the deferred proof of Theorem 3.13 in Section 3.7.

3.2 Tree decomposition

We now give the tree decomposition discussed in the proof outline (see Figure 3.2). This is a modified version of a result of Kathapurkar and Montgomery (see [19]); though its proof is very similar, we include it for completeness.

Lemma 3.1. *Let $1/n \ll 1/m \ll \eta, 1/q$ with $q \geqslant 2$. Then, any n-vertex tree T contains forests $T_0 \subseteq T_1 \subseteq T_2 \subseteq T_3 \subseteq T_4 = T$, such that T_3 is a tree, and the following properties hold.*

C1 $|T_0| \leqslant \eta n$.

C2 *T_1 is formed from T_0 by vertex-disjointly attaching a tree S_v to each $v \in V(T_0)$, so that, for each $v \in V(T_0)$, $S_v - v$ is a forest with each component tree having size at most m.*

C3 *T_2 is the disjoint union of T_1 and a forest with each component tree having size at most m.*

C4 *T_3 is formed from T_2 by connecting components by paths of length q.*

C5 $|V(T_4) \setminus V(T_2)| \leqslant \eta n$.

Proof. Choose $\varepsilon > 0$ and $k \in \mathbb{N}$ such that $1/m \ll \varepsilon \ll 1/k \ll \eta, 1/q$. Fix an arbitrary vertex $t \in V(T)$. We start by finding a subtree T' of T which includes t and has few leaves, and is such that $T - V(T')$ is a forest of components each having size at most m. We do this by including in T' every vertex which appears on the path in T from t to many other vertices. That is, for each $v \in V(T)$, let $w(v)$ be the number of vertices $u \in V(T)$ whose path from t to u includes v (in particular, v is such a vertex). Let T' be the subgraph of T induced on all the vertices $v \in V(T)$ with $w(v) \geqslant m + 1$.

For each $v \in V(T')$, let S_v be the tree containing v in $T - (V(T') \setminus \{v\})$. Note that $S_v - v$ is a forest with each component tree having size at most m. Indeed, suppose T'' is a component of $S_v - v$, and let v' be the neighbour of v in T''. Since every path from a

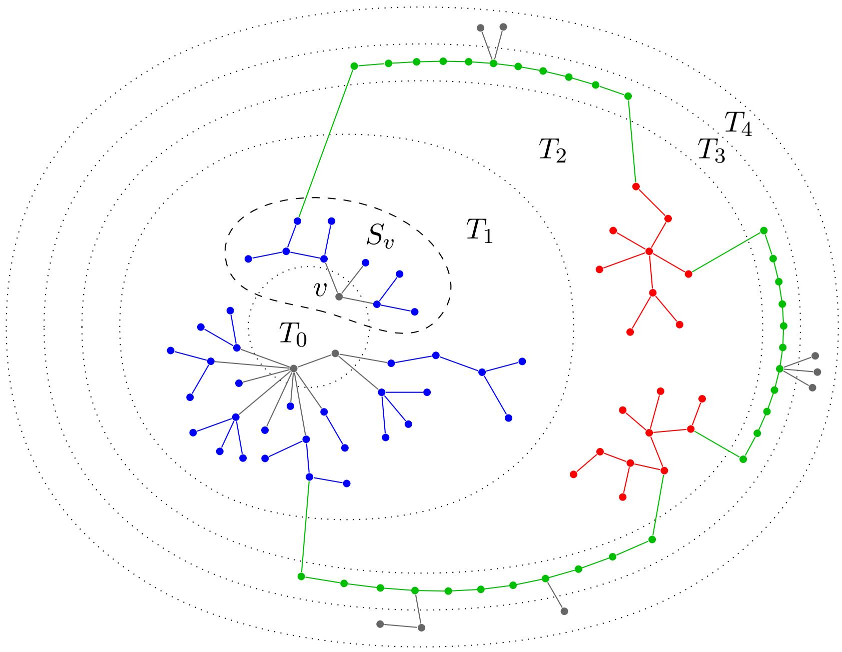

Figure 3.2: A simplified example of the tree decomposition $T_0 \subseteq T_1 \subseteq T_2 \subseteq T_3 \subseteq T_4 = T$ described by Lemma 3.1. In this illustration, the forest T_0 consists of a single edge together with an isolated vertex, and there are three paths, each with length $q = 12$, connecting components of T_2 to form the tree T_3.

vertex $u \in V(T'')$ to t in T goes through v' (and then v), we have that $m \geqslant w(v') \geqslant |T''|$ (and, in fact, the final inequality is an equality). Observe further that, for any leaf v of T', $|S_v - v| = w(v) - 1 \geqslant m$, and, therefore, T' can have at most $n/m \leqslant \varepsilon n$ leaves.

Recall that a subpath P of T' is called a bare path (in T') if all of the internal vertices v of P have $d_{T'}(v) = 2$, and we denote by $T' - P$ the graph formed from T' by removing all the edges and internal vertices of P. Using [19, Lemma 2.8], find in T' vertex disjoint bare paths $P_1, \ldots, P_r$ with length k such that

$$|T' - P_1 - \ldots - P_r| \leqslant 6k \cdot \varepsilon n + 2n/(k+1) \leqslant \eta n/4 \tag{3.1}$$

Note that $r \leqslant n/k$. For each path P_i, if possible, find within P_i a path P_i' with length at least $k - 2\eta^3 k$, such that, letting X_i, Y_i be the subpaths of P_i' induced by the first and last $q - 1$ vertices of P_i', the following hold.

(i) $\sum_{v\in V(X_i)}|S_v|, \sum_{v\in V(Y_i)}|S_v| \leqslant \eta k/4$.

(ii) Letting Q_i be the component of $T - X_i - Y_i$ containing $P'_i - X_i - Y_i$, we have $|Q_i| \leqslant m$.

Say, with relabelling, these paths are $P'_1, \ldots, P'_{r'}$. Let $T_0 = T' - P'_1 - \ldots - P'_{r'}$. We will show that $|T_0| \leqslant \eta n$. Consider first the number of paths P_i which do not have length $q-2$ subpaths X_i, Y_i, each contained within $\eta^3 k$ of each end of P_i, and for which $\sum_{v\in V(X_i)}|S_v|, \sum_{v\in V(Y_i)}|S_v| \leqslant \eta k/4$. Each such path contains at least $\lfloor \eta^3 k/(q-1)\rfloor$ disjoint length $q-2$ subpaths Z satisfying $\sum_{v\in V(Z)}|S_v| > \eta k/4$, and so the number of such paths is at most

$$\frac{n}{\lfloor \eta^3 k/(q-1)\rfloor(\eta k/4)} \leqslant \eta n/4k.$$

Of the remaining paths, at most n/m may fail to produce a P'_i due to having $|Q_i| > m$. Thus, we have $r' \geqslant r - \eta n/4k - n/m \geqslant r - \eta n/2k$.

Note that, for each $i \in [r']$, $|V(P_i) \setminus V(P'_i)| \leqslant 2\eta^3 k$. Therefore

$$|T_0| \leqslant |T' - P_1 - \ldots - P_r| + k(r - r') + r'(2\eta^3 k) \leqslant \eta n/4 + k(\eta n/2k) + r(2\eta^3 k) \leqslant \eta n,$$

and hence **C1** holds. Let $T_1 = T[\cup_{v\in V(T_0)} V(S_v)]$. Recall that for each $v \in V(T')$, $S_v - v$ is a forest with each component tree having size at most m. Therefore, **C2** holds. Let $T_2 = T_1 \cup (\cup_{i\in[r']} Q_i)$, and note that **C3** holds. Note that

$$|V(T) \setminus V(T_2)| = \sum_{i\in[r']}\sum_{v\in V(X_i)\cup V(Y_i)} |S_v| \leqslant 2r(\eta k/4) \leqslant \eta n,$$

and hence **C5** holds. Let $T_3 = T[V(T_2) \cup (\cup_{i\in[r']}(V(X_i) \cup V(Y_i)))]$ and note that **C4** holds. Finally, the only vertices missing from T_3 are those in $S_v - v$ for each $v \in \cup_{i\in[r']}(V(X_i) \cup V(Y_i))$, and hence T_3 is a tree. □

3.3 Regularity

Our embeddings use the regularity lemma for digraphs, by now a well-established tool in the study of tournaments (see, for example, [21, 22, 25]). As with the regularity lemma for graphs, this partitions most of the vertices of a tournament into clusters so that edges behave pseudorandomly between most pairs of clusters. We will now recall the notation needed to state the regularity lemma for digraphs.

Let G be a digraph. For disjoint subsets $A, B \subseteq V(G)$, define the *directed density* from A to B to be

$$d(A, B) = \frac{|E(A, B)|}{|A||B|},$$

where $E(A, B)$ denotes the set of edges of G directed from A towards B. Note that, if G is a tournament, then $d(B, A) = 1 - d(A, B)$. We say that (A, B) forms an *ε-regular pair* if, for every $X \subseteq A$ such that $|X| \geqslant \varepsilon|A|$ and every $Y \subseteq B$ such that $|Y| \geqslant \varepsilon|B|$, we have $|d(X, Y) - d(A, B)| \leqslant \varepsilon$. Note that, for tournaments, $|d(X, Y) - d(A, B)| \leqslant \varepsilon$ if and only if $|d(Y, X) - d(B, A)| \leqslant \varepsilon$. We say that (A, B) forms an ε-regular pair of density at least μ if, in addition to forming an ε-regular pair, we also have $d(A, B) \geqslant \mu$.

We will use the following directed version of Szemerédi's regularity lemma proved by Alon and Shapira [1].

Theorem 3.2 (Regularity lemma for digraphs)**.** *Let $1/r_2 \ll 1/r_1 \ll \varepsilon$. Every digraph on a vertex set V of order at least r_1 partitions as $V = V_0 \cup V_1 \cup \ldots \cup V_r$, with $r_1 \leqslant r \leqslant r_2$, satisfying the following.*

D1 $|V_0| \leqslant \varepsilon|V|$.

D2 $|V_1| = \ldots = |V_r|$.

D3 *All but at most εr^2 pairs (V_i, V_j) with $1 \leqslant i < j \leqslant r$ are ε-regular.*

We now state the definition of an *ε-regular partition.* For convenience, we use a slightly different definition of an ε-regular partition of a tournament than is directly produced by Theorem 3.2, but which is gained through the removal of few clusters (see Corollary 3.4).

Definition 3.3. *An* ε-regular partition *of a tournament* G *is a partition* $V(G) = V_1 \cup \ldots \cup V_r$ *with* $|V_1| = \ldots = |V_r|$ *such that, for each fixed* $i \in [r]$*,* (V_i, V_j) *forms an* ε*-regular pair for all but at most* $\sqrt{\varepsilon}r$ *many* $j \in [r]$*.*

Corollary 3.4. *Let* $\alpha > \beta > 0$ *and* $1/n \ll 1/r_2 \ll 1/r_1 \ll \varepsilon \ll \beta$*. Let* G *be a* $(1+\alpha)n$*-vertex tournament. Then, there is a subtournament* $G' \subseteq G$ *with* $|G'| \geqslant (1+\alpha-\beta)n$*, and an* ε*-regular partition* $V(G') = V_1 \cup \ldots \cup V_r$ *with* $r_1 \leqslant r \leqslant r_2$*.*

Proof. Given a tournament G, using Theorem 3.2, take a partition $V(G) = V_0 \cup V_1 \cup \ldots \cup V_{\bar{r}}$, with $2r_1 \leqslant \bar{r} \leqslant r_2$, satisfying **D1**-**D3**. By reordering, we may suppose there is some r with $0 \leqslant r \leqslant \bar{r}$ such that, for each fixed $i \in [\bar{r}]$, (V_i, V_j) forms an ε-regular pair with at least $(1 - \sqrt{\varepsilon}/2)\bar{r}$ many $j \in [\bar{r}]$ if and only if $i \in [r]$. By **D3**, we find $(\bar{r} - r)\sqrt{\varepsilon}\bar{r}/2 \leqslant \varepsilon\bar{r}^2$, and hence $r \geqslant (1 - 2\sqrt{\varepsilon})\bar{r} \geqslant r_1$. Let $G' = G[V_1 \cup \ldots \cup V_r]$. The desired properties for G' then follow by noting that $|V(G) \setminus V(G')| \leqslant |V_0| + \frac{\bar{r}-r}{\bar{r}}|G| \leqslant (\varepsilon + 2\sqrt{\varepsilon})|G| \leqslant \beta n$, and that $\sqrt{\varepsilon}\bar{r}/2 \leqslant \sqrt{\varepsilon}r$. □

We will use the following simple proposition on vertex degrees in ε-regular partitions.

Proposition 3.5. *Let* $\varepsilon, \mu > 0$ *and* $r, m \in \mathbb{N}$*. Suppose* G *is a tournament with disjoint subsets* $W, V_1, \ldots, V_r \subseteq V(G)$ *of size* $|W| = |V_1| = \ldots = |V_r| = m$*, such that* (W, V_i) *is an* ε*-regular pair of density at least* μ *for each* $1 \leqslant i \leqslant r$*. Fix a subset* $U \subseteq \cup_{i \in [r]} V_i$*. Then, all but at most* εm *vertices of* W *have at least* $(\mu - \varepsilon)(|U| - \varepsilon rm)$ *out-neighbours in* U*.*

Proof. Let Z be the set of vertices of W which have fewer than $(\mu - \varepsilon)(|U| - \varepsilon rm)$ out-neighbours in U, and suppose that $|Z| \geqslant \varepsilon m$. Then, for each $i \in [r]$, because (W, V_i) is an ε-regular pair of density at least μ, there are at least $|Z|(\mu - \varepsilon)(|U \cap V_i| - \varepsilon m)$ edges directed from Z to $U \cap V_i$, noting that this is trivial if $|U \cap V_i| \leqslant \varepsilon m$. Therefore, there are at least $|Z|(\mu - \varepsilon)(|U| - \varepsilon rm)$ edges directed from Z to U. However, from the definition of Z, the number of edges from Z to U is less than $|Z|(\mu - \varepsilon)(|U| - \varepsilon rm)$, a contradiction. □

Our proofs will often allocate the vertices of a tree to the clusters of a regularity partition, before applying variations of standard regularity methods to embed these vertices so that they are (mostly) embedded to their assigned cluster. For this we will use, in part, the following simple proposition, which embeds a tree from an assignment in this way, provided that the tree is small and also that the vertices of the tree are not assigned to too many different clusters.

Proposition 3.6. *Let $1/m \ll \varepsilon \ll \beta, \mu, 1/\ell$. Suppose G is a tournament with subsets $V_1, \ldots, V_\ell \subseteq V(G)$ of size $|V_1| = \ldots = |V_\ell| = m$, and, for $j \in [\ell]$, let $U_j \subseteq V_j$ have size $|U_j| \geqslant \beta m$. Let T be an oriented tree with $|T| \leqslant \varepsilon m$, and suppose $\varphi : V(T) \to [\ell]$ is such that if $uv \in E(T)$ and $\varphi(u) \neq \varphi(v)$, then $(V_{\varphi(u)}, V_{\varphi(v)})$ is an ε-regular pair of density at least μ. Then, there is an embedding $\psi : T \to G$ with $\psi(v) \in U_{\varphi(v)}$ for each $v \in V(T)$.*

Proof. Let $V(T) = Y_1 \cup \ldots \cup Y_r$ be a partition such that

- for each $i \in [r]$, $T[Y_i]$ is a connected component of $T[\phi^{-1}(j)]$ for some $j \in [\ell]$, and
- for each $i \in [r]$, $T[Y_1 \cup \ldots \cup Y_i]$ is a tree.

Let $s \in \{0\} \cup [r]$ be maximal such that, if $T_s = T[Y_1 \cup \ldots \cup Y_s]$, then there is an embedding $\psi : T_s \to G$ with $\psi(v) \in U_{\varphi(v)}$ for every $v \in V(T_s)$, and, for every $v \in V(T_s)$ and $j \in [\ell]$ for which $(V_{\varphi(v)}, V_j)$ is an ε-regular pair, we have $d^+_G(\psi(v), U_j) \geqslant (d(V_{\varphi(v)}, V_j) - \varepsilon)\beta m$ and $d^-_G(\psi(v), U_j) \geqslant (d(V_j, V_{\varphi(v)}) - \varepsilon)\beta m$. Suppose, for contradiction, that $s < r$. If $s = 0$, then let $y \in Y_1$ be arbitrary and set $Z_1 = U_{\varphi(y)}$. If instead we have $s > 0$, then let $x \in V(T_s)$, $y \in Y_{s+1}$ and $\diamond \in \{+, -\}$ be such that $y \in N^\diamond_T(x)$, and set $Z_{s+1} = N^\diamond_G(\psi(x), U_{\varphi(y)})$. In either case, we find that $|Z_{s+1}| \geqslant \beta\mu m/2$ and $Z_{s+1} \subseteq U_{\varphi(y)}$. For each $j \in [\ell]$ such that $(V_{\varphi(y)}, V_j)$ is an ε-regular pair, all but at most εm vertices z of Z_{s+1} satisfy $d^+_G(z, U_j) \geqslant (d(V_{\varphi(y)}, V_j) - \varepsilon)\beta m$ and all but at most εm vertices z of Z_{s+1} satisfy $d^-_G(z, U_j) \geqslant (d(V_j, V_{\varphi(y)}) - \varepsilon)\beta m$. Therefore, as $\varepsilon \ll \beta, \mu, 1/\ell$, there is a subset $Z'_{s+1} \subseteq Z_{s+1} \setminus \psi(V(T_s))$ with $|Z'_{s+1}| \geqslant \beta\mu m/4$, such that, for every $z \in Z'_{s+1}$ and $j \in [\ell]$ for which $(V_{\varphi(y)}, V_j)$ is an ε-regular pair, we have $d^+_G(z, U_j) \geqslant (d(V_{\varphi(y)}, V_j) - \varepsilon)\beta m$ and $d^-_G(z, U_j) \geqslant (d(V_j, V_{\varphi(y)}) - \varepsilon)\beta m$. But then, by Theorem 1.8, there is a copy of $T[Y_{s+1}]$ in $G[Z'_{s+1}]$, and so we can extend ψ to cover Y_{s+1}, a contradiction to the maximality of s. □

As is common, given an ε-regular partition $V_1 \cup \ldots \cup V_r$ of a tournament G, we will consider the *reduced digraph* R for the partition which has $V(R) = [r]$, and $ij \in E(R)$ exactly when (V_i, V_j) is an ε-regular pair with density comfortably larger than ε. We will sometimes delete edges arbitrarily from R so that there is at most 1 edge between any pair of vertices. As a small proportion of pairs of clusters in an ε-regular partition may not form an ε-regular pair, this will not necessarily result in a tournament. For this, we define an *ε-almost tournament*, as follows.

Definition 3.7. *An* ε-almost tournament *R is an oriented graph in which, for each $v \in V(R)$, there are at most $\varepsilon|R|$ vertices $u \in V(R)$ with $vu \notin E(R)$ and $uv \notin E(R)$.*

We will use the following simple property of ε-almost tournaments, which shows they each have some vertex with a good number of both in- and out-neighbours.

Proposition 3.8. *Let R be an ε-almost tournament on r vertices. Then, there exists a vertex $v \in V(R)$ such that $d^+_R(v), d^-_R(v) \geqslant \frac{r-1}{4} - \varepsilon r$.*

Proof. Let H be any tournament with $V(H) = V(R)$ such that $R \subseteq H$, and let $m = \frac{r-1}{4}$. Any set of $2m+1$ vertices in H contains a vertex with out-degree at least m and a vertex with in-degree at least m. Therefore, all but at most $2m$ vertices of H have out-degree at least m, and all but at most $2m$ vertices of H have in-degree at least m. Therefore, as $n > 4m$, there is some $v \in V(H)$ with $d^+_H(v), d^-_H(v) \geqslant \frac{r-1}{4}$. Then, $v \in V(R)$ satisfies $d^+_R(v), d^-_R(v) \geqslant \frac{r-1}{4} - \varepsilon r$. □

3.4 Theorem 1.4: embedding the core and attached small trees

In this section, following the proof outline in Section 3.1, we embed T_0 and T_1 for Theorem 1.4. In the embeddings we may assume that T_0 is connected (i.e., that it is a tree not just a forest), and so our embedding of T_0 and T_1 will follow by identifying T_1 with T in the following theorem.

Theorem 3.9. *Let $1/n \ll \eta \ll \bar{\alpha} < 1$. Suppose T is an n-vertex k-leaf oriented tree with a subtree $T_0 \subseteq T$, such that $|T_0| \leqslant \eta n$ and every component of $T - V(T_0)$ has size at most ηn. Then, any $((1+\bar{\alpha})n + k)$-vertex tournament contains a copy of T.*

Note that, within this section, T will refer to trees satisfying the hypotheses of Theorem 3.9 (i.e., those usually denoted as T_1 elsewhere in this chapter).

We require, for Theorem 3.9, the components of $T - V(T_0)$ to be bounded above by ηn, for some appropriately small η. This is not required for its application, where these components will have constant size (see **C2** earlier), but this small linear bound follows at no additional cost. As discussed in Section 3.1, to embed the core T_0 and extend this embedding to T, we will first allocate the vertices of the tree to regularity clusters. This allocation requires care beyond that in previous embeddings of trees in tournaments (see [21, 22, 25]) as the large degree of some vertices in the tree require edges not just to have sufficient density for regularity embedding techniques to be effective (i.e., $\varepsilon \ll \mu$ in Proposition 3.6), but sufficient density for potentially linearly many neighbours of a vertex to be embedded within the same regularity cluster. For this, we find it convenient to consider an ε-regular partition of clusters $V_1 \cup \ldots \cup V_r$ as a weighted complete looped digraph D with vertex set $[r]$ and edge weights $d(e) \in [0, 1]$, $e \in E(D)$, indicating the edge density between ε-regular pairs of clusters. We call the sets of edge weights we typically encounter *ε-complete*, as follows.

Definition 3.10. *Given a complete looped digraph D on vertex set $[r]$, we say edge weights $d(e) \in [0, 1]$, $e \in E(D)$, are* ε-complete *if the following holds.*

- **E** *For each $j \in [r]$, $d(j, j) = 1$ and, for all but at most εr values of $i \in [r] \setminus \{j\}$, $d(i, j) + d(j, i) = 1$.*

From this perspective, an allocation of the vertices of an oriented tree T to the regularity clusters is a map from $V(T)$ to the vertices of a complete looped digraph D with edge weights representing the density of edges between regularity clusters. Considering the role of μ in Proposition 3.6, it is important that whenever endpoints of an edge of

T are allocated to different clusters, those clusters form an ε-regular pair of density at least μ. When D is introduced in the proof later, we will adjust the weights on some edges to ensure that $d(i, j) = 0$ whenever $d(V_i, V_j) < \mu$. Thus, a valid allocation will be a homomorphism from T to D, defined as follows.

Definition 3.11. *Given a digraph H and a complete looped digraph D with associated edge-weights $d(e) \in [0, 1]$, $e \in E(D)$, we say that a function $\phi : V(H) \to V(D)$ is a* homomorphism from H to D *if $d(\phi(v), \phi(w)) > 0$ whenever $vw \in E(H)$.*

We need to find such a homomorphism from T to D satisfying additional properties, such as a limit to how many vertices are assigned to each cluster. The allocation we find for Theorem 3.9 will always assign the vertices of T_0 to a single cluster, whose index we call j_t, and then distribute the vertices of the components of $T - V(T_0)$ across the other regularity clusters. Rather than considering the components of $T - V(T_0)$ directly, we will work with a small vertex-weighted digraph H, which represents an average of the components of $T - T(V_0)$. We will find a probability distribution $\mathcal{D}$ on the set of homomorphisms from H to D so that when we assign vertices of each component of $T - V(T_0)$ according to an independent sampling of $\mathcal{D}$, then, with high probability, the resulting homomorphism from T to D has the required properties. Working in this randomised setting allows us to obtain this homomorphism concisely and without the need to consider the many distinct oriented trees which may appear as components of $T - V(T_0)$.

The existence of the probability distribution $\mathcal{D}$ is asserted by Theorem 3.13 below, which is used in this section as a starting point for Theorem 3.9. The proof of Theorem 3.13 itself is the most involved part of this chapter, and so we defer this to Section 3.7. Before stating Theorem 3.13, and also explaining the statement in more detail, we first define the digraph H used to represent the components of $T - V(T_0)$ (see also Figure 3.3). To simplify the notation relating to H, we make use of the following definition.

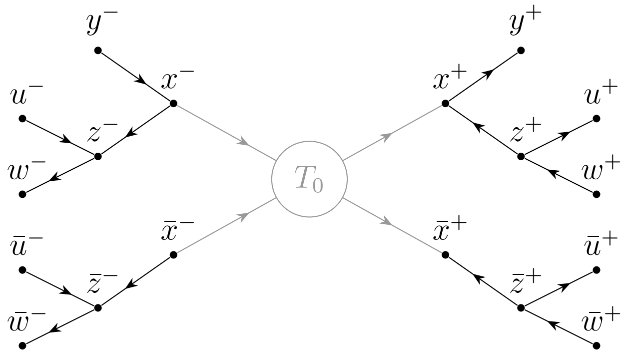

Figure 3.3: The fully-looped oriented forest H (with looped edges omitted).

Definition 3.12. *Say that a digraph F is a* fully-looped oriented forest *if F has a looped edge on every vertex, and the deletion of all looped edges from F leaves an oriented forest.*

Let H be the fully-looped oriented forest with vertex and edge sets given by

$$\begin{aligned} V(H) &= \left\{x^+, y^+, z^+, u^+, w^+, \bar{x}^+, \bar{z}^+, \bar{u}^+, \bar{w}^+, x^-, y^-, z^-, u^-, w^-, \bar{x}^-, \bar{z}^-, \bar{u}^-, \bar{w}^-\right\}, \\ E(H) &= \left\{ \begin{array}{c} x^+y^+, z^+x^+, z^+u^+, w^+z^+, \bar{z}^+\bar{x}^+, \bar{z}^+\bar{u}^+, \bar{w}^+\bar{z}^+, \\ y^-x^-, x^-z^-, u^-z^-, z^-w^-, \bar{x}^-\bar{z}^-, \bar{u}^-\bar{z}^-, \bar{z}^-\bar{w}^- \end{array} \right\} \cup \{vv : v \in V(H)\}. \end{aligned} \tag{3.2}$$

For each $\diamond \in \{+,-\}$, let $X^\diamond = \{x^\diamond, \bar{x}^\diamond\}$. Let $X = X^+ \cup X^-$.

We are now ready to state Theorem 3.13. Note that, for the function $\beta : V(H) \to [0,1]$, if $A \subseteq V(H)$ we will often write $\beta(A)$ to mean $\sum_{v\in A} \beta(v)$ and $\beta(v_1, \dots, v_k)$ to mean $\beta(\{v_1, \dots, v_k\})$.

Theorem 3.13. *Let $1/r \ll \varepsilon \ll \mu \ll \alpha < 1$. Let H be the fully-looped oriented forest with vertex and edge sets given by (3.2). For each $\diamond \in \{+,-\}$, set $X^\diamond = \{x^\diamond, \bar{x}^\diamond\}$, and set $X = X^+ \cup X^-$. Let $\beta : V(H) \to [0,1]$ be a function satisfying $\sum_{v\in V(H)} \beta(v) = 1$ with $\beta(y^+) \geqslant \beta(x^+)$ and $\beta(y^-) \geqslant \beta(x^-)$, and, for every $v \in V(H)$, $\beta(v) \geqslant \mu$. Let D be a complete looped digraph on vertex set $[r]$ with ε-complete edge weights $d(e)$ for $e \in E(D)$. Let*

$$\gamma = \max\{\beta(x^+, \bar{x}^+), \beta(z^+, \bar{z}^+)\} + \max\{\beta(x^-, \bar{x}^-), \beta(z^-, \bar{z}^-)\}. \tag{3.3}$$

Then, there is a fixed $j_t \in [r]$ and a probability distribution $\mathcal{D}$ on the set of functions from

$V(H)$ to $V(D)$, such that, if ϕ is sampled according to $\mathcal{D}$, then the following properties hold.

F1 *With probability 1, ϕ is a homomorphism from H to D, and $j_t \notin \phi(\{x^+, \bar{x}^+, x^-, \bar{x}^-\})$.*

F2 *For each $j \in [r]$, $\mathbb{E}(\beta(\phi^{-1}(j))) \leqslant \frac{1+\gamma+\alpha}{r}$.*

F3 *For each $j \in [r]$, either*

F3.1 *$\mathbb{E}(\beta(\phi^{-1}(j) \cap X^+)) \leqslant d(j_t, j) \cdot \frac{1+\gamma+\alpha}{r}$ and $\mathbb{E}(\beta(\phi^{-1}(j) \cap X)) \leqslant d(j, j_t) \cdot \frac{1+\gamma+\alpha}{r}$, or*

F3.2 *$\mathbb{E}(\beta(\phi^{-1}(j) \cap X^-)) \leqslant d(j, j_t) \cdot \frac{1+\gamma+\alpha}{r}$ and $\mathbb{E}(\beta(\phi^{-1}(j) \cap X)) \leqslant d(j_t, j) \cdot \frac{1+\gamma+\alpha}{r}$.*

F4 *With probability 1, we have $|\phi(e)| = 2$ for every non-looped edge e of H.*

The technical nature of Theorem 3.13 is a result of the complications involved in finding an appropriate allocation of the vertices of $T - V(T_0)$ to the regularity clusters represented by D. Having chosen a single cluster (indexed by j_t) for the embedding of T_0, the restriction on where a vertex u in $T - V(T_0)$ can be embedded depends on the path from T_0 to u in T, and its edge directions. However, it will turn out that all we need to consider is what proportion of the vertices in $T - V(T_0)$ have paths from T_0 beginning with any given oriented path of length at most 3. Accordingly, in the application of Theorem 3.13, each vertex u of $T - V(T_0)$ will be associated to a vertex of H depending on the orientation of the first few edges on the path from T_0 to u. Then, each vertex v of H will be given weight $\beta(v)$ roughly equal to the proportion of vertices of $T - V(T_0)$ associated to v. It is in this sense that the digraph H together with the weight function β represents an average of the components of $T - V(T_0)$. Vertices in X represent the vertices connected by an edge to T_0 in T, which may therefore be neighbours of any very high degree vertices in T_0, and so we need to pay particular attention to how often they are assigned to each regularity cluster.

Identifying H and β in this way helps contextualise the statement of Theorem 3.13. Suppose j_t is fixed as the index corresponding to the image of T_0, and ϕ is sampled according to $\mathcal{D}$. By **F1**, we will have with probability 1 that ϕ is a homomorphism, so that the regularity properties can be used to embed any edges assigned between two regularity clusters, and, for convenience, no vertex in X is assigned to j_t. **F2** ensures that (on average) not too much weight is allocated to a single cluster. **F3** ensures that (on average) the weight of vertices in X^+ (i.e., those which need to be attached by an out-edge to T_0, which is allocated to j_t), or X^-, allocated to each cluster is not too much, where the limit is dictated by the appropriate density from that cluster to j_t, or from j_t to that cluster. Finally, **F4** is present to later ensure that vertices of $N_T(V(T_0))$ are allocated to a different cluster to their neighbours, which will assist with the embedding process.

We use the parameter γ (see (3.3) in Theorem 3.13) to control the total size of components we can embed using ϕ relative to the size of the tournament from which D is derived. As we use this for Theorem 1.4 it should be related to the number of leaves. In the application of Theorem 3.13, the weight on the vertices with base label 'x' or 'z' is distributed so that it can be bounded based on the number of leaves of the original tree (and in certain cases uses a lower bound than that required for Theorem 1.4).

We now sketch the proof of Theorem 3.9 from Theorem 3.13. In this proof, we first define a homomorphism f from $T - V(T_0)$ to H, and then use f to define an appropriate weight function $\beta : V(H) \to [0, 1]$. Then, taking an ε-regular partition of a tournament G with vertex classes $V_1, \dots, V_r$, we choose the appropriate edge-weighted complete looped digraph D. Applying Theorem 3.13, we obtain $j_t \in [r]$ and the probability distribution $\mathcal{D}$. We then embed T_0 into a subset of good vertices inside V_{j_t}. By independently sampling ϕ according to $\mathcal{D}$ for each component of $T - V(T_0)$, we then get a homomorphism $\hat{\varphi}$ from $T - V(T_0)$ to D. We will see later (in Claim 3.15) that $\hat{\varphi}$ is a good guide for assigning vertices of $T - V(T_0)$ to the clusters $V_1, \dots, V_r$.

Because T_0 may have vertices with high degree, we need to allocate room in the clusters for vertices in $N_T(V(T_0))$ before embedding the rest of $T - V(T_0)$. However, if we fix the images of $N_T(V(T_0))$, then the embedding of the components of $N_T(V(T_0))$ is restricted in a way that possibly prevents us from being able to exactly follow the allocation given by $\hat{\varphi}$. To handle this, we consider large components and small components separately. Precisely, using S_x to denote the component of $T - V(T_0)$ containing $x \in N_T(V(T_0))$, we partition $N_T(V(T_0)) = X_0 \cup Y_0$ such that S_x is at most constant-sized whenever $x \in X_0$ (and larger than constant-sized whenever $x \in Y_0$). We then embed as much of $\cup_{x \in X_0 \cup Y_0} V(S_x)$ as possible. Given $x \in X_0$, if it is not possible to embed S_x according to $\hat{\varphi}$, then we may still be able to find an embedding for S_x by switching its allocation with an identical component. Indeed, if a significant number of $x \in X_0$ are yet to have their corresponding component embedded, then the bound on $|S_x|$ for $x \in X_0$ helps to find a suitable identical pair. Thus, we can extend the embedding to cover most of $\cup_{x \in X_0} V(S_x)$. On the other hand, Y_0 is small, and so the corresponding larger components can be all be embedded greedily using specially reserved sets for their roots. Finally, any remaining components of $T - V(T_0)$ that are not embedded can then be handled by greedily embedding them to a random subset $U \subseteq V(G)$ reserved at the beginning of the proof.

Proof of Theorem 3.9. Let $\alpha = \bar{\alpha}/35$ and introduce constants $\mu, \varepsilon, r_1, r_2$ such that $\eta \ll 1/r_2 \ll 1/r_1 \ll \varepsilon \ll \mu \ll \alpha$. For each $x \in N_T(V(T_0))$, let S_x be the component of $T - V(T_0)$ containing x.

Let G be a $((1 + 35\alpha)n + k)$-vertex tournament, and note that $n \leqslant |G| \leqslant 3n$. Let $U \subseteq V(G)$ be a random subset, with elements from $V(G)$ chosen independently at random with probability 2α, and let V' be the set of vertices $v \in V(G) \setminus U$ with $d^{\pm}_G(v, U) \geqslant 4\alpha^2 n$. By Proposition 1.20, we may proceed assuming that $|U| \geqslant \alpha n$ and $|V'| \geqslant ((1+11\alpha)n+k)$.

Define $f_0 : N_T(V(T_0)) \to \{x^+, \bar{x}^+, x^-, \bar{x}^-\}$ as follows. For each $\diamond \in \{+, -\}$ and $v \in N^{\diamond}_T(V(T_0))$, set $f_0(v) = x^{\diamond}$ if $N^{\diamond}_T(v) \setminus V(T_0) \neq \emptyset$, and set $f_0(v) = \bar{x}^{\diamond}$ if $N^{\diamond}_T(v) \setminus V(T_0) = \emptyset$. Then, let $f : V(T) \setminus V(T_0) \to V(H)$ be the homomorphism from $T - V(T_0)$ to H extending f_0 such that $f^{-1}(\{x^+, \bar{x}^+, x^-, \bar{x}^-\}) = N_T(V(T_0))$, $f^{-1}(\{z^+, \bar{z}^+\}) = N^-_T(N^+_T(V(T_0))) \setminus V(T_0)$, and $f^{-1}(\{z^-, \bar{z}^-\}) = N^+_T(N^-_T(V(T_0))) \setminus V(T_0)$. (Note that this homomorphism is unique, as each vertex in $(N^-_T(N^+_T(V(T_0))) \cup N^+_T(N^-_T(V(T_0)))) \setminus V(T_0)$ has only one viable candidate for its image among $\{z^+, \bar{z}^+, z^-, \bar{z}^-\}$, and once those vertices have their images fixed, each remaining vertex has only one viable candidate for its image among $\{y^+, u^+, w^+, \bar{u}^+, \bar{w}^+, y^-, u^-, w^-, \bar{u}^-, \bar{w}^-\}$.)

Let $\beta : V(H) \to [0,1]$ be given by setting, for each $v \in V(H)$,

$$\beta(v) = \frac{|f^{-1}(v)| + 2\mu n}{|V(T) \setminus V(T_0)| + 36\mu n}, \tag{3.4}$$

and note that, because $|V(H)| = 18$, β is a function satisfying $\sum_{v \in V(H)} \beta(v) = 1$, and $\beta(v) \geqslant \mu$ for every $v \in V(H)$. Set

$$\gamma = \max\{\beta(x^+, \bar{x}^+), \beta(z^+, \bar{z}^+)\} + \max\{\beta(x^-, \bar{x}^-), \beta(z^-, \bar{z}^-)\}. \tag{3.5}$$

We remark that, for each $\diamond \in \{+,-\}$, if $v \in f^{-1}(x^\diamond)$, then there is some $v' \in N_T^\diamond(v) \setminus V(T_0)$ with $f(v') = y^\diamond$. Therefore, $\beta(y^+) \geqslant \beta(x^+)$ and $\beta(y^-) \geqslant \beta(x^-)$. We also remark that, for each $x \in N_T(V(T_0))$, the number of leaves of T appearing in S_x is at least $\max\{1, |f^{-1}(\{z^+, \bar{z}^+, z^-, \bar{z}^-\}) \cap V(S_x)|\}$, and hence

$$\begin{aligned} k &\geqslant \sum_{x \in N_T(V(T_0))} \max\{1, |f^{-1}(\{z^+, \bar{z}^+, z^-, \bar{z}^-\}) \cap V(S_x)|\} \\ &\geqslant \max\{|N_T^+(V(T_0))|, |f^{-1}(\{z^+, \bar{z}^+\})|\} + \max\{|N_T^-(V(T_0))|, |f^{-1}(\{z^-, \bar{z}^-\})|\} \\ &= \max\{|f^{-1}(\{x^+, \bar{x}^+\})|, |f^{-1}(\{z^+, \bar{z}^+\})|\} + \max\{|f^{-1}(\{x^-, \bar{x}^-\})|, |f^{-1}(\{z^-, \bar{z}^-\})|\}. \end{aligned}$$

Therefore, by (3.4) and (3.5), $\gamma \leqslant \alpha + k/n$ and hence $|V'| \geqslant (1 + \gamma + 10\alpha) \cdot n$.

By Corollary 3.4, there is some r with $r_1 \leqslant r \leqslant r_2$ and disjoint subsets $V_1, \ldots, V_r \subseteq V'$, with $|V_1| = \ldots = |V_r| \geqslant (1 + \gamma + 9\alpha) \cdot n/r$, such that $V_1 \cup \ldots \cup V_r$ is an ε-regular partition of $G[V_1 \cup \ldots \cup V_r]$. Let D be a complete looped digraph on vertex set $[r]$ with edge weights $d(e)$, $e \in E(D)$ given by setting $d(j,j) = 1$ for every $j \in [r]$, $d(j,j') = 0$ for every $jj' \in E(D)$ for which $j \neq j'$ and $(V_j, V_{j'})$ is not an ε-regular pair, and, for every

$jj' \in E(D)$ for which $(V_j, V_{j'})$ forms an ε-regular pair, setting

$$d(j,j') = \begin{cases} 1 & \text{if } d(V_j,V_{j'}) > 1-\mu, \\ d(V_j,V_{j'}) & \text{if } \mu \leqslant d(V_j,V_{j'}) \leqslant 1-\mu, \\ 0 & \text{if } d(V_j,V_{j'}) < \mu. \end{cases}$$

We remark that the edge weights $d(e)$, $e \in E(D)$ are $\sqrt{\varepsilon}$-complete, and, for $j \neq j'$, if $d(j,j') > 0$, then $(V_j, V_{j'})$ is an ε-regular pair with density satisfying $d(V_j, V_{j'}) \geqslant \mu$ and $d(V_j, V_{j'}) \geqslant (1-\mu) \cdot d(j,j')$.

By Theorem 3.13 applied to β and D, there is some $j_t \in [r]$ and probability distribution $\mathcal{D}$ on the set of functions from $V(H)$ to $V(D)$, such that, if ϕ is sampled according to $\mathcal{D}$, then **F1**-**F4** hold. Let J_1 be the set of $j \in [r]$ for which we have **F3.1**, and let $J_2 = [r] \setminus J_1$, so that **F3.2** holds for every $j \in J_2$. For $j \in [r]$, let $U_j, W_j \subseteq V_j$ be disjoint subsets with $|U_j| = (1+\gamma+4\alpha) \cdot n/r$ and $|W_j| = 3\alpha \cdot n/r$. Let Z be the set of $z \in V_{j_t} \setminus (U_{j_t} \cup W_{j_t})$ such that the following holds for ϕ with probability at least $1-\sqrt{\varepsilon}$.

$$\begin{aligned} d^+_G(z, U_{\phi(x^+)}) &\geqslant d(j_t, \phi(x^+)) \cdot (1+\gamma+3\alpha) \cdot n/r, & d^+_G(z, W_{\phi(x^+)}) &\geqslant 2\alpha\mu \cdot n/r, \\ d^+_G(z, U_{\phi(\bar{x}^+)}) &\geqslant d(j_t, \phi(\bar{x}^+)) \cdot (1+\gamma+3\alpha) \cdot n/r, & d^+_G(z, W_{\phi(\bar{x}^+)}) &\geqslant 2\alpha\mu \cdot n/r, \\ d^-_G(z, U_{\phi(x^-)}) &\geqslant d(\phi(x^-), j_t) \cdot (1+\gamma+3\alpha) \cdot n/r, & d^-_G(z, W_{\phi(x^-)}) &\geqslant 2\alpha\mu \cdot n/r, \\ d^-_G(z, U_{\phi(\bar{x}^-)}) &\geqslant d(\phi(\bar{x}^-), j_t) \cdot (1+\gamma+3\alpha) \cdot n/r, & d^-_G(z, W_{\phi(\bar{x}^-)}) &\geqslant 2\alpha\mu \cdot n/r. \end{aligned} \tag{3.6}$$

Claim 3.14. *$|Z| \geqslant \alpha \cdot n/r$.*

Proof of Claim 3.14. Let $\bar{Z}$ be the set of $z \in V_{j_t}$ such that (3.6) fails with probability at least $\sqrt{\varepsilon}$. If $|\bar{Z}| < \alpha \cdot n/r$, then, as $|V_{j_t} \setminus (U_{j_t} \cup W_{j_t})| \geqslant 2\alpha \cdot n/r$, the claim follows. So assume for contradiction that $|\bar{Z}| \geqslant \alpha \cdot n/r$.

Let Ω be the set of homomorphisms $\bar{\phi} : H \to D$ such that $j_t \notin \bar{\phi}(\{x^+, \bar{x}^+, x^-, \bar{x}^-\})$ and $d(j_t, \bar{\phi}(x^+))$, $d(j_t, \bar{\phi}(\bar{x}^+))$, $d(\bar{\phi}(x^-), j_t))$, and $d(\bar{\phi}(\bar{x}^-), j_t)$ are all positive, and hence, by our choice of $d(e)$, $e \in E(D)$, are all at least μ. Note that, by **F1**, $\mathbb{P}(\phi \in \Omega) = 1$. Given $\bar{\phi} \in \Omega$, let $B_{\bar{\phi}}$ be the set of $z \in V_{j_t}$ such that (3.6) fails for $\phi = \bar{\phi}$. We claim that

$|B_{\bar\phi}| \leqslant 24\varepsilon \cdot n/r$ for every $\bar\phi \in \Omega$. Indeed, if $\bar\phi \in \Omega$, then $(V_{j_t}, V_{\bar\phi(x^+)})$ is an ε-regular pair of density $d(V_{j_t}, V_{\bar\phi(x^+)}) \geqslant \min\{d(j_t, \bar\phi(x^+)), 1-\mu\} \geqslant \mu$, and so the number of $z \in V_{j_t}$ for which we do not have $d^+_G(z, U_{\bar\phi(x^+)}) \geqslant d(j_t, \bar\phi(x^+)) \cdot (1+\gamma+3\alpha) \cdot n/r$ is at most $\varepsilon|V_{j_t}| \leqslant 3\varepsilon \cdot n/r$ (using, for example, Proposition 3.5, with $r' = 1$ and $\mu' = d(V_{j_t}, V_{\bar\phi(x^+)})$). More generally, if $\bar\phi \in \Omega$, then $(V_{j_t}, V_{\bar\phi(x^+)})$, $(V_{j_t}, V_{\bar\phi(\bar x^+)})$, $(V_{\bar\phi(x^-)}, V_{j_t})$, and $(V_{\bar\phi(\bar x^-)}, V_{j_t})$ all form ε-regular pairs (of density at least μ), and thus each one of the inequalities of (3.6) fails for at most $3\varepsilon \cdot n/r$ many $z \in V_{j_t}$. Hence we have $|B_{\bar\phi}| \leqslant 8 \cdot 3\varepsilon \cdot n/r = 24\varepsilon \cdot n/r$ for every $\bar\phi \in \Omega$, as claimed. But then

$$\alpha\sqrt{\varepsilon} \cdot n/r \leqslant |\bar Z| \cdot \sqrt{\varepsilon} \leqslant \sum_{\bar\phi \in \Omega} \mathbb{P}(\phi = \bar\phi) \cdot |B_{\bar\phi}| \leqslant 24\varepsilon \cdot n/r,$$

a contradiction as $\varepsilon \ll \alpha$. ⊡

Note that, by Claim 3.14 and as $\eta \ll \alpha, 1/r_2$, $|Z| \geqslant 3\eta n \geqslant 3|T_0|$. Therefore, using Theorem 1.8, let $\psi : T_0 \to G$ be an embedding so that $\psi(V(T_0)) \subseteq Z$. For each $x \in N_T(V(T_0))$, let $z_x \in Z$ be the image under ψ of the unique neighbour of x in $V(T_0)$. Our aim now is to extend ψ to cover the components S_x, $x \in N_T(V(T_0))$, with each $\psi(x)$ in the appropriate in- or out-neighbourhood of z_x.

Given $v \in V(T) \setminus V(T_0)$, let $x(v) \in N_T(V(T_0))$ be the unique vertex such that $v \in V(S_{x(v)})$. For each $x \in N_T(V(T_0))$, choose a homomorphism $\phi_x : H \to D$ by sampling ϕ, conditioned on the event that (3.6) holds for $z = z_x$. Define a function $\hat\varphi : V(T) \setminus V(T_0) \to [r]$ by setting $\hat\varphi(v) = \phi_{x(v)}(f(v))$. We remark that $\hat\varphi$ is a homomorphism from $T - V(T_0)$ to D, with $|\hat\varphi(V(S_x))| \leqslant |H|$ for every $x \in N_T(V(T_0))$.

Let X_0 be the set of $x \in N_T(V(T_0))$ with $|S_x| \leqslant 1/\mu^3$, and let $Y_0 = N_T(V(T_0)) \setminus X_0$, so that $|S_x| > 1/\mu^3$ whenever $x \in Y_0$. Note that $|Y_0| \leqslant \mu^3 n$. Roughly speaking, we will try to embed each $v \in V(T) \setminus V(T_0)$ into $V_{\hat\varphi(v)}$, with each $v \in Y_0$ embedded into $W_{\hat\varphi(v)}$ and each $v \in V(T) \setminus (V(T_0) \cup Y_0)$ embedded into $U_{\hat\varphi(v)}$. This motivates the following claim, which we will prove later. For this, for each $j \in [r]$ and $\diamond \in \{+,-\}$, let $X_j^\diamond$ (respectively, $Y_j^\diamond$) be the set of vertices in X_0 (respectively, Y_0) which are $\diamond$-neighbours of $V(T_0)$ and allocated to V_j by $\hat{\varphi}$. That is, for each $j \in [r]$ and $\diamond \in \{+,-\}$, let $X_j^\diamond = X_0 \cap f^{-1}(X^\diamond) \cap \hat{\varphi}^{-1}(j)$ and $Y_j^\diamond = Y_0 \cap f^{-1}(X^\diamond) \cap \hat{\varphi}^{-1}(j)$.

Claim 3.15. *With probability at least* $3/4$*, the following properties hold.*

G1 *For every* $j \in [r]$, $|\hat{\varphi}^{-1}(j)| \leqslant (1+\gamma+3\alpha) \cdot n/r$.

G2 *For every* $j \in [r]$ *and* $x \in X_j^+ \cup X_j^-$,

G2.1 *if* $j \in J_1$*, then* $d_G^+(z_x, U_j) \geqslant |X_j^+|$ *if* $x \in X_j^+$ *and* $d_G^-(z_x, U_j) \geqslant |X_j^+ \cup X_j^-|$ *if* $x \in X_j^-$*;*

G2.2 *if* $j \in J_2$*, then* $d_G^-(z_x, U_j) \geqslant |X_j^-|$ *if* $x \in X_j^-$ *and* $d_G^+(z_x, U_j) \geqslant |X_j^+ \cup X_j^-|$ *if* $x \in X_j^+$*.*

G3 $|Y_j^+ \cup Y_j^-| \leqslant \alpha\mu \cdot n/r$ *for every* $j \in [r]$.

We therefore proceed with the assumption that properties **G1-G3** hold. Extend ψ to cover X_0 as follows.

- For each $j \in J_1$, using **G2.1**, greedily extend ψ to first cover X_j^+, and then to cover X_j^-, so that $\psi(X_j^+ \cup X_j^-) \subseteq U_j$.
- For each $j \in J_2$, using **G2.2** greedily extend ψ to first cover X_j^-, and then to cover X_j^+, so that $\psi(X_j^+ \cup X_j^-) \subseteq U_j$.

Next, let $X' \subseteq X_0 \cup Y_0$ be a maximal set such that there exists a homomorphism φ from $T - V(T_0)$ to D and an extension of ψ covering $\cup_{x \in X'} V(S_x)$ such that the following properties hold.

H1 $\varphi(v) = \hat{\varphi}(v)$ for every $v \in X_0 \cup (\cup_{x \in Y_0} V(S_x))$.

H2 $|\varphi(V(S_x))| \leqslant |H|$ for every $x \in X_0 \cup Y_0$.

H3 $|\varphi^{-1}(j)| = |\hat{\varphi}^{-1}(j)|$ for every $j \in [r]$.

H4 If $x \in X' \cap Y_0$ then $\psi(x) \in W_{\varphi(x)}$, and if $v \in (\cup_{x \in X'} V(S_x)) \setminus Y_0$ then $\psi(v) \in U_{\varphi(v)}$.

We remark that this is well-defined, as we may take $X' = \emptyset$ and $\varphi = \hat{\varphi}$.

For this maximal X', take (φ, ψ) so that **H1-H4** hold, and let $A = \psi(V(T_0) \cup X_0 \cup (\cup_{x \in X'} V(S_x)))$. Note that, by **G1**, **H3** and **H4**, we have

$$|U_j \setminus A| \geqslant \alpha \cdot n/r \tag{3.7}$$

for every $j \in [r]$. We now show that X' includes all of Y_0 and almost all of X_0.

Claim 3.16. *$Y_0 \subseteq X'$.*

Proof of Claim 3.16. For any $\diamond \in \{+, -\}$ and $x \in N_T^{\diamond}(V(T_0)) \cap Y_0$, we have

$$|N_G^{\diamond}(z_x, W_{\varphi(x)}) \setminus A| \overset{(3.6)}{\geqslant} 2\alpha\mu \cdot n/r - |Y_j^+ \cup Y_j^-| \overset{\mathbf{G3}}{\geqslant} \alpha\mu \cdot n/r.$$

So if $x \in Y_0 \setminus X'$, then, by Proposition 3.6 and (3.7), ψ can be extended to cover $V(S_x)$ with $\psi(x) \in W_{\varphi(x)} \setminus A$ and $\psi(v) \in U_{\varphi(v)} \setminus A$ whenever $v \in V(S_x) \setminus \{x\}$, contradicting the maximality of X'. So we must have $Y_0 \subseteq X'$. ⊡

Claim 3.17. *$|X_0 \setminus X'| \leqslant \mu^4 n$.*

Proof of Claim 3.17. For each $m \in \mathbb{N}$, let $g(m)$ denote the number of rooted oriented trees with at most m vertices. Suppose, for contradiction, that $|X_0 \setminus X'| > \mu^4 n$. Then there is some $j \in [r]$ with $|(X_j^+ \cup X_j^-) \setminus X'| > \mu^4 \cdot n/r$. Therefore, there is some rooted oriented tree S such that, if X_j^S is the set of $x \in (X_j^+ \cup X_j^-) \setminus X'$ for which S_x is isomorphic to S, then $|X_j^S| \geqslant (\mu^4/g(\lfloor 1/\mu^3 \rfloor)) \cdot n/r$.

Choose $x_1 \in X_j^S$ arbitrarily. By Proposition 3.6 and (3.7), there is a copy of S_{x_1} in G, with each $v \in V(S_{x_1}) \setminus \{x_1\}$ copied to $U_{\varphi(v)} \setminus A$ and x_1 copied to $\psi(X_j^S)$, and let $x_2 \in X_j^S$ be such that $\psi(x_2)$ is the image of x_1 in this copy. Because S_{x_1} and S_{x_2} are isomorphic, we may regard this as a copy of S_{x_2}, and use this copy to extend ψ to cover $V(S_{x_2})$.

Let ρ be an automorphism of $T - V(T_0)$ with $\rho(S_{x_1}) = S_{x_2}$, $\rho(S_{x_2}) = S_{x_1}$, and $\rho(v) = v$ whenever $v \notin V(S_{x_1}) \cup V(S_{x_2})$. Note that $\psi(v) \in U_{\varphi(\rho(v))}$ whenever $v \in (\cup_{x \in X' \cup \{x_2\}} V(S_x)) \setminus Y_0$, and so $\varphi \circ \rho$ is a homomorphism from $T - V(T_0)$ to D also satisfying **H1**-**H4**. So using this extension of ψ and the homomorphism $\varphi \circ \rho$, we may add x_2 to X', a contradiction. $\boxdot$

We now have an embedding of a subtree $T[\psi^{-1}(A)] \subseteq T$ into $G[V']$, where, using Claims 3.16 and 3.17,

$$|V(T) \setminus \psi^{-1}(A)| \leqslant \sum_{x \in (X_0 \cup Y_0) \setminus X'} |S_x| \leqslant \mu n.$$

Recall that we also have $d^{\pm}_G(v, U) \geqslant 4\alpha^2 n \geqslant 3\mu n$ for every $v \in V'$. Therefore, by Corollary 1.9, this embedding can be extended to an embedding of T into G with the vertices of $V(T) \setminus \psi^{-1}(A)$ embedded into U. All that remains now is to prove Claim 3.15.

Proof of Claim 3.15. We will prove that each of the properties **G1**-**G3** fails with probability at most 1/16, and so the claim then follows.

G1: As each ϕ_x was chosen previously by sampling ϕ conditioned on an event which holds with probability at least $(1 - \sqrt{\varepsilon})$, we have that for any $v \in V(T) \setminus V(T_0)$, $j \in [r]$,

$$\mathbb{P}(\phi_{x(v)}(f(v)) = j) \leqslant (1 - \sqrt{\varepsilon})^{-1} \mathbb{P}(\phi(f(v)) = j). \tag{3.8}$$

Thus, we find that, for any $w \in V(H)$, $j \in [r]$,

$$\begin{aligned}
\mathbb{E}(|\hat{\varphi}^{-1}(j) \cap f^{-1}(w)|) &= \sum_{v \in f^{-1}(w)} \mathbb{P}(\phi_{x(v)}(f(v)) = j) \\
&\overset{(3.8)}{\leqslant} \sum_{v \in f^{-1}(w)} (1 - \sqrt{\varepsilon})^{-1} \mathbb{P}(\phi(f(v)) = j) \\
&\overset{(3.4)}{\leqslant} (1 + \sqrt{\mu}) \cdot \mathbb{E}(\beta(\phi^{-1}(j) \cap \{w\})) \cdot n.
\end{aligned} \tag{3.9}$$

For any $j \in [r]$, we have

$$\mathbb{E}(|\hat{\varphi}^{-1}(j)|) = \sum_{w \in V(H)} \mathbb{E}(|\hat{\varphi}^{-1}(j) \cap f^{-1}(w)|)$$
$$\overset{(3.9)}{\leqslant} \mathbb{E}(\beta(\phi^{-1}(j))) \cdot n + \alpha \cdot n/r \overset{\mathbf{F2}}{\leqslant} (1+\gamma+2\alpha) \cdot n/r. \tag{3.10}$$

Therefore, as

$$\sum_{x \in X_0 \cup Y_0} |S_x|^2 \leqslant \sum_{x \in X_0 \cup Y_0} |S_x| \cdot \max_{x \in X_0 \cup Y_0} |S_x| \leqslant \eta n^2, \tag{3.11}$$

we find that

$$\mathbb{P}(|\hat{\varphi}^{-1}(j)| > (1+\gamma+3\alpha) \cdot n/r) \overset{(3.10)}{\leqslant} \mathbb{P}(||\hat{\varphi}^{-1}(j)| - \mathbb{E}(|\hat{\varphi}^{-1}(j)|)| \geqslant \alpha \cdot n/r)$$
$$\overset{\text{Theorem 1.19}}{\leqslant} 2\exp\left(-\frac{2\alpha^2 \cdot n^2/r^2}{\sum_{x \in X_0 \cup Y_0} |S_x|^2}\right) \overset{(3.11)}{\leqslant} 2\exp\left(-\frac{2\alpha^2}{\eta r^2}\right),$$

and so the probability that **G1** fails is at most $2r \cdot \exp(-2\alpha^2/\eta r^2) < 1/16$.

G2: We first note that, for any $j \in [r]$ and $\diamond \in \{+,-\}$,

$$\mathbb{E}(|X_j^\diamond|) = \sum_{w \in X^\diamond} \mathbb{E}(|\hat{\varphi}^{-1}(j) \cap f^{-1}(w)|) \overset{(3.9)}{\leqslant} (1+\sqrt{\mu}) \cdot \mathbb{E}(\beta(\phi^{-1}(j) \cap X^\diamond)) \cdot n. \tag{3.12}$$

Also,

$$\mathbb{P}(||X_j^\diamond| - \mathbb{E}(|X_j^\diamond|)| \geqslant \mu^2 \cdot n/r) \overset{\text{Theorem 1.19}}{\leqslant} 2\exp\left(-\frac{2\mu^4 \cdot n^2/r^2}{|X_0 \cap f^{-1}(X^\diamond)|}\right) \leqslant 2\exp\left(-\frac{2\mu^4}{r^2} \cdot n\right).$$

Therefore, with probability at least $1 - 4r \cdot \exp(-(2\mu^4/r^2) \cdot n) > 15/16$, we have

$$||X_j^\diamond| - \mathbb{E}(|X_j^\diamond|)| \leqslant \mu^2 \cdot n/r \qquad \text{for every } j \in [r], \diamond \in \{+,-\}. \tag{3.13}$$

Thus, it is enough to show that **G2** follows from (3.13). Indeed, for $j \in J_1$, if $\mathbb{P}(|X_j^+| > 0), \mathbb{P}(|X_j^-| > 0) > 0$, then $d(j_t, j), d(j, j_t) > \mu$, and so for any $x \in X_j^+$ we have

$$|X_j^+| \overset{(3.13)}{\leqslant} \mathbb{E}(|X_j^+|) + \mu^2 \cdot n/r \overset{(3.12),\mathbf{F3.1}}{\leqslant} (1+\sqrt{\mu}) d(j_t, j)(1+\gamma+\alpha) \cdot n/r + \mu^2 \cdot n/r \overset{(3.6)}{\leqslant} d_G^+(z_x, U_j),$$

and for any $x \in X_j^-$ we have

$$|X_j^+ \cup X_j^-| \overset{(3.13)}{\leqslant} \mathbb{E}(|X_j^+ \cup X_j^-|) + 2\mu^2 \cdot n/r$$
$$\overset{(3.12),\mathbf{F3.1}}{\leqslant} (1+\sqrt{\mu})d(j,j_t)(1+\gamma+\alpha)\cdot n/r + 2\mu^2\cdot n/r \overset{(3.6)}{\leqslant} d_G^-(z_x, U_j),$$

and so **G2.1** holds. If instead $|X_j^+| = 0$ with probability 1 or $|X_j^-| = 0$ with probability 1, then the same conclusion holds. Similarly, if $j \in J_2$ then (3.13) implies **G2.2**.

G3: Note that if $x \in N_T(V(T_0))$ and $j \in [r]$, then, because $\beta(f(x)) \geqslant \mu$,

$$\mathbb{P}(\hat{\varphi}(x) = j) = \mathbb{P}(\phi_x(f(x)) = j) \overset{(3.8)}{\leqslant} (1-\sqrt{\varepsilon})^{-1}\mathbb{P}(\phi(f(x)) = j)$$
$$= (1-\sqrt{\varepsilon})^{-1}\frac{1}{\beta(f(x))}\cdot \mathbb{E}(\beta(\phi^{-1}(j)\cap\{f(x)\})) \overset{\mathbf{F2}}{\leqslant} 6/\mu r,$$

and so, for any $j \in [r]$, we have $\mathbb{E}(|Y_j^+ \cup Y_j^-|) \leqslant 6\mu^2 \cdot n/r$. Therefore, for any $j \in [r]$,

$$\mathbb{P}(|Y_j^+ \cup Y_j^-| > \alpha\mu\cdot n/r) \leqslant \mathbb{P}(||Y_j^+ \cup Y_j^-| - \mathbb{E}(|Y_j^+ \cup Y_j^-|)| > \mu^2\cdot n/r)$$
$$\overset{\text{Theorem 1.19}}{\leqslant} 2\exp\left(-\frac{2\mu^4\cdot n^2/r^2}{|Y_0|}\right) \leqslant 2\exp\left(-\frac{2\mu^4}{r^2}\cdot n\right),$$

and so, the probability that **G3** fails is at most $r\cdot\exp\left(-(2\mu^4/r^2)\cdot n\right) < 1/16$. ⊡ □

3.5 Theorem 1.5: embedding the core and attached small trees

In this section, following the proof outline in Section 3.1, we embed T_0 and T_1 for Theorem 1.5, doing so in the form of the following result, Theorem 3.18. (This compares to our work in Section 3.4 for Theorem 1.4, proving Theorem 3.9.)

Theorem 3.18. *Let $1/n \ll \eta \ll \alpha$. Suppose T is an n-vertex oriented tree with a subtree $T_0 \subseteq T$, such that $|T_0| \leqslant \eta n$ and T is formed from T_0 by attaching to each vertex v of T_0 a tree S_v with $|S_v| \leqslant \eta n$. Then, any $(1+\alpha)n$-vertex tournament contains a copy of T.*

Note that there is no direct maximum degree imposed on T in Theorem 3.18, but as (exactly) one tree is attached to each vertex in T_0 to get T, it follows that $\Delta(T) \leqslant 2\eta n$. As with Theorem 3.9, the proof of Theorem 3.18 is broken into two main parts – in Section 3.5.1 we allocate the vertices of T to regularity clusters, before embedding the vertices according to this allocation in Section 3.5.2.

3.5.1 Allocating vertices for Theorem 3.18

To allocate the vertices of an oriented tree T from Theorem 3.18 to regularity clusters in some ε-regular partition $V_1 \cup \ldots \cup V_r$, we first find an assignment of the vertices of T to the vertices of a simpler 'caterpillar-like' digraph (see Figure 3.4). This assignment maps the vertices of the core $T_0 \subseteq T$ into a small transitive tournament, with the components of $T - V(T_0)$ assigned to an in- or out-leaf from this transitive tournament according to the direction of the edge from T_0 to the component. The number of in- and out-leaves from each transitive tournament vertex is chosen so that the number of vertices of $V(T) \setminus V(T_0)$ mapped onto each one is approximately even.

We ultimately find the 'caterpillar-like' digraph within the reduced digraph R for an ε-regular partition $V_1 \cup \ldots \cup V_r$ (see Section 3.3), and therefore we wish to find the 'caterpillar-like' digraph in any ε-almost tournament R. The method for finding such a 'caterpillar-like' digraph is presented in Lemma 3.19, which is then applied, to a weight function naturally arising from the simplification of T discussed above, to produce a full description of the 'caterpillar-like' digraph in Corollary 3.20. The transitive tournament of the 'caterpillar-like' digraph is found with vertex set $\{j_1, j_2, \ldots, j_s\}$ (where condition **J2** in Corollary 3.20 guarantees it is a transitive tournament), with sets of out-leaves I_i^+ and in-leaves I_i^- of j_i, for each $i \in [s]$. The condition **J3** in Corollary 3.20 ensures there are enough in- and out-leaves to allow the approximately even distribution of $V(T) \setminus V(T_0)$ in the simplification of T.

Lemma 3.19. *Let $\varepsilon > 0$ and $\bar{s}, m, r \in \mathbb{N}$. Let $n_i^+, n_i^- \in \mathbb{N}$, $i \in [\bar{s}]$, be such that $m \leqslant n_i^+ + n_i^- \leqslant 4m$ for each $i \in [\bar{s}]$. Suppose that R is an oriented graph on $[r]$ in which, for each $j \in [r]$,*

$$d_R^+(j) + d_R^-(j) \geqslant |R| - m \geqslant (25 + 1000 \log \bar{s})m + \sum_{i \in [\bar{s}]} (n_i^+ + n_i^-). \tag{3.14}$$

Then, there is some $s \in [\bar{s}]$ for which there exists $0 = i_0 < i_1 < \ldots < i_{s-1} < i_s = \bar{s}$, and subsets $\{j_\ell\}, I_\ell^+, I_\ell^- \subseteq [r]$ for $\ell \in [s]$, all disjoint, with the following properties.

I1 *$j_{\ell_1} \to_R j_{\ell_2}$ whenever $\ell_1 < \ell_2$.*

I2 *For each $\ell \in [s]$ and $\diamond \in \{+, -\}$, we have $I_\ell^\diamond \subseteq N_R^\diamond(j_\ell)$, and*

$$|I_\ell^\diamond| = \sum_{i=i_{\ell-1}+1}^{i_\ell} n_i^\diamond.$$

Proof. Fix $m \in \mathbb{N}$ and $\varepsilon > 0$. We will show, by strong induction on $\bar{s}$, that the lemma holds for each $\bar{s} \geqslant 1$.

First, suppose $\bar{s} = 1$. It follows from (3.14) that R is a $(1/25)$-almost tournament with $|R| \geqslant 25m$, and therefore, by Proposition 3.8, there is some $j_1 \in [r]$ such that $d_R^+(j_1), d_R^-(j_1) \geqslant |R|/5 \geqslant 4m$. If we set $I_1^\diamond \subseteq N_R^\diamond(j_1)$ with $|I_1^\diamond| = n_1^\diamond$ for $\diamond \in \{+, -\}$, then all the required properties are satisfied.

Suppose then that $\bar{s} > 1$. It follows from (3.14) that R is an $(1/25)$-almost tournament with $|R| \geqslant 25m$, and therefore, by Proposition 3.8, there is some $j \in [r]$ such that $d_R^+(j), d_R^-(j) \geqslant |R|/5$. Now, by (3.14), we have $d_R^+(j) + d_R^-(j) \geqslant \sum_{i \in [\bar{s}]} (n_i^+ + n_i^-)$. Therefore, at least one of $\sum_{i \in [\bar{s}]} n_i^+ \leqslant d_R^+(j)$ or $\sum_{i \in [\bar{s}]} n_i^- \leqslant d_R^-(j)$ holds. If both inequalities hold, then the desired result follows by taking $s = 1$, $j_1 = j$, and $I_1^\diamond \subseteq N_R^\diamond(j)$ with $|I_1^\diamond| = \sum_{i \in [\bar{s}]} n_i^\diamond$ for each $\diamond \in \{+, -\}$. Otherwise, by directional duality, we may assume that $\sum_{i \in [\bar{s}]} n_i^+ \leqslant d_R^+(j)$ and $\sum_{i \in [\bar{s}]} n_i^- > d_R^-(j)$.

Then, let $s' \in [\bar{s}-1]$ be maximal such that

$$\sum_{i\in[s']} n_i^- \leqslant d_R^-(j).$$

Note that, as

$$d_R^-(j) \geqslant \frac{|R|}{5} \overset{(3.14)}{\geqslant} \frac{1}{5}\sum_{i\in[\bar{s}]}(n_i^+ + n_i^-) \geqslant \frac{\bar{s}m}{5}$$

and $n_i^- \leqslant (n_i^+ + n_i^-) \leqslant 4m$ for each $i \in [\bar{s}]$, we have $s' \geqslant \bar{s}/20$. Furthermore, by the maximality of s', we have

$$\sum_{i\in[s']} n_i^- \geqslant d_R^-(j) - 4m. \tag{3.15}$$

Let $i_0 = 0$ and $i_1 = s'$. Let $I_1^- \subseteq d_R^-(j)$ have size $\sum_{i\in[s']} n_i^-$. Using that $d_R^+(j) \geqslant \sum_{i\in[\bar{s}]} n_i^+$, let $I_1^+ \subseteq d_R^+(j)$ have size $\sum_{i\in[s']} n_i^+$ and let $I = N_R^+(j) \setminus I_1^+$.

Now, $\bar{s} - s' \leqslant \bar{s} - \bar{s}/20 = 19\bar{s}/20$ so that $1000\log(\bar{s}-s') \leqslant -5 + 1000\log\bar{s}$, and hence

$$\begin{aligned}
|I| = d_R^+(j) - |I_I^+| &\overset{(3.15)}{\geqslant} d_R^+(j) - |I_1^+| + d_R^-(j) - |I_1^-| - 4m \\
&\overset{(3.14)}{\geqslant} (25 + 1000\log\bar{s})m + \sum_{i\in[\bar{s}]}(n_i^+ + n_i^-) - |I_1^+| - |I_1^-| - 4m \\
&= (21 + 1000\log\bar{s})m + \sum_{i\in[\bar{s}]\setminus[s']}(n_i^+ + n_i^-) \\
&\geqslant (26 + 1000\log(\bar{s}-s'))m + \sum_{i\in[\bar{s}]\setminus[s']}(n_i^+ + n_i^-).
\end{aligned}$$

Let $R' = R[I]$, and note that, for each $j \in V(R')$, by (3.14) we have $d_{R'}^+(j) + d_{R'}^-(j) \geqslant |R'| - m = |I| - m$, so that, in combination with the above calculation,

$$d_{R'}^+(j) + d_{R'}^-(j) \geqslant |R'| - m \geqslant (25 + 1000\log(\bar{s}-s'))m + \sum_{i\in[\bar{s}]\setminus[s']}(n_i^+ + n_i^-),$$

for each $j \in V(R')$. Therefore, by the inductive hypothesis for $\bar{s} - s'$, there is (with relabelling) some $s \in [\bar{s}]$ for which there exists $s' = i_1 < i_2 < \ldots < i_s = \bar{s}$ and subsets $\{j_\ell\}, I_\ell^+, I_\ell^- \subseteq V(R') = N_R^+(j) \setminus I_1^+$ for $\ell \in [s] \setminus [1]$, all disjoint, such that $j_{\ell_1} \to_R j_{\ell_2}$

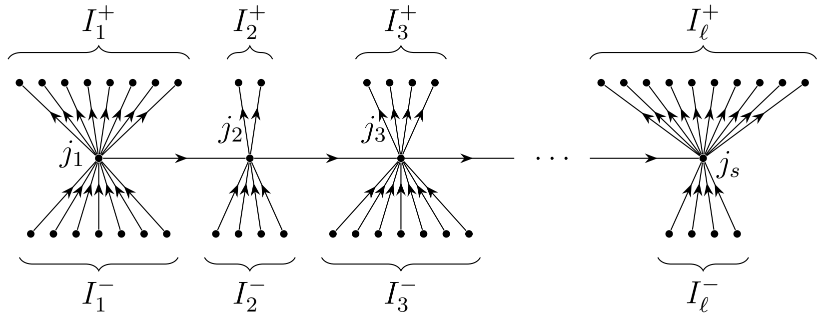

Figure 3.4: A 'caterpillar-like' digraph, as appearing in Lemma 3.19 and Corollary 3.20. While the other edges are omitted for legibility, $j_{\ell_1} \to j_{\ell_2} \to \ldots \to j_s$ is the underlying directed path of a transitive tournament.

whenever $\ell_1 < \ell_2$, and, for each $\ell \in [s]$ and $\diamond \in \{+,-\}$, we have $|I_\ell^\diamond| = \sum_{i=i_{\ell-1}+1}^{i_\ell} n_i^\diamond$. Thus, the required properties are satisfied, completing the proof. □

Corollary 3.20. *Let $1/n \ll \varepsilon, \eta, 1/r \ll \alpha \leqslant 1$. Suppose T is an n-vertex oriented tree with a subtree $T_0 \subseteq T$, such that $|T_0| \leqslant \eta n$ and T is formed from T_0 by attaching to each vertex v of T_0 a tree S_v^+ in which v only has out-neighbours and a tree S_v^- in which v only has in-neighbours, so that $|S_v^+|, |S_v^-| \leqslant \eta n$. Let R be an ε-almost tournament with vertex set $[r]$.*

Then, there is some $s \leqslant \alpha/100\varepsilon$ for which there exists a partition $V(T_0) = X_1 \cup \ldots \cup X_s$ and subsets $\{j_\ell\}, I_\ell^+, I_\ell^- \subseteq [r]$ for $\ell \in [s]$, all disjoint, with the following properties.

J1 *There are no edges of T_0 directed from X_i to X_j with $i, j \in [s]$ and $i > j$.*

J2 *$j_{\ell_1} \to_R j_{\ell_2}$ whenever $\ell_1 < \ell_2$.*

J3 *For each $\ell \in [s]$ and $\diamond \in \{+,-\}$, we have $I_\ell^\diamond \subseteq N_R^\diamond(j_\ell)$, and*

$$|I_\ell^\diamond| \geqslant \frac{r}{(1+\alpha/4)n} \cdot \sum_{v \in X_\ell} |S_v^\diamond|.$$

Proof. Pick $c \geqslant 2\eta$ such that $\varepsilon, 1/r \ll c \ll \alpha$. Let $\bar{m} = cn$. Let $n_0 = |T_0|$ and let $v_1, \ldots, v_{n_0}$ order $V(T_0)$ such that $i < j$ whenever $v_i \to_T v_j$. Let $\bar{s}$ be the largest integer for which there are integers $0 = k_0 < k_1 < \ldots < k_{\bar{s}} \leqslant n_0$ such that $\bar{m} \leqslant \sum_{k=k_{\ell-1}+1}^{k_\ell}(|S_{v_k}^+|+$

$|S^-_{v_k}|) \leqslant 2\bar{m}$ for each $\ell \in [\bar{s}]$. Now, as $|S^+_{v_k}|+|S^-_{v_k}| \leqslant \bar{m}$ for each $k \in [n_0]$, we must have by this maximality that $\sum_{k=k_{\bar{s}}+1}^{n_0}(|S^+_{v_k}| + |S^-_{v_k}|) < \bar{m}$, and therefore, as T has n vertices, we have that $\bar{s} \geqslant n/3\bar{m} = 1/3c \geqslant 1$. Furthermore, setting $W_\ell = \{v_{k_{\ell-1}+1}, \dots, v_{k_\ell}\}$ for each $\ell \in [\bar{s}-1]$ and $W_{\bar{s}} = \{v_{k_{\bar{s}-1}+1}, \dots, v_{n_0}\}$, we have, for each $\ell \in [\bar{s}]$, that

$$\bar{m} \leqslant \sum_{v \in W_\ell} (|S^+_v| + |S^-_v|) \leqslant 3\bar{m}.$$

Finally, note that

$$\frac{n}{3\bar{m}} \leqslant \bar{s} \leqslant \frac{2n}{\bar{m}}. \tag{3.16}$$

Now, for each $i \in [\bar{s}]$, let

$$n_i^\diamond = \left\lceil \frac{r}{(1+\alpha/4)n} \sum_{v \in W_i} |S^\diamond_v| \right\rceil.$$

Let $m = r\bar{m}/n(1+\alpha/4)$, so that $cr/2 \leqslant m \leqslant cr$ and, for each $i \in [\bar{s}]$, $m \leqslant n_i^+ + n_i^- \leqslant 4m$. From (3.16), we have $r/4m \leqslant \bar{s} \leqslant 2r/m$. Therefore, as $\bar{s} \geqslant 1/3c$ and $1/r \ll c \ll \alpha$, we have

$$2\bar{s} + (26 + 1000\log \bar{s})m \leqslant \frac{4r}{m} + \left(\frac{10^5 \log \bar{s}}{\bar{s}}\right) r \leqslant \frac{8}{c} + \frac{\alpha r}{16} \leqslant \frac{\alpha r}{8}. \tag{3.17}$$

Note that

$$\sum_{i \in [\bar{s}]}(n_i^+ + n_i^-) \leqslant 2\bar{s} + \frac{r}{(1+\alpha/4)n} \cdot \sum_{i \in [\bar{s}]} \sum_{v \in W_i} (|S^+_v| + |S^-_v|) \leqslant 2\bar{s} + \frac{r(1+\eta)}{(1+\alpha/4)} \leqslant 2\bar{s} + (1-\alpha/8)r,$$

as $\eta, 1/r \ll \alpha$, so that, by (3.17), we have

$$r \geqslant (26 + 1000\log\bar{s})m + \sum_{i \in [\bar{s}]}(n_i^+ + n_i^-).$$

Finally, we have $m \geqslant cr/2 \geqslant \varepsilon r$, so that, as R is an ε-almost tournament, for each $v \in V(R)$, we have $d^+_R(v) + d^-_R(v) \geqslant |R| - \varepsilon|R| \geqslant |R| - m$.

Thus, by Lemma 3.19, there is some $s \in [\bar{s}]$ for which there exists $0 = i_0 < i_1 < \ldots <$

$i_{s-1} < i_s = \bar{s}$, and subsets $\{j_\ell\}, I^+_\ell, I^-_\ell \subseteq [r]$ for $\ell \in [s]$, all disjoint, such that **I1** and **I2** hold. Letting $X_\ell = \cup_{i=j_{\ell-1}}^{j_\ell} W_i$ for each $\ell \in [s]$ then gives the required partition. □

3.5.2 Embedding vertices for Theorem 3.18

We can now prove Theorem 3.18 using Corollary 3.20. We first prove Lemma 3.21 which can embed the vertices of T that have been mapped to a single vertex of the small transitive tournament and its in- and out-neighbours in the 'caterpillar-like' digraph (here corresponding to V_0, with in-neighbours corresponding to $V_1^+, \dots, V_k^+$ and out-neighbours corresponding to $V_1^-, \dots, V_\ell^-$), before using this repeatedly for each vertex of the small transitive tournament produced by Corollary 3.20 to prove Theorem 3.18.

Lemma 3.21. *Fix $\alpha \geqslant \beta > 0$, $\mu > 0$ and let $1/m \ll \eta \ll 1/r \ll \varepsilon \ll \gamma \ll \mu, \beta$. Let G be a tournament. Suppose, for some $k, \ell \leqslant r$, there are disjoint subsets $V_0, V_1^+, \dots, V_k^+, V_1^-, \dots, V_\ell^-$ of $V(G)$, all of size $(1+\alpha)m$, such that (V_0, V_i^+) is an ε-regular pair of density at least μ for $i \in [k]$, and (V_i^-, V_0) is an ε-regular pair of density at least μ for $i \in [\ell]$.*

Suppose T is an oriented tree with a subtree $T_0 \subseteq T$, such that $|T_0| \leqslant \eta m$, and T is formed from T_0 by attaching to each vertex v of T_0 trees S_v^+, S_v^- with $d^-_{S_v^+}(v) = 0$, $d^+_{S_v^-}(v) = 0$, and $|S_v^+|, |S_v^-| \leqslant \eta m$.

Let $W \subseteq V_0$ be a set with $|W| \geqslant \gamma m$, and let $U_i^+ \subseteq V_i^+$, $i \in [k]$, and $U_i^- \subseteq V_i^-$, $i \in [\ell]$ be sets such that $\sum_{i\in[k]} |U_i^+| \geqslant \sum_{v \in V(T_0)} |S_v^+| + k\beta m$ and $\sum_{i\in[\ell]} |U_i^-| \geqslant \sum_{v\in V(T_0)} |S_v^-| + \ell\beta m$.

Then, there is a copy of T in G, with T_0 copied to W and $T - V(T_0)$ copied to $U_1^+ \cup \dots \cup U_k^+ \cup U_1^- \cup \dots \cup U_\ell^-$.

Proof. For the smallest possible p, take a partition $V(T_0) = X_1 \cup \dots \cup X_p$ such that, for each $j \in [p]$, $T_0[X_1 \cup \dots \cup X_j]$ is a tree, $\sum_{v\in X_j} |S_v^+| \leqslant k\beta\mu m/4$, and $\sum_{v \in X_j} |S_v^-| \leqslant \ell\beta\mu m/4$. This is possible for $p = |T_0|$, so a smallest such p will exist. We in fact claim that $p \leqslant 32/\beta\mu$. Indeed, for this smallest possible p, take a partition that minimises $\sum_{j\in[p]} j|X_j|$. Suppose there is some $j' < p$ for which both $\sum_{v\in X_{j'}} |S_v^+| \leqslant k\beta\mu m/8$ and $\sum_{v\in X_{j'}} |S_v^-| \leqslant \ell\beta\mu m/8$.

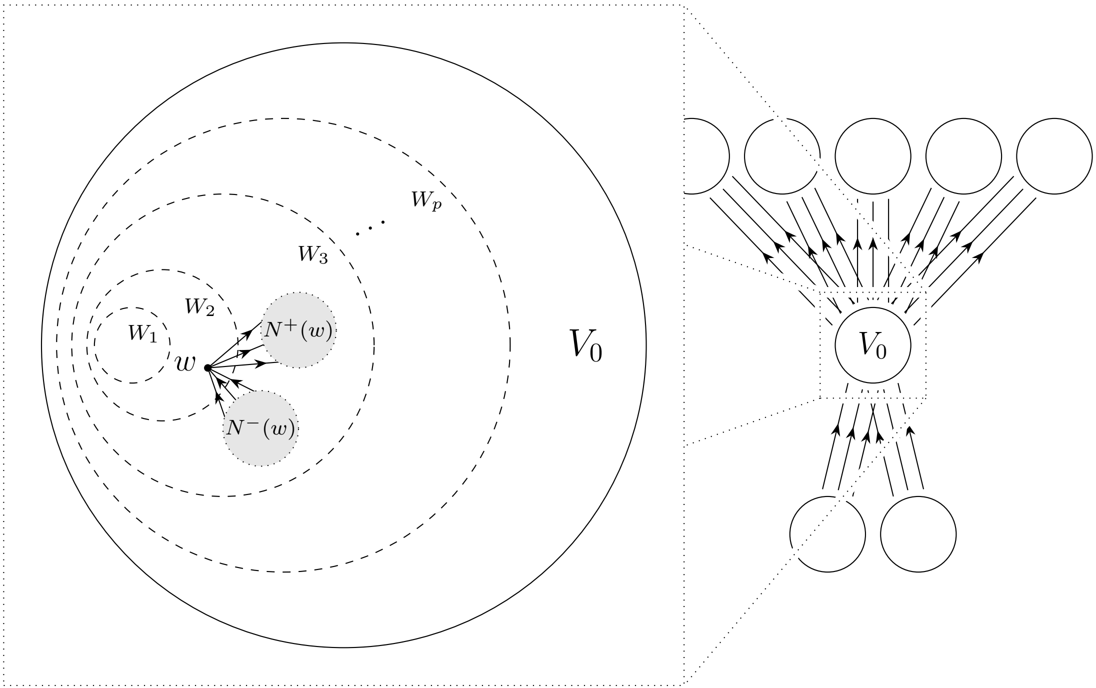

Figure 3.5: The sets $W_1 \subseteq \ldots \subseteq W_p$ in the proof of Lemma 3.21. The sets are chosen so that each vertex $w \in W_j$ has sufficiently many in- and out-neighbours in W_{j+1}.

Let $x \in X_{j'+1}$ be such that $T_0[X_1 \cup \ldots \cup X_{j'} \cup \{x\}]$ is a tree. Then moving x from $X_{j'+1}$ to $X_{j'}$ produces a partition which contradicts the minimality of $\sum_{j \in [p]} j|X_j|$. Thus we have

$$p \leqslant 1 + \frac{\sum_{v \in V(T_0)} |S_v^+|}{(k\beta\mu m/8)} + \frac{\sum_{v \in V(T_0)} |S_v^-|}{(\ell\beta\mu m/8)} \leqslant 1 + \frac{8\sum_{i \in [k]} |U_i^+|}{k\beta\mu m} + \frac{8\sum_{i \in [\ell]} |U_i^-|}{\ell\beta\mu m} \leqslant 32/\beta\mu.$$

Now find vertex sets $W_1 \subseteq W_2 \subseteq \ldots \subseteq W_p = W$ such that $|W_j| \geqslant |W_{j+1}|/8$ for each $j \in [p-1]$, and $d^{\pm}(w, W_{j+1}) \geqslant |W_{j+1}|/8$ for each $j \in [p-1]$, $w \in W_j$. This is possible by starting with W_p and iteratively using that fact that at most $|W_{j+1}|/4$ vertices w of W_{j+1} have $d^+(w, W_{j+1}) \leqslant |W_{j+1}|/8$, and at most $|W_{j+1}|/4$ vertices w of W_{j+1} have $d^-(w, W_{j+1}) \leqslant |W_{j+1}|/8$.

We will now embed T in p stages as follows. At stage j, suppose we have already embedded $T[\cup_{j'=1}^{j-1} \cup_{v \in X_{j'}} (V(S_v^+) \cup V(S_v^-))]$. For each $v \in X_1 \cup \ldots \cup X_{j-1}$ in turn, consider the forest F_v^+ consisting of trees of $T_0[X_j]$ attached to v by out-neighbours of v, and suppose v has already been copied to some $w \in W_{j-1}$ (for the case $j = 1$, regard all of $T_0[X_j]$ as components attached to a single auxiliary vertex v by out-neighbours, where

v has already been copied to an auxiliary vertex w satisfying $W_1 \subseteq N^+_G(w)$). Let $Z^+_{j,v}$ be the set of unoccupied out-neighbours of w in W_j which each have at least $3k\beta\mu m/4$ unoccupied out-neighbours in $\cup_{i\in[k]}U^+_i$ as well as $3\ell\beta\mu m/4$ unoccupied in-neighbours in $\cup_{i\in[\ell]}U^-_i$. Because there are always at least $k\beta m$ unoccupied vertices in $\cup_{i\in[k]}U^+_i$ and $\ell\beta m$ unoccupied vertices in $\cup_{i\in[\ell]}U^-_i$, Proposition 3.5 implies $|Z^+_{j,v}| \geqslant |N^+(w, W_j)| - |T_0| - 2\varepsilon m \geqslant |W_1|/8 - 3\varepsilon m \geqslant |W|/8^{p+1} \geqslant 3\eta m$. Therefore, by Theorem 1.8, there is a copy of F^+_v in $Z^+_{j,v}$. Then, for each $v' \in V(F_v)$, if v' has now been copied to w', find a copy of $S^+_{v'} - v'$ in the unoccupied vertices of $N^+(w', \cup_{i\in[k]}U^+_i)$. Because $w' \in Z^+_{j,v}$, and only at most $\sum_{v''\in X_j, v''\neq v'} |S^+_{v''}|$ additional vertices of $N^+(w', \cup_{i\in[k]}U^+_i)$ may become occupied since choosing $Z^+_{j,v}$, at least $3k\beta\mu m/4 - \sum_{v''\in X_j, v''\neq v'} |S^+_{v''}| \geqslant 3|S^+_{v'}|$ vertices of $N^+(w', \cup_{i\in[k]}U^+_i)$ remain unoccupied, allowing the copy of $S^+_{v'} - v'$ to be found using Theorem 1.8. Similarly, find a copy of $S^-_{v'} - v'$ in the unoccupied vertices of $N^-(w', \cup_{i\in[\ell]}U^-_i)$. We then do the same for the forest F^-_v consisting of trees of $T_0[X_j]$ attached to $X_1 \cup \ldots \cup X_{j-1}$ by in-neighbours. Performing this process for each $v \in X_1 \cup \ldots \cup X_{j-1}$ completes stage j of the embedding procedure. Upon the completion of stage p, we obtain a copy of T in G, with T_0 copied to W and $T - V(T_0)$ copied to $U^+_1 \cup \ldots \cup U^+_k \cup U^-_1 \cup \ldots \cup U^-_\ell$. □

We now combine Lemma 3.19 and Lemma 3.21 to prove Theorem 3.18.

Proof of Theorem 3.18. Set $\beta = \alpha/4$, $\mu = 1/2$, and introduce constants ε, r_1, r_2 such that $\eta \ll 1/r_2 \ll 1/r_1 \ll \varepsilon \ll \beta$. Let G be a $(1+\alpha)n$-vertex tournament. By Corollary 3.4, there is a subtournament $G' \subseteq G$ with $|G'| \geqslant (1+3\beta)n$, and an ε-regular partition $V(G') = V_1 \cup \ldots \cup V_r$ with $r_1 \leqslant r \leqslant r_2$. Let R be a $\sqrt{\varepsilon}$-almost tournament with vertex set $[r]$, such that (V_i, V_j) is an ε-regular pair of density at least μ whenever $i \to_R j$. Fix disjoint subsets $U_j, W_j \subseteq V_j$ for each $j \in [r]$ with $|U_j| = (1+2\beta)\cdot n/r$ and $|W_j| = \beta \cdot n/r$.

For each $v \in V(T_0)$, let $S^+_v \subseteq S_v$ be the subtree of S_v induced by the vertices whose path from v begins with an out-edge, and let $S^-_v \subseteq S_v$ be the subtree of S_v induced by the vertices whose path from v begins with an in-edge. Note that we have $|S^+_v|, |S^-_v| \leqslant \eta n$ for every $v \in V(T_0)$. By Corollary 3.20, there is some $s \leqslant \alpha/100\varepsilon$ for which there exists a partition $V(T_0) = X_1 \cup \ldots \cup X_s$ and subsets $\{j_\ell\}, I_\ell^+, I_\ell^- \subseteq [r]$ for $\ell \in [s]$, all disjoint, satisfying properties **J1-J3**. In particular, for each $\ell \in [s]$ and $\diamond \in \{+,-\}$, we have

$$\sum_{v \in X_\ell} |S_v^\diamond| + |I_\ell^\diamond|\beta \cdot n/r \overset{\mathbf{J3}}{\leqslant} |I_\ell^\diamond|(1+2\beta) \cdot n/r = \sum_{j \in I_\ell^\diamond} |U_j^\diamond| \tag{3.18}$$

Set $\gamma = \beta\mu/8r$. Using **J2**, Proposition 3.5, and $s \leqslant \alpha/100\varepsilon$, for each $\ell \in [s]$ at most $s\varepsilon(1+\alpha)\cdot n/r \leqslant 2\gamma n$ vertices w of W_{j_ℓ} have either some $\ell' > \ell$ for which $d^+(w, W_{j_{\ell'}}) \leqslant 4\gamma n$, or some $\ell' < \ell$ for which $d^-(w, W_{j_{\ell'}}) \leqslant 4\gamma n$. Therefore, we may take subsets $W'_{j_\ell} \subseteq W_{j_\ell}$ for $\ell \in [s]$ such that, $d^+(w, W'_{j_{\ell_2}}) \geqslant 2\gamma n$ whenever $\ell_2 > \ell_1$ and $w \in W'_{j_\ell}$, and $d^-(w, W'_{j_{\ell_2}}) \geqslant 2\gamma n$ whenever $\ell_2 < \ell_1$ and $w \in W'_{j_\ell}$.

Now obtain a partition $V(T_0) = Y_1 \cup \ldots \cup Y_\tau$ such that

- for each $t \in [\tau]$, $T[Y_t]$ is a connected component of $T_0[X_\ell]$ for some $\ell \in [s]$, and
- for each $t \in [\tau]$, $T_0[Y_1 \cup \ldots \cup Y_t]$ is a tree.

We will now embed T into G so that X_ℓ is copied to W'_{j_ℓ} for each $\ell \in [s]$, and $\cup_{v \in X_\ell} V(S_v^\diamond)$ is copied to $\cup_{j \in I_\ell^\diamond} U_j^\diamond$ for each $\ell \in [s]$ and $\diamond \in \{+,-\}$. The embedding is given in τ stages as follows. Let C_0 be the empty graph. Suppose after stage $t-1$, we have embedded $T[\cup_{v \in Y_1 \cup \ldots \cup Y_{t-1}} V(S_v)]$ to get C_{t-1}. Let $\ell \in [s]$ be such that $Y_t \subseteq X_\ell$. If $t = 1$, set $A_t = W'_{j_\ell}$. Otherwise, if $t > 1$, let y_t be the unique vertex of $Y_1 \cup \ldots \cup Y_{t-1}$ with a neighbour in Y_t, let $\diamond \in \{+,-\}$ be such that the neighbour in Y_t is a $\diamond$-neighbour, let z_t be the image of y_t in C_{t-1}, and set $A_t = N^\diamond(z_t, W'_{j_\ell}) \setminus V(C_{t-1})$. Note that in both cases we find $|A_t| \geqslant \gamma n$. Also, we find for $\diamond \in \{+,-\}$ that

$$\sum_{j \in I_\ell^\diamond} |U_j^\diamond \setminus V(C_{t-1})| \overset{(3.18)}{\geqslant} \sum_{v \in Y_t} |S_v^\diamond| + |I_\ell^\diamond|\beta n/r.$$

Therefore, by **I2** and Lemma 3.21, there is a copy of $T[Y_t \cup (\cup_{v \in Y_t} S_v)]$ in G with Y_t copied to $A_t \subseteq W'_{j_\ell}$, and $(\cup_{v \in Y_t} V(S_v^\diamond)) \setminus Y_t$ copied to $(\cup_{j \in I_\ell^\diamond} U_j^\diamond) \setminus V(C_{t-1})$ for $\diamond \in \{+,-\}$. Thus we obtain a copy of T after stage τ. $\square$

3.6 Proof of Theorem 1.4 and Theorem 1.5

Recall the decomposition of our tree T from Section 3.1 as $T_0 \subseteq T_1 \subseteq T_2 \subseteq T_3 \subseteq T_4 = T$. In Sections 3.4 and 3.5 respectively, we showed how to embed T_0 and extend this to T_1 for both Theorems 1.4 and 1.5. In this section, we will show how a copy of T_1 can be extended to a copy of T, completing the proof of both theorems. As noted in the proof outline, the main challenge here is to embed the vertices in $V(T_3) \setminus V(T_2)$, where these vertices form paths with constant length between vertices in T_2. Indeed, firstly, $T_2 - V(T_1)$ is a forest of constant-sized components not directly connected to T_1 (see **C3** in Lemma 3.1), which can be embedded greedily using, for example, Theorem 1.8. Secondly, to reach T_4 from T_3 we add small tree components on to T_3, which is already connected. This can be done by reserving a small random subset of vertices U (using Proposition 1.20) and carrying out the rest of the embedding in the vertices with sufficient out- and in-degree to U. Such an embedding can then be completed greedily, giving an embedding of $T_4 = T$.

Thus, most of this section will be dedicated to showing how we can extend a copy of T_2 to a copy of T_3 (using a method effective for both Theorems 1.4 and 1.5). Recall that T_3 is obtained from T_2 by attaching paths of fixed length by their endpoints (see **C4** in Lemma 3.1), but such that the total number of vertices contained in such paths is only a small proportion of the resulting tree (see **C5** in Lemma 3.1). Thus, with a copy of T_2 already found, we will often wish to find paths of a fixed length between certain attachment points. By ensuring these attachment points have plenty of out- and in-neighbours, we need only to be able to connect linear-sized sets with paths of fixed but small length, while avoiding some small set of vertices already used in some paths. As we will see, paths with changes of direction are comparatively easy to find, so we only consider whether we can find such paths so that they are directed paths. We will call tournaments with this connection property *well-connected*, as follows.

Definition 3.22. *We say a tournament G is (a,b,ℓ)-*well-connected* if, for every $A_1, A_2 \subseteq V(G)$ with $|A_1|, |A_2| \geqslant a$ and $B \subseteq V(G)$ with $|B| \leqslant b$, there is a directed path in $V(G) \setminus B$*

from A_1 to A_2 with length ℓ.

In Lemma 3.28, we will see that both of our main theorems hold if the tournament G is well-connected. Of course, not every tournament is well-connected, but, in Lemma 3.26 we will see that any tournament that is not well-connected contains a bipartition of most of its vertices, so that all the relevant edges are directed in the same direction across the bipartition. Through the repeated application of Lemma 3.26, we can then decompose the vertices of any tournament as $V(G) = B \cup W_1 \cup \ldots \cup W_r$, so that B is small, all possible edges are directed from W_i to W_j for $1 \leqslant i < j \leqslant r$, and each $G[W_i]$ is either small or well-connected (see Lemma 3.27). We then assign the vertices of T to the sets $W_1, \ldots, W_r$, so that any edge of T assigned between some W_i and W_j with $i < j$ is to be embedded as directed from W_i into W_j. Thus, we can embed the vertices of T assigned to W_i into $G[W_i]$ independently for each $i \in [r]$, while knowing the other edges of T can then be embedded. As noted in the proof sketch, this is a streamlined version of techniques by Kühn, Mycroft and Osthus [21, 22]. In [21, 22], a notion of robust out-expansion is used, from which our well-connected property can be derived. As we do not need any other results of robust out-expansion (most notably, we do not use a Hamilton, or almost-spanning, cycle in the reduced digraph), we use the well-connected property directly. This allows the decomposition of [22, Lemma 5.2] to be simplified to find bipartitions with all the edges directed from one side to another, rather than just most of the edges.

In Section 3.6.1, we will prove a number of results on well-connected tournaments, including the tournament decomposition discussed above. Then, in Section 3.6.2, after showing our main results hold for well-connected tournaments (i.e., Lemma 3.28), we prove both Theorems 1.4 and 1.5.

3.6.1 Well-connected tournaments

We start by proving two simple properties of well-connected tournaments in Lemma 3.23. The first is that removing a small number of vertices from a well-connected tournament maintains some (potentially slightly weaker) connection property. The second shows that (a, b, ℓ)-well-connected tournaments robustly contain paths of length ℓ, regardless of the desired orientation of the paths' edges. While Definition 3.22 only refers to directed paths, the results of Section 1.3 show that a path with at least one change of direction can be found between two sufficiently large subsets of any tournament, covering all other cases.

Lemma 3.23. *Let $a, b, \ell \geqslant 0$, and suppose G is a (a, b, ℓ)-well-connected tournament.*

(i) *If $C \subseteq V(G)$ has size $c \leqslant b$, then $G - C$ is $(a, b - c, \ell)$-well-connected.*

(ii) *Suppose P is an oriented path of length ℓ, and $A_1, A_2, B \subseteq V(G)$ satisfy $|A_1|, |A_2| \geqslant a$, $|B| \leqslant b$. If $a \geqslant b + \ell + 3$, then there is a copy of P in $G - B$, with its first vertex in A_1 and its last vertex in A_2.*

Proof. First, fix a subset $C \subseteq V(G)$ with size c. Then, if $A_1, A_2, B \subseteq V(G)$ satisfy $|A_1|, |A_2| \geqslant a$ and $|B| \leqslant b - c$, then, because G is (a, b, ℓ)-well-connected, there is a directed path in $V(G) \setminus (B \cup C)$ from A_1 to A_2 with length ℓ. Therefore, $G - C$ is $(a, b - c, \ell)$-well-connected and (i) holds.

Next, suppose P is an oriented path of length ℓ, and $A_1, A_2, B \subseteq V(G)$ satisfy $|A_1|, |A_2| \geqslant a$, $|B| \leqslant b$. If P is a directed path, then, because G is (a, b, ℓ)-well-connected there is a copy of P in $G - B$ with first vertex in A_1 and last vertex in A_2. On the other hand, if P has at least two blocks, then by Corollary 1.14, there is a copy of P in $G[(A_1 \cup A_2) \setminus B]$, with first vertex in A_1 and last vertex in A_2. Therefore, (ii) holds. □

We will need to set aside a random subset of vertices to use to attach paths to T_2 to obtain T_3. We need therefore to show that random subsets of well-connected tournaments can be used to find connecting paths of this sort. To do this, we use median orders, the main embedding tool of Chapter 2. We recall that a median order is an ordering $v_1, \ldots, v_n$ of the vertices of a tournament that maximises the number of pairs $i < j$ with $v_i \to v_j$. While median orders have several useful properties applicable to embedding trees in tournaments (see Section 2.1), here we only require Lemma 2.4, which is restated below with slightly different notation.

Lemma 3.24. *Suppose G is a tournament with a median order $v_1, \dots, v_n$. Then, for any $1 \leqslant i < j \leqslant n$ with $j - i \geqslant 7$, and $A \subseteq V(G) \setminus \{v_i, v_j\}$ with $|A| \leqslant (j - i - 7)/6$, there is a directed v_i, v_j-path in $G - A$ with length 3.*

We are now ready to state and prove our lemma, showing that, with high probability, a random subset of vertices in a well-connected tournament induces a well-connected tournament, as follows.

Lemma 3.25. *Let $1/n \ll \eta \ll 1/\ell \ll \varepsilon \ll p$. Suppose G is a $(\varepsilon n, \eta n, \ell)$-well-connected tournament with $|G| \leqslant 3n$, and that $U \subseteq V(G)$ is a random subset with vertices included uniformly at random with probability p. Then, with high probability, $G[U]$ is $(6\varepsilon n, \eta^2 n, \ell + 6)$-well-connected.*

Proof. Let $v_1, \dots, v_m$ be a median order for G. Let W_1 and W_2 respectively denote the first and last εn vertices of the median order. Let V' be the middle $m - 4\varepsilon n$ vertices of the median order.

It is enough to show that, with high probability, for every $v, w \in V'$, there are at least $2\eta^2 n$ internally vertex-disjoint directed v, w-paths with length $\ell + 6$ and with all internal vertices in U. Indeed, then for any $A_1, A_2 \subseteq U$ with $|A_1|, |A_2| \geqslant 6\varepsilon n$ and $B \subseteq U$ with $|B| \leqslant \eta^2 n$, there is some $v \in (V' \cap A_1) \setminus B$ and $w \in (V' \cap A_2) \setminus (B \cup \{v\})$, and hence at least $2\eta^2 n$ internally vertex-disjoint directed paths from $A_1 \setminus B$ to $A_2 \setminus B$ in $G[U]$ with length $\ell + 6$. Of these paths, at most $\eta^2 n$ contain some internal vertex in B, and so there is some directed path in $U \setminus B$ from A_1 to A_2 of length $\ell + 6$, thus demonstrating $G[U]$ is $(6\varepsilon n, \eta^2 n, \ell + 6)$-well-connected.

Fix $v, w \in V'$. Because G is $(\varepsilon n, \eta n, \ell)$-well-connected, and $|W_1|, |W_2| \geqslant \varepsilon n$, we can greedily find at least $\eta n / 2\ell$ disjoint directed paths in $V(G) \setminus \{v, w\}$ from W_2 to W_1 with length ℓ. Using Lemma 3.24, we can greedily and disjointly connect v to the first vertex of each path by a directed path of length 3, while avoiding all other vertices used so far. Indeed, at least εn vertices in the median order lie between v and the first vertex of each path, while the total number of vertices to be avoided each time is at most $\eta n \leqslant (\varepsilon n - 7)/6$.

Similarly, we can also disjointly connect the last vertex of each path to w by a directed path of length 3, also avoiding any vertex used previously. Therefore, we have at least $\eta n/2\ell$ internally disjoint directed paths in $V(G)$ from v to w, each with length $\ell+6$.

Let $X_{v,w}$ be the number of these directed v,w-paths which additionally have all internal vertices in U, and note that $X_{v,w}$ is a binomial variable with $\mathbb{E}X_{v,w} \geqslant p^{\ell+5}\eta n/2\ell > 3\eta^2 n$. From Lemma 1.18, we have

$$\mathbb{P}(X_{v,w} \leqslant 2\eta^2 n) \leqslant \mathbb{P}(|X_{v,w} - \mathbb{E}X_{v,w}| \geqslant \mathbb{E}X_{v,w}/3) \leqslant 2\exp\left(-\mathbb{E}X_{v,w}/27\right) \leqslant 2\exp\left(-\eta^2 n/9\right).$$

Thus, the probability that the desired property fails is at most $18n^2\exp\left(-\eta^2 n/9\right)$, and so, as $1/n \ll \eta$, the conclusion of the lemma holds with high probability. $\square$

Next we will show that, if a tournament is not well-connected, then, except for a small subset of vertices, we may partition the vertices in two so that all the edges between the parts are directed into the same part.

Lemma 3.26. *Let $\varepsilon > 0$, $\ell \in \mathbb{N}$, and $\eta \ll \varepsilon, 1/\ell$. Suppose G is a tournament with $|G| \leqslant 3n$ that is not $(\varepsilon n, \eta n, \ell)$-well-connected. Then, there is a partition $V(G) = W_1 \cup W_2 \cup B$ so that $|W_1|, |W_2| \geqslant \varepsilon n/2$, $|B| \leqslant 4\ell^{-1}n$, and $x \to y$ for every $x \in W_1$, $y \in W_2$.*

Proof. Using that G is not $(\varepsilon n, \eta n, \ell)$-well-connected, let $A_1, A_2, B_0 \subseteq V(G)$ be sets such that $|A_1|, |A_2| \geqslant \varepsilon n$, $|B_0| \leqslant \eta n$, and there is no directed path in $V(G) \setminus B$ from A_1 to A_2 with length ℓ. Construct a chain of subsets $U_0 \subseteq U_1 \subseteq \ldots \subseteq U_\ell$ as follows. Let $U_0 = A_1 \setminus B_0$, and, for $i \in [\ell]$, let U_i be the set of vertices $x \in V(G)$ such that there is a directed path from A_1 to x in $V(G) \setminus B_0$ with length at most i. We remark that $|(U_r \setminus U_{r-1}) \cap A_2| \leqslant \ell + 1$ for any $r \leqslant \ell$, else, by taking a directed path of length $\ell - r$ in $(U_r \setminus U_{r-1}) \cap A_2$ together with a path of length r from A_1 to that path's initial vertex, we would be able to find a directed path in $V(G) \setminus B_0$ from A_1 to A_2 of length ℓ. In particular, we have $|U_r \cap A_2| \leqslant (\ell+1)^2$ for any $r \leqslant \ell$.

Let $r \in [\ell]$ be minimal such that $|U_r \setminus U_{r-1}| \leqslant 3\ell^{-1}n$. Set $W_2 = U_{r-1}$, $B = B_0 \cup (U_r \setminus U_{r-1})$, and $W_1 = V(G) \setminus (U_r \cup B_0)$, so that $W_1 \cup W_2 \cup B$ is a partition of $V(G)$.

Because $A_1 \setminus B_0 \subseteq W_2$, we have $|W_2| \geqslant \varepsilon n/2$. Because $A_2 \setminus (U_r \cup B_0) \subseteq W_1$, we have $|W_1| \geqslant \varepsilon n - (\ell+1)^2 - \eta n \geqslant \varepsilon n/2$. From the choice of r, we have $|B| \leqslant 3\ell^{-1}n + \eta n \leqslant 4\ell^{-1}n$. Finally, the fact that $x \to y$ for every $x \in W_1$, $y \in W_2$ follows from the definition of U_r and U_{r-1}. □

Using a repeated application of Lemma 3.26, we are now ready to state and prove the tournament decomposition referred to at the start of this section.

Lemma 3.27. *Suppose $\eta \ll \varepsilon$ and let $\ell = \lceil \varepsilon^{-3} \rceil$. Suppose G is a tournament with $|G| \leqslant 3n$. Then, there is a partition $V(G) = B \cup W_1 \cup \ldots \cup W_r$ so that $|B| \leqslant \varepsilon n$ and the following properties hold.*

K1 *If $1 \leqslant i < j \leqslant r$ and $x \in W_i$, $y \in W_j$, then $x \to y$.*

K2 *For $i \in [r]$, if $|W_i| \geqslant \sqrt{\varepsilon} n$, then $G[W_i]$ is $(\varepsilon n, \eta n, \ell)$-well-connected.*

Proof. Initially, set $B^{(1)} = \emptyset$ and $W_1^{(1)} = V(G)$. Then, for $r \geqslant 1$, do the following. We are given a partition $V(G) = B^{(r)} \cup W_1^{(r)} \cup \ldots \cup W_r^{(r)}$ with $|B^{(r)}| \leqslant 5r\varepsilon^3 n$, such that $|W_i^{(r)}| \geqslant \varepsilon n/2$ for each $i \in [r]$, and, if $1 \leqslant i < j \leqslant r$ and $x \in W_i^{(r)}$, $y \in W_j^{(r)}$, then $x \to y$. If we have that $G[W_i^{(r)}]$ is $(\varepsilon n, \eta n, \ell)$ well-connected whenever $|W_i^{(r)}| \geqslant \sqrt{\varepsilon} n$, then set $B = B^{(r)}$ and $W_i = W_i^{(r)}$ for $i \in [r]$. Otherwise, let $j \in [r]$ be such that $W_j^{(r)}$ is not $(\varepsilon n, \eta n, \ell)$-well-connected, with $|W_j^{(r)}|$ maximal (so $|W_j^{(r)}| \geqslant \sqrt{\varepsilon} n$). By Lemma 3.26, there is a partition $W_j^{(r)} = U_1 \cup U_2 \cup B_r$ so that $|U_1|, |U_2| \geqslant \varepsilon n/2$, $|B_r| \leqslant 4\ell^{-1}n \leqslant 5\varepsilon^3 n$, and $x \to y$ for every $x \in U_1$ and $y \in U_2$. We then set

$$B^{(r+1)} = B^{(r)} \cup B_r$$

$$W_i^{(r+1)} = \begin{cases} W_i^{(r)} & \text{if } 1 \leqslant i < j \\ U_1 & \text{if } i = j \\ U_2 & \text{if } i = j+1 \\ W_{i-1}^{(r)} & \text{if } j+1 < i \leqslant r+1 \end{cases}$$

We remark that $V(G) = B^{(r+1)} \cup W_1^{(r+1)} \cup \ldots \cup W_{r+1}^{(r+1)}$ is a partition with $|B^{(r+1)}| \leqslant 5(r+1)\varepsilon^2 n$, such that $|W_i^{(r+1)}| \geqslant \varepsilon n/2$ for each $i \in [r]$, and, if $1 \leqslant i < j \leqslant r+1$ and $x \in W_i^{(r+1)}$, $y \in W_j^{(r+1)}$, then $x \to y$, and so the procedure may continue.

On the r^{th} iteration of this procedure, the largest $|W_i^{(r)}|$ that is not $(\varepsilon n, \eta n, \ell)$-well-connected has size at most $3n - (r-1) \cdot \varepsilon n/2$, and so the procedure will terminate after at most $6\varepsilon^{-1}$ iterations, at which point we find $|B^{(r)}| \leqslant 30\varepsilon^2 n \leqslant \varepsilon n$. □

3.6.2 Proof of Theorem 1.4 and Theorem 1.5

We first prove that our two main theorems hold when the tournament is well-connected.

Lemma 3.28. *Suppose $1/n \ll \eta \ll \varepsilon \ll \alpha$ and let $\ell = \lceil \varepsilon^{-3} \rceil$. Suppose G is a tournament which is $(\varepsilon n, 5\eta^{1/4} n, \ell)$-well-connected, and that T is an n-vertex oriented tree.*

(1) *Suppose that $|G| = ((1+\alpha)n + k)$ where k is the number of leaves of T. Then, G contains a copy of T.*

(2) *Suppose that c is a constant such that $1/n \ll c \ll \eta$, that $|G| = (1+\alpha)n$, and that $\Delta(T) \leqslant cn$. Then, G contains a copy of T.*

Proof. The proof for each statement of this theorem is nearly identical, so here we will present a proof for (1), and explain in the footnotes any places where the proof for (2) differs.

Fix $\alpha > 0$ and introduce a constant m such that $1/n \ll 1/m \ll \eta \ll \varepsilon \ll \alpha$. Fix an n-vertex k-leaf oriented tree T and let G be a $((1+\alpha)n + k)$-vertex tournament which is $(\varepsilon n, 5\eta^{1/4} n, \ell)$-well-connected. We will show that G contains a copy of T, thus proving (1).[1]

Let $U_0 \subseteq V(G)$ be a random subset, with elements from $V(G)$ chosen independently at random with probability $2\sqrt{\eta}$, and let W_0 be the set of vertices v in $V(G) \setminus U_0$ with

[1]For (2), fix $\alpha > 0$ and introduce a constant m such that $1/n \ll c \ll 1/m \ll \eta \ll \varepsilon \ll \alpha$. Fix an n-vertex oriented tree T with $\Delta(T) \leqslant cn$ and let G be a $(1+\alpha)n$-vertex tournament which is $(\varepsilon n, 5\eta^{1/4} n, \ell)$-well-connected. We will show that G contains a copy of T, thus proving (2).

$d^{\pm}(v, U_0) \geqslant 4\eta n$. By Proposition 1.20, we have that $|V(G) \setminus W_0| \leqslant 24\sqrt{\eta}n$ with high probability. $V(G) \setminus U_0$ may be regarded as a random subset of $V(G)$ with elements chosen independently at random with probability $1-2\sqrt{\eta}$, and so, by Lemma 3.25, we have that $G[V(G) \setminus U_0]$ is $(6\varepsilon n, 25\sqrt{\eta}n, \ell+6)$-well-connected with high probability. Therefore, we may proceed assuming that $|V(G) \setminus W_0| \leqslant 24\sqrt{\eta}n$, and, using Lemma 3.23 (i), that $G[W_0]$ is $(6\varepsilon n, \sqrt{\eta}n, \ell+6)$-well-connected.

Let $U_1 \subseteq W_0$ be a random subset, with elements from W_0 chosen independently at random with probability $\alpha/36$, and let W_1 be the set of vertices v in $W_0 \setminus U_1$ with $d^{\pm}(v, U_1) \geqslant 36\varepsilon n$. By Proposition 1.20, we have that $|W_1 \setminus W_0| \leqslant \alpha n/3$ with high probability, and, by Lemma 3.25, we have that $G[U_1]$ is $(36\varepsilon n, \eta n, \ell+12)$-well-connected with high probability. Therefore, we may proceed assuming that $|W_1| \geqslant ((1+\alpha/2)n+k)$, and that $G[U_1]$ is $(36\varepsilon n, \eta n, \ell+12)$-well-connected. [1]

Let $q = \ell + 14$. By Lemma 3.1, there exist forests $T_0 \subseteq T_1 \subseteq T_2 \subseteq T_3 \subseteq T_4 = T$, such that T_3 is a tree and properties **C1-C5** hold. By Theorem 3.9, $G[W_1]$ contains a copy, C_1 say, of T_1.[2] By Theorem 1.8 applied iteratively to the components of $T_2 - V(T_1)$, $G[W_1] - V(C_1)$ then contains a copy of $T_2 - V(T_1)$, which, taken together with C_1, gives a copy, C_2 say, of T_2.

Let $P_1, \dots, P_r$ be the paths of length $\ell+14$ attached to T_2 to obtain T_3. For $i \in [r]$, let x_i, y_i be the endvertices of P_i, let $P_i' = P_i - x_i - y_i$ (so that P_i' has length $\ell+12$), and let x_i', y_i' be the images of x_i, y_i in C_2. For each $i \in [r]$ in turn, using Lemma 3.23 (ii), there is a copy Q_i of P_i' in the unoccupied vertices of $G[U_1]$, with first vertex in $N^{\diamond_1}(x_i', U_1)$ and last vertex in $N^{\diamond_2}(y_i', U_1)$, where $\diamond_1, \diamond_2 \in \{+,-\}$ are taken so that $x_i' Q_i y_i'$ gives a copy of P_i. We remark that we may always proceed as, by **C5**, the total number of vertices being embedded into U_1 is at most $|T_3 \setminus T_2| \leqslant \eta n$, and $G[U_1]$ is $(36\varepsilon n, \eta n, \ell+12)$-well-connected with $d^{\pm}(v, U_1) \geqslant 36\varepsilon n$ for every $v \in W_1$. Thus, we obtain a copy, C_3 say, of T_3 in $G[W_0]$. Finally, using Corollary 1.9, C_3 can be extended to a copy of $T_4 = T$ in G, with the

[1] For (2), we may proceed assuming that $|W_1| \geqslant (1+\alpha/2)n$, and that $G[U_1]$ is $(36\varepsilon n, \eta n, \ell+12)$-well-connected.

[2] For (2), as $\Delta(T) \leqslant cn$, each tree S_v of **C2** satisfies $|S_v| \leqslant cmn + 1 \leqslant \eta n$. By Theorem 3.18, $G[W_1]$ contains a copy, C_1 say, of T_1.

vertices of $V(T_4) \setminus V(T_3)$ copied to U_0. □

To finish the proof of both our main results simultaneously we will use *superadditive set functions*. In the proof of each case, we define a function $f_T : \mathcal{P}(V(T)) \to \mathbb{N}_0$ on the power set of $V(T)$, where, for $A \subseteq V(T)$, $f_T(A)$ may be interpreted as representing a rough upper bound on the number of vertices required to guarantee a copy of $T[A]$, according to each of the theorems. The only limitation on f_T required for the proof to work is for it to be *superadditive*.

Definition 3.29. *Given a set X, we say that a set function $f : \mathcal{P}(X) \to \mathbb{N}_0$ is* superadditive *if $f(A \cup B) \geqslant f(A) + f(B)$ for any disjoint sets $A, B \subseteq X$.*

In particular, we will use the property that, if $f : \mathcal{P}(X) \to \mathbb{N}_0$ is a superadditive set function and $X = A_1 \cup \ldots \cup A_r$ is a partition, then

$$f(X) \geqslant \sum_{i \in [r]} f(A_i). \tag{3.19}$$

We also remark that superadditive set functions are increasing, in the sense that if $A \subseteq B$, then $f(A) \leqslant f(B)$.

Given a tree T, we will have one particular superadditive set function $f_T : \mathcal{P}(V(T)) \to \mathbb{N}_0$ for each main theorem. For Theorem 1.4, we take $f_T(A) = |A| + k(A) - 2s(A)$, where $k(A)$ denotes the number of leaves of the forest $T[A]$ (and isolated vertices count as two leaves), and $s(A)$ denotes the number of components of the forest $T[A]$. For Theorem 1.5, we take $f_T(A) = |A|$. To see that the first function is superadditive, let $A, B \subseteq V(T)$ be disjoint sets, and compare the forest $T[A \cup B]$ to the forest $T[A] \cup T[B]$. $T[A] \cup T[B]$ can be reached from $T[A \cup B]$ by removing the edges with one endpoint in each of A and B one at a time. Each time an edge is removed, the total number of vertices remains the same, the total number of leaves increases by at most 2, and the total number of components increases by 1. Thus we find that $f_T(A \cup B) \geqslant f_T(A) + f_T(B)$.

We are now ready to prove Theorems 1.4 and 1.5 using Lemma 3.28. The proof for each theorem is nearly identical, so we will present a proof for Theorem 1.4, and explain in the footnotes any places where the proof for Theorem 1.5 differs. In each case, using Lemma 3.27, we will have a partition of most of the vertex set of the tournament G into sets $W_1, \dots, W_r$ such that, if $u \in W_i$ and $v \in W_j$ with $i < j$, then $uv \in E(G)$, and such that each $G[W_i]$ is well-connected if W_i is not too small (i.e., **K1** and **K2** hold). It remains to find a good way to partition the tree T to embed it across this decomposition.

For each $i \in [r]$, if $|W_i| \geqslant \sqrt{\varepsilon}n$, then let $w_i = (1 - \alpha/4)|W_i|$, and otherwise let $w_i = |W_i|$. For each $i \in [r]$, we want to assign a set $U_i \subseteq V(T)$ satisfying $f_T(U_i) \leqslant w_i$ to embed in W_i. First, if $|W_r| \geqslant \sqrt{\varepsilon}n$, then we order the vertices of T as $v_1, \dots, v_n$ so that all edges of T go forward in this ordering, and let U_r be a set of vertices at the end of the ordering satisfying $(1 - \varepsilon)w_i \leqslant f_T(U_i) \leqslant w_i$ (in this case, we will be able to embed $T[U_r]$ into $G[W_r]$ by **K2** and Lemma 3.28). If $|W_r| < \sqrt{\varepsilon}n$, then, if possible, we let U_r be w_r out-leaves of T, and if it is not possible then we stop. If we have not stopped, then we remove U_r from T and repeat this procedure to find U_{r-1}, and so on. Note that if this stops then either we have assigned vertices for each W_i, or the remaining forest has at most $\sqrt{\varepsilon}n$ out-leaves. In the latter case we carry out a similar assignment for $W_1, W_2, \dots$. When this stops either all the vertices have been assigned or the remaining forest has at most $\sqrt{\varepsilon}n$ in-leaves as well as at most $\sqrt{\varepsilon}n$ out-leaves — such a forest we can embed with $O(\sqrt{\varepsilon}n)$ spare vertices using Theorem 1.2.

Proof of Theorem 1.4 (with appropriate alterations for Theorem 1.5 indicated). Fix $\alpha > 0$, and note that we may additionally assume that $\alpha \leqslant 1$. Introduce constants ε, η, n_0 such that $1/n_0 \ll \eta \ll \varepsilon \ll \alpha$. Given a tree T, let $f_T : \mathcal{P}(V(T)) \to \mathbb{N}_0$ be the superadditive set function defined by $f_T(A) = |A| + k(A) - 2s(A)$, where $k(A)$ denotes the number of leaves of the forest $T[A]$ (and isolated vertices count as two leaves), and $s(A)$ denotes the number of components of the forest $T[A]$.[1]

Let $n \geqslant n_0$. Fix an n-vertex k-leaf oriented tree T and let G be a $((1+\alpha)n+k)$-vertex tournament, so that $f_T(V(T)) + \alpha n \leqslant |G| \leqslant 3n$. We will show that G contains a copy of

[1]For (2), fix $\alpha > 0$ and introduce constants $\varepsilon, \eta, c, n_0$ such that $1/n_0 \ll c \ll \eta \ll \varepsilon \ll \alpha$. Given a tree T, let $f_T : \mathcal{P}(V(T)) \to \mathbb{N}_0$ be the superadditive set function defined by $f_T(A) = |A|$.

T, thus proving the theorem.[1]

By Lemma 3.27, there is a partition $V(G) = B \cup W_1 \cup \ldots \cup W_r$ so that $|B| \leqslant \varepsilon n$ and the properties **K1** and **K2** hold. For each $i \in [r]$, if $|W_i| \geqslant \sqrt{\varepsilon} n$, then let $w_i = (1 - \alpha/4)|W_i|$, and otherwise let $w_i = |W_i|$. Partition $[r]$ into intervals I^-, I, I^+ (in that order), so that I is minimal subject to there being disjoint sets $U_i \subseteq V(T)$, $i \in I^- \cup I^+$, for which the following hold.

L1 For each $i \in I^- \cup I^+$, $(1 - \varepsilon)w_i \leqslant f_T(U_i) \leqslant w_i$.

L2 There are no edges from $\cup_{i \in I^+} U_i$ to $V(T) \setminus (\cup_{i \in I^+} U_i)$ in T.

L3 There are no edges from $V(T) \setminus (\cup_{i \in I^-} U_i)$ to $\cup_{i \in I^-} U_i$ in T.

L4 If $|W_i| < \sqrt{\varepsilon} n$, then there are no edges in $T[U_i]$.

L5 If $i, j \in I^- \cup I^+$ with $i < j$, then there are no edges from U_j to U_i in T.

Note that this is possible as $I = [r]$ is a valid partition. Let $T' = T - \cup_{i \in I^+ \cup I^-} U_i$ and $W = \cup_{i \in I} W_i$. We will show that, for each $i \in I^- \cup I^+$, $G[W_i]$ contains a copy of $T[U_i]$, and $G[W]$ contains a copy of T'. Putting these together then gives a copy of T, by **L2**, **L3**, **L5**, and **K1**.

For each $i \in I^- \cup I^+$, if $|W_i| \geqslant \sqrt{\varepsilon} n$, then

$$|U_i| \geqslant f_T(U_i)/2 \overset{\mathbf{L1}}{\geqslant} (1 - \varepsilon)w_i/2 \geqslant \sqrt{\varepsilon} n/4, \tag{3.20}$$

and

$$f_T(U_i) + (\alpha/4) \cdot |U_i| \leqslant (1 + \alpha/4) \cdot f_T(U_i) \overset{\mathbf{L1}}{\leqslant} (1 + \alpha/4)w_i \leqslant |W_i|,$$

and so $G[W_i]$ contains a copy of $T[U_i]$ by **K2** and Lemma 3.28 (noting that (3.20) gives the required lower bound on $|T[U_i]|$ for the application of Lemma 3.28). On the other hand, if $|W_i| < \sqrt{\varepsilon} n$, then by **L4**, $G[W_i]$ contains a copy of $T[U_i]$, noting that we have $f_T(U_i) = |U_i|$ in this case.

[1]For (2), fix an n-vertex oriented tree T with $\Delta(T) \leqslant cn$ and let G be a $(1 + \alpha)n$-vertex tournament, so that $f_T(V(T)) + \alpha n = |G| \leqslant 3n$. We will show that G contains a copy of T, thus proving the theorem.

It is left to show that $G[W]$ contains a copy of T'. Note that this is trivial if $I = \emptyset$, and so we can assume $I \neq \emptyset$ and label $j_1, j_2 \in [r]$ so that I is the interval from j_1 to j_2. Also, note that, because $\sum_{i\in I^-\cup I^+} f_T(U_i) \leqslant 2n$,

$$
\begin{aligned}
|W| &= |G| - |B| - \sum_{i\in I^-\cup I^+} |W_i| \\
&\overset{\textbf{L1}}{\geqslant} f_T(V(T)) + \alpha n - \varepsilon n - (1-\alpha/4)^{-1}(1-\varepsilon)^{-1} \sum_{i\in I^-\cup I^+} f_T(U_i) \\
&\geqslant f_T(V(T)) - \sum_{i\in I^-\cup I^+} f_T(U_i) + \alpha n/4 \overset{(3.19)}{\geqslant} f_T(V(T')) + \alpha n/4 \geqslant |T'| + \alpha n/4.
\end{aligned}
$$

If $|W_{j_2}| \geqslant \sqrt{\varepsilon} n$, then we must have $f_T(V(T')) < (1-\varepsilon)w_{j_2}$, otherwise we could order $V(T')$ as $v_1, \ldots, v_{|T'|}$ so that all edges of T' go forward in this ordering, and define $U_{j_2} = \{v_s, \ldots, v_{|T'|}\}$ for some s chosen such that $(1-\varepsilon)w_{j_2} \leqslant f_T(U_{j_2}) \leqslant w_{j_2}$, a contradiction to the minimality of I. Thus, if $|W_{j_2}| \geqslant \sqrt{\varepsilon} n$, then $G[W_{j_2}]$, and hence $G[W]$, contains a copy of T' by Lemma 3.28. Similarly, if $|W_{j_1}| \geqslant \sqrt{\varepsilon} n$, then $G[W_{j_1}]$, and hence $G[W]$, contains a copy of T'. We must have then that $|W_{j_1}| < \sqrt{\varepsilon} n$ and $|W_{j_2}| < \sqrt{\varepsilon} n$. Thus, by the minimality of I, T' has at most $w_{j_2} \leqslant \sqrt{\varepsilon} n$ out-leaves and at most $w_{j_1} \leqslant \sqrt{\varepsilon} n$ in-leaves. As $|W| \geqslant |T'| + \alpha n/4$, $G[W]$ then contains a copy of T' by Theorem 1.2, as required. □

3.7 Proof of Theorem 3.13

In this section we prove Theorem 3.13, which, in the notation in Section 3.1, finds an index j_t for a regularity cluster for the core T_0 of a tree, and a random homomorphism of a fixed digraph H with vertex weight function β representing an average component of $T_1 - V(T_0)$ (where here, and throughout this section, we use the term *random homomorphism* to refer to any random variable taking values in the set of all possible homomorphisms, in the sense of Theorem 3.13). For convenience, we restate the definition of H (see Figure 3.3) and Theorem 3.13. Let H be the fully-looped oriented forest with vertex and edge sets given by

$$V(H) = \left\{x^+, y^+, z^+, u^+, w^+, \bar{x}^+, \bar{z}^+, \bar{u}^+, \bar{w}^+, x^-, y^-, z^-, u^-, w^-, \bar{x}^-, \bar{z}^-, \bar{u}^-, \bar{w}^-\right\},$$
$$E(H) = \left\{\begin{array}{c} x^+y^+, z^+x^+, z^+u^+, w^+z^+, \bar{z}^+\bar{x}^+, \bar{z}^+\bar{u}^+, \bar{w}^+\bar{z}^+, \\ y^-x^-, x^-z^-, u^-z^-, z^-w^-, \bar{x}^-\bar{z}^-, \bar{u}^-\bar{z}^-, \bar{z}^-\bar{w}^- \end{array}\right\} \cup \{vv : v \in V(H)\}. \tag{3.2}$$

For each $\diamond \in \{+,-\}$, let $X^\diamond = \{x^\diamond, \bar{x}^\diamond\}$. Let $X = X^+ \cup X^-$.

Theorem 3.13. *Let $1/r \ll \varepsilon \ll \mu \ll \alpha < 1$. Let H be the fully-looped oriented forest with vertex and edge sets given by (3.2). For each $\diamond \in \{+,-\}$, set $X^\diamond = \{x^\diamond, \bar{x}^\diamond\}$, and set $X = X^+ \cup X^-$. Let $\beta : V(H) \to [0,1]$ be a function satisfying $\sum_{v \in V(H)} \beta(v) = 1$ with $\beta(y^+) \geqslant \beta(x^+)$ and $\beta(y^-) \geqslant \beta(x^-)$, and, for every $v \in V(H)$, $\beta(v) \geqslant \mu$. Let D be a complete looped digraph on vertex set $[r]$ with ε-complete edge weights $d(e)$ for $e \in E(D)$. Let*

$$\gamma = \max\{\beta(x^+, \bar{x}^+), \beta(z^+, \bar{z}^+)\} + \max\{\beta(x^-, \bar{x}^-), \beta(z^-, \bar{z}^-)\}. \tag{3.3}$$

Then, there is a fixed $j_t \in [r]$ and a probability distribution $\mathcal{D}$ on the set of functions from $V(H)$ to $V(D)$, such that, if ϕ is sampled according to $\mathcal{D}$, then the following properties hold.

F1 *With probability 1, ϕ is a homomorphism from H to D, and $j_t \notin \phi(\{x^+, \bar{x}^+, x^-, \bar{x}^-\})$.*

F2 *For each $j \in [r]$, $\mathbb{E}(\beta(\phi^{-1}(j))) \leqslant \frac{1+\gamma+\alpha}{r}$.*

F3 *For each $j \in [r]$, either*

F3.1 $\mathbb{E}(\beta(\phi^{-1}(j) \cap X^+)) \leqslant d(j_t, j) \cdot \frac{1+\gamma+\alpha}{r}$ *and* $\mathbb{E}(\beta(\phi^{-1}(j) \cap X)) \leqslant d(j, j_t) \cdot \frac{1+\gamma+\alpha}{r}$,

or

F3.2 $\mathbb{E}(\beta(\phi^{-1}(j) \cap X^-)) \leqslant d(j, j_t) \cdot \frac{1+\gamma+\alpha}{r}$ *and* $\mathbb{E}(\beta(\phi^{-1}(j) \cap X)) \leqslant d(j_t, j) \cdot \frac{1+\gamma+\alpha}{r}$.

F4 *With probability 1, we have $|\phi(e)| = 2$ for every non-looped edge e of H.*

To prove Theorem 3.13, we first make two key simplifications before dividing into three critical cases. Our first simplification is to work only with the vertices of H representing components attached by an out-edge from T_0 (see Figure 3.3). Let H^+ and H^- be the subdigraphs of H induced on the vertices with $+$ and $-$ in the superscript, respectively (i.e., the right and the left parts of H in Figure 3.3). Considering each possible location $j \in [r]$ for j_t in Theorem 3.13, either a) j has enough weight on its out-edges that a random embedding of H^- can be extended relatively easily (with perhaps some modification) to one of H satisfying our requirements, or b) j has enough weight on its in-edges to similarly extend a random embedding of H^+. If many $j \in [r]$ satisfy a), then we may randomly embed H^- into the weighted looped digraph D induced on these j. If not, then enough $j \in [r]$ satisfy b), so that we may randomly embed H^+ into the weighted looped digraph D induced on these j. By appealing to directional duality if necessary, this allows us to prove a simplified version of Theorem 3.13 with weight only on H^+ (see Section 3.7.4).

Our second simplification to Theorem 3.13 is to drop the condition **F4**; we later show this condition can be recovered without undue difficulty. These two simplifications of Theorem 3.13 result in Theorem 3.34, which we state in Section 3.7.2 after introducing a notational framework of 'distillations' in Section 3.7.1 in order to have a concise and consistent language for the proofs in this section. The proof of Theorem 3.34 varies depending on the weight distribution β on the vertices in H. This falls into three main cases, which we also state in Section 3.7.2, in the form of Lemmas 3.35, 3.37 and 3.38, before deducing Theorem 3.34 from these cases. We then prove the lemma for each of these cases in Section 3.7.3, before finally deducing Theorem 3.13 from Theorem 3.34 in Section 3.7.4.

3.7.1 Distillations

We prove Theorem 3.13 from three specific cases where, roughly speaking, H is replaced by simpler subgraphs of H. In order to have a concise and consistent language for proving these cases, we will use the notion of a *distillation*, as follows.

Definition 3.30. *A* distillation *is a triple $\mathcal{F} = (F, X, \beta)$, where F is a fully-looped oriented forest, $X \subseteq V(F)$ is a set containing precisely one vertex in each component of F, and $\beta : V(F) \to [0,1]$ satisfies $\sum_{v \in V(F)} \beta(v) = 1$.*

In Theorem 3.13, we have a distillation (H, X, β) which we used to represent the average component of $T - V(T_0)$ for Theorem 3.9. There is some flexibility in how we could have chosen this distillation — for example we could move all the weight from y^+ to x^+, or from u^+ to z^+ and still have a useful distillation of the average component if we can find a matching random homomorphism. However, H is the smallest digraph that records enough structure in the average component to allow every relevant distillation to have a matching random homomorphism. The construction of the random homomorphism falls into three cases depending on the distribution of the weight — in each case we can move weight off some vertices (different in each case) to simplify the digraph in the distillation for which we find a random homomorphism.

To describe which simplifications of distillations are valid in this way formally, and prove this validity, we will define a transitive relation $\hookrightarrow$ between distillations. Very roughly, given two distillations $\mathcal{F}_0$ and $\mathcal{F}_1$, if $\mathcal{F}_0 \hookrightarrow \mathcal{F}_1$, then we can move weight in $\mathcal{F}_1$ (and possibly delete vertices) to transform it into $\mathcal{F}_0$. Formally, we define the relation as follows.

Definition 3.31. *Given distillations $\mathcal{F}_i = (F_i, X_i, \beta_i)$ for $i \in \{0, 1\}$, say $\mathcal{F}_0 \hookrightarrow \mathcal{F}_1$ if there is a random homomorphism $\rho : F_0 \to F_1$ with the following properties.*

M1 *With probability 1, $\rho(X_0) \subseteq X_1$.*

M2 $\mathbb{E}(\beta_0(\rho^{-1}(v))) = \beta_1(v)$ *for every $v \in V(F_1)$.*

Finally, we need a notion of which distillations are useful — i.e., which distillations have a matching random homomorphism with properties like those in Theorem 3.13. It will be convenient to consider a small collection of distillations and allow a sampling of the random homomorphism to take any one of them as its domain, and so we define the following notion of γ-goodness on sets of distillations. Roughly speaking, γ corresponds to the extra proportion of vertices we need to embed the tree, as in the use of Theorem 3.13.

Definition 3.32. *Given $\gamma \geqslant 0$, and distillations $\mathcal{F}_i = (F_i, X_i, \beta_i)$, $i \in [m]$, we say $\{\mathcal{F}_i\}_{i=1}^m$ is* γ-good *if the following holds for any fixed $\alpha > 0$: if $1/r \ll \varepsilon \ll \alpha$ and D is a complete looped digraph on vertex set $[r]$ with ε-complete edge weights $d(e)$, $e \in E(D)$, then there exists some $j_t \in [r]$ and a random $(\phi, i(\phi))$ with the following properties.*

N1 *With probability 1, we have that $i(\phi) \in [m]$, that ϕ is a homomorphism from $F_{i(\phi)}$ to D, and that $j_t \notin \phi(X_{i(\phi)})$.*

N2 *For each $j \in [r]$, $\mathbb{E}(\beta_{i(\phi)}(\phi^{-1}(j))) \leqslant \frac{1+\gamma+\alpha}{r}$.*

N3 *For each $j \in [r]$, $\mathbb{E}(\beta_{i(\phi)}(\phi^{-1}(j) \cap X_{i(\phi)})) \leqslant d(j_t, j) \cdot \frac{1+\gamma+\alpha}{r}$.*

Note that if $\{\mathcal{F}_i\}_{i=1}^m$ is γ-good for some $\gamma \geqslant 0$, then $\{\mathcal{F}_i\}_{i=1}^m$ is γ'-good for every $\gamma' \geqslant \gamma$. In addition, a set of distillations is γ-good if and only if it contains a non-empty subset which is γ-good.

Finally here, we prove the following key lemma that confirms that if a distillation can be simplified via the relation $\hookrightarrow$ to each one of a family of distillations which are collectively γ-good, then that original distillation is γ-good, as follows.

Lemma 3.33. *Let $\gamma \geqslant 0$, and suppose $\mathcal{F}$ and $\mathcal{G}_1, \dots, \mathcal{G}_m$ are distillations such that $\mathcal{F} \hookrightarrow \mathcal{G}_i$ for every $i \in [m]$. If $\{\mathcal{G}_i\}_{i=1}^m$ is γ-good, then $\{\mathcal{F}\}$ is γ-good.*

Proof. Let $\mathcal{F} = (F, X, \beta)$ and $\mathcal{G}_i = (G_i, X_i, \beta_i)$ for each $i \in [m]$. For each $i \in [m]$, let $\rho_i : F \to G_i$ be a random homomorphism realising $\mathcal{F} \hookrightarrow \mathcal{G}_i$.

Take $1/r \ll \varepsilon \ll \alpha$, and let D be a complete looped digraph on vertex set $[r]$ with ε-complete edge weights $d(e)$, $e \in E(D)$. Let $j_t \in [r]$ and $(\phi, i(\phi))$ realise that $\{\mathcal{G}_i\}_{i=1}^m$ is γ-good in the case of D. Define $(\psi, k(\psi))$ as follows. First, sample $(\phi, i(\phi))$. Then, with $i(\phi)$ now fixed, sample $\rho_{i(\phi)}$, and set $\psi = \phi \circ \rho_{i(\phi)}$. Let $k(\psi) = 1$ with probability 1, and note that **N1** holds for $(\psi, k(\psi))$.

For $i \in [m]$ let A_i be the event $\{i(\phi) = i\}$. Then, by the law of total expectation,

$$\mathbb{E}(\beta(\psi^{-1}(j))) = \sum_{i\in[m]} \mathbb{P}(A_i) \cdot \mathbb{E}(\beta(\psi^{-1}(j)) \mid A_i) = \sum_{i\in[m]} \mathbb{P}(A_i) \cdot \mathbb{E}(\beta(\rho_i^{-1}(\phi^{-1}(j))) \mid A_i)$$
$$\overset{\mathbf{M2}}{=} \sum_{i\in[m]} \mathbb{P}(A_i) \cdot \mathbb{E}(\beta_i(\phi^{-1}(j)) \mid A_i) = \mathbb{E}(\beta_{i(\phi)}(\phi^{-1}(j))),$$

so **N2** holds for $(\psi, k(\psi))$, and

$$\begin{aligned}
\mathbb{E}(\beta(\psi^{-1}(j) \cap X)) &= \sum_{i\in[m]} \mathbb{P}(A_i) \cdot \mathbb{E}(\beta(\psi^{-1}(j) \cap X) \mid A_i) \\
&= \sum_{i\in[m]} \mathbb{P}(A_i) \cdot \mathbb{E}(\beta(\rho_i^{-1}(\phi^{-1}(j)) \cap X) \mid A_i) \\
&\leqslant \sum_{i\in[m]} \mathbb{P}(A_i) \cdot \mathbb{E}(\beta(\rho_i^{-1}(\phi^{-1}(j) \cap \rho_i(X))) \mid A_i) \\
&\overset{\mathbf{M1}}{\leqslant} \sum_{i\in[m]} \mathbb{P}(A_i) \cdot \mathbb{E}(\beta(\rho_i^{-1}(\phi^{-1}(j) \cap X_i)) \mid A_i) \\
&\overset{\mathbf{M2}}{=} \mathbb{E}(\beta_{i(\phi)}(\phi^{-1}(j) \cap X_{i(\phi)})),
\end{aligned}$$

so **N3** holds for $(\psi, k(\psi))$. □

3.7.2 Statement of overarching theorem and subcases

Let H_0 be the fully-looped oriented forest with vertices $\{x, y, z, u, w, \bar{x}, \bar{z}, \bar{u}, \bar{w}\}$ and non-looped edges $\{xy, zx, zu, wz, \bar{z}\bar{x}, \bar{z}\bar{u}, \bar{w}\bar{z}\}$, noting that this is the subdigraph of H defined at the start of this section restricted to the vertices with $+$ in the superscript, and with vertices labelled more concisely (see also Figure 3.3). As noted at the start of this section, we will first prove a version of Theorem 3.13 for this subdigraph of H, without the condition **F4**, before deducing Theorem 3.13 from this in Section 3.7.4. Having introduced our relevant notation, we can now state this version of Theorem 3.13 concisely, as follows.

Theorem 3.34. *Let $\beta_0 : V(H_0) \to [0,1]$ be a function with $\sum_{v \in V(H_0)} \beta_0(v) = 1$ and $\beta_0(y) \geqslant \beta_0(x)$, and set $X_0 = \{x, \bar{x}\}$. Set $\gamma = \max\{\beta_0(x,\bar{x}), \beta_0(z,\bar{z})\}$. Let $\mathcal{H}_0 = (H_0, X_0, \beta_0)$. Then, $\{\mathcal{H}_0\}$ is γ-good.*

As mentioned before, the proof of Theorem 3.34 depends on the weight distribution β_0 on H_0. Dividing into cases, solving them, and showing they combine to prove this theorem is no easy task. Doing so while additionally motivating the choice of these cases is more difficult still. However, while we do concentrate on giving as clear and concise a proof of Theorem 3.34 as possible, we will give some motivation behind the cases by relating them to an embedding of a tree T into a tournament G.

In particular, our notation is designed to make the cases as efficient as possible to check, rather than explain the larger understanding that is necessary to produce these cases. To motivate the cases more directly, we now recall the discussion in Section 3.1. Our aim is to use Theorem 3.13 to embed a tree T with a small core T_0, where $T - V(T_0)$ is a collection of small components. To do this we take the tournament G and find a regularity partition $V_1 \cup \ldots \cup V_r$. We then want to make a careful choice of $j_t \in [r]$ and embed T_0 into V_{j_t} before distributing the components of $T - V(T_0)$ across the clusters of the regularity partition. The choice of j_t restricts which component any vertex in $v \in V(T) \setminus V(T_0)$ can be embedded to. For example, if the path from T_0 to v in T is a directed path towards v, and $v \in V_i$, then there must be a directed path from V_{j_t} to V_i of edges with positive weight in the reduced digraph obtained from the regularity partition. Fortunately, for our cases we need to consider at most the direction of the first three edges on the path from T_0 to v (and the first edge will always be directed away from T_0).

Very roughly, we first divide into two cases corresponding to the following situations, where, for example, we use the notation of a $(++)$-path from u to v to be a length two path from u to v comprised of two edges directed forward from u to v, with other notation used similarly.

- For most of the vertices $v \in V(T) \setminus V(T_0)$, if T_0 is embedded to V_{j_t}, $i \in [r]$, and there is a $(++)$-path from V_{j_t} to V_i in D, then we could embed v to V_i.

- For most of the vertices $v \in V(T) \setminus V(T_0)$, if T_0 is embedded to V_{j_t}, $i \in [r]$, and there is a $(+-)$-path from V_{j_t} to V_i in D then we could embed v to V_i.

These cases correspond roughly to Lemma 3.35 and Lemma 3.36, respectively. We then further subdivide the latter case into two cases, essentially replacing the $(+-)$-path with a $(+-+)$-path and a $(+--)$-path, respectively. This gives us cases 1, 2, and 3 which, in terms of the weight distribution β on H correspond to the following roughly-defined three cases:

1. Most of the weight not on $\{x, \bar{x}\}$ is on y.

2. Most of the weight not on $\{x, \bar{x}\}$ is on $\{y, u, \bar{u}\}$ (but Case 1 does not apply).

3. Most of the weight not on $\{x, \bar{x}\}$ is on $\{z, w, \bar{z}, \bar{w}\}$.

We now use our concept of distillations and the relation $\hookrightarrow$ to state the lemmas corresponding to these cases and combine them to prove Theorem 3.34. We first divide Theorem 3.34 into two lemmas – based on the distribution of β_0, we distill H_0 into H_1 or H_2, where for the former we remove the vertices $\{u, w, \bar{u}, \bar{w}\}$ and in the latter we remove $\{\bar{x}, \bar{z}, \bar{u}, \bar{w}\}$. This gives Lemma 3.35 (corresponding to Case 1 above) and Lemma 3.36. We then break Lemma 3.36 into two further lemmas, Lemma 3.37 and 3.38 which correspond respectively to Case 2 and 3 above, where in each case we have a set of distillations rather than simplifying to just one distillation. The structure of this division is depicted in Figure 3.7.

Let H_1 be the fully-looped oriented forest with vertices $\{x, y, z, \bar{x}, \bar{z}\}$ and non-looped edges $\{xy, zx, \bar{z}\bar{x}\}$.

Lemma 3.35. *Let $\mathcal{H} = (H_1, \{x, \bar{x}\}, \beta)$ be a distillation with $\beta(y) \geqslant \beta(z, \bar{z}), \beta(x)$. Then, $\{\mathcal{H}\}$ is $\beta(x, \bar{x})$-good.*

Let H_2 be the fully-looped oriented forest with vertices $\{x, y, z, u, w\}$ and non-looped edges $\{xy, zx, zu, wz\}$.

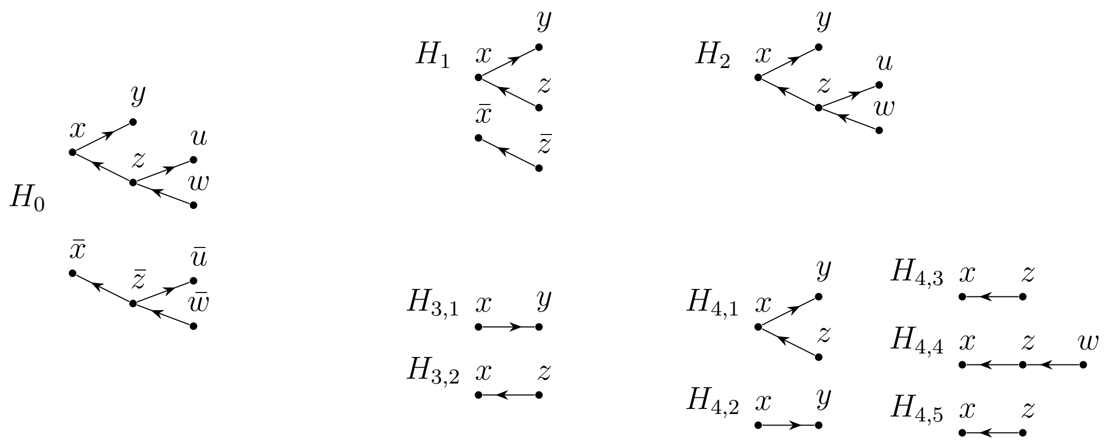

Figure 3.6: The underlying digraphs of the distillations described in this section (with looped edges omitted).

Lemma 3.36. *Let $\mathcal{H} = (H_2, \{x\}, \beta)$ be a distillation with $\beta(y) \leqslant \beta(z, u, w)$. Then, $\{\mathcal{H}\}$ is $(\max\{\beta(x), \beta(z)\})$-good.*

Let $H_{3,1}$ be the fully-looped oriented forest with vertices $\{x, y\}$ and non-looped edges $\{xy\}$. Let $H_{3,2}$ be the fully-looped oriented forest with vertices $\{x, z\}$ and non-looped edges $\{zx\}$.

Lemma 3.37. *Let $\beta_x \in [0, 1]$ and, set $\beta_1(x) = \beta_2(x) = (1 + \beta_x)/2$ and $\beta_1(y) = \beta_2(z) = (1 - \beta_x)/2$. For $i \in [2]$, set $\mathcal{H}_i = (H_{3,i}, \{x\}, \beta_i)$. Then, $\{\mathcal{H}_i\}_{i=1}^{2}$ is β_x-good.*

Let $H_{4,1}$ be the fully-looped oriented forest with vertices $\{x, y, z\}$ and non-looped edges $\{xy, zx\}$. Let $H_{4,2}$ be the fully-looped oriented forest with vertices $\{x, y\}$ and non-looped edges $\{xy\}$. Let $H_{4,3}$ be the fully-looped oriented forest with vertices $\{x, z\}$ and non-looped edges $\{zx\}$. Let $H_{4,4}$ be the fully-looped oriented forest with vertices $\{x, z, w\}$ and non-looped edges $\{zx, wz\}$. Let $H_{4,5} = H_{4,3}$.

Lemma 3.38. *Let $\beta_x, \beta_y, \beta_z, \beta_u, \beta_w \in [0, 1]$ have sum 1 and $\beta_y + \beta_u \leqslant \beta_z + \beta_w$. Let $\gamma = \max\{\beta_x, \beta_z\}$. Take the following weight functions $\beta_i : V(H_{4,i}) \to [0, 1]$ for $i \in [5]$.*

$$\beta_1(y) = \max\{\beta_x + \beta_y - \gamma, 0\} \qquad \beta_1(z) = \min\{\beta_w + \beta_z, 1 - \beta_z\} \qquad \beta_1(x) = 1 - \beta_1(y) - \beta_1(z),$$

$$\beta_2(x) = \beta_3(x) = \beta_x + \beta_z + \beta_w \qquad \beta_2(y) = \beta_3(z) = \beta_y + \beta_u,$$

$$\beta_4(x) = \min\{\beta_x+\beta_y+\beta_u, \max\{\beta_x+\beta_y, \beta_z\}\} \qquad \beta_4(w) = \beta_w \qquad \beta_4(z) = 1-\beta_4(x)-\beta_4(w),$$

$$\beta_5(x) = \beta_x + \beta_y \qquad \beta_5(z) = \beta_z + \beta_u + \beta_w.$$

For $i \in [5]$, set $\mathcal{H}_i = (H_{4,i}, \{x\}, \beta_i)$. Then, $\{\mathcal{H}_i\}_{i=1}^5$ is γ-good.

We remark that each set of weights defined in Lemma 3.38 sum to 1, either by the choice of $\beta_1(x)$ or $\beta_4(z)$ or as $\beta_x + \beta_y + \beta_z + \beta_u + \beta_w = 1$. Furthermore, from the choices they can all immediately be seen to be non-negative except for $\beta_1(x)$ and $\beta_4(z)$, but this we can also show, as follows.

First note that

$$\begin{aligned}\beta_4(z) &= 1 - \min\{\beta_x + \beta_y + \beta_u + \beta_w, \max\{\beta_x + \beta_y + \beta_w, \beta_z + \beta_w\}\} \\ &= \max\{\beta_z, \min\{\beta_u + \beta_z, \beta_x + \beta_y + \beta_u\}\}. \end{aligned} \tag{3.21}$$

Therefore, $\beta_4(z) \geqslant 0$. For use later, we will show that $\beta_1(x) \geqslant \beta_4(z)$, which also then confirms that $\beta_1(x) \geqslant 0$.

First suppose that $\beta_z \leqslant \beta_x + \beta_y$. Then, $1 - \beta_z = \beta_u + \beta_y + \beta_x + \beta_w \geqslant \beta_w + \beta_z$, so that $\beta_1(z) = \beta_w + \beta_z$, and hence

$$\beta_1(x) = 1 - \beta_1(y) - \beta_1(z) = 1 - (\beta_x + \beta_y - \gamma) - (\beta_w + \beta_z) = \beta_u + \gamma \geqslant \beta_u + \beta_z.$$

On the other hand, if $\beta_z > \beta_x + \beta_y$, then $\beta_1(y) = 0$, so that

$$\beta_1(x) = 1 - \beta_1(z) = \max\{1 - (\beta_w + \beta_z), \beta_z\} = \max\{\beta_u + \beta_y + \beta_x, \beta_z\}.$$

Therefore, in both cases, we have $\beta_1(x) \geqslant \beta_z$ and either $\beta_1(x) \geqslant \beta_u + \beta_z$ or $\beta_1(x) \geqslant \beta_u + \beta_y + \beta_x$. Thus, by (3.21), we have

$$\beta_1(x) \geqslant \beta_4(z). \tag{3.22}$$

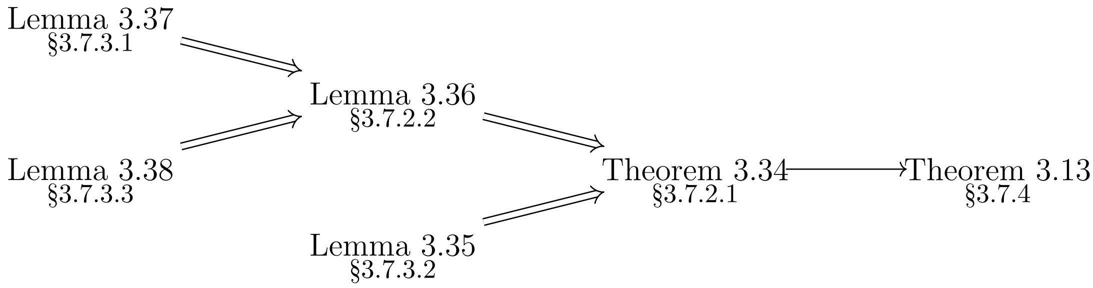

Figure 3.7: An overview of how the results of this section combine to prove Theorem 3.13. Each implication denoted by $\implies$ indicates a suitable application of Lemma 3.33.

We will now outline how Lemmas 3.35, 3.36, 3.37, and 3.38 together imply Theorem 3.34 (see Figure 3.7). First, we will show that Theorem 3.34 follows from Lemmas 3.35 and 3.36. We will then show that Lemma 3.36 follows from Lemmas 3.37 and 3.38. The proof of each of these implications is a straightforward case of verifying certain $\hookrightarrow$ relations between the relevant distillations hold according to the different values β may take, and applying Lemma 3.33 to deduce γ-goodness. All that will remain then is to prove Lemmas 3.35, 3.37, and 3.38, which we do in Section 3.7.3, and then deduce Theorem 3.13 from Theorem 3.34, which we do in Section 3.7.4.

3.7.2.1 Proof of Theorem 3.34 using Lemmas 3.33, 3.35, and 3.36

Using Lemma 3.33, it is simple to deduce Theorem 3.34 from Lemmas 3.35 and 3.36.

Proof of Theorem 3.34. Define $\rho_1 : H_0 \to H_1$ by setting $\rho_1(x) = x$, $\rho_1(y) = y$, $\rho_1(z, u, w) = z$, $\rho_1(\bar{x}) = \bar{x}$ and $\rho_1(\bar{z}, \bar{u}, \bar{w}) = \bar{z}$. Define $\rho_2 : H_0 \to H_2$ by setting $\rho_2(x, \bar{x}) = x$, $\rho_2(y) = y$, $\rho_2(z, \bar{z}) = z$, $\rho_2(u, \bar{u}) = u$ and $\rho_2(w, \bar{w}) = w$.

For each $i \in [2]$, let $\beta_i : V(H_i) \to [0, 1]$ be given by $\beta_i(v) = \beta_0(\rho_i^{-1}(v))$. Let $\mathcal{H}_1 = (H_1, \{x, \bar{x}\}, \beta_1)$ and $\mathcal{H}_2 = (H_2, \{x\}, \beta_2)$. Note that, for each $i \in [2]$, the homomorphism $\rho_i : H_0 \to H_i$ realises $\mathcal{H}_0 \hookrightarrow \mathcal{H}_i$.

If $\beta_0(y) \geqslant \beta_0(z, u, w, \bar{z}, \bar{u}, \bar{w})$, then, by Lemma 3.35, $\{\mathcal{H}_1\}$ is $\beta_0(x, \bar{x})$-good. On the other hand, if $\beta_0(y) \leqslant \beta_0(z, u, w, \bar{z}, \bar{u}, \bar{w})$, then, by Lemma 3.36, $\{\mathcal{H}_2\}$ is

$(\max\{\beta_0(x,\bar{x}),\beta_0(z,\bar{z})\})$-good. In either case we find $\{\mathcal{H}_0\}$ is γ-good, by Lemma 3.33. □

3.7.2.2 Proof of Lemma 3.36 using Lemmas 3.33, 3.37, and 3.38

To prove Lemma 3.36 follows from Lemmas 3.37 and 3.38 using Lemma 3.33 requires more checking due to the larger sets of distillations, but this is straightforward, as follows.

Proof of Lemma 3.36. For $v \in V(H_2)$, let $\beta_v = \beta(v)$. Note that we have $\beta_y \leqslant \beta_z+\beta_u+\beta_w$. Define $\mathcal{H}_{3,i}$, $i \in [2]$ and $\mathcal{H}_{4,i}$, $i \in [5]$ as described in Lemmas 3.37 and 3.38.

We will later prove the following two claims.

Claim 3.39. *If $\beta_y+\beta_u \geqslant \beta_z+\beta_w$, then $\mathcal{H} \hookrightarrow \mathcal{H}_{3,i}$ for $i \in [2]$.*

Claim 3.40. *If $\beta_y+\beta_u \leqslant \beta_z+\beta_w$, then $\mathcal{H} \hookrightarrow \mathcal{H}_{4,i}$ for $i \in [5]$.*

If $\beta_y+\beta_u \geqslant \beta_z+\beta_w$, then $\mathcal{H}$ is γ-good by Lemma 3.37, Lemma 3.33 and Claim 3.39. Otherwise, if $\beta_y+\beta_u \leqslant \beta_z+\beta_w$, then $\mathcal{H}$ is γ-good by Lemma 3.38, Lemma 3.33 and Claim 3.40. Therefore, it remains only to prove Claims 3.39 and 3.40.

Proof of Claim 3.39. For $i \in [2]$, let $\beta_i : V(H_{3,i}) \to [0,1]$ be defined as in Lemma 3.37.

To realise $\mathcal{H} \hookrightarrow \mathcal{H}_{3,1}$: If $\beta_y+\beta_u = 0$ (and hence, $\beta_1(y)=0$) then let $p_1 = 0$, and otherwise let

$$p_1 = \frac{\beta_1(y)}{\beta_y+\beta_u} = \frac{1-\beta_x}{2(\beta_y+\beta_u)} = \frac{\beta_y+\beta_u+\beta_z+\beta_w}{2(\beta_y+\beta_u)} \leqslant 1.$$

Define $\rho_1 : H_2 \to H_{3,1}$ by $\rho_1(x,z,w) = x$ and setting $\rho_1(y,u) = y$ with probability p_1, and otherwise setting $\rho_1(x,y,z,u,w) = x$.

To realise $\mathcal{H} \hookrightarrow \mathcal{H}_{3,2}$: If $\beta_z+\beta_u+\beta_w = 0$ (and hence, $\beta_2(z)=0$) then let $p_2=0$, and otherwise let

$$p_2 = \frac{\beta_2(z)}{\beta_z+\beta_u+\beta_w} = \frac{1-\beta_x}{2(\beta_z+\beta_u+\beta_w)} = \frac{\beta_y+\beta_z+\beta_u+\beta_w}{2(\beta_z+\beta_u+\beta_w)} \leqslant 1.$$

Define $\rho_2 : H_2 \to H_{3,2}$ by $\rho_2(x,y) = x$ and setting $\rho_2(z,u,w) = z$ with probability p_2, and otherwise setting $\rho_2(x,y,z,u,w) = x$. ⊡

Proof of Claim 3.40. For $i \in [5]$, let $\beta_i : V(H_{4,i}) \to [0,1]$ be defined as in Lemma 3.38.

To realise $\mathcal{H} \hookrightarrow \mathcal{H}_{4,1}$: If $\beta_y = 0$ (and hence, $\beta_1(y) = 0$) then let $p_1 = 0$, and otherwise let

$$p_1 = \frac{\beta_1(y)}{\beta_y} = \frac{\max\{\beta_x + \beta_y - \max\{\beta_x, \beta_z\}, 0\}}{\beta_y} \leqslant \frac{\beta_y}{\beta_y} = 1.$$

If $\beta_z + \beta_u + \beta_w = 0$ (and hence, $\beta_1(z) = 0$) then let $p_1' = 0$, and otherwise let

$$p_1' = \frac{\beta_1(z)}{\beta_z + \beta_u + \beta_w} \leqslant \frac{\beta_w + \beta_z}{\beta_z + \beta_u + \beta_w} \leqslant 1.$$

Define $\rho_1 : H_0 \to H_{4,1}$ by $\rho_1(x) = x$, and independently at random with probability p_1 setting $\rho_1(y) = y$ and otherwise setting $\rho_1(y) = x$, and independently at random with probability p_1' setting $\rho_1(z, u, w) = z$ and otherwise setting $\rho_1(z, u, w) = x$.

To realise $\mathcal{H} \hookrightarrow \mathcal{H}_{4,2}$: Define $\rho_2 : H_2 \to H_{4,2}$ by $\rho_2(x, z, w) = x$ and $\rho_2(y, u) = y$.

To realise $\mathcal{H} \hookrightarrow \mathcal{H}_{4,3}$: If $\beta_z + \beta_w + \beta_u = 0$ (and hence, $\beta_3(z) = 0$) then let $p_3 = 0$, and otherwise let

$$p_3 = \frac{\beta_3(z)}{\beta_z + \beta_w + \beta_u} = \frac{\beta_y + \beta_u}{\beta_z + \beta_w + \beta_u} \leqslant 1.$$

Define $\rho_3 : H_2 \to H_{4,3}$ by $\rho_3(x, y) = x$ and setting $\rho_3(z, w, u) = z$ with probability p_3, and otherwise setting $\rho_3(z, w, u) = x$.

To realise $\mathcal{H} \hookrightarrow \mathcal{H}_{4,4}$: From (3.21) we have $\beta_z \leqslant \beta_4(z) \leqslant \beta_z + \beta_u$. Using this, if $\beta_u = 0$ (and hence, $\beta_4(z) = \beta_z$) then let $p_4 = 0$, and otherwise let

$$p_4 = \frac{\beta_4(z) - \beta_z}{\beta_u},$$

so that $0 \leqslant p_4 \leqslant 1$ and $p_4\beta_u + \beta_z = \beta_4(z)$. Define $\rho_4 : H_2 \to H_{4,4}$ by $\rho_4(x, y) = x$, $\rho_4(z) = z$, $\rho_4(w) = w$, and setting $\rho_4(u) = z$ with probability p_4, and otherwise setting $\rho_4(u) = x$.

To realise $\mathcal{H} \hookrightarrow \mathcal{H}_{4,5}$: Define $\rho_5 : H_2 \to H_{4,5}$ by $\rho_5(x, y) = x$ and $\rho_5(z, u, w) = z$. $\boxdot$ $\square$

3.7.3 Proofs of the three cases

We are now ready to prove Lemmas 3.35, 3.37, and 3.38, thus completing the proof of Theorem 3.34. We give these in order of difficulty, first proving Lemma 3.37, followed by Lemma 3.35 and finally Lemma 3.38, with an informal motivating discussion preceding each proof.

3.7.3.1 Proof of Lemma 3.37

In the following proof of Lemma 3.37, we describe the random $(\phi, i(\phi))$ realising the β_x-goodness of the set $\{\mathcal{H}_i\}_{i=1}^2$. We assume (by relabelling) that $j_t = r$ has at least average out-edge weight, and describe a simple random homomorphism based on these edge weights. As this proof is relatively easy to check we do not motivate this further, but comment on it in the motivation for the other two cases. In the proof, we will use $N_D(j)$ to denote the set of $j' \in V(D)$ with $d(j,j') + d(j',j) > 0$.

Proof of Lemma 3.37. Let $\gamma = \beta_x$. Let $1/r \ll \varepsilon \ll \alpha$, and let D be a complete looped digraph on vertex set $[r]$ with ε-complete edge weights $d(e)$, $e \in E(D)$. We will find a random $(\phi, i(\phi))$ satisfying **N1-N3**. By relabelling, we can assume that

$$\sum_{j\in[r-1]} d(r,j) \geqslant (\tfrac{1}{2} - 2\varepsilon)\cdot r.$$

For $j \in [r-1]$, choose $0 \leqslant d_j \leqslant d(r,j)$ so that $\sum_{j\in[r-1]} d_j = (\frac{1}{2} - 2\varepsilon)\cdot r$.

Define $(\phi, i(\phi))$ randomly as follows. First, choose $\phi(x) \in [r-1]$ at random, so that $\phi(x) = j$ with probability $d_j/(\frac{1}{2} - 2\varepsilon)\cdot r$. Then choose $j' \in N_D(\phi(x))$ at random so that $j' = j$ with probability $(1-d_j)/\sum_{j\in N_D(\phi(x))}(1-d_j)$. If $d(\phi(x), j') > 0$, then set $i(\phi) = 1$ and $\phi(y) = j'$. Otherwise, set $i(\phi) = 2$ and $\phi(z) = j'$. Note that, in either case, ϕ is a homomorphism from $H_{3,\phi(i)}$ to D. By identifying $j_t = r$, **N1** holds. We now note that, for $j \in [r-1]$,

$$\mathbb{E}(\beta_{i(\phi)}(\phi^{-1}(j) \cap \{x\})) = \frac{(1+\beta_x)}{2} \cdot \frac{d_j}{(\frac{1}{2} - 2\varepsilon) \cdot r} \leqslant d_j \cdot \frac{1+\gamma+\alpha}{r},$$

and, because $\sum_{j \in N_D(j')}(1-d_j) \geqslant (\frac{1}{2} - 3\varepsilon) \cdot r$ for any $j' \in [r-1]$,

$$\mathbb{E}(\beta_{i(\phi)}(\phi^{-1}(j) \cap \{y,z\})) \leqslant \frac{(1-\beta_x)}{2} \cdot \frac{(1-d_j)}{(\frac{1}{2} - 3\varepsilon) \cdot r} \leqslant (1-d_j) \cdot \frac{1+\gamma+\alpha}{r}.$$

Thus, we deduce **N2** and **N3** hold. □

3.7.3.2 Proof of Lemma 3.35

Describing our proofs of the two remaining cases directly is difficult, in part because we are finding homomorphisms to a weighted digraph so we can apply our results to a regularity partition. What we do instead is describe an embedding of a tree T into a tournament G that follows all the major steps in our proof in an analogous, but more comprehensible, way. The embedding we describe is plausible but lacks detail and ignores several subtleties that influence the formal proof – our aim is to give a step by step embedding (see steps 1 to 8 below, and also Figure 3.8) in a simplified set-up that, by comparison, makes the proof of Lemma 3.35 easier to follow.

For this simplified set-up, assume that we have a tree T containing a vertex t with only out-neighbours in T, such that $T - \{t\}$ consists of small components. Assume further we have a tournament G, which is larger than T, into which we are attempting to find an embedding ψ of T. As t has many out-neighbours in T, an obvious choice for $\psi(t)$ is a vertex in G with maximal out-degree – say v_t is such a vertex and set $\psi(t) = v_t$. Let $A = N^+_G(v_t)$ and $B = V(G) \setminus (A \cup \{v_t\})$. Any component of $T - \{t\}$ can be embedded into $G[A]$ to extend the embedding to cover that component (using, for example, Theorem 1.8), but there is not necessarily enough room in A to embed all of the components at once, and so the challenge is to embed components so that many vertices in B are also used.

Let $\hat{H}_1$ have vertex set $\{x, y, z\}$ and edge set $\{xy, zx\}$. For each component of $T - \{t\}$, map the out-neighbour of t to x, any vertex whose path in T from t begins with two out-edges to y, and any vertex whose path in T from t begins with an out-edge then an in-edge to z. Thus, all vertices are mapped to $V(\hat{H}_1)$. We always want to embed the vertices of T mapped to x into A, so that they are out-neighbours of v_t. If all the edges between A and B in G are directed from A to B then, given a component of $T - \{t\}$ we could embed vertices mapped to y into B and vertices mapped to z into A. On the other hand, if all the edges between A and B in G are directed from B to A then, given a component, we could embed vertices mapped to z into B and vertices mapped to y into A. In practice, we expect the edges between A and B to meet neither extreme, and that some components can be embedded with vertices in B using edges directed from A to B and some using edges directed from B to A.

It may be be that all, or almost all, the edges between A and B in G are directed from A to B. Here, it is crucial that Lemma 3.35 covers the case corresponding to components having enough vertices mapped to y in order to place plenty of vertices into B. The other extreme, where almost all the edges between A and B in G are directed from B to A, cannot occur unless B is very small, otherwise we can find a vertex in B with high enough out-degree, both in $G[B]$ and into A, so that it has higher out-degree than v_t.

In trees corresponding to Case 1 (i.e., those for which we use Lemma 3.35), more vertices are mapped to y than z, so we prefer to embed components with the vertices mapped to y embedded in B. Our embedding is then via the following steps (where we first recap the embedding of t), and also sketched in Figure 3.8.

1. Embed t to a vertex v_t with a largest out-neighbourhood in G, and let $A = N^+_G(v)$ and $B = V(G) \setminus (A \cup \{v_t\})$.

2. Embed as many components of $T - \{t\}$ mapped to $\{x, y, z\} \subseteq V(H_1)$ as possible with the vertex mapped to $x \in V(H_1)$ in A and either a) all other vertices embedded into B or b) the vertices mapped to y embedded into B and the vertices mapped to z embedded into A.

3. Subject to this, choose the embedding maximising the number of components satisfying a).

4. Let A_0 be the unused vertices in A and B_0 be the unused vertices in B.

5. If B_0 is small (smaller than the number of leaves of T, say), then we do not need to use these vertices and can greedily find the remaining components within A_0. Thus, we assume B_0 is not too small and that we have not found all our components.

6. A_0 then cannot be too small as we always embedded enough vertices in B compared to A.

7. There cannot be many edges directed from A_0 to B_0, for otherwise another component could be embedded with vertices mapped to y embedded in B_0.

8. Using a sequence of deductions from the maximality of our component embeddings, we can then find large sets A_1 and B_1 with $A_0 \subseteq A_1 \subseteq A$ and $B_0 \subseteq B_1 \subseteq B$ so that the edges in G are mostly directed from B_1 into A_1, before concluding there is some vertex in B_1 with out-degree approximately $|A_1| + |B_1|/2$, which will be higher than the out-degree of v_t, $|A|$, giving a contradiction.

An example deduction is the following: for components satisfying b), the image of the vertex mapped to x must have few in-neighbours in B_0, for otherwise the vertices mapped to z could be embedded into B_0 to increase the number of embedded components satisfying a). Consequently, there cannot be many edges from A_0 to the images of vertices mapped to y in components satisfying b), else we could move the images of such vertices into B_0 and embed a new component using the freed up space. Thus we can add the images of vertices mapped to y in components satisfying b) to B_1.

We now describe the proof of Lemma 3.35 in comparison to these steps. For the lemma, T contains a small core $T_0 \subseteq T$ (corresponding to t above) and we have a distillation $(H_1, \{x, \bar{x}\}, \beta)$ representing an average component of $T - V(T_0)$, where H_1 is the fully-looped oriented forest with vertex set $\{x, y, z, \bar{x}, \bar{z}\}$ and edge set $\{xy, zx, \bar{z}\bar{x}\}$. We have a

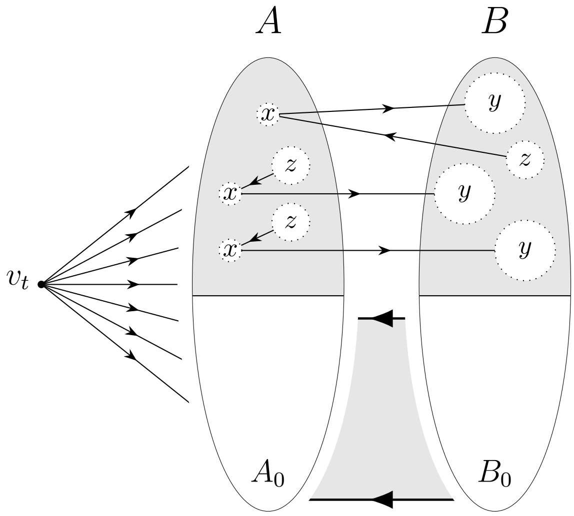

Figure 3.8: An example of how we aim to embed components of $T - \{t\}$ into the sets A and B in the simplified set-up. For this case, where more vertices in the components are mapped to y, we at first aim to embed these vertices mapped to y in B. Once this is no longer possible, we find that edges between the leftover vertices are mostly directed from B_0 towards A_0.

complete looped weighted digraph D which represents a regularity partition, and choose j_t to maximise the weight on the out-edges from j_t in D (cf. 1. above). By relabelling, we assume $j_t = r$. Instead of having a partition $A \cup B$ of the other vertices $j \in V(D)$, each vertex is duplicated and lies in both A and B with a weight, representing the proportion of that vertex that is in the out- and in-neighbourhood of j_t respectively (i.e., the proportions $d(j_t, j)$ and $d(j, j_t)$).

Instead of embedding components, we find homomorphisms $\phi_1, \dots, \phi_s$ (for some appropriate s) from H_1, before ultimately picking at random from these homomorphisms to get our required random homomorphism. These homomorphisms effectively allocate space within regularity clusters (represented by vertices of D) to embed a batch of components of $T - V(T_0)$, and we similarly aim to allocate as much space in B as possible. To do this, we first find as many as many homomorphisms $\hat{\phi}_1, \dots, \hat{\phi}_{s_0}$ from $\hat{H}_1 = H_1[\{x, y, z\}]$ as possible so that, ideally, y and z are both embedded into B (see condition **O1** in the proof), and, failing this, at least y is embedded into B (see **O2**), while x is always embedded to A (cf. 2. and 3. above). In doing so, we always ensure that the total weight assigned to any one vertex of D is not too much (see **O3**). Maximising the number of such homomorphisms (s_0), and then the number for which **O1** is relevant, in fact will allow us to find the remaining homomorphisms $\hat{\phi}_{s_0+1}, \ldots, \hat{\phi}_s$ before extending them to homomorphisms from H_1 (cf. 5. above). We prove this by assuming it cannot be done and steadily deducing a series of claimed properties of D that ultimately allow us to find a vertex with more weight on its out-edges in D than j_t, a contradiction.

Proof of Lemma 3.35. Let $\gamma = \beta(x, \bar{x})$. Let $1/r \ll \varepsilon \ll \alpha$. We remark that $\gamma \leqslant 1$, and we may also assume that $\alpha \leqslant 1$, so we have $(1 + \gamma + \alpha) \leqslant 3$.

Let D be a complete looped digraph on vertex set $[r]$ with ε-complete edge weights $d(e)$, $e \in E(D)$. We will find a random $(\phi, i(\phi))$ satisfying **N1**-**N3**. By relabelling, we can assume that

$$\sum_{j\in[r-1]} d(r,j) = \max_{i\in[r]} \sum_{j\in[r]\setminus\{i\}} d(i,j) \geqslant (\tfrac{1}{2} - 2\varepsilon)\cdot r. \tag{3.23}$$

Take two new disjoint vertex sets $A = \{a_1, \ldots, a_{r-1}\}$ and $B = \{b_1, \ldots, b_{r-1}\}$. Let $\bar{D}$ be the weighted complete looped digraph on $A \cup B$ in which the edges $a_i b_j$, $a_i a_j$, $b_i b_j$ and $b_i a_j$ have weight $d(i,j)$.

Let s be such that $1/s \ll 1/r$. For each $i \in [r-1]$, let $w_{a_i} = d(r,i)\cdot(1+\gamma+\alpha)\cdot s/r$ and $w_{b_i} = (1 - d(r,i))\cdot(1+\gamma+\alpha)\cdot s/r$. Note that

$$\sum_{v\in A\cup B} w_v \geqslant (1 + \beta(x,\bar{x}) + 7\alpha/8)\cdot s, \tag{3.24}$$

and

$$\sum_{v\in B} w_v = \sum_{i\in[r-1]} (1 - d(r,i))\cdot(1+\gamma+\alpha)\cdot s/r \overset{(3.23)}{<} (\tfrac{1}{2} + \tfrac{1}{2}\beta(x,\bar{x}) + 3\alpha/4)\cdot s. \tag{3.25}$$

We aim to find homomorphisms $\phi_1, \ldots, \phi_s : H_1 \to \bar{D}$, with $\phi_i(x), \phi_i(\bar{x}) \in A$ for each $i \in [s]$ and $\sum_{i\in[s]} \beta(\phi_i^{-1}(v)) \leqslant w_v$ for every $v \in V(\bar{D})$. Then if $\phi : H_1 \to D$ is the natural homomorphism induced by a uniform random selection from $\{\phi_1, \dots, \phi_s\}$, the conclusion of the theorem will hold.

Let $\hat{H}_1 = H_1[\{x, y, z\}]$. Let $s_0 \leqslant s$ be the largest integer for which there exist homomorphisms $\hat{\phi}_1, \dots, \hat{\phi}_{s_0} : \hat{H}_1 \to \bar{D}$ and indicators $j_1, \dots, j_{s_0} \in [2]$ such that the following properties hold.

O1 For each $i \in [s_0]$ with $j_i = 1$, we have $\hat{\phi}_i(x) \in A$, $\hat{\phi}_i(y) \in B$, and $\hat{\phi}_i(z) \in B$.

O2 For each $i \in [s_0]$ with $j_i = 2$, we have $\hat{\phi}_i(x) \in A$, $\hat{\phi}_i(y) \in B$, and $\hat{\phi}_i(z) \in A$.

O3 For each $v \in A \cup B$, $\sum_{i \in [s_0]} \beta(\hat{\phi}_i^{-1}(v)) \leqslant w_v$.

Subject to this, maximise the number of $i \in [s_0]$ with $j_i = 1$. Let I_1 be the set of $i \in [s_0]$ with $j_i = 1$, and let I_2 be the set of $i \in [s_0]$ with $j_i = 2$. For each $v \in A \cup B$, let $\hat{w}_v = \sum_{i \in [s_0]} \beta(\hat{\phi}_i^{-1}(v))$, so that, by **O3**, we have $\hat{w}_v \leqslant w_v$.

Note that

$$\sum_{v \in A \cup B} \hat{w}_v = \sum_{v \in A \cup B} \sum_{i \in [s_0]} \beta(\hat{\phi}_i^{-1}(v)) = \sum_{i \in [s_0]} \beta(\hat{\phi}_i^{-1}(A \cup B)) = \beta(x, y, z) \cdot s_0. \tag{3.26}$$

Let B_0 be the set of $v \in B$ with $w_v - \hat{w}_v \geqslant 1$. Let A_0 be the set of $v \in A$ with $w_v - \hat{w}_v \geqslant 2$, noting that we are placing a slightly stronger condition on the definition of A_0 to enable a switching argument later on (see the end of the proof of Claim 3.45).

We now show that we are done, unless $\sum_{v \in B}(w_v - \hat{w}_v)$ is not too small.

Claim 3.41. *Either there exists a random $(\phi, i(\phi))$ satisfying* **N1**-**N3**, *or*

$$\sum_{v \in B}(w_v - \hat{w}_v) > (\beta(x, \bar{x}) + 3\alpha/4) \cdot s. \tag{3.27}$$

Proof of Claim 3.41. Suppose that $\sum_{v\in B}(w_v - \bar{w}_v) \leqslant (\beta(x,\bar{x}) + 3\alpha/4)\cdot s$. Then

$$\begin{aligned}\sum_{v\in A}(w_v - \hat{w}_v) &= \sum_{v\in A\cup B}(w_v - \hat{w}_v) - \sum_{v\in B}(w_v - \hat{w}_v)\\ &\overset{(3.24),(3.26)}{\geqslant} (1 + \beta(x,\bar{x}) + 7\alpha/8)\cdot s - \beta(x,y,z)\cdot s_0 - (\beta(x,\bar{x}) + 3\alpha/4)\cdot s\\ &= (1+\alpha/8)\cdot s - \beta(x,y,z)\cdot s_0\\ &= \beta(\bar{x},\bar{z})\cdot s_0 + \beta(x,y,z,\bar{x},\bar{z})\cdot(s - s_0) + (\alpha/8)\cdot s. \end{aligned} \tag{3.28}$$

Greedily extend the homomorphisms $\hat{\phi}_1,\dots,\hat{\phi}_{s_0} : \hat{H}_1 \to \bar{D}$ to homomorphisms $\phi_1,\dots,\phi_{s_0}$: $H_1 \to \bar{D}$, so that, for every $i \in [s_0]$, $\phi_i|_{\{x,y,x\}} = \hat{\phi}_i$, and $\phi_i(\bar{x}), \phi_i(\bar{z}) \in A$. Then, greedily choose homomorphisms $\phi_{s_0+1},\dots,\phi_s$ so that, for every $i \in [s]\setminus[s_0]$, $\phi_i(V(H_1)) = \{a_j\}$ for some $a_j \in A$. These steps are possible, while also ensuring that $\sum_{i\in[s]}\beta(\phi_i^{-1}(v)) \leqslant w_v$ for every $v \in V(\bar{D})$, due to (3.28). Then, by defining $(\phi, i(\phi))$ by sampling ϕ from $\phi_1,\dots,\phi_s$ uniformly at random (identifying the result as a map $V(H_1) \to V(D)$ in the natural way) and setting $i(\phi) = 1$, we obtain a random $(\phi, i(\phi))$ satisfying **N1-N3**. ⊡

Thus, we may now assume that (3.27) holds, and hence also $|B_0| \geqslant 2\varepsilon r$. In particular, as $\hat{\phi}_i(y) \in B$ for each $i \in [s_0]$, we have

$$\beta(y)\cdot s_0 \leqslant \sum_{v\in B}\hat{w}_v \overset{(3.27)}{<} \sum_{v\in B} w_v - (\beta(x,\bar{x}) + 3\alpha/4)\cdot s \overset{(3.25)}{<} \frac{1}{2}(1 - \beta(x,\bar{x}))\cdot s \leqslant \beta(y)\cdot s,$$

and so we have that $s_0 < s$.

Claim 3.42. *If $i \in I_2$ and $v \in B_0$, then $d(v, \hat{\phi}_i(x)) = 0$. Hence, by* **E**, *given $i \in I_2$, there is some $v \in B_0$ with $d(\hat{\phi}_i(x), v) = 1$.*

Proof of Claim 3.42. Let $i \in I_2$, $v \in B_0$, and suppose that $d(v, \hat{\phi}_i(x)) > 0$. Then we may instead set $\hat{\phi}_i(z) = v$ and $j_i = 1$ and observe that **O1**–**O3** still hold. As this increases $|I_1|$, this is a contradiction. ⊡

Given $v \in B$, let $\bar{w}_v = \beta(y) \cdot |\{i \in I_2 : \hat{\phi}_i(y) = v\}|$. We remark that $\bar{w}_v \leqslant \hat{w}_v$, and

$$\sum_{j \in [r-1]} \bar{w}_{b_j} \overset{\mathbf{O1,O2}}{\geqslant} \sum_{j \in [r-1]} \hat{w}_{a_j} - \beta(x) \cdot s. \tag{3.29}$$

Let B_y be the set of $v \in B$ for which $\bar{w}_v \geqslant 2$.

Claim 3.43. *If $i, i' \in I_2$ are such that $i \neq i'$, then $d(\hat{\phi}_{i'}(y), \hat{\phi}_i(x)) = 0$. Hence, $d(v, \hat{\phi}_i(x)) = 0$ whenever $i \in I_2$ and $v \in B_y$.*

Proof of Claim 3.43. Let $i, i' \in I_2$ be such that $i \neq i'$, and suppose that $d(\hat{\phi}_{i'}(y), \hat{\phi}_i(x)) > 0$. Let $v' = \hat{\phi}_{i'}(y)$. By Claim 3.42, there is some $v \in B_0$ such that $d(\hat{\phi}_{i'}(x), v) = 1$. Then, because $\beta(y) \geqslant \beta(z)$, we may instead set $\hat{\phi}_i(z) = v'$, $\hat{\phi}_{i'}(y) = v$, and $j_i = 1$, increasing $|I_1|$, a contradiction. ⊡

Claim 3.44. *If $i \in I_2$, then $d(v, \hat{\phi}_i(x)) > 0$ for at least εr many $v \in A_0$.*

Proof of Claim 3.44. Let $i \in I_2$. Suppose that there are fewer than εr many $v \in A_0$ for which $d(v, \hat{\phi}_i(x)) > 0$. So, using Claim 3.42 and Claim 3.43, $d(v, \hat{\phi}_i(x)) = 0$ for all but at most εr many $v \in A_0 \cup B_0 \cup B_y$. Then, by **E**, for all but at most $2\varepsilon r$ many $v \in A_0 \cup B_0 \cup B_y$, we have $d(\hat{\phi}_i(x), v) = 1$.

Let j' be such that $\hat{\phi}_i(x) = a_{j'}$. Let J be the set of $j \in [r-1]$ such that $(w_{a_j} - \hat{w}_{a_j}) + (w_{b_j} - \hat{w}_{b_j}) + \bar{w}_{b_j} \geqslant 5$. If $j \in J$, then either $a_j \in A_0$ or $b_j \in B_0 \cup B_y$. So $d(j', j) = 1$ for all but at most $2\varepsilon r$ many $j \in J$, and hence $\sum_{j \in [r] \setminus \{j'\}} d(j', j) \geqslant |J| - 3\varepsilon r$. Noting that $(w_{a_j} - \hat{w}_{a_j}) + (w_{b_j} - \hat{w}_{b_j}) + \bar{w}_{b_j} \leqslant w_{a_j} + w_{b_j} = (1 + \gamma + \alpha) \cdot s/r$ for any $j \in [r-1]$, we have

$$\begin{aligned}
\sum_{j \in [r] \setminus \{j'\}} d(j', j) \cdot (1 + \gamma + \alpha) \cdot s/r &\geqslant |J| \cdot (1 + \gamma + \alpha) \cdot s/r - 3\varepsilon(1 + \gamma + \alpha) \cdot s \\
&\geqslant \sum_{j \in [r-1]} [(w_{a_j} - \hat{w}_{a_j}) + (w_{b_j} - \hat{w}_{b_j}) + \bar{w}_{b_j}] - 5r - 9\varepsilon s \\
&\overset{(3.27)}{\geqslant} \sum_{j \in [r-1]} w_{a_j} + \sum_{j \in [r-1]} (\bar{w}_{b_j} - \hat{w}_{a_j}) + (\beta(x, \bar{x}) + \alpha/2) \cdot s \\
&\overset{(3.29)}{\geqslant} \sum_{j \in [r-1]} w_{a_j} + (\beta(\bar{x}) + \alpha/2) \cdot s > \sum_{j \in [r-1]} d(r, j) \cdot (1 + \gamma + \alpha) \cdot s/r,
\end{aligned}$$

contradicting (3.23). $\boxdot$

Let I_Y be the set of $i \in [s_0]$ such that there exist distinct $i_0, \dots, i_\ell \in [s_0]$ with $i_0 = i$, such that

- $d(\hat{\phi}_{i_{k-1}}(x), \hat{\phi}_{i_k}(y)) > 0$ for $k \in [\ell]$, and
- $d(\hat{\phi}_{i_\ell}(x), v) > 0$ for some $v \in B_0$.

We remark that $d(\hat{\phi}_i(x), v) = 0$ whenever $i \notin I_Y, v \in B_0$, and also that $d(\hat{\phi}_i(x), \hat{\phi}_{i'}(y)) = 0$ whenever $i \notin I_Y, i' \in I_Y$.

Let A_1 be the set of $v \in A$ with $w_v - \sum_{i \in I_Y} \beta(\hat{\phi}_i^{-1}(v) \cap \{x\}) \geqslant 1$, and let B_1 be the set of $v \in B$ with $w_v - \hat{w}_v + \sum_{i \in I_Y} \beta(\hat{\phi}_i^{-1}(v) \cap \{y\}) \geqslant 1$.

Claim 3.45. *If $u \in A_1$ and $v \in B_1$, then $d(u, v) = 0$.*

Proof of Claim 3.45. Let $u \in A_1$ and $v \in B_1$, and suppose for contradiction that $d(u, v) > 0$.

If $\hat{\phi}_i(x) = u$ for some $i \notin I_Y$, then we must have $d(u, v') = 0$ for every $v' \in B_0$, else $i \in I_Y$. So in particular, we would have $v \in B_1 \setminus B_0$, and hence $\hat{\phi}_{i'}(y) = v$ for some $i' \in I_Y$. But then $d(\hat{\phi}_i(x), \hat{\phi}_{i'}(y)) > 0$ for some $i \notin I_Y$, $i' \in I_Y$, a contradiction.

Therefore, we may assume that $\sum_{i \in [s_0] \setminus I_Y} \beta(\hat{\phi}_i^{-1}(u) \cap \{x\}) = 0$, and hence

$$w_u - \hat{w}_u + \sum_{i \in I_2} \beta(\hat{\phi}_i^{-1}(u) \cap \{z\}) = w_u - \sum_{i \in [s_0]} \beta(\hat{\phi}_i^{-1}(u) \cap \{x\}) \geqslant 1. \tag{3.30}$$

Set $\hat{\phi}_{s_0+1}(x), \hat{\phi}_{s_0+1}(z) = u$ and $\hat{\phi}_{s_0+1}(y) = v$, and set $j_{s_0+1} = 2$.

Let $I_Z \subseteq I_2$ be a minimal set such that $\hat{\phi}_i(z) = u$ for $i \in I_Z$ and $w_u - \hat{w}_u + \sum_{i \in I_Z} \beta(\hat{\phi}_i^{-1}(u) \cap \{z\}) \geqslant 1$, noting that (3.30) shows such a choice is possible. By minimality of I_Z, we have $\sum_{i \in I_Z} \beta(\hat{\phi}_i^{-1}(u) \cap \{z\}) \leqslant 2$. Using Claim 3.44, choose $u_i \in A_0 \setminus \{u\}$ for $i \in I_Z$ with $d(u_i, u) > 0$.

If $w_v - \hat{w}_v \geqslant \beta(y)$, then set $\ell = 0$, $i_\ell = s_0 + 1$, and $v^\star = v$. Otherwise, we find there is some $i \in I_Y$ with $\hat{\phi}_i(y) = v$, and so let $i_0, \dots, i_\ell \in [s_0]$ be distinct with $i_0 = i$, and $v^\star \in B_0$

be such that $d(\hat{\phi}_{i_{k-1}}(x), \hat{\phi}_{i_k}(y)) > 0$ for $k \in [\ell]$ and $d(\hat{\phi}_{i_\ell}(x), v^\star) > 0$. In either case, set $v_{i_{k-1}} = \hat{\phi}_{i_k}(y)$ for $k \in [\ell]$, and set $v_{i_\ell} = v^\star$.

Now, setting $\hat{\phi}_i(z) = u_i$ for $i \in I_Z$ and $\hat{\phi}_{i_k}(y) = v_{i_k}$ for $k \in \{0\} \cup [\ell]$ yields a contradiction to the maximality of s_0, proving the claim. ⊡

Let J_A be the set of $j \in [r-1]$ with $a_j \in A_1$ and J_B be the set of $j \in [r-1]$ with $b_j \in B_1$. By Claim 3.45, J_A and J_B are disjoint. Let $j' \in J_B$ be such that $\sum_{j \in J_B} d(j', j)$ is maximised. So by Claim 3.45,

$$\sum_{j \in [r] \setminus \{j'\}} d(j', j) \geqslant |J_A| + \frac{1}{2}|J_B| - 2\varepsilon r. \tag{3.31}$$

Also, because $\beta(y) \geqslant \beta(x)$,

$$\begin{aligned} \sum_{v \in B} \left(w_v - \hat{w}_v + \sum_{i \in I_Y} \beta(\hat{\phi}_i^{-1}(v) \cap \{y\}) \right) &\overset{(3.27)}{>} (\beta(x, \bar{x}) + 3\alpha/4) \cdot s + \beta(y) \cdot |I_Y| \\ &\geqslant 2\beta(x) \cdot |I_Y| + (3\alpha/4) \cdot s = 2 \sum_{v \in A} \sum_{i \in I_Y} \beta(\hat{\phi}_i^{-1}(v) \cap \{x\}) + (3\alpha/4) \cdot s. \end{aligned} \tag{3.32}$$

Therefore,

$$\begin{aligned} &\sum_{j \in [r] \setminus \{j'\}} d(j', j) \cdot (1 + \gamma + \alpha) \cdot s/r \overset{(3.31)}{\geqslant} \left(|J_A| + \tfrac{1}{2}|J_B| - 2\varepsilon r\right) \cdot (1 + \gamma + \alpha) \cdot s/r \\ &= |J_A| \cdot (1 + \gamma + \alpha) \cdot s/r + \frac{1}{2}|J_B| \cdot (1 + \gamma + \alpha) \cdot s/r - 2\varepsilon(1 + \gamma + \alpha) \cdot s \\ &\geqslant \sum_{v \in A} (w_v - \sum_{i \in I_Y} \beta(\hat{\phi}_i^{-1}(v) \cap \{x\})) + \frac{1}{2} \sum_{v \in B} \left(w_v - \hat{w}_v + \sum_{i \in I_Y} \beta(\hat{\phi}_i^{-1}(v) \cap \{y\}) \right) - 10\varepsilon s \\ &\overset{(3.32)}{>} \sum_{v \in A} w_v + (3\alpha/8) \cdot s > \sum_{j \in [r-1]} d(r, j) \cdot (1 + \gamma + \alpha) \cdot s/r, \end{aligned}$$

contradicting (3.23). □

3.7.3.3 Proof of Lemma 3.38

In this section, we will prove Lemma 3.38. Similarly as in Section 3.7.3.2, we will outline our strategy in a simplified setting, along with a depiction in Figure 3.9, so that this outline may guide the reader through the technical proof. For this, let T again be a tree containing a vertex t with only out-neighbours in T, such that $T - \{t\}$ consists of small components. Furthermore, let G be a tournament with more vertices than T into which we are attempting to find an embedding ψ of T. We proceed initially with a very similar strategy to that described at the start of Section 3.7.3.2, as follows.

Let v_t be a vertex in G with maximal out-degree and set $\psi(t) = v_t$. Let $A = N^+(v_t)$ and $B = V(G) \setminus (A \cup \{v_t\})$. Just as before, we now aim to embed components of $T - \{t\}$ so that they can be attached correctly to v_t but so that as many vertices as possible lie in B. For the trees relevant for Lemma 3.35, if we carefully maximised the number of components we embedded, then we covered enough vertices in B that we were able to finish the embedding by embedding the remaining components into the unused vertices in A. The problem here is that Lemma 3.38 covers trees for which this might not be possible. To see this, consider again $\hat{H}_1$ with vertex set $\{x, y, z\}$ and edge set $\{xy, zx\}$, and, for each component of $T - \{t\}$, map the out-neighbour of t to x, and map the other vertices to y or z according to the direction of the first edge of their path from t in the component as before. If all edges between A and B in G are directed from A to B then the only vertices we can embed into B are those mapped to y, and in the trees relevant to Lemma 3.38 there may be few or even none of these! That is, it simply may not be the case that we can embed T with t embedded to v_t as before.

Nevertheless, with t embedded to v_t, we attempt to embed as many components as possible subject to a careful maximisation as before. In particular, we always have the vertex mapped to x embedded into A and vertices mapped to z embedded into B. Previously, we then made a sequence of deductions that led us to find sets $A_1 \subseteq A$ and $B_1 \subseteq B$ such that almost all the edges of G were directed from B_1 to A_1, and this led to a contradiction (see step 8 in Section 3.7.3.2). For Lemma 3.38, we again make a (here, simpler) sequence of deductions. Roughly, if A_0 and B_0 are the vertices in A and B respectively which are not in the image of any component we have embedded, then G must have almost all the possible edges directed from A_0 into B_0 as well as certain other properties. The vertices in B_0 are then the vertices we struggled to cover when embedding components of the tree. However, if we pick a typical vertex v'_t in A_0 and try instead to embed the tree starting with embedding t to v'_t, then it is easy to cover vertices in B_0 with components in $T - \{t\}$ as most of these vertices are in the out-neighbourhood of v'_t. Even better, the vertices in $A_0 \setminus \{v'_t\}$ can also be used relatively easily as there are many edges from A_0 to B_0 and, now the embedding of t is changed, components of $T - \{t\}$ can be embedded with vertices mapped to $\{x, y\}$ embedded into B_0 and vertices mapped to z embedded into A_0 (and for the trees covered by Lemma 3.38 significantly many vertices are mapped to z). Of course, the remaining vertices of G – those which had components embedded to them – may now be hard to cover, but, from our maximised component embedding and subsequent deductions, we will have information about how we can adjust the embedded components to attach them instead to the new image, v'_t, of t while still using roughly the same vertices for that component. This is not always simple, and we allocate some additional vertices in B_0 to each embedded component before finding a new version of that component using the old vertices for that component and possibly also the additional vertices allocated from B_0. That we can allocate some additional vertices in this fashion relies on the fact that G is larger than T, so we can afford not to use every vertex in G in the final embedding.

The portioning of the vertices in B_0 to allow the rearrangement and reattachment of the components requires quite some delicacy, resulting ultimately in the choice of the functions $\beta_1, \ldots, \beta_5$ in Lemma 3.38. This choice can be motivated, but only at some length and difficulty, so instead we concentrate on writing a proof that can be directly verified. As in Section 3.7.3.2, in the actual setting, with a tree T with small core T_0 in mind, given a distillation of an average component, we find homomorphisms to a weighted looped digraph which will represent a regularity partition. We then pick from

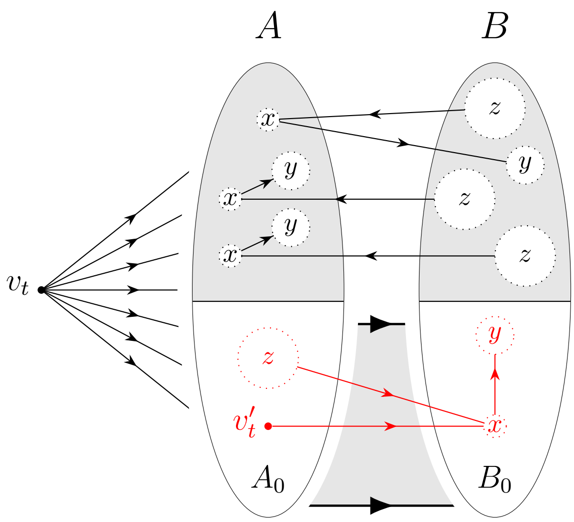

Figure 3.9: In our simplified set-up, for the case corresponding to Lemma 3.38, we may instead begin by aiming to embed vertices of $T - \{t\}$ mapped to z in B. The component in the lower half of the diagram illustrates how leftover space in A_0 and B_0 may be used efficiently via a new embedding v'_t for t.

these homomorphisms randomly to get a random homomorphism. These homomorphisms again essentially allocate room for components of $T - V(T_0)$ in the regularity clusters corresponding to the weighted digraph D.

To find these homomorphisms, we begin by selecting a vertex $j_t \in V(D)$ with at least the average amount of weight on its out-edges. By relabelling, we assume $j_t = r$. Similarly to the proof of Lemma 3.35, we again duplicate vertices of D to get weighted vertex sets A and B representing the proportion of each vertex that is in the out- and in-neighbourhood of j_t respectively, and aim to find homomorphisms which allocate plenty of space in B. To do this, we find maximally many homomorphisms from $H_{4,1}$ to D satisfying certain conditions (**P1**–**P4** in the proof), and further optimising over two additional conditions (stated just after **P4**), to get homomorphisms $\phi_1, \dots, \phi_{s_0}$. Assuming we do not have enough homomorphisms to easily find the remaining ones we wanted (see Claim 3.47), we make a sequence of deductions based on this assumption and the maximality (Claim 3.46 and Claim 3.48) about the weight distribution on D. Letting $A_0 \subseteq A$ and $B_0 \subseteq B$ be the sets of vertices with unallocated weight, an example deduction is that almost all of the weight on the edges of D between A_0 and B_0 is on edges directed from A_0 to B_0 (for otherwise we would have found at least one more homomorphism). We then choose a new vertex $j_t' \in [r]$, with $a_{j_t'} \in A_0$, to use instead of $j_t = r$ by selecting it so that $a_{j_t'}$ has sufficient out-weight to certain vertices in the image of the homomorphisms we have found (just before Claim 3.49) – this will allow us later to rearrange these homomorphisms to embed x into the out-neighbourhood of j_t' instead of j_t. While it may not necessarily be possible to ensure the desired conditions on j_t' hold with respect to every homomorphism chosen so far, Claim 3.49 confirms that there is a choice of j_t' with the required conditions for rearrangement holding with respect to most of them – by relabelling, these will be the homomorphisms $\phi_1, \dots, \phi_{s_1}$. We add a dummy vertex q with weight $\beta_1(q) = \max\{\beta_x, \beta_z\}$ to $H_{4,1}$, and then extend $\phi_1, \dots, \phi_{s_1}$ to homomorphisms $\psi_1, \dots, \psi_{s_1}$ also covering q (see **S1**, **S2** and Claim 3.50). The role played by q is to reserve additional weight in the out-neighbourhood of j_t', which may be required for the desired rearrangement. Each ψ_i then represents some allocated space, for which we then find homomorphisms $\phi_{i,j}$ from $H_{4,2}, H_{4,3}$ or $H_{4,4}$ which use a proportion of this space (in either Claim 3.51 or Claim 3.52) – a proportion matching the space a tree must cover in the tournament following an eventual application of the lemma (that is, the proportion $|T|/|G|$). The homomorphisms $\phi_{i,j}$ potentially leave plenty of weight remaining on vertices in A_0 and B_0, but, as noted above in the simplified setting, these vertices are easiest to use for new homomorphisms as there is a lot of weight on edges from j_t' to B_0 and from B_0 to A_0 in D. We therefore find more homomorphisms, this time from $H_{4,5}$, to use enough of this weight (that is, so that the proportion of the weight used is at least the matching proportion $|T|/|G|$). Finally, then, we have a collection of homomorphisms from which to pick our random homomorphism to complete the proof.

Proof of Lemma 3.38. Let $\gamma = \max\{\beta_x, \beta_z\}$. Let $1/r \ll \varepsilon \ll \alpha$. We remark that $\gamma \leqslant 1$, and we may also assume that $\alpha \leqslant 1$, so we have $(1 + \gamma + \alpha) \leqslant 3$.

Let D be a complete looped digraph on vertex set $[r]$ with ε-complete edge weights $d(e)$, $e \in E(D)$. We will find a random $(\phi, i(\phi))$ satisfying **N1**-**N3**. By relabelling, we can assume that

$$\sum_{j\in[r-1]} d(r,j) \geqslant (\tfrac{1}{2} - 2\varepsilon)\cdot r. \tag{3.33}$$

Take two new disjoint vertex sets $A = \{a_1, \ldots, a_{r-1}\}$ and $B = \{b_1, \ldots, b_{r-1}\}$. Let $\bar{D}$ be the weighted complete looped digraph on $A \cup B$ in which the edges a_ib_j, a_ia_j, b_ib_j and b_ia_j have weight $d(i,j)$.

Let s be such that $1/s \ll 1/r$. For each $i \in [r-1]$, let $w_{a_i} = d(r,i)\cdot(1+\gamma+\alpha)\cdot s/r$ and $w_{b_i} = (1-d(r,i))(1+\gamma+\alpha)\cdot s/r$. Note that

$$\sum_{v\in A\cup B} w_v \geqslant (1+\gamma+\alpha/2)\cdot s, \tag{3.34}$$

and

$$\sum_{v\in A} w_v = \sum_{i\in[r-1]} d(r,i)\cdot(1+\gamma+\alpha)\cdot s/r \overset{(3.33)}{\geqslant} \frac{1}{2}(1+\gamma+\alpha/2)\cdot s. \tag{3.35}$$

Let $s_0 \leqslant s$ be the largest integer for which there exist homomorphisms $\phi_1, \ldots, \phi_{s_0} : H_{4,1} \to \bar{D}$ and indicators $j_1, \ldots, j_{s_0} \in [2]$ such that the following properties hold.

P1 For each $i \in [s_0]$, $d(\phi_i(y), \phi_i(z)) + d(\phi_i(z), \phi_i(y)) > 0$.

P2 For each $i \in [s_0]$ with $j_i = 1$, we have $\phi_i(x) \in A$, $\phi_i(y) \in B$, and $\phi_i(z) \in B$.

P3 For each $i \in [s_0]$ with $j_i = 2$, we have $\phi_i(x) \in A$, $\phi_i(y) \in A$, and $\phi_i(z) \in B$.

P4 For each $v \in A \cup B$, $\sum_{i\in[s_0]} \beta_1(\phi_i^{-1}(v)) \leqslant w_v$.

Given s_0, take $\phi_1, \ldots \phi_{s_0}$ and $j_1, \ldots, j_{s_0}$ such that the number of $i \in [s_0]$ with $j_i = 1$ is maximised. For each $v \in A \cup B$, let $\tilde{w}_v = \sum_{i\in[s_0]} \beta_1(\phi_i^{-1}(v))$, so that, by **P4**, we have $\tilde{w}_v \leqslant w_v$.

Note that

$$\sum_{v\in A\cup B} \tilde{w}_v = \sum_{v\in A\cup B}\sum_{i\in[s_0]} \beta_1(\phi_i^{-1}(v)) = \sum_{i\in[s_0]} \beta_1(\phi_i^{-1}(A\cup B)) = s_0. \tag{3.36}$$

Let A_0 be the set of $v \in A$ with $w_v - \tilde{w}_v \geqslant 1$. Let B_0 be the set of $v \in B$ with $w_v - \tilde{w}_v \geqslant 1$. Subject to the value of s_0, and the value of $|\{i \in [s_0] : j_i = 1\}|$, assume that

$$|\{(i, v) : i \in [s_0], v \in A_0, d(\phi_i(y), v) > 0\}| \tag{3.37}$$

is minimised.

Suppose that $s_0 < s$, for otherwise we are done by letting $i(\phi) = 1$ and picking ϕ from $\{\phi_i : i \in [s]\}$ uniformly at random. Note that, as $1 - \gamma \leqslant 1 - \beta_x = \beta_y + \beta_u + \beta_z + \beta_w \leqslant 2(\beta_z + \beta_w)$, we have

$$\begin{aligned} \frac{1+\gamma}{2} - \beta_1(x, y) = \frac{1+\gamma}{2} - (1 - \beta_1(z)) &= \min\{\beta_w + \beta_z, 1 - \beta_z\} - \frac{1-\gamma}{2} \\ &\geqslant \min\left\{\frac{1-\gamma}{2}, 1 - \gamma\right\} - \frac{1-\gamma}{2} = 0. \end{aligned} \tag{3.38}$$

Therefore,

$$\begin{aligned} \sum_{v \in A_0} (w_v - \tilde{w}_v) &\geqslant \sum_{v \in A} (w_v - \tilde{w}_v) - r = \sum_{v \in A} w_v - r - \sum_{i \in I_1} \beta_1(x) - \sum_{i \in I_2} \beta_1(x, y) \\ &\geqslant \sum_{v \in A} w_v - r - \beta_1(x, y) \cdot s_0 \overset{(3.35)}{\geqslant} \frac{1}{2}(1 + \gamma + \alpha/4) \cdot s - \beta_1(x, y) \cdot s_0 \\ &= \left(\frac{1+\gamma}{2}\right) \cdot (s - s_0) + \left(\frac{1+\gamma}{2} - \beta_1(x, y)\right) \cdot s_0 + (\alpha/8) \cdot s \\ &\overset{(3.38)}{\geqslant} \left(\frac{1+\gamma}{2}\right) \cdot (s - s_0) + (\alpha/8) \cdot s, \end{aligned} \tag{3.39}$$

and hence, $|A_0| \geqslant (\alpha/32) \cdot r$. In addition, we have

$$\begin{aligned} \sum_{v \in A_0 \cup B_0} (w_v - \tilde{w}_v) \geqslant \sum_{v \in A \cup B} (w_v - \tilde{w}_v) - 2r &\overset{(3.34),(3.36)}{\geqslant} (1 + \gamma + \alpha/2) \cdot s - s_0 - 2r \\ &\geqslant (s - s_0) + (\gamma + \alpha/4) \cdot s. \end{aligned} \tag{3.40}$$

We now show there are few pairs from B_0 to A_0 with positive weight.

Claim 3.46. *If $u \in A_0$ and $v \in B_0$, then $d(v, u) = 0$.*

Proof of Claim 3.46. If $u \in A_0$, $v \in B_0$, are such that $d(v,u) > 0$, then we may set $\phi_{s_0+1}(x), \phi_{s_0+1}(y) = u$, $\phi_{s_0+1}(z) = v$, and $i_{s_0+1} = 2$, contradicting the maximality of s_0. ⊡

We now show that we are done unless $\sum_{v \in B_0}(w_v - \tilde{w}_v)$ is not too small.

Claim 3.47. *Either there exists a random* $(\phi, i(\phi))$ *satisfying* **N1**-**N3**, *or*

$$\sum_{v \in B_0}(w_v - \tilde{w}_v) \geqslant (\gamma + \alpha/16) \cdot s + \beta_1(y) \cdot (s - s_0). \tag{3.41}$$

Proof of Claim 3.47. Suppose that (3.41) does not hold. If $\sum_{v \in B_0}(w_v - \tilde{w}_v) \leqslant (\gamma + \alpha/8) \cdot s$, then set $s_1 = s_0$. Otherwise, let $s_1 \in [s] \setminus [s_0]$ be maximal such that $\sum_{v \in B_0}(w_v - \tilde{w}_v) \geqslant (\gamma + \alpha/16) \cdot s + \beta_1(y) \cdot (s_1 - s_0)$. Note that $s_1 < s$, else (3.41) holds. By considering the cases $s_1 = s_0$ and $s_1 > s_0$ separately (using the maximality of s_1 in the latter), we deduce that

$$\sum_{v \in B_0}(w_v - \tilde{w}_v) \leqslant (\gamma + \alpha/8) \cdot s + \beta_1(y) \cdot (s_1 - s_0), \tag{3.42}$$

and hence,

$$\sum_{v \in A_0}(w_v - \tilde{w}_v) \overset{(3.40),(3.42)}{\geqslant} \beta_1(x,z) \cdot (s_1 - s_0) + (s - s_1) + (\alpha/8) \cdot s.$$

Therefore, using Claim 3.46, we may greedily choose homomorphisms $\phi_{s_0+1}, \ldots, \phi_s : H_{4,1} \to \bar{D}$ such that the following properties hold.

- **Q1** For each $i \in [s_1] \setminus [s_0]$, we have $\phi_i(x) \in A$, $\phi_i(y) \in B$, and $\phi_i(z) \in A$.
- **Q2** For each $i \in [s] \setminus [s_1]$ we have $\phi_i(x) \in A$, $\phi_i(y) \in A$, and $\phi_i(z) \in A$.
- **Q3** For each $v \in A \cup B$, $\sum_{i \in [s]} \beta(\phi_i^{-1}(v)) \leqslant w_v$.

Thus, by defining $(\phi, i(\phi))$ by sampling ϕ from $\phi_1, \ldots, \phi_s$ uniformly at random (identifying the result as a map $V(H_{4,1}) \to V(D)$ in the natural way) and setting $i(\phi) = 1$, we obtain a random $(\phi, i(\phi))$ satisfying **N1-N3**. ⊡

Thus, we may now assume that (3.41) holds, and hence also $|B_0| \geqslant (\alpha/64) \cdot r$. Next we show that each $i \in [s_0]$ satisfies (at least) one of two properties, **R1** or **R2**.

Claim 3.48. *For each $i \in [s_0]$, at least one of the following holds.*

R1 $d(\phi_i(z), v) = 0$ *for every* $v \in A_0$.

R2 $d(v, \phi_i(x)) = 0$ *for every* $v \in B_0$, $d(\phi_i(y), v) = 0$ *for every* $v \in A_0$, *and* $j_i = 1$.

Proof of Claim 3.48. Let $i \in [s_0]$. Assume there is some $u \in A_0$ with $d(\phi_i(z), u) > 0$, for otherwise **R1** holds.

Now, if there is some $v \in B_0$ with $d(v, \phi_i(x)) > 0$, set $\phi_{s_0+1}(x), \phi_{s_0+1}(y) = u$ and $\phi_{s_0+1}(z) = \phi_i(z)$, and then switch $\phi_i(z) = v$. This contradicts the maximality of s_0.

Thus, $d(v, \phi_i(x)) = 0$ for every $v \in B_0$. As $|B_0| \geqslant (\alpha/64) \cdot r$, we may now fix some $v \in B_0$ with $d(\phi_i(x), v) > 0$ and $d(v, \phi_i(z)) + d(\phi_i(z), v) > 0$, by **E**. Then we must have that $j_i = 1$, else we could switch $\phi_i(y) = v$ and $j_i = 1$, contradicting the maximality of $|\{i \in [s_0] : j_i = 1\}|$.

Now, note that, by Claim 3.46, $|\{u' \in A_0 : d(v, u') > 0\}| = 0$. Therefore, if $|\{u' \in A_0 : d(\phi_i(y), u') > 0\}| > 0$, we could switch $\phi_i(y) = v$ to reduce $\sum_{i' \in [s_0]} |\{u' \in A_0 : d(\phi_{i'}(y), u') > 0\}|$ while leaving A_0 unmodified, a contradiction to the minimisation of (3.37). Thus, $|\{u' \in A_0 : d(\phi_i(y), u') > 0\}| = 0$, and hence **R2** holds. ⊡

Given $u \in A_0$, let $I_1(u)$ be the set of $i \in [s_0]$ which satisfy **R1** and are such that $d(u, \phi_i(z)) = 1$ and $\phi_i^{-1}(u) = \emptyset$, and let $I_2(u)$ be the set of $i \in [s_0] \setminus I_1(u)$ which satisfy **R2** and are such that $d(u, \phi_i(y)) = 1$ and $\phi_i^{-1}(u) = \emptyset$. Pick $j'_t \in \{j \in [r-1] : a_j \in A_0\}$ such that $|I_1(a_{j'_t})| + |I_2(a_{j'_t})|$ is maximised, and set $I_1 = I_1(a_{j'_t})$, $I_2 = I_2(a_{j'_t})$. By relabelling, we may assume that $I_1 \cup I_2 = [s_1]$ for some $s_1 \leqslant s_0$.

Claim 3.49. $s_1 \geqslant (1 - \sqrt{\varepsilon}) s_0$.

Proof of Claim 3.49. If $i \in [s_0]$ satisfies **R1**, then, by **E**, $d(u, \phi_i(z)) = 1$ holds for all but at most εr many $u \in A_0$. Similarly, if $i \in [s_0]$ satisfies **R2**, then $d(u, \phi_i(y)) = 1$ holds for all but at most εr many $u \in A_0$. In addition, for each $i \in [s_0]$, we have $\phi_i^{-1}(u) \neq \emptyset$ for at most three $u \in A_0$. Therefore, for every $i \in [s_0]$, we have $i \in I_1(u) \cup I_2(u)$ for all but at most $2\varepsilon r + 3$ many $u \in A_0$. In particular,

$$\sum_{u \in A_0} (|I_1(u)| + |I_2(u)|) \geqslant s_0 \cdot (|A_0| - 3\varepsilon r).$$

Noting that $|I_1| + |I_2| \geqslant \frac{1}{|A_0|} \sum_{u \in A_0} (|I_1(u)| + |I_2(u)|)$ and $|A_0| \geqslant (\alpha/32) \cdot r$, we deduce $s_0 - |I_1| - |I_2| \leqslant \sqrt{\varepsilon} s_0$, and hence the claim. ⊡

Let $B_1 \subseteq B_0$ be the subset of vertices $v \in B_0$ with $d(a_{j'_t}, v) = 1$, and note, by Claim 3.46 and **E**, that $|B_0 \setminus B_1| \leqslant \varepsilon r$. Let q be a new vertex disjoint from $V(H_{4,1})$ and add it to $H_{4,1}$ to get $H'_{4,1}$. Let $\beta_1(q) = \gamma$. Let $s'_1 \leqslant s_1$ be maximal for which, for each $i \in [s'_1]$, we can define $\psi_i(q) \in B_1$ such that the following hold.

S1 For each $i \in [s'_1]$ and $u \in \{x, y, z\}$, $d(\psi_i(q), \phi_i(u)) + d(\phi_i(u), \psi_i(q)) > 0$.

S2 For each $v \in B$,

$$\tilde{w}_v + \gamma \cdot |\{i \in [s'_1] : \psi_i(q) = v\}| \leqslant w_v. \tag{3.43}$$

Let $\psi_i : H'_{4,1} \to \bar{D}$ be defined by this choice of $\psi_i(q)$ and by setting $\psi_i(v) = \phi_i(v)$ for each $v \in V(H_{4,1})$.

Claim 3.50. $s'_1 = s_1$.

Proof of Claim 3.50. Suppose for contradiction that $s'_1 < s_1$. Let B'_1 be the set of $v \in B_1$ such that $d(v, \phi_{s'_1+1}(u)) + d(\phi_{s'_1+1}(u), v) > 0$ for every $u \in \{x, y, z\}$. Because the edge weights $d(e)$, $e \in E(D)$, are ε-complete and $|B_0 \setminus B_1| \leqslant \varepsilon r$, we have $|B_0 \setminus B'_1| \leqslant 4\varepsilon r$. Therefore, we have

$$\begin{aligned}\sum_{v \in B'_1} (w_v - \tilde{w}_v - \gamma \cdot |\{i \in [s'_1] : \psi_i(q) = v\}|) &\geqslant \sum_{v \in B_0} (w_v - \tilde{w}_v) - \sum_{v \in B'_1 \setminus B_0} w_v - \gamma \cdot s'_1 \\ &\overset{(3.41)}{\geqslant} (\gamma + \alpha/16) \cdot s - 12\varepsilon \cdot s - \gamma \cdot s'_1 \geqslant (\alpha/32) \cdot s \geqslant \gamma \cdot r\end{aligned}$$

Thus, there is some $v \in B_1'$ such that $\tilde{w}_v + \gamma \cdot |\{i \in [s_1'] : \psi_i(q) = v\}| + \gamma \leqslant w_v$. But then setting $\psi_{s_1'+1}(q) = v$ contradicts the maximality of s_1'. $\boxdot$

Let m satisfy $\varepsilon \ll 1/m \ll \alpha$.

Claim 3.51. *For each $i \in I_1$, there are homomorphisms $\phi_{i,j}$ and indicators $k_{i,j} \in \{2,3\}$, $j \in [m-1]$, such that, for each $j \in [m-1]$, $\phi_{i,j}$ is a homomorphism from $H_{4,k_{i,j}}$ to $\bar{D}[\psi_i(V(H_{4,1}'))]$, and the following hold.*

T1 $d(a_{j_t'}, \phi_{i,j}(x)) = 1$.

T2 *For each $v \in \psi_i(V(H_{4,1}'))$, $\beta_1(\psi_i^{-1}(v)) \geqslant \sum_{j \in [m-1]} \beta_{k_{i,j}}(\phi_{i,j}^{-1}(v))/m$.*

Proof of Claim 3.51. Fixing $i \in I_1$, for each $j' = 1, \dots, m-1$ in turn, choose a homomorphism $\phi_{i,j'}$ from $H_{4,2}$ or $H_{4,3}$ to $\bar{D}[\psi_i(V(H_{4,1}'))]$ such that the following properties hold.

(i) $\beta_1(\psi_i^{-1}(v)) \geqslant \sum_{j \in [j']} \beta_{k_{i,j}}(\phi_{i,j}^{-1}(v))/m$ for each $v \in \psi_i(H_{4,1}')$.

(ii) $\phi_{i,j'}(x) \in \{\psi_i(z), \psi_i(q)\}$.

(iii) (a) $\phi_{i,j'}(y) \in \{\psi_i(x), \psi_i(y)\}$ if $\phi_{i,j'}$ is a homomorphism from $H_{4,2}$.

(b) $\phi_{i,j'}(z) \in \{\psi_i(x), \psi_i(y)\}$ if $\phi_{i,j'}$ is a homomorphism from $H_{4,3}$.

Then **T1** holds, through the definition of either I_1 (if $\phi_{i,j'}(x) = \psi_i(z)$) or B_1 (if $\phi_{i,j'}(x) = \psi_i(q)$).

Note that this is possible because, as

$$\beta_1(z,q) = \min\{\beta_w + \beta_z, 1 - \beta_z\} + \max\{\beta_x, \beta_z\} \geqslant \min\{\beta_w + \beta_z + \beta_x, 1\} = \beta_2(x) = \beta_3(x),$$

there is enough room in $\{\psi_i(z), \psi_i(q)\}$ for $\phi_{i,j'}(x)$, and, as

$$\beta_1(x,y) = 1 - \beta_1(z) \geqslant 1 - (\beta_w + \beta_z) = \beta_x + \beta_u + \beta_y \geqslant \beta_2(y) = \beta_3(z).$$

there is enough room in $\{\psi_i(x), \psi_i(y)\}$ for $\phi_{i,j'}(y)$. We also use that there is weight in at least one direction between any pair from $\{\psi_i(z), \psi_i(q)\}$ and $\{\psi_i(x), \psi_i(y)\}$, where the direction of the edge gives whether we embed $H_{4,2}$ or $H_{4,3}$. $\boxdot$

Claim 3.52. *For each $i \in I_2$, there are homomorphisms $\phi_{i,j}$, $j \in [m-1]$, such that, for each $j \in [m-1]$, $\phi_{i,j}$ is a homomorphism from $H_{4,4}$ to $\bar{D}[\psi_i(V(H'_{4,1}))]$, and the following holds.*

U1 $d(a_{j'_t}, \phi_{i,j}(x)) = 1$.

U2 *For each $v \in \psi_i(V(H'_{4,1}))$, $\beta_1(\psi_i^{-1}(v)) \geqslant \sum_{j\in[m-1]} \beta_4(\phi_{i,j}^{-1}(v))/m$.*

Proof of Claim 3.52. Fixing $i \in I_2$, for each $j' = 1, \dots, m-1$ in turn, choose a homomorphism $\phi_{i,j'}$ from $H_{4,4}$ to $\bar{D}[\psi_i(V(H'_{4,1}))]$ such that $\beta_1(\psi_i^{-1}(v)) \geqslant \sum_{j\in[j']} \beta_4(\phi_{i,j}^{-1}(v))/m$ for each $v \in \psi_i(H'_{4,1})$, $\phi_{i,j'}(x) \in \{\psi_i(y), \psi_i(q)\}$, $\phi_{i,j'}(z) = \psi_i(x)$ and $\phi_{i,j'}(w) = \psi_i(z)$. Then **U1** holds, through the definition of either I_2 (if $\phi_{i,j'}(x) = \psi_i(y)$) or B_1 (if $\phi_{i,j'}(x) = \psi_i(q)$).

Note that this is possible using the following. As $\beta_1(y)+\beta_1(q) = \beta_1(y)+\gamma = \max\{\beta_x+\beta_y, \gamma\} = \max\{\beta_x + \beta_y, \beta_z\} \geqslant \beta_4(x)$, there is enough room in $\{\psi_i(y), \psi_i(q)\}$ for $\phi_{i,j'}(x)$. As $\beta_1(z) \geqslant \min\{\beta_w, 1-\beta_z\} \geqslant \beta_w = \beta_4(w)$, there is enough room in $\psi_i(z)$ for $\phi_{i,j'}(w)$. Finally, recall from (3.22), that $\beta_4(z) \leqslant \beta_1(x)$, so there is enough room in $\psi_i(x)$ for $\phi_{i,j'}(z)$. Because $d(\psi_i(x), \psi_i(y)), d(\psi_i(z), \psi_i(x)) > 0$ (as $\psi_i : H'_{4,1} \to \bar{D}$ is a homomorphism) and $d(\psi_i(x), \psi_i(q)) > 0$ (by **R2** and **S1**), we also have that $\phi_{i,j'} : H_{4,4} \to \bar{D}[\psi_i(V(H'_{4,1}))]$ is a homomorphism, as required. $\boxdot$

For each $i \in I_2$ and $j \in [m-1]$, let $k_{i,j} = 4$. For each $v \in A \cup B$, let

$$\begin{aligned}\hat{w}_v &= \frac{1}{m}\sum_{i\in[s_1]}\sum_{j\in[m-1]} \beta_{k_{i,j}}(\phi_{i,j}^{-1}(v)) \overset{\textbf{T2},\textbf{U2}}{\leqslant} \sum_{i\in[s_1]} \beta_1(\psi_i^{-1}(v)) \\ &= \sum_{i\in[s_1]} \beta_1(\phi_i^{-1}(v)) + \gamma \cdot |\{i \in [s_1] : \psi_i(q) = v\}| \\ &= \tilde{w}_v + \gamma \cdot |\{i \in [s_1] : \psi_i(q) = v\}| \overset{(3.43)}{\leqslant} w_v. \end{aligned} \tag{3.44}$$

We note that

$$\begin{aligned}
\sum_{v\in A_0\cup B_1}(w_v-\hat{w}_v) &\geqslant \sum_{v\in A_0\cup B_0}(w_v-\hat{w}_v)-3\varepsilon\cdot s\\
&\overset{(3.44)}{\geqslant} \sum_{v\in A_0\cup B_0}(w_v-\tilde{w}_v)-3\varepsilon\cdot s-\sum_{v\in A_0\cup B_0}\gamma\cdot|\{i\in[s_1]:\psi_i(q)=v\}|\\
&\overset{(3.40)}{\geqslant} s-s_0+(\gamma+\alpha/8)\cdot s-\gamma\cdot s_1\\
&\overset{\text{Claim 3.49}}{\geqslant} (1+\gamma)\cdot(s-s_1)+(\alpha/16)\cdot s. \qquad (3.45)
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\sum_{v\in B_1}(w_v-\hat{w}_v) &\geqslant \sum_{v\in B_0}(w_v-\hat{w}_v)-3\varepsilon\cdot s\\
&\overset{(3.44)}{\geqslant} \sum_{v\in B_0}(w_v-\tilde{w}_v)-3\varepsilon\cdot s-\sum_{v\in B_0}\gamma\cdot|\{i\in[s_1]:\psi_i(q)=v\}|\\
&\overset{(3.41)}{\geqslant} \beta_1(y)\cdot(s-s_0)+(\gamma+\alpha/32)\cdot s-\gamma\cdot s_1\\
&\overset{\text{Claim 3.49}}{\geqslant} (\gamma+\beta_1(y))\cdot(s-s_1)+(\alpha/64)s. \qquad (3.46)
\end{aligned}$$

Take a maximal set $J\subseteq([s]\times[m])\setminus([s_1]\times[m-1])$ for which there are homomorphisms $\phi_{i,j}:H_{4,5}\to\bar{D}[A_0\cup B_1]$, $(i,j)\in J$ such that the following hold.

V1 For each $(i,j)\in J$, $\phi_{i,j}(x)\in B_1$.

V2 For each $v\in A\cup B$, $\sum_{(i,j)\in J}\beta_5(\phi_{i,j}^{-1}(v))/m\leqslant w_v-\hat{w}_v$.

Subject this choice of J, maximise

$$|\{(i,j)\in J:\phi_{i,j}(z)\in A_0\}|. \qquad (3.47)$$

Claim 3.53. $J=([s]\times[m])\setminus([s_1]\times[m-1])$.

Proof of Claim 3.53. Suppose, for contradiction, that there is some $(i',j')\in(([s]\times[m])\setminus([s_1]\times[m-1]))\setminus J$. We must then have, for each $v\in B_1$, that $\sum_{(i,j)\in J}\beta_5(\phi_{i,j}^{-1}(v))/m\geqslant w_v-\hat{w}_v-1/m$, for otherwise we can take the homomorphism $\phi_{i',j'}$ sending $H_{4,5}$ to v.

Therefore, we have

$$\sum_{v\in A_0}\left(w_v-\hat{w}_v-\tfrac{1}{m}-\sum_{(i,j)\in J}\beta_5(\phi_{i,j}^{-1}(v))/m\right)\overset{(3.45)}{\geqslant}\gamma\cdot(s-s_1)+(\alpha/16)\cdot s-\tfrac{|A_0|}{m}-\tfrac{|B_1|}{m}$$
$$\geqslant(\alpha/32)\cdot s.$$

Thus, there must be at least $2\varepsilon r$ vertices $v\in A_0$ with $\sum_{(i,j)\in J}\beta_5(\phi_{i,j}^{-1}(v))/m\leqslant w_v-\hat{w}_v-1/m$. Therefore, by Claim 3.46, if there is some $(i,j)\in J$ with $\phi_{i,j}(z)\notin A_0$, then we could move $\phi_{i,j}(z)$ into A_0, and increase the value of (3.47). Thus, we must have $\phi_{i,j}(z)\in A_0$ for each $(i,j)\in J\setminus([s_1]\times[m-1])$.

Therefore, using that $\beta_5(x)\leqslant\gamma+\beta_1(y)$ and $|J|\leqslant(s-s_1)m+s$,

$$\begin{aligned}0&\geqslant\sum_{v\in B_1}\left(w_v-\hat{w}_v-\tfrac{1}{m}-\sum_{(i,j)\in J}\beta_5(\phi_{i,j}^{-1}(v))/m\right)\\&\geqslant\sum_{v\in B_1}(w_v-\hat{w}_v)-|B_1|/m-((s-s_1)m+s)\cdot\beta_5(x)/m\\&\geqslant\sum_{v\in B_1}(w_v-\hat{w}_v)-(s-s_1)\cdot(\gamma+\beta_1(y))-s/m-|B_1|/m\overset{(3.46)}{>}0,\end{aligned}$$

a contradiction. Therefore, $J=([s]\times[m])\setminus([s_1]\times[m-1])$. ⊡

For each $(i,j)\in([s]\times[m])\setminus([s_1]\times[m-1])$, let $k_{i,j}=5$. Select $(\phi,i(\phi))$ uniformly at random from $(\phi_{i,j},k_{i,j})$, $(i,j)\in[s]\times[m]$. □

3.7.4 Proof of Theorem 3.13

We are now ready to complete this section by proving Theorem 3.13. To give a brief overview of this proof, we again turn to our simplified situation: assume we are trying to embed a tree T in a tournament G and suppose we have $t\in V(T)$ so that $T-\{t\}$ consists of small components. Unlike in Sections 3.7.3.2 and 3.7.3.3, t can have both in- and out-neighbours in T. Let T^+ and T^- be the trees covering the edges of T, intersecting only on t, so that t has only out-neighbours in T^+ and only in-neighbours in T^-. As G is a tournament with distinctly more vertices than T, each vertex $v \in V(G)$ either has enough out-neighbours in G that we can embed the components of $T^+ - \{t\}$ greedily into $N^+_G(v)$ or the components of $T^- - \{t\}$ greedily into $N^-_G(v)$. If we partition $V(G) = V^+ \cup V^-$ so that the former holds for vertices in V^+ and the latter holds for vertices in V^-, then, from this partition, either $G[V^+]$ will be large enough to embed T^- (using our previous methods) or $G[V^-]$ will be large enough to embed T^+. By directional duality, we can assume that the latter case holds. This allows us to find a copy of T^+ in G with t embedded to v_t, a vertex of G which has enough in-neighbours for us to greedily embed the components of $T^- - \{t\}$. Of course, some of these in-neighbours may be occupied already by the embedding of T^+, but, by embedding T^+ in such a way to cover minimally many of these in-neighbours we will have there are enough in-neighbours to embed the components of $T^- - \{t\}$, and complete the embedding.

For Theorem 3.13, we do this in the setting of distillations, random homomorphisms and a weighted looped digraph D. We ultimately wish to find a random homorphism from H, where H is the fully-looped oriented forest with vertex and edge sets given by

$$V(H) = \left\{x^+, y^+, z^+, u^+, w^+, \bar{x}^+, \bar{z}^+, \bar{u}^+, \bar{w}^+, x^-, y^-, z^-, u^-, w^-, \bar{x}^-, \bar{z}^-, \bar{u}^-, \bar{w}^-\right\},$$

$$E(H) = \left\{ \begin{array}{c} x^+y^+, z^+x^+, z^+u^+, w^+z^+, \bar{z}^+\bar{x}^+, \bar{z}^+\bar{u}^+, \bar{w}^+\bar{z}^+, \\ y^-x^-, x^-z^-, u^-z^-, z^-w^-, \bar{x}^-\bar{z}^-, \bar{u}^-\bar{z}^-, \bar{z}^-\bar{w}^- \end{array} \right\} \cup \{vv : v \in V(H)\}.$$

For each $\diamond \in \{+, -\}$, let $X^\diamond = \{x^\diamond, \bar{x}^\diamond\}$, and let $X = X^+ \cup X^-$.

Note that $H_0 \cong H[\{x^+, y^+, z^+, u^+, w^+, \bar{x}^+, \bar{z}^+, \bar{u}^+, \bar{w}^+\}]$. Therefore, we will assume equality here by letting, for example, $x^+ = x$. In addition, let $H_0' = H - V(H_0)$, and note that H_0' is itself isomorphic to a copy of H_0 with all edges reversed. Here H_0 and H_0' correspond to T^+ and T^- in the sketch above.

Instead of partitioning $V(G)$ as $V^+ \cup V^-$ we partition $V(D)$ as $J^+ \cup J^-$ in a similar manner, and assume, by directional duality, that J^- is large enough that we can apply Theorem 3.34 to get a random homomorphism of H_0 into $D[J^-]$ satisfying **W1**–**W3** below (comparable to the embedding of T^+ into $G[V^-]$ in the sketch above) before minimising a certain property (comparable to the embedding of T^+ using minimally many in-neighbours of v_t above). We then use the minimisation of the random homomorphism to extend it to cover H_0', so that we have a random homorphism of H into D. Finally, we adjust this random homomorphism to get the additional condition **F4** which we dropped for Theorem 3.34, completing the proof.

Proof of Theorem 3.13. For $\diamond \in \{+,-\}$, let

$$\lambda^\diamond = \beta(x^\diamond, y^\diamond, z^\diamond, u^\diamond, w^\diamond, \bar{x}^\diamond, \bar{z}^\diamond, \bar{u}^\diamond, \bar{w}^\diamond)$$

so that $\lambda^+ + \lambda^- = 1$, and let

$$\gamma^\diamond = \max\{\beta(x^\diamond, \bar{x}^\diamond), \beta(z^\diamond, \bar{z}^\diamond)\}/\lambda^\diamond.$$

For $\diamond \in \{+,-\}$, define

$$r^\diamond = \left\lceil \frac{\lambda^\diamond(1+\gamma^\diamond+\alpha/16)}{1+\gamma+\alpha/4} \cdot r \right\rceil,$$

so that $r^+ + r^- \leqslant (1-\varepsilon)\cdot r$.

Let K be an ε-almost tournament with vertex set $[r]$, such that $d(i,j) \geqslant 1/2$ whenever $i \to_K j$. Partition $[r]$ as $J^+ \cup J^-$ such that $d_K^\diamond(j) \geqslant r^\diamond$ whenever $j \in J^\diamond$. Note that we either have $|J^+| \geqslant r^-$ or $|J^-| \geqslant r^+$. By directional duality, we may assume that $|J^-| \geqslant r^+$.

Let $\beta_0 : V(H_0) \to [0,1]$ be given by $\beta_0(v) = \beta(v)/\lambda^+$, and note that $\sum_{v\in V(H_0)} \beta_0(v) = 1$ and $\beta_0(y^+) \geqslant \beta_0(x^+)$. By Theorem 3.34, if $\mathcal{H}_0 = (H_0, X^+, \beta_0)$, then $\{\mathcal{H}_0\}$ is γ^+-good. Therefore, because $\beta = \lambda^+ \cdot \beta_0$ and $\lambda^+ \cdot \frac{1+\gamma^++\alpha/16}{r^+} \leqslant \frac{1+\gamma^++\alpha/4}{r}$, there exists some $j_t \in J^-$ and random $\psi : H_0 \to D[J^-]$ such that the following hold.

W1 With probability 1, we have that ψ is a homomorphism from H_0 to D, and that $j_t \notin \psi(X^+)$.

W2 For each $j \in [r]$, $\mathbb{E}(\beta(\psi^{-1}(j))) \leqslant \frac{1+\gamma+\alpha/4}{r}$.

W3 For each $j \in [r]$, $\mathbb{E}(\beta(\psi^{-1}(j) \cap X^+)) \leqslant d(j_t, j) \cdot \frac{1+\gamma+\alpha/4}{r}$.

Fix such a $j_t \in J^-$. Let $A = N^+_K(j_t)$ and $B = N^-_K(j_t)$. Take a random $\psi : H_0 \to D$ satisfying **W1-W3** so that $\mathbb{E}(\beta(\psi^{-1}(B)))$ is minimised.

Claim 3.54. $|B| \cdot \frac{1+\gamma+\alpha/4}{r} - \mathbb{E}(\beta(\psi^{-1}(B))) - \mathbb{E}(\beta(\psi^{-1}(B) \cap X^+)) \geqslant \lambda^- + \beta(X^-) + \alpha/8$.

Proof of Claim 3.54. First, if $|A| \cdot \frac{1+\gamma+\alpha/4}{r} - \mathbb{E}(\beta(\psi^{-1}(A)) - \mathbb{E}(\beta(\psi^{-1}(A) \cap X^+)) \leqslant 0$, then

$$\begin{aligned}|B| \cdot \tfrac{1+\gamma+\alpha/4}{r} - \mathbb{E}(\beta(\psi^{-1}(B))) - \mathbb{E}(\beta(\psi^{-1}(B) \cap X^+)) &\geqslant |D| \cdot \tfrac{1+\gamma+\alpha/4}{r} - 3\varepsilon - \lambda^+ - \beta(X^+) \\ &\geqslant 1 + \gamma + \alpha/8 - \lambda^+ - \beta(X^+) \geqslant \lambda^- + \beta(X^-) + \alpha/8.\end{aligned}$$

So we may assume that $|A| \cdot \frac{1+\gamma+\alpha/4}{r} - \mathbb{E}(\beta(\psi^{-1}(A))) - \mathbb{E}(\beta(\psi^{-1}(A) \cap X^+)) > 0$, else the claim is proven. In particular, we may assume there is some $j \in A$ such that, if

$$\begin{aligned}p &:= \frac{1}{2}\left(\frac{1+\gamma+\alpha/4}{r} - \mathbb{E}(\beta(\psi^{-1}(j))) - \mathbb{E}(\beta(\psi^{-1}(j) \cap X^+))\right) \\ &\leqslant d(j_t, j) \cdot \frac{1+\gamma+\alpha/4}{r} - \mathbb{E}(\beta(\psi^{-1}(j) \cap X^+)),\end{aligned}$$

then $0 < p < 1$. In addition, we may assume that $\mathbb{E}(\beta(\psi^{-1}(B))) > 0$, else the claim follows immediately from the definition of J^-. Then, however, if we define $\hat{\psi} : H_0 \to D$ by setting $\hat{\psi}(V(H_0)) = j$ with probability p and sampling ψ otherwise, we find $\hat{\psi}$ satisfies **W1-W3**, but $\mathbb{E}(\beta(\hat{\psi}^{-1}(B))) = (1-p) \cdot \mathbb{E}(\beta(\psi^{-1}(B)))$, a contradiction to the minimality of $\mathbb{E}(\beta(\psi^{-1}(B)))$. ⊡

Consider the random $\hat{\phi} : H \to D$ defined by sampling ψ to determine $\hat{\phi}|_{V(H_0)}$, and independently choosing $\hat{\phi}(V(H_0')) \in B$ at random so that

$$\mathbb{P}(\hat{\phi}(V(H_0')) = j) = p_j := \frac{\frac{1+\gamma+\alpha/4}{r} - \mathbb{E}(\beta(\psi^{-1}(j))) - \mathbb{E}(\beta(\psi^{-1}(j) \cap X^+))}{|B| \cdot \frac{1+\gamma+\alpha/4}{r} - \mathbb{E}(\beta(\psi^{-1}(B))) - \mathbb{E}(\beta(\psi^{-1}(B) \cap X^+))}.$$

We remark that for every $j \in B$, we have

$$\begin{aligned}&p_j \cdot \max\{\lambda^-, 2\beta(X^-)\} \leqslant p_j \cdot (\lambda^- + \beta(X^-)) \\ &\qquad \overset{\text{Claim 3.54}}{\leqslant} \tfrac{1+\gamma+\alpha/4}{r} - \mathbb{E}(\beta(\psi^{-1}(j))) - \mathbb{E}(\beta(\psi^{-1}(j) \cap X^+)). \qquad (3.48)\end{aligned}$$

Therefore, if $j \in B$, we have

$$\mathbb{E}(\beta(\hat{\phi}^{-1}(j))) = \mathbb{E}(\beta(\psi^{-1}(j))) + p_j \cdot \lambda^- \overset{(3.48)}{\leqslant} \frac{1+\gamma+\alpha/4}{r},$$
$$\mathbb{E}(\beta(\hat{\phi}^{-1}(j) \cap X^+)) \overset{\mathbf{W3}}{\leqslant} d(j_t, j) \cdot \frac{1+\gamma+\alpha/4}{r},$$
$$\mathbb{E}(\beta(\hat{\phi}^{-1}(j) \cap X)) \leqslant \mathbb{E}(\beta(\psi^{-1}(j) \cap X^+)) + p_j \cdot \beta(X^-)$$
$$\overset{(3.48)}{\leqslant} \frac{1}{2}\left(\frac{1+\gamma+\alpha/4}{r}\right) \leqslant d(j, j_t) \cdot \frac{1+\gamma+\alpha/4}{r},$$

whereas if $j \in [r] \setminus B$, we have

$$\mathbb{E}(\beta(\hat{\phi}^{-1}(j))) = \mathbb{E}(\beta(\psi^{-1}(j))) \overset{\mathbf{W2}}{\leqslant} \frac{1+\gamma+\alpha/4}{r},$$
$$\mathbb{E}(\beta(\hat{\phi}^{-1}(j) \cap X^-)) = 0 \leqslant d(j, j_t) \cdot \frac{1+\gamma+\alpha/4}{r},$$
$$\mathbb{E}(\beta(\hat{\phi}^{-1}(j) \cap X)) = \mathbb{E}(\beta(\psi^{-1}(j) \cap X^+)) \overset{\mathbf{W3}}{\leqslant} d(j_t, j) \cdot \frac{1+\gamma+\alpha/4}{r}.$$

Take $\bar{\phi} : H \to D$ with $\sum_{e \in E(H)} \mathbb{P}(|\bar{\phi}(e)| = 1)$ minimal, such that the following properties hold.

X1 With probability 1, we have that $\bar{\phi}$ is a homomorphism from H to D, and that $j_t \notin \bar{\phi}(\{x^+, \bar{x}^+, x^-, \bar{x}^-\})$.

X2 For each $j \in [r]$, $\mathbb{E}(\beta(\bar{\phi}^{-1}(j))) \leqslant \frac{1+\gamma+\alpha/2}{r} - \frac{\alpha^2}{r} \cdot \sum_{e \in E(H)} \mathbb{P}(|\bar{\phi}(e)| = 1)$.

X3 For each $j \in [r]$, either

X3.1 $\mathbb{E}(\beta(\bar{\phi}^{-1}(j) \cap X^+)) \leqslant d(j_t, j) \cdot \frac{1+\gamma+\alpha/2}{r}$ and $\mathbb{E}(\beta(\bar{\phi}^{-1}(j) \cap X)) \leqslant d(j, j_t) \cdot \frac{1+\gamma+\alpha/2}{r}$,

or

X3.2 $\mathbb{E}(\beta(\bar{\phi}^{-1}(j) \cap X^-)) \leqslant d(j, j_t) \cdot \frac{1+\gamma+\alpha/2}{r}$ and $\mathbb{E}(\beta(\bar{\phi}^{-1}(j) \cap X)) \leqslant d(j_t, j) \cdot \frac{1+\gamma+\alpha/2}{r}$.

$\bar{\phi}$ is well-defined, as we may take $\bar{\phi} = \hat{\phi}$.

We will shortly prove the following claim.

Claim 3.55. $\mathbb{P}(|\bar{\phi}(e)| = 1) \leqslant \varepsilon^{1/4}/|E(H)|$ *for every non-looped edge* e *of* H.

From Claim 3.55 it follows that $\mathbb{P}(|\bar{\phi}(e)| = 2$ for every non-looped edge e of $H) \leqslant \varepsilon^{1/4}$. Then, if we take ϕ to be $\bar{\phi}$ conditioned on the event that $|\bar{\phi}(e)| = 2$ for every non-looped edge e of H, we have

$$\mathbb{E}(\beta(\phi^{-1}(j) \cap \{v\})) \leqslant (1 - \varepsilon^{1/4})^{-1} \cdot \mathbb{E}(\beta(\bar{\phi}^{-1}(j) \cap \{v\}))$$

for every $j \in [r], v \in V(H)$, thus ϕ satisfies the conclusion of the theorem. So it only remains to prove Claim 3.55.

Proof of Claim 3.55. Suppose for contradiction that there is some non-looped edge e of H with $\mathbb{P}(|\bar{\phi}(e)| = 1) \geqslant \varepsilon^{1/4}/|E(H)|$. Let $e = v_1 v_2$. For $i \in [2]$, let H_i be the component of $H - e$ containing v_i. Assume, by directional duality, that $X \cap V(H_2) = \emptyset$.

Note that, because $\beta(v_1, v_2) \geqslant 2\mu$, **X2** implies that $\mathbb{P}(\bar{\phi}(e) = \{j\}) \leqslant \frac{2}{\mu r}$ for every $j \in [r]$. So, if J is the set of $j \in [r]$ for which $\mathbb{P}(\bar{\phi}(e) = \{j\}) \geqslant \frac{\sqrt{\varepsilon}}{r}$, then $|J| \geqslant \sqrt{\varepsilon} \cdot r$, else we find $\mathbb{P}(|\bar{\phi}(e)| = 1) = \sum_{j \in [r]} \mathbb{P}(\bar{\phi}(e) = \{j\}) \leqslant \frac{2\sqrt{\varepsilon}}{\mu} + \sqrt{\varepsilon} < \varepsilon^{1/4}/|E(H)|$.

Let m be the number of possible homomorphisms $H \to D$. Choose homomorphisms $\phi_j, j \in J$ such that $\phi_j(e) = \{j\}$ and $\mathbb{P}(\bar{\phi} = \phi_j) \geqslant \frac{\sqrt{\varepsilon}}{mr}$ for every $j \in J$ (these can be found as the edge weights of D are ε-complete).

Set $s = \lceil 1/\alpha^3 \rceil$. Let $j_1, \ldots, j_{s+1} \in J$ be distinct such that $d(j_i, j_{i+1}) > 0$ for every $i \in [s]$. Let $k \in [2]$ be random, with distribution coupled with $\bar{\phi}$ such that $\mathbb{P}(k = 2) = \frac{(s+1)\sqrt{\varepsilon}}{mr}$, and that $\mathbb{P}(\bar{\phi} = \phi_{j_i} \mid k = 2) = \frac{\sqrt{\varepsilon}}{mr}$ for every $i \in [s+1]$.

Define a random $\psi : H \to D$ as follows. Sample $(\bar{\phi}, k)$, choose $i \in [s+1]$ uniformly at random, and set

$$\psi(v) = \begin{cases} \bar{\phi}(v) & \text{if } k = 1, \\ \phi_{j_i}(v) & \text{if } k = 2,\ i = s+1, \\ \phi_{j_i}(v) & \text{if } k = 2,\ i \in [s],\ v \in V(H_1), \\ \phi_{j_{i+1}}(v) & \text{if } k = 2,\ i \in [s],\ v \in V(H_2). \end{cases}$$

We then find $\mathbb{P}(|\psi(e)| = 1) = \mathbb{P}(|\bar{\phi}(e)| = 1) - \frac{s\sqrt{\varepsilon}}{mr}$, yet $\mathbb{E}(\beta(\psi^{-1}(j) \cap X)) = \mathbb{E}(\beta(\bar{\phi}^{-1}(j) \cap X))$ and $\mathbb{E}(\beta(\psi^{-1}(j))) \leqslant \mathbb{E}(\beta(\bar{\phi}^{-1}(j))) + \frac{\sqrt{\varepsilon}}{mr}$ for every $j \in [r]$, a contradiction to the minimality of $\sum_{e \in E(H)} \mathbb{P}(|\bar{\phi}(e)| = 1)$. $\boxdot$ $\square$

CHAPTER 4

UNAVOIDABLE SUBGRAPHS OF INFINITE TOURNAMENTS

In this chapter we extend the study of oriented subgraphs of tournaments to an infinite setting. First, recall from Chapter 1 the discussion of the following theorem.

Theorem 1.7. *Let H be a countably-infinite oriented graph. The following are equivalent:*

(i) *H is acyclic, locally-finite, and has no infinite directed paths.*

(ii) *H is contained in every countably-infinite tournament.*

(iii) *H is a spanning subgraph of every countably-infinite tournament.*

Of course, (ii) follows from (iii) trivially, and (i) follows easily from (ii) by considering tournaments on $\mathbb{N}$ with all edges oriented forward (or with all edges oriented backward). Thus, the proof of Theorem 1.7 will mostly consist of showing (i) implies (iii). After stating some further notation required for the results of this chapter, we sketch and prove the theorem in Section 4.2. We then take the infinite setting further in Section 4.3, where we consider whether certain oriented graphs which are not contained in every countably-infinite tournament can still be guaranteed to appear in a tournament on $\mathbb{N}$ if we impose a condition on the lower density of the tournament's *forward* edges (i.e., those edges (i, j) such that $i < j$).

4.1 Notation

An (oriented) graph is *locally-finite* if every vertex is incident with finitely many edges. The infinite directed path with exactly one vertex of in-degree 0 is called the *infinite forward directed path* and the infinite directed path with exactly one vertex of out-degree 0 is called the *infinite backward directed path*.

Say that an oriented graph H is *weakly-connected* if the underlying graph is connected. Given oriented graphs H and G, say that H is a *spanning subgraph* of G if there exists a bijective embedding $\phi : H \to G$. For each $\diamond \in \{+, -\}$, the *common $\diamond$-neighbourhood* of a set $X \subseteq V(G)$ is $\hat{N}^\diamond(X) = \bigcap_{v \in X} N^\diamond(v)$.

Given an acyclic oriented graph H and $u \in V(H)$, let

$$\Gamma^+(u) = \{v \in V(H) : \text{there exists a directed path from } u \text{ to } v\}$$

and let

$$\Gamma^-(u) = \{v \in V(H) : \text{there exists a directed path from } v \text{ to } u\}$$

(equivalently, $\Gamma^\diamond(u)$ is the $\diamond$-neighbourhood of u in the transitive, reflexive closure of H).

Given a strict total order $\tau = (V, \prec)$, let K_τ be the tournament on V where $(u, v) \in E(K_\tau)$ if and only if $u \prec v$. We write τ^* to be the converse of τ; that is, $\tau^* = (V, \succ)$. We say that K_τ and K_{τ^*} are the transitive tournaments of type-τ.

We use the von Neumann definition of ordinals where an ordinal is the strictly well-ordered set of all smaller ordinals. Thus, given an ordinal λ, the definition of K_λ and K_{λ^*} is given in the previous paragraph. As is standard, we let ω be the first infinite ordinal and ω_1 be the first uncountable ordinal. Given ordinals α, β, we define $\beta \cdot \alpha$ to be the order type of $(\beta \times \alpha, \prec)$, where $(i_1, j_1) \prec (i_2, j_2)$ if either $j_1 < j_2$ or $j_1 = j_2$ and $i_1 < i_2$.

4.2 Proof of Theorem 1.7

The most involved part of the proof of Theorem 1.7 is showing that, for any countably-infinite acyclic locally-finite graph H that has no infinite directed paths, and any countably-infinite tournament K on $\mathbb{N}$, there is a bijective embedding $\phi : H \to K$. To construct this ϕ, we first partition $V(K) = V^+ \cup V^-$ so that, for any finite non-empty subsets $X \subseteq V^+$ and $Y \subseteq V^-$, $\hat{N}^+(X) \cap \hat{N}^-(Y)$ is infinite. The aim then is to extend an embedding of H finitely many vertices at a time, each time covering at least the next vertex of K (see **Z1** later), while requiring that whenever $v \in V(H)$ is embedded to some $u \in V^+$, then $N^-_H(v)$ is embedded at the same time as v (and whenever $v \in V(H)$ is embedded to some $u \in V^-$, then $N^+_H(v)$ is embedded at the same time as v). It is therefore useful to partition $V(H)$ into finite sets, with each set having only out-edges or only in-edges to the rest of H, so that each set can be embedded simultaneously to ensure the above requirement holds. This motivates the following definition.

Definition 4.1. *Given a countably-infinite acyclic weakly-connected oriented graph H, a* ±-partition *of H is a partition $\{C_i : i \in \mathbb{N}\}$ of $V(H)$ such that the following properties hold:*

- **Y1** *For all $i \in \mathbb{N}$, C_i is finite and non-empty.*
- **Y2** *For all $(u, v) \in E(H)$, there exists $i \in \mathbb{N}$ such that $\{u, v\} \subseteq C_i \cup C_{i+1}$.*
- **Y3** *If i is odd, then every vertex in C_i has in-degree 0 to $C_{i-1} \cup C_{i+1}$, and if i is even, then every vertex in C_i has out-degree 0 to $C_{i-1} \cup C_{i+1}$.*
- **Y4** *If i is odd, then there exists a vertex in C_i with in-degree 0 in H, and if i is even there exists a vertex in C_i with out-degree 0 in H.*

If i is odd, we say that C_i has type $+$, *and if i is odd, we say that C_i has* type $-$.

Likewise one can define a *∓-partition* by switching every instance of in/out in the above definition. We note that a similar definition for finite oriented trees was given by Dross and Havet [10].

Lemma 4.2. *Let H be a countably-infinite oriented graph. If H is weakly-connected, acyclic, locally-finite, and has no infinite directed paths, then for every vertex v of in-degree 0, H has a $\pm$-partition with $C_1 = \{v\}$, and for every vertex v of out-degree 0, H has a $\mp$-partition with $C_1 = \{v\}$.*

Proof. Since H is acyclic and has no infinite directed paths, the set of vertices with in-degree 0 is non-empty; let v be a vertex of in-degree 0 and set $C_1 = \{v\}$. For even $i \geqslant 1$, let $C_i = \left(\bigcup_{v \in C_{i-1}} \Gamma^+(v)\right) \setminus (C_{i-1} \cup C_{i-2})$, and for odd $i \geqslant 1$, let $C_i = \left(\bigcup_{v \in C_{i-1}} \Gamma^-(v)\right) \setminus (C_{i-1} \cup C_{i-2})$. Note that since H is locally-finite, and has no infinite directed paths, each C_i is finite. In addition, because H is weakly-connected, $\{C_i : i \in \mathbb{N}\}$ is a partition of $V(H)$, and each C_i is non-empty. Therefore, **Y1** holds.

Suppose, for some $i < j$, that $(u, v) \in E(H)$ with $u \in C_i$ and $v \in C_j$. Then, we must have that i is odd (else $v \in C_i$) and $j = i + 1$. On the other hand, if $(v, u) \in E(H)$ is such that $u \in C_i$ and $v \in C_j$ for some $i < j$, then we must have that i is even (else $v \in C_i$) and $j = i + 1$. Therefore, we deduce that **Y2** and **Y3** hold.

Finally, note that since C_i is finite and H is acyclic, $H[C_i]$ has a vertex u_i of in-degree 0 in $H[C_i]$ and a vertex v_i of out-degree 0 in $H[C_i]$. Thus, by **Y3**, if i is even, then v_i has out-degree 0 in H, and if i is odd, then u_i has in-degree 0 in H. Therefore, **Y4** holds, and $\{C_i : i \in \mathbb{N}\}$ is a $\pm$-partition with $C_1 = \{v\}$.

Likewise, by directional duality, for every vertex v of out-degree 0, H has a $\mp$-partition with $C_1 = \{v\}$. □

We are now ready to prove Theorem 1.7.

Proof of Theorem 1.7. First note that if H has a cycle, then $H \not\subseteq K_\omega$ and $H \not\subseteq K_{\omega^*}$. If H has a vertex of infinite in-degree, then $H \not\subseteq K_\omega$, and if H has a vertex of infinite out-degree, then $H \not\subseteq K_{\omega^*}$. If H has an infinite directed path with the first vertex having out-degree 0, then $H \not\subseteq K_\omega$, and if H has an infinite directed path with the first vertex having in-degree 0, then $H \not\subseteq K_{\omega^*}$. Therefore, (ii) implies (i). In addition, (iii) implies (ii) trivially. Therefore, all that remains is to show (i) implies (iii).

So suppose H is acyclic, locally-finite, and has no infinite directed paths. If H is not weakly-connected, we can make it so while maintaining the three properties (say by choosing a vertex v_i from each component H_i of H and adding an antidirected path on $v_1, v_2, \dots$). Let K be a countably-infinite tournament and let $(u_i)_{i\in\mathbb{N}}$ be an enumeration of $V(K)$. Define $*_1, *_2, *_3, \dots$ inductively by

$$*_i = \begin{cases} + & \text{if } \left(\bigcap_{j=1}^{i-1} N^{*_j}(u_j)\right) \cap N^+(u_i) \text{ is infinite,} \\ - & \text{otherwise.} \end{cases}$$

Let $V^+ = \{u_i \in V(K) : *_i = +\}$ and let $V^- = \{u_i \in V(K) : *_i = -\}$. The key property is that for all $\diamond, * \in \{+,-\}$ and all finite non-empty subsets $X \subseteq V^\diamond$ and $Y \subseteq V^*$, $\hat{N}^\diamond(X) \cap \hat{N}^*(Y)$ is infinite. (A more standard approach to assigning the $*_i$ would have been to choose a non-principal ultrafilter on $\mathbb{N}$ and let $*_i = \diamond$ if and only if $N^\diamond(u_i)$ is in the ultrafilter. We note that our assignment of $*_i$ without the use of ultrafilters is inspired by the proof of [23, Lemma 1].)

If $*_1 = +$, then we choose a vertex $v_1 \in V(H)$ with in-degree 0 and apply Lemma 4.2 to get a $\pm$-partition $\{C_i : i \in \mathbb{N}\}$ of H with $C_1 = \{v_1\}$. If $*_1 = -$, then we choose a vertex $v_1 \in V(H)$ with out-degree 0 and apply Lemma 4.2 to get a $\mp$-partition $\{C_i : i \in \mathbb{N}\}$ of H with $C_1 = \{v_1\}$. We may suppose by directional duality that $*_1 = +$ and thus we choose a vertex $v_1 \in V(H)$ with in-degree 0 and apply Lemma 4.2 to get a $\pm$-partition $\{C_i : i \in \mathbb{N}\}$ of H with $C_1 = \{v_1\}$. Finally, define

$$\diamond_i = \begin{cases} + & \text{if i is odd,} \\ - & \text{if i is even} \end{cases}$$

and note that $\diamond_i$ simply describes the type of the set C_i.

We construct a sequence $i_1 \leqslant i_2 \leqslant \dots$, growing an embedding $\phi : H[\cup_{i\in[i_j]} C_i] \to K$ as we do so, such that following properties hold for every $j \in \mathbb{N}$.

Z1 $\{u_1, \dots, u_j\} \subseteq \phi(\cup_{i\in[i_j]} C_i)$.

Z2 $\phi(C_{i_j}) \subseteq V^{\diamond_{i_j}}$.

If such a sequence exists, then by **Z1**, the resulting embedding $\phi : H \to K$ proves the theorem.

We initially set $i_1 = 1$ and $\phi(v_1) = u_1$. Then, given i_{j-1} and $\phi : H[\cup_{i\in[i_{j-1}]}C_i] \to K$ satisfying **Z1** and **Z2**, we proceed as follows.

If $u_j \in \phi(\cup_{i\in[i_{j-1}]}C_i)$, then set $i_j = i_{j-1}$ (trivially, **Z1** and **Z2** are satisfied). Otherwise, by **Z2** we have that $U_{j-1} := N^{*_j}(u_j) \cap \hat{N}^{\diamond_{i_{j-1}}}(\phi(C_{i_{j-1}}))$ is infinite. If $U_{j-1} \cap V^+$ is infinite, set i_j to be the smallest integer at least $i_{j-1}+5$ with $\diamond_{i_j} = +$ (i.e., the smallest odd integer at least $i_{j-1}+5$). Otherwise, $U_{j-1} \cap V^-$ is infinite and we set i_j to be the smallest integer at least $i_{j-1}+5$ with $\diamond_{i_j} = -$ (i.e., the smallest even integer at least $i_{j-1}+5$). We now embed the acyclic finite subgraph $H[C_{i_{j-1}+1} \cup \ldots \cup C_{i_j}]$ into the infinite tournament $K[\{u_j\} \cup (U_{j-1} \cap V^{\diamond_{i_j}})]$ in such a way that if $*_j = \diamond_{i_{j-1}}$, then we will choose a vertex $v_j \in C_{i_{j-1}+2}$ which only has $*_j$-neighbours and embed v_j to u_j, and if $*_j \neq \diamond_{i_{j-1}}$, then we will choose a vertex $v_j \in C_{i_{j-1}+3}$ which only has $*_j$-neighbours and embed v_j to u_j. Thus **Z1** is satisfied. Also note that by construction, every vertex in C_{i_j} is embedded into $V^{\diamond_{i_j}}$, so **Z2** is satisfied.

(In the above paragraph, it is instructive to have a specific example, so suppose $\phi(C_{i_{j-1}}) \subseteq V^+$ (i.e., i_{j-1} is odd), $u_j \in V^-$, and $(N^-(u_j) \cap \hat{N}^+(\phi(C_{i_{j-1}})) \cap V^-$ is infinite. In this case we would set $i_j = i_{j-1}+5$ (note that i_j is even), embed a vertex from $C_{i_{j-1}+3}$ ($i_{j-1}+3$ is also even) with in-degree 0 to u_j and embed the rest of $C_{i_{j-1}+1} \cup \cdots \cup C_{i_j}$ into $(N^-(u_j) \cap \hat{N}^+(\phi(C_{i_{j-1}})) \cap V^-$. Note that since i_j is even and $\phi(C_{i_j}) \subseteq V^-$, **Z2** is satisfied.) □

4.3 Forward edge density

By Theorem 1.7, if we are given a countably-infinite oriented graph H such that H is acyclic, locally-finite, and has no infinite directed paths, then for every countably-infinite tournament K, we have $H \subseteq K$. However, if we drop one or more of these conditions on

H, what conditions could we impose on the tournament K to still ensure $H \subseteq K$? One approach to this question is to consider the density of the forward (or backward) edges in the tournament K.

For an infinite tournament K on $\mathbb{N}$, define, for each n,

$$d^+(K[n]) = \frac{|\{(i,j) \in E(K) : 1 \leqslant i < j \leqslant n\}|}{\binom{n}{2}},$$

and define the *forward density* of K to be

$$d^+(K) = \liminf_{n \to \infty} d^+(K[n]).$$

Given an oriented graph H, define $\overrightarrow{\rho}(H)$ to be the smallest $\rho \in [0,1]$ such every tournament K on $\mathbb{N}$ with $d^+(K) > \rho$ contains a copy of H. By Theorem 1.7, we know that if H is acyclic, locally-finite, and has no infinite directed paths, then $\overrightarrow{\rho}(H) = 0$. On the other hand, if H contains a cycle, a vertex of infinite in-degree, or an infinite backward directed path, then H does not appear in K_ω, and so we have $\overrightarrow{\rho}(H) = 1$. In addition, if H contains a vertex of infinite out-degree, then we also have $\overrightarrow{\rho}(H) = 1$, due to the existence of tournaments with forward density 1 in which every vertex has finite out-degree (such as the tournament on $\mathbb{N}$ with forward edges given by $\{(i,j) : j \leqslant 2^i\}$).

The only oriented graphs then left to consider are those which are acyclic, locally-finite, and contain an infinite forward directed path (but no infinite backward directed path). A natural case to consider then is to determine the value of $\overrightarrow{\rho}(P)$, where P is the infinite forward directed path. Notably, we have $\overrightarrow{\rho}(P)$ strictly between 0 and 1.

Theorem 4.3. *Let P be the infinite forward directed path. Then, $\overrightarrow{\rho}(P) = 3/4$.*

Note that obtaining a lower bound of $\overrightarrow{\rho}(P) \geqslant 1/2$ is easy. Let $I_k = [k!] \setminus [(k-1)!]$ for $k \geqslant 2$, and let K be the tournament on $\mathbb{N}$ with all edges with both endpoints some I_k oriented forward and all other edges oriented backward. Then, $d^+(K) = 1/2$, but, because $K \cong K_{\omega^*}$, K contains no copy of P.

Proving an upper bound strictly below 1, and also sharpening the lower bound, is more involved. To do this, we will reduce the problem to finding the smallest possible forward density of a large class of P-free tournaments on $\mathbb{N}$, defined as follows. Given an ordinal λ and an injection $f : \mathbb{N} \to \lambda$, define K_{f^*} to be the tournament on $\mathbb{N}$ with

$$E(K_{f^*}) = \{(i, j) : f(i) > f(j)\}.$$

Because λ is well ordered, no tournament defined in this way contains a copy of P. In addition, the forward density of K_{f^*} can be easily related to the density of *inversions* of f. Precisely, if we define, for sets $A, B \subseteq \mathbb{N}$,

$$I_f[A, B] = |\{(i, j) \in A \times B : i < j \text{ and } f(i) > f(j)\}|$$

and also $I_f[A] = I_f[A, A]$ and $I_f[n] = I_f[[n]]$, then we have

$$d^+(K_{f^*}) = \liminf_{n\to\infty} I_f[n]/\tbinom{n}{2}. \tag{4.1}$$

The quantity $\liminf_{n\to\infty} I_f[n]/\binom{n}{2}$ can be thought of as an infinite analogue of the inversion number of a permutation (see, for example, [20]), especially if we consider a bijection $f : \mathbb{N} \to \omega$, which may be regarded as a permutation of $\mathbb{N}$.

We note that the earlier P-free tournament showing $\overrightarrow{\rho}(P) \geqslant 1/2$ can be realised as K_{f^*} for some appropriate $f : \mathbb{N} \to \omega$. In fact, given any P-free tournament K on $\mathbb{N}$, we can construct an injection $h : \mathbb{N} \to \omega_1$ such that $d^+(K_{h^*}) \geqslant d^+(K)$. From this we obtain the following correspondence.

Lemma 4.4. *Define*

$$C = \sup\left\{\liminf_{n\to\infty} I_f[n]/\tbinom{n}{2} : \ f : \mathbb{N} \to \omega_1 \text{ is an injection}\right\}.$$

Then $\overrightarrow{\rho}(P) = C$.

Proof. Because $\overrightarrow{\rho}(P) \geqslant d^+(K_{f^*})$ for any injection $f : \mathbb{N} \to \omega_1$, we have, by (4.1), that $\overrightarrow{\rho}(P) \geqslant C$. Therefore it is enough to show that for any P-free tournament K on $\mathbb{N}$ there is an injection $f : \mathbb{N} \to \omega_1$ with $d^+(K_{f^*}) \geqslant d^+(K)$, and hence

$$\begin{aligned}\overrightarrow{\rho}(P) &= \sup\{d^+(K) : K \text{ is } P\text{-free}\}\\ &\leqslant \sup\{d^+(K_{f^*}) : \ f : \mathbb{N} \to \omega_1 \text{ is an injection}\} \overset{(4.1)}{=} C.\end{aligned}$$

So suppose K is a P-free tournament on $\mathbb{N}$. For $i, j \in \mathbb{N}$, say that j is a forward out-neighbour of i if $i < j$ and $i \to_K j$. Define $A_0 = \emptyset$. Given an ordinal α, define $A_{\alpha+1} \supseteq A_\alpha$ to be the set of $x \in \mathbb{N}$ such that every forward out-neighbour of x is in A_α. If α is a limit ordinal, define $A_\alpha = \cup_{\beta<\alpha} A_\beta$. Define $\partial A_\alpha = A_{\alpha+1} \setminus A_\alpha$. Let λ be the smallest ordinal with $\partial A_\lambda = \emptyset$, and note that λ is well-defined and $\lambda < \omega_1$ (because we can inject $\lambda \to \mathbb{N}$ by sending $\alpha < \lambda$ to the smallest $x \in \mathbb{N}$ with $x \in \partial A_\alpha$). If $A_\lambda \neq \mathbb{N}$, then K contains a copy of P (any $x_i \notin A_\lambda$ has a forward out-neighbour $x_{i+1} \notin A_\lambda$). Therefore, we have a partition $\mathbb{N} = \cup_{\alpha<\lambda} \partial A_\alpha$. Given $i \in \mathbb{N}$, let $\alpha(i) < \lambda$ be the unique ordinal with $i \in \partial A_{\alpha(i)}$. Define an injection $f : \mathbb{N} \to \omega \cdot \lambda < \omega_1$ by $f(i) = (i, \alpha(i))$. Note that, if $i < j$ and $i \to_K j$, then $\alpha(i) > \alpha(j)$, and hence $f(i) > f(j)$. Therefore, $d^+(K) \leqslant d^+(K_{f^*})$. □

With Lemma 4.4, determining the value of $\overrightarrow{\rho}(P)$ is now equivalent to this natural question of what is the maximum 'inversion density' that can be attained by an injection $f : \mathbb{N} \to \omega_1$. Thus, Theorem 4.3 is now an immediate consequence of the following result.

Theorem 4.5.

(i) *If* $f : \mathbb{N} \to \omega_1$ *is an injection, then* $\liminf_{n\to\infty} I_f[n]/\binom{n}{2} \leqslant 3/4$.

(ii) *There is an injection* $g : \mathbb{N} \to \omega$ *such that* $\liminf_{n\to\infty} I_g[n]/\binom{n}{2} = 3/4$.

To prove Theorem 4.5 (ii), for the sake of convenience we will actually describe an injection $g : \mathbb{N} \to \omega^2$; however, because each set of the form $\omega \times \{n\}$ will only be hit by g finitely many times, the order type of $g(\mathbb{N})$ will be ω, and so we could have equivalently constructed (with some added technicalities) an injection $g : \mathbb{N} \to \omega$.

Proof of Theorem 4.5 (ii). Given $n \in \mathbb{N}$, let a_n be the smallest odd number such that $n \in [(a_n + 2)^2!]$ and let b_n be the smallest even number such that $n \in [(b_n + 2)^2!]$. Define functions $g_0, g_1 : \mathbb{N} \to \omega^2$ as follows. (We recall here that for $(n_1, n_2), (m_1, m_2) \in \omega^2$ we have $(n_1, n_2) < (m_1, m_2)$ if $n_2 < m_2$, or if $n_2 = m_2$ and $n_1 < m_1$.)

$$g_0(n) = ((a_n + 2)^2! - n, a_n) \qquad\qquad g_1(n) = ((b_n + 2)^2! - n, b_n)$$

We note that $\liminf_{n\to\infty} I_{g_0}[n]/\binom{n}{2} = \liminf_{n\to\infty} I_{g_1}[n]/\binom{n}{2} = 1/2$ ($K_{g_0^*}$ and $K_{g_1^*}$ are both similar to the P-free tournament realizing $\overrightarrow{\rho}(P) \geqslant 1/2$ that was described earlier); however, we can selectively choose values from either g_0 or g_1 to construct an injection g with $\liminf_{n\to\infty} I_g[n]/\binom{n}{2} \geqslant 3/4$.

Let $c_n = \max\{a_n, b_n\}$ so that $n \in [(c_n + 1)^2!]$, and let $q_n \in [2c_n + 1]$ be minimal such that $n \in [(c_n^2 + q_n)!]$. Finally, let $r_n \in [2c_n]$ be such that $n \equiv r_n \mod 2c_n$. Then set

$$g(n) = \begin{cases} g_0(n) & \text{if } c_n \text{ is odd and } r_n < q_n, \\ g_1(n) & \text{if } c_n \text{ is odd and } r_n \geqslant q_n, \\ g_1(n) & \text{if } c_n \text{ is even and } r_n < q_n, \\ g_0(n) & \text{if } c_n \text{ is even and } r_n \geqslant q_n. \end{cases}$$

The specific details of this technical definition are not very important, and it is perhaps most useful to understand g by referring to Figures 4.1 and 4.2.

We will now show $\liminf_{n\to\infty} I_g[n]/\binom{n}{2} \geqslant 3/4$. Given sets $A, B \subseteq \mathbb{N}$, define

$$\overline{I}_g[A, B] = |\{(i, j) \in A \times B : i < j \text{ and } g(i) < g(j)\}|,$$

and for integers $n_1, n_2 \in \mathbb{N}$, define $\overline{I}_g[n_1, n_2] = \overline{I}_g[[n_1], [n_2]]$. Suppose $n \in \mathbb{N}$ is sufficiently large so that $c_n \geqslant 2$. If $q_n = 1$, then

$$\overline{I}_g[n] = \overline{I}_g[(c_n^2 - 1)!, n] \leqslant \left(\frac{(c_n^2 - 1)!}{n}\right) \cdot n^2 \leqslant \frac{n^2}{c_n^2}. \tag{4.2}$$

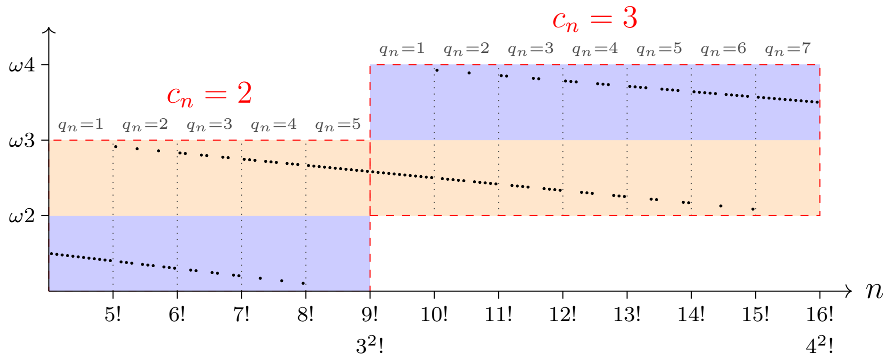

Figure 4.1: A representation of the values taken by g for n up to $4^2!$, with the blue region indicating points where $g(n) = g_0(n)$ and the orange region indicating points where $g(n) = g_1(n)$. While the columns here shown as equal width, each corresponding interval of $\mathbb{N}$ is of course actually much larger than everything coming before, and so $I_g[n]/\binom{n}{2}$ may be approximated using local behaviour only. Thus, if q_n is close to 1 or $2c_n+1$ we have $I_g[n]/\binom{n}{2} \approx 1$, whereas if q_n is close to $(2c_n+1)/2$ we have $I_g[n]/\binom{n}{2} \approx 3/4$.

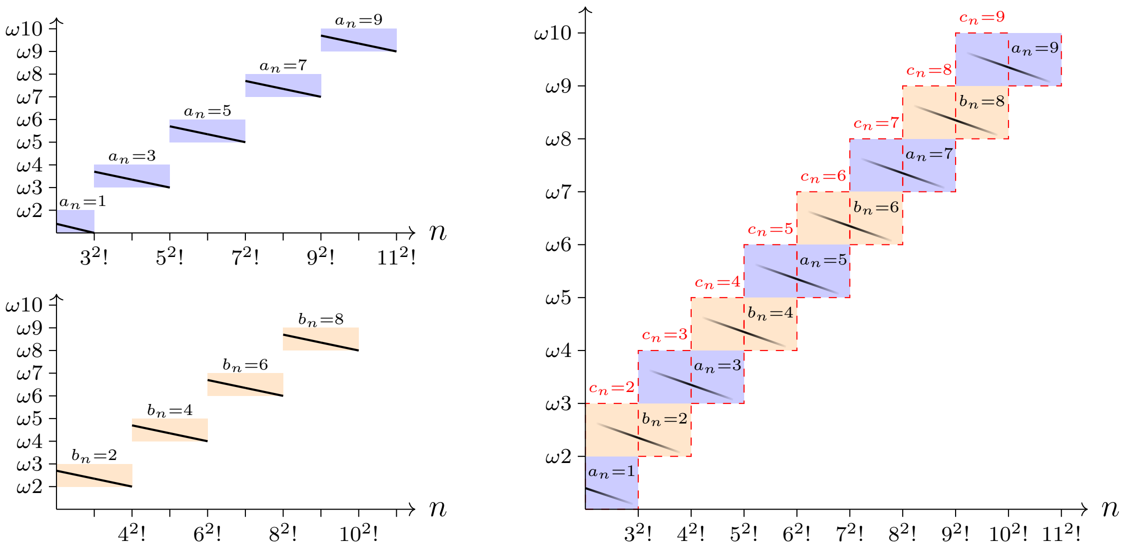

Figure 4.2: A representation of the values taken by g for n up to $11^2!$, with the blue region indicating points where $g(n) = g_0(n)$ and the orange region indicating points where $g(n) = g_1(n)$. Here, the opacity of the black lines on the right hand axis is in correspondence with density of the points in Figure 4.1.

On the other hand, if $q_n > 1$, then, letting A_n be the set of $n' \in [n] \setminus [(c_n^2 + q_n - 2)!]$ such that $r_i < q_i$ and B_n be the set of $n' \in [n] \setminus [(c_n^2 + q_n - 2)!]$ such that $r_i \geqslant q_i$, we have

$$\begin{aligned}\overline{I}_g[n] = \overline{I}_g[(c_n^2 + q_n - 2)!, n] + \overline{I}_g[A_n, B_n] &\leqslant \frac{n^2}{c_n^2} + \sum_{n' \in A_n} |B_n \cap ([n] \setminus [n'])| \\ &\leqslant \frac{n^2}{c_n^2} + \sum_{n' \in A_n} \left(\frac{2c_n + 2 - q_n}{2c_n}\right)(n - n') \leqslant \frac{n^2}{c_n^2} + \frac{1}{2}\left(\frac{2c_n + 2 - q_n}{2c_n}\right)\left(\frac{q_n}{2c_n}\right) \cdot n^2 \\ &\leqslant \left(\frac{1}{4} + \frac{100}{c_n^2}\right) \cdot \frac{1}{2} n^2. \end{aligned} \tag{4.3}$$

Therefore, we have

$$\liminf_{n\to\infty} I_g[n]/\binom{n}{2} = 1 - \limsup_{n\to\infty} \overline{I}_g[n]/\binom{n}{2} \overset{(4.2),(4.3)}{\geqslant} 3/4,$$

as required. □

Finally, to prove Theorem 4.5 (i), we must show that every injection $f : \mathbb{N} \to \omega_1$ has $\liminf_{n\to\infty} I_f[n]/\binom{n}{2} \leqslant 3/4$. To do this, given an injection $f : \mathbb{N} \to \omega_1$ we will define a sequence of ordinals $\alpha_1, \alpha_2, \ldots$ in $\{0,1\} \cdot \omega$, where α_n can be considered to be the median of the set $f([n])$. The key property is that every time we find $\alpha_n > \alpha_{n-1}$, we have that $f(n') > f(n)$ for fewer than $(n-1)/2$ many $n' < n$; that is, at every n where the median increases, there are more non-inversions ending with n than there are inversions. We then identify a large finite interval $[n_1]\setminus[n_0]$ with $n_1 \gg n_0$ in which for every $n \in [n_1]\setminus[n_0]$ where the median decreases, there is some $n' \in [n_1]\setminus[n_0]$ with $n' > n$ where the median increases back to the same value. From this, we deduce at least $(n_1 - n_0)/2$ of the $n \in [n_1] \setminus [n_0]$ give rise to an increase of the median, and also that these median increases are not skewed towards the beginning of $[n_1] \setminus [n_0]$, thus the number of inversions satisfies $I_f[n_1] \lesssim \frac{3}{4}\binom{n_1}{2}$.

Proof of Theorem 4.5 (i). Let $f : \mathbb{N} \to \omega_1$ be an injection. Note that, for each n, $f([n])$ is a subset of ω_1 of size n. Given n, define the *median* α_n to be the unique element of $\{0,1\} \times f([n]) \subseteq \{0,1\} \times \omega_1$ such that $\lfloor n/2 \rfloor$ elements of $\{1\} \times f([n])$ are greater than α_n and $\lfloor n/2 \rfloor$ elements of $\{0\} \times f([n])$ are less than α_n (where here, $\{0,1\} \times \omega_1$ is ordered according to the ordinal product $\{0,1\} \cdot \omega_1$). Note that $\{0,1\}$ essentially acts as a tiebreaker here, as it will later be important that $\alpha_{n+1} \neq \alpha_n$ holds for every $n \in \mathbb{N}$.

Let $N \in \mathbb{N}$ and $\varepsilon > 0$. We will show that there exists $n_1 > N$ such that $I_f[n_1] < (\frac{3}{4} + \varepsilon)\binom{n_1}{2}$, thus proving $\liminf_{n\to\infty} I_f[n]/\binom{n}{2} \leqslant 3/4$.

Claim 4.6. *There exists $n_0, n_1 \in \mathbb{N}$ with $N < n_0 \leqslant \varepsilon n_1$, such that $\alpha_{n_1} = \max\{\alpha_{n_0}, \alpha_{n_0+1}, \ldots, \alpha_{n_1}\}$.*

Proof of Claim 4.6. For $j \geqslant 0$, define $m_j = \lceil \left(\frac{2}{\varepsilon}\right)^j (N+1) \rceil$. For $j \geqslant 0$, define $\gamma_j = \max\{\alpha_{m_j}, \alpha_{m_j+1}, \ldots, \alpha_{m_{j+1}-1}\}$. We must have $\gamma_j \leqslant \gamma_{j+1}$ for some j, otherwise the set $\{\gamma_1, \gamma_2, \ldots\}$ contains no minimal element. For this j, let $n_0 = m_j > N$, and let $n_1 \geqslant m_{j+1}$ be minimal such that $\alpha_{n_1} \geqslant \gamma_j$. We then have $n_0 = m_j \leqslant \varepsilon m_{j+1} \leqslant \varepsilon n_1$ and $\alpha_{n_1} = \max\{\alpha_{n_0}, \alpha_{n_0+1}, \ldots, \alpha_{n_1}\}$. $\boxdot$

Let A be the set of $n \in [n_1] \setminus [n_0]$ such that $\alpha_n > \alpha_{n-1}$, and let B be the set of $n \in [n_1] \setminus [n_0]$ such that $\alpha_n < \alpha_{n-1}$. Because $\alpha_n \neq \alpha_{n+1}$ for every $n \in \mathbb{N}$, $A \cup B$ is a partition of $[n_1] \setminus [n_0]$. We note that for any $n \in B$, there exists $n' \in [n_1] \setminus [n]$ with $\alpha_{n'} = \alpha_{n-1}$. Given $n \in B$, we define $\sigma(n)$ to be the smallest such n'. We remark that $\sigma(n) \in A$ for any $n \in B$. We also note that $\sigma : B \to A$ is an injection. Therefore, we have

$$\begin{aligned}(1-\varepsilon)\binom{n_1}{2} &\leqslant \sum_{n\in[n_1]\setminus[n_0]} (n-1) = \sum_{n\in A}(n-1) + \sum_{n\in B}(n-1) \\ &\leqslant \sum_{n\in A}(n-1) + \sum_{n\in B}(\sigma(n)-1) \leqslant 2\sum_{n\in A}(n-1).\end{aligned} \tag{4.4}$$

Finally, observe that if $n \in A$, then, because $\alpha_n > \alpha_{n-1}$, we have $|\{n' \in [n-1] : f(n') < f(n)\}| \geqslant \frac{1}{2}(n-1)$. Hence,

$$\begin{aligned}I_f[n_1] &= \binom{n_1}{2} - \sum_{n\in[n_1]} |\{n' \in [n-1] : f(n') < f(n)\}| \\ &\leqslant \binom{n_1}{2} - \sum_{n\in A} |\{n' \in [n-1] : f(n') < f(n)\}| \leqslant \binom{n_1}{2} - \frac{1}{2}\sum_{n\in A}(n-1) \\ &\overset{(4.4)}{\leqslant} \left(\tfrac{3}{4}+\varepsilon\right)\binom{n_1}{2},\end{aligned}$$

as required. □

There are still other interesting questions that can be asked on what forward density is required to ensure a copy of an infinite oriented graph.

Question 4.7. *Let H be a countably-infinite acyclic oriented graph such that H is locally-finite, has no infinite backward directed path, but does have an infinite forward directed path.*

(i) *Determine $\overrightarrow{\rho}(H)$.*

(ii) *For which $d \in [\frac{3}{4}, 1]$ does there exist such an oriented graph H with $\overrightarrow{\rho}(H) = d$?*

LIST OF REFERENCES

[1] N. Alon and A. Shapira. Testing subgraphs in directed graphs. *Journal of Computer and System Sciences*, 69:354–382, 2004.

[2] J. Balogh and A. Lamaison. Ramsey upper density of infinite graph factors. *arXiv preprint arXiv:2010.13633*, 2020.

[3] A. Benford, L. DeBiasio, and P. Larson. Unavoidable structures in infinite tournaments. *arXiv preprint arXiv2301.04636*, 2023.

[4] A. Benford and R. Montgomery. Trees with few leaves in tournaments. *Journal of Combinatorial Theory, Series B*, 155:141–170, 2022.

[5] A. Benford and R. Montgomery. Trees with many leaves in tournaments. *arXiv preprint arXiv2207.06384*, 2022.

[6] S. Ceroi and F. Havet. Trees with three leaves are $(n+1)$-unavoidable. *Discrete Applied Mathematics*, 141:19–39, 2004.

[7] J. Corsten, L. DeBiasio, A. Lamaison, and R. Lang. Upper density of monochromatic infinite paths. *Advances in Combinatorics*, page 10810, 2019.

[8] J. Corsten, L. DeBiasio, and P. McKenney. Density of monochromatic infinite subgraphs II. *arXiv preprint arXiv:2007.14277*, 2020.

[9] L. DeBiasio and P. McKenney. Density of monochromatic infinite subgraphs. *Combinatorica*, 39(4):847–878, 2019.

[10] F. Dross and F. Havet. On the unavoidability of oriented trees. *Journal of Combinatorial Theory, Series B*, 151:83–110, 11 2021.

[11] A. El Sahili. Trees in tournaments. *Journal of Combinatorial Theory, Series B*, 92:183–187, 2004.

[12] P. Erdős and J. W. Moon. On sets of consistent arcs in a tournament. *Canadian Mathematical Bulletin*, 8(3):269–271, 1965.

[13] A. Frieze and M. Karoński. *Random Graphs and Networks: A First Course*. Cambridge University Press, 2023.

[14] R. Häggkvist and A. Thomason. Trees in tournaments. *Combinatorica*, 11:123–130, 1991.

[15] F. Havet. Trees in tournaments. *Discrete Mathematics*, 243(1):121–134, 2002.

[16] F. Havet. On unavoidability of trees with k leaves. *Graphs and Combinatorics*, 19:101–110, 2003.

[17] F. Havet and S. Thomassé. Median orders of tournaments: A tool for the second neighborhood problem and Sumner's conjecture. *Journal of Graph Theory*, 35(4):244–256, 2000.

[18] F. Havet and S. Thomassé. Oriented Hamiltonian paths in tournaments: a proof of Rosenfeld's conjecture. *Journal of Combinatorial Theory, Series B*, 78(2):243–273, 2000.

[19] A. Kathapurkar and R. Montgomery. Spanning trees in dense directed graphs. *Journal of Combinatorial Theory, Series B*, 156:223–249, 2022.

[20] D. E. Knuth. *The Art of Computer Programming, Volume 3 (2nd Edition): Sorting and Searching*. Addison Wesley Longman Publishing Co., Inc., USA, 1998.

[21] D. Kühn, R. Mycroft, and D. Osthus. An approximate version of Sumner's universal tournament conjecture. *Journal of Combinatorial Theory, Series B*, 101(6):415–447, 2011.

[22] D. Kühn, R. Mycroft, and D. Osthus. A proof of Sumner's universal tournament conjecture for large tournaments. *Proceedings of the London Mathematical Society*, 102(4):731–766, 2010.

[23] A. Lamaison. Ramsey upper density of infinite graphs. *arXiv preprint arXiv:2003.06329*, 2020.

[24] X. Lu, D.-W. Wang, and C. Wong. On avoidable and unavoidable claws. *Discrete Mathematics*, 184(1):259–265, 1998.

[25] R. Mycroft and T. Naia. Unavoidable trees in tournaments. *Random Structures & Algorithms*, 53(2):352–385, 2018.

[26] K. Reid and N. Wormald. Embedding oriented n-trees in tournaments. *Studia Sci. Math. Hungar*, 18(2-4):377–387, 1983.

[27] V. Rödl, A. Ruciński, and E. Szemerédi. A Dirac-type theorem for 3-uniform hypergraphs. *Combinatorics, Probability & Computing*, 15(1-2):229, 2006.

[28] M. Rosenfeld. Antidirected Hamiltonian paths in tournaments. *Journal of Combinatorial Theory, Series B*, 12(1):93–99, 1972.

[29] A. Thomason. Paths and cycles in tournaments. *Transactions of the American Mathematical Society*, 296:167–180, 1986.