Escape from a potential well with a randomly switching boundary.

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Abstract. We consider diffusion in a potential well with a boundary that randomly switches between absorbing and reflecting and show how the switching boundary affects the classical escape theory. Using the theory of stochastic hybrid systems, we derive boundary value problems for the mean first passage time and splitting probability and find explicit solutions in terms of the spectral decomposition of the associated differential operator. Further, using a more probabilistic approach, we prove asymptotic formulae for these statistics in the small diffusion limit. In particular, we show that the statistical behavior depends critically on the gradient of the potential near the switching boundary and we derive corrections to Kramers' reaction rate theory.

1. Introduction

Determining the rate of escape from a metastable state is important for understanding the non-equilibrium behavior of many different physical, chemical, and biological processes. A classical framework for analyzing escape problems is Kramers' reaction rate theory [1, 2], in which the thermally activated escape from a metastable state is formulated in terms of the one-dimensional Brownian motion of a fictitious particle along a reaction coordinate leading from an initial to a final locally stable state. In order to overcome the energetic barrier separating the two states, the particle has to extract energy from its surroundings, which is typically an extremely rare event since the activation energy tends to be much larger than the thermal energy. Hence, the particle will make many unsuccessful attempts before eventually overcoming the barrier separating the two states. As a result, the particle loses any memory of its initial state, and the waiting time in the initial potential well will be random with an exponential distribution whose average coincides with the inverse of the decay rate. In practice, the mean first passage time to escape from a potential well is calculated by placing absorbing boundaries at one or both ends of the well and solving a backwards Fokker-Planck equation [3].

In this paper, we consider an extension of Kramers' reaction rate theory to a Brownian particle moving in a potential well in which one of its boundaries randomly switches between absorbing and reflecting. This type of scenario is common in cell biology, where a macromolecule diffuses in some bounded intracellular domain that contains one or more narrow channels within the boundary of the domain; each channel is controlled by a stochastic gate that switches between an open and closed state [4]. (A distinct but related problem is the flow of an ion through an open channel with a fluctuating potential [5].) For simplicity, we focus on a one-dimensional domain. In the presence of a randomly switching boundary, the resulting stochastic differential equation (SDE) becomes a hybrid stochastic process in which one has to keep track of the discrete random state of the boundary as well as the random position of the Brownian particle. Recently, we have analyzed such a hybrid system in the case of a flat potential (pure Brownian motion) [6]. In particular, we determined the splitting or hitting probability that r non-interacting Brownian particles all exit at the switching boundary; although the particles are non-interacting, statistical correlations arise due to the fact that they all move in the same randomly switching environment. We also showed how the hitting probability is equivalent to the r-th moment of a stochastic diffusion equation with a switching boundary. This type of piecewise deterministic PDE has also been analyzed by Lawley et al. [7] using the theory of random iterative systems.

The paper is organized as follows. In section 2, we briefly review the classical Kramers' rate theory for a one-dimensional Brownian particle in a potential well. We then extend the theory to the case of a switching boundary, using the theory of stochastic hybrid systems (section 3). We consider a bounded domain $x \in [-L, L]$ with a switching boundary at $x = L$ and either (i) a fixed absorbing (open) or (ii) a fixed reflecting (closed) boundary condition at $x = -L$. We derive a general expression for the hitting probability to escape at $x = L$ in case (i) and the mean first passage time (MFPT) to escape at $x = L$ in case (ii). We show that both involve a correction to the classical result, which is obtained by starting the Brownian particle at $x = L$ with the right-hand boundary in the closed state. The corrections can be expressed in terms of the Green's function of the associated Fokker-Planck differential operator.

In section 4, we explicitly calculate the Green's function expressions for the hitting probability and MFPT in the case of a square well potential. We then use the theory of stopping times to obtain approximations to the hitting probability and MFPT for a general smooth potential well in the small diffusion limit (section 5); the corresponding Green's function cannot be calculated explicitly in this case. In particular, we show that the effect of a switching boundary on Kramers' reaction rate depends on the gradient of the potential in a neighborhood of the boundary. (More technical aspects of the proofs are collected in the appendix.)

2. One-dimensional SDE with fixed boundaries

Consider the one-dimensional SDE

$$dX(t) = -V'(X)dt + \sqrt{2D}dW(t), \tag{2.1}$$

where $W(t)$ is a Wiener process with

$$\langle dW(t)\rangle = 0, \quad \langle dW(t)dW(t')\rangle = \delta(t-t')dt\,dt',$$

D is a diffusion coefficient. Equation (2.1) represents a Brownian particle moving in an external potential V. A classical escape problem consists of restricting the dynamics to an interval $[-L, L]$ with absorbing boundaries at either end. One is then interested in determining the hitting probability that the particle reaches $x = L$ before $x = -L$, say, and the corresponding conditional mean first passage time. The analysis of escape problems is usually developed in terms of the Fokker-Planck (FP) equation for the conditional probability density $p(x,t) = p(x,t|y,0)$ with $-L < y < L$ fixed. In the case of the SDE (2.1), the FP equation is given by

$$\frac{\partial p}{\partial t} = \frac{\partial V'(x)p(x,t)}{\partial x} + D\frac{\partial^2 p(x,t)}{\partial x^2}, \tag{2.2}$$

such that

$$p(-L,t) = p(L,t) = 0, \quad p(x,0) = \delta(x-y). \tag{2.3}$$

We assume that the particle starts at position $y \in [-L, L]$ at time $t = 0$. Let $\tau_L(y)$ denote the first passage or hitting time that the particle reaches $x = L$ before being killed at $x = -L$, and conversely for $\tau_{-L}(y)$. We can then define the hitting probability of reaching $x = L$ first according to

$$h(y) = \mathbb{P}[\tau_L(y) < \tau_{-L}(y)] = \mathbb{E}[1_{\tau_L(y)<\tau_{-L}(y)}]. \tag{2.4}$$

Another quantity of interest is the (conditional) mean first passage time

$$T(y) = \mathbb{E}[\tau_L(y)1_{\tau_L(y)<\tau_{-L}(y)}]. \tag{2.5}$$

Introducing the Laplace transform of the hitting time,

$$\mathcal{H}(y,\lambda) = \mathbb{E}[\mathrm{e}^{-\lambda\tau_L(y)}1_{\tau_L(y)<\tau_{-L}(y)}], \tag{2.6}$$

we see that

$$h(y) = \mathcal{H}(y,0), \quad T(y) = -\left.\frac{d}{d\lambda}\mathcal{H}(y,\lambda)\right|_{\lambda=0}. \tag{2.7}$$

Let $S_L(y,t)$ denote the probability that the particle exits at $x = L$ after time t, having started at the point y, $-L < y < L$. Then

$$S_L(y,t) = \int_t^\infty J(L,t'|y,0)dt'$$

with J the probability flux

$$J(x,t|y,0) = -V'(x)p(x,t|y,0) - D\frac{\partial p(x,t|y,0)}{\partial x}.$$

It follows that $f_L(y,t) \equiv \partial_t S_L(y,t) = -J(L,t|y,0)$ is the hitting time density and

$$\mathcal{H}(y,\lambda) = -\int_0^\infty \mathrm{e}^{-\lambda t} J(L,t|y,0)dt. \tag{2.8}$$

In order to derive a differential equation for $\mathcal{H}$, we introduce the backward FP equation for $q(y,t) = p(x,t|y,0)$ with x fixed,

$$\frac{\partial q(y,t)}{\partial t} = -V'(y)\frac{\partial q(y,t)}{\partial y} + D\frac{\partial^2 q(y,t)}{\partial y^2}, \tag{2.9}$$

such that

$$q(-L,t) = q(L,t) = 0, \quad q(y,0) = \delta(x-y). \tag{2.10}$$

It immediately follows that $J(L,t|y,0)$ also satisfies the backward FP equation and, hence, differentiating equation (2.8) shows that

$$-V'(y)\frac{\partial \mathcal{H}(y,\lambda)}{\partial y} + D\frac{\partial^2 \mathcal{H}(y,\lambda)}{\partial y^2} = \lambda \mathcal{H}(y,\lambda), \tag{2.11}$$

supplemented by the boundary conditions

$$\mathcal{H}(L,\lambda) = 1, \quad \mathcal{H}(-L,\lambda) = 0. \tag{2.12}$$

We have used the result

$$\begin{aligned} \int_0^\infty \mathrm{e}^{-\lambda t}\partial_t J(L,t|y,0)dt &= \lim_{t\to 0} J(L,t|y,0) + \lambda \int_0^\infty \mathrm{e}^{-\lambda t} J(L,t|y,0)dt \\ &= \lambda \mathcal{H}(y,\lambda). \end{aligned}$$

Finally, combining equations (2.7) and (2.11) we see that the hitting probability and mean hitting time satisfy the differential equations

$$-V'(y)\frac{\partial h(y)}{\partial y} + D\frac{\partial^2 h(y)}{\partial y^2} = 0, \quad h(L) = 1, \quad h(-L) = 0, \tag{2.13}$$

and

$$-V'(y)\frac{\partial T(y)}{\partial y} + D\frac{\partial^2 T(y)}{\partial y^2} = -h(y), \quad T(-L) = T(L) = 0. \tag{2.14}$$

If the left-hand boundary is taken to be reflecting rather than absorbing, then $h(y) = 1$ for all y, and we recover the well-known equation for the MFPT T to reach $x = L$ starting from y,

$$-V'(y)\frac{\partial T(y)}{\partial y} + D\frac{\partial^2 T(y)}{\partial y^2} = -1, \quad T(L) = T'(-L) = 0. \tag{2.15}$$

Equation (2.13) has the solution

$$h(y) = \left[\int_{-L}^y \mathrm{e}^{V(x)/D}dx\right]\left[\int_{-L}^L \mathrm{e}^{V(x)/D}dx\right]^{-1}, \tag{2.16}$$

whereas (2.15) has the solution

$$T(y) = \frac{1}{D}\int_y^L \int_{-L}^z \mathrm{e}^{[V(z)-V(x)]/D}dx\,dz. \tag{2.17}$$

3. One-dimensional SDE with a switching boundary

Now suppose that the right-hand boundary of the one-dimensional SDE (2.1) randomly switches between absorbing and reflecting, whereas the left-hand boundary is fixed. In order to keep track of the boundary state, we introduce the discrete random variable $n(t) \in \{0, 1\}$ such that

$$dX(t) = -V'(X)dt + \sqrt{2D}dW(t), \tag{3.1}$$

with an absorbing boundary at $x = L$ if $n(t) = 0$ and a reflecting boundary at $x = L$ if $n(t) = 1$. The boundary at $x = -L$ is taken to be absorbing for all t. Assume that transitions between the two states are given by the two-state Markov process, $n = 0, 1$

$$0 \underset{\alpha}{\overset{\beta}{\rightleftharpoons}} 1 \tag{3.2}$$

with fixed transition rates α, β. Setting $p_n(x, t) = \mathbb{E}[p(x, t)1_{n(t)=n}]$, the FP equation (2.2) is replaced by the differential Chapman-Kolmogorov (CK) equation‡

$$\frac{\partial p_n(x, t)}{\partial t} = \frac{\partial}{\partial x}[V'(x)p_n(x, t)] + D\frac{\partial^2 p_n(x, t)}{\partial x^2} + \sum_{m=0,1} A_{nm}p_m(x, t), \tag{3.3}$$

with $\mathbf{A}$ the matrix

$$\mathbf{A} = \left(\begin{array}{cc} -\beta & \alpha \\ \beta & -\alpha \end{array} \right). \tag{3.4}$$

Equation (3.3) is supplemented by the boundary conditions

$$p_0(L, t) = 0, \quad -V'(L)p_1(L, t) - D\left.\frac{\partial p_1(x, t)}{\partial x}\right|_{x=L} = 0, \quad p_0(-L, t) = p_1(-L, t) = 0,$$

and the initial condition

$$p_n(x, 0) = \delta(x - y)\rho_n,$$

where ρ_n is the stationary measure of the ergodic two-state Markov process generated by the matrix $\mathbf{A}$,

$$\sum_{m=0,1} A_{nm}\rho_m = 0, \quad \rho_0 = \frac{\alpha}{\alpha + \beta}, \quad \rho_1 = \frac{\beta}{\alpha + \beta}. \tag{3.5}$$

Since we are interested in first passage time problems, we need to consider the backward CK equation for $q_m(y, t) = \mathbb{E}[q(y, t)1_{n(0)=m}]$ with x fixed:

$$\frac{\partial q_m(y, t)}{\partial t} = \mathbb{L}_{\rm FP}q_m(y, t) + \sum_{n=0,1} A^{\top}_{mn}q_n(x, t), \tag{3.6}$$

with linear operator

$$\mathbb{L}_{\rm FP} = -V'\frac{d}{dy} + D\frac{d^2}{dy^2} \tag{3.7}$$

‡ There are a few different ways to understand the origin of the CK equation (3.3), in which the discrete label n associated with the switching boundary also occurs within the interior of the domain. First, for a single Brownian particle, we could equivalently consider a non-switching absorbing boundary at $x = L$ and a particle that switches between two conformational states; the particle can only escape the domain when in state $n = 0$. Thus $p_n(x, t)$ becomes the joint probability density that the particle is in state $x(t) = x, n(t) = n$. An alternative derivation of the CK equation would be to write down the Fokker-Planck equation corresponding to equation (3.1), which has a switching boundary and then to take first moments of the resulting stochastic PDE [6].

and boundary conditions

$$q_0(L,t)=0, \quad -\left.\frac{\partial q_1(y,t)}{\partial x}\right|_{y=L}=0, \quad q_0(-L,t)=q_1(L,t)=0.$$

Note that $\mathbb{L}_{\rm FP}$ is non-Hermitian with respect to the inner product

$$\langle f,g\rangle=\int_{-L}^{L} f(x)g(x)dx$$

with f and g satisfying the same Neumann or Dirichlet boundary conditions at $x=\pm L$.

3.1. Hitting probability

Following along similar lines to section 2.1, we can derive a differential equation for the hitting probability of exiting at $x=L$ before $x=-L$, given that the particle started in state (y,m) at $t=0$:

$$\mathbb{L}_{\rm FP}h_m(y)+\sum_{n=0,1}A_{mn}^{\top}h_n(y)=0, \quad m=0,1 \tag{3.8}$$

with boundary conditions

$$h_0(L)=1, \quad \partial_y h_1(L)=0, \quad h_0(-L)=h_1(-L)=0.$$

Performing the change of variables $\pi_n(y)=\rho_n h_n(y)$, equation (3.8) becomes the pair of equations

$$\mathbb{L}_{\rm FP}\pi_0(y)-\beta\pi_0(y)+\alpha\pi_1(y)=0, \tag{3.9a}$$

$$\mathbb{L}_{\rm FP}\pi_1(y)+\beta\pi_0(y)-\alpha\pi_1(y)=0, \tag{3.9b}$$

with

$$\pi_0(L)=\rho_0, \quad \partial_y\pi_1(L)=0, \quad \pi_0(-L)=\pi_1(-L)=0.$$

Adding equations (3.9a), (3.9b) and setting $\pi(y)=\pi_0(y)+\pi_1(y)$ gives

$$\mathbb{L}_{\rm FP}\pi(y)=0, \quad \pi(L)=\rho_0+\pi_1(L), \quad \pi(-L)=0. \tag{3.10}$$

Suppose for the moment that $\pi_1(L)$ is known so that, from section 2.1,

$$\pi(y)=(\rho_0+\pi_1(L))h(y), \tag{3.11}$$

with $h(y)$ the hitting probability (2.16) in the case that the right-hand boundary is always absorbing. Equation (3.11) can be interpreted as follows. First note that $\pi(y)$ is the hitting probability of being absorbed at $x=L$ rather than $x=-L$ given that it starts in the state n with probability ρ_n, $n=0,1$. In order for this event to occur, the Brownian particle must first hit the end $x=L$ before the end $x=-L$, which is given by the probability $h(y)$. Once it reaches $x=L$, either the boundary is open with probability ρ_0 or it is closed. If it is closed then the hitting probability that it is subsequently absorbed at $x=L$ is given by $\pi_1(L)$.

Substituting (3.11) into equations (3.9a), (3.9b) and using $\pi_0=\pi-\pi_1$, we obtain the following inhomogeneous equation for π_1:

$$\mathbb{L}_{\rm FP}\pi_1(y)-(\alpha+\beta)\pi_1(y)=-\beta\pi(y). \tag{3.12}$$

Let $G_D(x,y)$ denote the Green's function of the adjoint of the linear operator on the left-hand side of equation (3.12), assuming it exists. That is,

$$[\mathbb{L}_{\rm FP,y}^{\dagger}-(\alpha+\beta)]G_D(x,y)=-\delta(x-y), \tag{3.13}$$

with

$$\partial_y G_D(x, L) = G_D(x, -L) = 0.$$

We then have the formal solution

$$\pi_1(x) = \beta \int_{-L}^{L} G_D(x, y)\pi(y)dy. \tag{3.14}$$

Finally using equation (3.11), we obtain the following self-consistency condition for $\pi_1(L)$:

$$\pi_1(L) = \beta(\rho_0 + \pi_1(L)) \int_{-L}^{L} G_D(L, y)h(y)dy,$$

which yields

$$\pi_1(L) = \frac{\beta\rho_0\Gamma_D}{1 - \beta\Gamma_D}, \quad \Gamma_D = \int_{-L}^{L} G_D(L, y)h(y)dy. \tag{3.15}$$

At first sight, it looks like $\pi_1(L)$ is a singular function of β. However, the Green's function G_D itself depends on $\alpha+\beta$, see equation (3.26). In the limit of slow switching, $\alpha, \beta \to 0$, we find that $\pi_1(L) \to 0$, and thus $\pi(y) \to \rho_0 h(y)$. This means that if the system starts out with the right-hand boundary open, then the boundary is still open almost surely when the particle first hits the boundary. On the other-hand, $\lim_{\alpha,\beta\to\infty} \beta G_D(x, y) = \rho_1\delta(x - y)$ so that

$$\pi_1(L) \to \frac{\rho_1\rho_0 h(L)}{1 - \rho_1 h(L)} = \frac{\rho_1\rho_0}{1 - \rho_1} = \rho_1.$$

3.2. Mean first passage time (MFPT)

Similarly, the MFPT for exiting at $x = L$ when there is a reflecting boundary at $x = -L$ satisfies the equation

$$\mathbb{L}_{\rm FP} T_m(y) + \sum_{n=0,1} A^{\top}_{mn} T_n(y) = -1, \tag{3.16}$$

with boundary conditions

$$T_0(L) = 0, \quad \partial_y T_1(L) = 0, \quad \partial_y T_m(-L) = 0, \quad m = 0, 1.$$

The MFPT equations (3.16) can be solved along similar lines to (3.6). Performing the change of variables

$$w_n(y) = \rho_n T_n(y),$$

equation (3.16) becomes the pair of equations

$$\mathbb{L}_{\rm FP} w_0(y) - \beta w_0(y) + \alpha w_1(y) = -\rho_0, \tag{3.17a}$$

$$\mathbb{L}_{\rm FP} w_1(y) + \beta w_0(y) - \alpha w_1(y) = -\rho_1, \tag{3.17b}$$

with

$$w_0(L) = 0, \quad \partial_y w_1(L) = 0, \quad \partial_y w_0(-L) = \partial_y w_1(-L) = 0.$$

Adding equations (3.17a), (3.17b), and setting $w(y) = w_0(y) + w_1(y)$ gives

$$\mathbb{L}_{\rm FP} w(y) = -1, \quad w(L) = w_1(L), \quad \partial_y w(-L) = 0. \tag{3.18}$$

For the given boundary conditions,

$$w(y) = w_1(L) + T(y), \tag{3.19}$$

where $T(y)$ is given by equation (2.17). That is, $T(y)$ is the expected time to reach the switching boundary for the first time starting from y, but there is now the possibility that the boundary is closed on arrival. Thus, the MFPT has an additional contribution $w_1(L)$, which is the MFPT to be absorbed at $x = L$ given that the particle starts at $x = L$ and the boundary is closed.

Substituting into equation (3.17b) and using $w_0 = w - w_1$, we obtain the following inhomogeneous equation for w_1:

$$\mathbb{L}_{\rm FP} w_1(y) - (\alpha + \beta) w_1(y) = -\beta w(y) - \rho_1. \tag{3.20}$$

Hence,

$$w_1(x) = \int_{-L}^{L} G_N(x, y)[\beta w(y) + \rho_1] dy, \tag{3.21}$$

with G_N similar to G_D except that $\partial_y G_N(x, -L) = 0$. Finally, since $w(y) = T(y) + w_1(L)$, we obtain the self-consistency condition

$$w_1(L) = [\rho_1 + \beta w_1(L)] \int_{-L}^{L} G_N(L, y) dy + \beta \int_{-L}^{L} G_N(L, y) T(y) dy,$$

that is,

$$w_1(L) = \frac{\rho_1 \Lambda_N + \beta \Gamma_N}{1 - \beta \Lambda_N} \tag{3.22}$$

with

$$\Lambda_N = \int_{-L}^{L} G_N(L, y) dy, \quad \Gamma_N = \int_{-L}^{L} G_N(L, y) T(y) dy. \tag{3.23}$$

3.3. Eigenvalue expansions

We will determine the Green's function G of equation (3.13) in terms of the eigenvalues and eigenfunctions of the linear operator $\mathbb{L}_{\rm FP}$. First, note that we can rewrite the linear operator (3.7) as

$$\mathbb{L}_{\rm FP} = -V'(y)\frac{d}{dy} + D\frac{d^2}{dy^2} = D\mathrm{e}^{V(y)/D}\frac{d}{dy}\mathrm{e}^{-V(y)/D}\frac{d}{dy}.$$

It follows that the eigenvalue equation

$$\mathbb{L}_{\rm FP}\phi_k = \lambda_k \phi_k$$

is equivalent to the Sturm-Liouville problem

$$\frac{d}{dy}\left[p(y)\frac{d}{dy}\phi_k(y)\right] = \lambda_k p(y)\phi_k(y),$$

with weight $p(y) = \exp[-V(y)/D]$. The standard theory of Sturm-Liouville operators then ensures that there exists a complete orthonormal set of eigenfunctions (with non-positive eigenvalues) for any potential $V(y)$ that is continuously differentiable in $[-L, L]$. More specifically, we can find a set of L^2-eigenfunctions $\{\phi_j\}$ that form an orthonormal basis with respect to the following weighted inner product

$$\langle f, g\rangle_* := \int_{-L}^{L} e^{-V(y)/D} f(y) g(y)\, dy.$$

Moreover,

$$\lambda_j \|\phi_j\|^2 = \langle \phi_j, \mathbb{L}_{\rm FP}\phi_j\rangle = -D\int_{-L}^{L}\left[\frac{d\phi_j(x)}{dx}\right]^2 {\rm e}^{-V(x)/D}dx \leq 0.$$

For the given boundary conditions there is no zero eigenvalue. It also useful to note that the linear operator $-\mathbb{L}$ with

$$\mathbb{L} = {\rm e}^{-V/2D}\mathbb{L}_{\rm FP}{\rm e}^{V/2D},$$

is Hermitian with respect to the standard L^2 norm and is identical in form to the Hamiltonian operator in quantum mechanics describing a particle in a potential well $U(x)$, since

$$\mathbb{L}\psi(x) = D\frac{d^2\psi(x)}{dx^2} - U(x)\psi(x), \quad U(x) = \frac{1}{4D}[V'(x)^2] - \frac{1}{2}V''(x).$$

Suppose that we now expand the solution of equation (3.12) for the hitting probability as the generalized Fourier series

$$\pi_1(x) = \sum_j c_j\phi_j(x), \quad c_j = \langle \phi_j, \pi_1\rangle_* = \int_{-L}^{L} e^{-V(y)/D}\pi_1(y)\phi_j(y)dy \tag{3.24}$$

with the eigenfunctions satisfying the boundary conditions

$$\partial_y\phi_j(L) = 0, \quad \phi_j(-L) = 0.$$

Substituting into equation (3.12) then yields the following solution for the coefficients c_j:

$$c_j = \frac{\beta}{\alpha+\beta-\lambda_j}\int_{-L}^{L}\pi(y)\phi_j(y)e^{-V(y)/D}\,dy. \tag{3.25}$$

Note that $\alpha+\beta$ is not an eigenvalue of $\mathbb{L}$, since it is positive. Comparison with equation (3.14) yields the following eigenvalue expansion of the Green's function G:

$$G_D(x,y) = \sum_j \frac{1}{\alpha+\beta-\lambda_j}\phi_j(y)\phi_j(x)e^{-V(y)/D}. \tag{3.26}$$

The same expansion holds for G_N except that now the eigenfunctions ϕ_j satisfy the reflecting boundary conditions

$$\partial_y\phi_j(-L) = \partial_y\phi_j(L) = 0.$$

In particular, we find that if $w_1 = \sum_j c_j\phi_j$, then plugging into equation (3.20) yields

$$c_j = \frac{1}{\alpha+\beta-\lambda_j}\int_{-L}^{L}[\beta w(y)+\rho_1]\phi_j(y)e^{-V(y)/D}\,dy.$$

Thus, comparison with equation (3.21) gives

$$G_N(x,y) = \sum_j \frac{1}{\alpha+\beta-\lambda_j}\phi_j(y)\phi_j(x)e^{-V(y)/D}. \tag{3.27}$$

4. Examples

We now consider a few example potentials for which the Green's functions can be calculated explicitly.

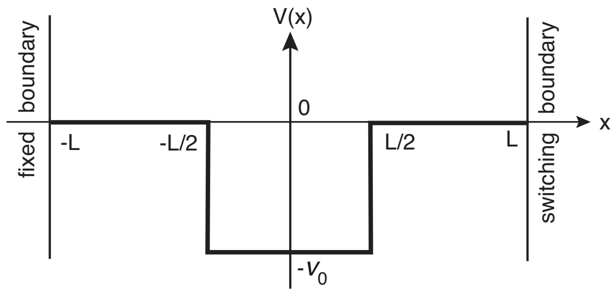

Figure 1. Metastable square-well potential

4.1. Pure Brownian motion ($V = 0$)

In the case that $V = 0$, we can solve equations (3.17*a*,3.17*b*) explicitly to find the mean first passage time conditioned on starting at position $y \in [-L, L]$

$$w_0(y) + w_1(y) = \frac{3L^2 - 2Ly - y^2}{2D} + C,$$

where

$$C = \frac{2\beta L \coth\left(2\sqrt{(\alpha+\beta)/D}L\right)}{\sqrt{D}\alpha\sqrt{\alpha+\beta}}. \tag{4.1}$$

As found in equation (3.19), we see that this is the usual mean first passage time for the process without boundary switching plus a constant $C = w_1(L)$. We can also solve equations (3.9*a*,3.9*b*) explicitly to calculate splitting probabilities. We have that

$$\pi_0(y) + \pi_1(y) = \frac{y+L}{2L}\left(1 + (\rho_1/\rho_0)(\xi 2L)^{-1}\tanh(\xi 2L)\right)^{-1}, \tag{4.2}$$

where $\xi = \sqrt{(\alpha+\beta)/D}$. As found in equation (3.11), we see that this is the usual splitting probability for the process without boundary switching multiplied by a constant, $\rho_0 + \pi_1(L)$. We will use equations (4.1) and (4.2) in section 5 below.

4.2. Metastable square-well potential

Consider the metastable potential well (Fig. 1)

$$V(x) = \begin{cases} -v_0, & \text{if} \quad |x| \le L/2 \\ 0, & \text{if} \quad L/2 < |x| \le L \\ +\infty, & \text{if} \quad |x| > L. \end{cases}$$

For simplicity, we will consider only the MFPT for this example, and thus we've chosen the conditions for V at $\pm L$ to correspond to equations (3.17*a*) and (3.17*b*). If we wanted V to correspond to the splitting probability problem in equations (3.9*a*) and (3.9*b*), then we would have taken $V(x) = -\infty$ for $x < L$.

4.2.1. Eigenvalues and eigenfunctions We seek eigenvalues and eigenfunctions for the operator $\mathbb{L} = \mathrm{e}^{-V/2D}\mathbb{L}_{\mathrm{FP}}\mathrm{e}^{V/2D}$,

$$\mathbb{L}\psi_j = \lambda_j \psi_j.$$

The discontinuities in V imply that the eigenfunctions must satisfy the following jump conditions at $\pm L/2$ (see [8] for details):

$$\begin{aligned}\lim_{\epsilon\to 0^+} \psi'(L/2+\epsilon) &= \lim_{\epsilon\to 0^+} \mathrm{e}^{v_0/2D}\psi'(L/2-\epsilon)\\ \lim_{\epsilon\to 0^+} \psi(L/2+\epsilon) &= \mathrm{e}^{-v_0/2D}\lim_{\epsilon\to 0^+} \psi(L/2-\epsilon).\end{aligned}$$

and

$$\begin{aligned}\lim_{\epsilon\to 0^+} \psi'(-L/2+\epsilon) &= \mathrm{e}^{-v_0/2D}\psi'(-L/2-\epsilon)\\ \lim_{\epsilon\to 0^+} \psi(-L/2+\epsilon) &= \lim_{\epsilon\to 0^+} \mathrm{e}^{v_0/2D}\psi(-L/2-\epsilon).\end{aligned}$$

Lastly, the eigenfunctions must satisfy the no flux boundary conditions

$$\psi'(L) = 0 = \psi'(-L).$$

As we have already established, all the eigenvalues are nonpositive, $\lambda \leq 0$. The eigenfunctions are of the form

$$\psi = \begin{cases} A^r\sin(ax) + B^r\cos(ax) & \text{if } L/2 < x < L\\ A^m\sin(ax) + B^m\cos(ax) & \text{if } -L/2 < x < L/2\\ A^l\sin(ax) + B^l\cos(ax) & \text{if } -L < x < -L/2,\end{cases}$$

where $a = \sqrt{\frac{-\lambda}{D}}$. The jump conditions and boundary conditions imply that

$$\begin{aligned}
&A^r\cos(aL) - B^r\sin(aL) &&= 0\\
&A^l\cos(aL) + B^l\sin(aL) &&= 0\\
&A^r\cos(aL/2) - B^r\sin(aL/2) - \mathrm{e}^{v_0/2D}[A^m\cos(aL/2) - B^m\sin(aL/2)] &&= 0\\
&A^r\sin(aL/2) + B^r\cos(aL/2) - \mathrm{e}^{-v_0/2D}[A^m\sin(aL/2) + B^m\cos(aL/2)] &&= 0\\
&A^m\cos(aL/2) + B^m\sin(aL/2) - \mathrm{e}^{-v_0/2D}[A^l\cos(aL/2) + B^l\sin(aL/2)] &&= 0\\
&-A^m\sin(aL/2) + B^m\cos(aL/2) - \mathrm{e}^{v_0/2D}[-A^l\sin(aL/2) + B^l\cos(aL/2)] &&= 0.
\end{aligned}$$

In order for this system to have a nontrivial solution $(A^r, B^r, A^m, B^m, A^l, B^l)$, we need the following determinant to be zero.

$$2\cosh\left(\frac{v_0}{2D}\right)\sin(aL)\left(\cos(aL)\cosh\left(\frac{v_0}{2D}\right) + \sinh\left(\frac{v_0}{2D}\right)\right) = 0.$$

Hence, we obtain the eigenvalues

$$\begin{aligned}
\lambda_{4n} &= -D(\pi/L)^2(2n)^2,\\
\lambda_{4n+2} &= -D(\pi/L)^2(2n+1)^2,\\
\lambda_{4n+1} &= -D(\pi/L)^2(2n+\nu)^2,\\
\lambda_{4(n+1)-1} &= -D(\pi/L)^2(2n-\nu)^2, \qquad n = 0,1,2,3,\ldots
\end{aligned}$$

where

$$\nu = \frac{1}{\pi}\arccos\big(-\tanh\big(\frac{v_0}{2D}\big)\big).$$

One finds that the normalized even eigenfunctions are

$$\begin{aligned}
\psi_0 &= \big(2L\mathrm{e}^{v_0/2D}\cosh\big(\frac{v_0}{2D}\big)\big)^{-1/2}e^{-V(x)/2D},\\
\psi_{4n} &= \big(Le^{\frac{v_0}{2D}}\cosh\big(\frac{v_0}{2D}\big)\big)^{-1/2}e^{-\frac{V(x)}{2D}}\cos(2nx\pi/L),
\end{aligned}$$

for $n = 1, 2, 3, \ldots$, and

$$\psi_{4n+2} = \left(Le^{-\frac{v_0}{2D}} \cosh\left(\frac{v_0}{2D}\right)\right)^{-1/2} e^{\frac{V(x)}{2D}} \cos((2n+1)x\pi/L),$$

for $n = 0, 1, 2, \ldots$, whereas the normalized odd eigenfunctions are

$$\psi_{4n+1} = \begin{cases} L^{-1/2}\cos((2n+\nu)(L-x)\pi/L) & \text{if } L/2 < x < L \\ L^{-1/2}\sin((2n+\nu)x\pi/L) & \text{if } -L/2 < x < L/2 \\ -L^{-1/2}\cos((2n+\nu)(L+x)\pi/L) & \text{if } -L < x < -L/2, \end{cases}$$

for $n = 0, 1, 2, \ldots$, and

$$\psi_{4n-1} = \begin{cases} L^{-1/2}\cos((2n-\nu)(L-x)\pi/L) & \text{if } L/2 < x < L \\ -L^{-1/2}\sin((2n-\nu)x\pi/L) & \text{if } -L/2 < x < L/2 \\ -L^{-1/2}\cos((2n-\nu)(L+x)\pi/L) & \text{if } -L < x < -L/2 \end{cases}$$

for $n = 1, 2, 3, \ldots$. Using these eigenvalues and eigenfunctions, we can numerically approximate $w_1(L)$. This is illustrated in Fig. 2(a), where we show numerical plots of $w_1(L)$ as a function of the well depth v_0 for fixed L, α, β. It can be seen that the contribution to the MFPT, $w_1(L)$, due to the switching boundary is a monotonically increasing function of v_0. In Fig. 2(b) we plot the relative error of the eigenfunction approximation of $w_1(L)$ versus the asymptotic formula derived in section 5 in the weak diffusion limit; the error approaches zero as $D \to 0$.

5. MFPT and splitting probability approximations

One of the disadvantages of the Green's function solutions for $w_1(L)$ and $\pi_1(L)$ constructed in section 3 is that there are only a limited number of potentials for which one can can explicitly calculate the eigenfunction expansions of the Green's functions.

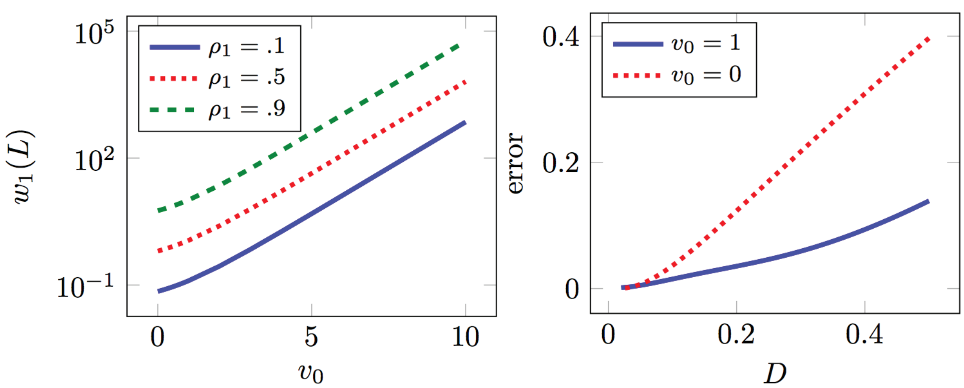

Figure 2. The plot on the left uses equations (3.22) and (3.27) and the eigenexpansion found in section 4 to give $w_1(L)$ for the square-well potential in Fig. 1 as a function of the well depth, v_0. We remark that the behavior of this function is consistent with our result for more regular potentials given in Theorem 2. The plot on the right plots the error of computing $w_1(L)$ by eigenexpansion versus the asymptotic formula in equation (5.10) as a function of the diffusion coefficient. If $\tilde{w}$ denotes $w_1(L)$ as computed by eigenexpansion and w denotes the expression in (5.10), then the error for $v_0 = 1$ is defined as $|\tilde{w}/w - 1|$. As a further check, in the case $v_0 = 0$, the error is $\max\{|\tilde{w}/w - 1|, |\tilde{w}/C - 1|, |C/w - 1|\}$ where C is as in (4.1).

In this section, we consider a method for approximating $w_1(L)$ and $\pi_1(L)$ in the small diffusion limit, which holds for a wide range of potentials, and we determine corrections to the classical Kramers' rate theory in the presence of a switching boundary.

As above, let $n(t) \in \{0, 1\}$ be a Markov jump process with jump rates as in equation (3.2) and suppose $X(t)$ is independent of $n(t)$ and satisfies

$$dX(t) = -V'(X)dt + \sqrt{2D}dW(t).$$

We assume that V is continuously differentiable on $[-L, L]$. Here, $X(0) = L$ and $n(0)$ is distributed according to the invariant measure of $n(t)$. That is, $n(0) = n$ with probability ρ_n with ρ_n defined in equation (3.5).

As in section 3 above, we are interested in the MFPT and splitting probability of the particle in the case that the right boundary is switching. Define the following stopping time§

$$T := \inf\{t \geq 0 : (X(t) = L) \wedge (n(t) = 0)\}.$$

We will use T to find the MFPT and splitting probability in the case that the right boundary is switching. In particular, notice that

$$\mathbb{E}[T] = w(L) = w_0(L) + w_1(L) = w_1(L).$$

But, in order to perform our analysis and to define stopping times that may be larger than T, we will impose reflecting boundary conditions at $x = \pm L$. That is, the particle is always diffusing in an environment with reflecting boundary conditions, but the stopping time T allows us to study the MFPT and splitting probability as if the right boundary was absorbing when $n(t) = 0$. Hence, we will still refer to T as an exit time even though the particle continues to diffuse in the interval after time T.

Let $\delta > 0$ be sufficiently small so that V' doesn't change signs in $[L-\delta, L]$. Define the stopping times

$$\begin{aligned}
\tau \;\; &:= \inf\{t \geq 0 : X(t) = L - \delta\}, \\
T_e &:= \inf\{t \geq 0 : X(\tau + t) = L\}, \\
T_r &:= \inf\{t \geq 0 : (X(\tau + T_e + t) = L) \wedge (n(\tau + T_e + t) = 0)\},
\end{aligned}$$

and the event that the particle reaches $L - \delta$ before time T,

$$A := \{\tau < T\}.$$

We remark that we will use B^c to denote the complement of a set B.

5.1. MFPT approximation

It's immediate that

$$\begin{aligned}
w_1(L) = \mathbb{E}[T] &= \mathbb{E}[T 1_{A^c}] + \mathbb{E}[\tau 1_A] + \mathbb{E}[T_e 1_A] + \mathbb{E}[T_r 1_A] \\
&= \mathbb{E}[T 1_{A^c}] + \mathbb{E}[\tau 1_A] + \mathbb{P}(A)\mathbb{E}[T_e] + \mathbb{E}[T_r 1_A], \qquad (5.1)
\end{aligned}$$

where the equality, $\mathbb{E}[T_e 1_A] = \mathbb{P}(A)\mathbb{E}[T_e]$, follows from the strong Markov property‖. By Lemma 2 in the Appendix,

$$\mathbb{E}[T_r 1_A] = \mathbb{E}[T]\mathbb{P}(N \cap A)/\rho_1,$$

§ A stopping time T is a random variable whose value is interpreted as the time (finite or infinite) at which a given stochastic process is terminated according to some stopping rule that depends on current and past states. A classical example of a stopping time is a first passage time.

‖ Recall that a stochastic process has the Markov property if the conditional probability distribution of future states of the process (conditional on both past and present states) depends only upon the present state, not on the sequence of events that preceded it. The term strong Markov property is similar to the Markov property, except that the "present" is defined in terms of a stopping time.

where N denotes the event that $n(\tau+T_e) = 1$. Plugging this into (5.1) and rearranging yields

$$\mathbb{E}[T] = \frac{\mathbb{E}[T1_{A^c}] + \mathbb{E}[\tau 1_A] + \mathbb{P}(A)\mathbb{E}[T_e]}{1 - \mathbb{P}(N \cap A)/\rho_1}. \tag{5.2}$$

If $T_e \to \infty$ in probability as $D \to 0$, then (5.2) gives the following approximation for $w_1(L)$. Here and throughout, we use the usual asymptotic notation, "$f(D) \sim g(D)$ as $D \to 0$" if $f(D)/g(D) \to 1$ as $D \to 0$.

Theorem 1. *If $T_e \to \infty$ in probability as $D \to 0$, then*

$$w_1(L) \sim \frac{\mathbb{E}[T1_{A^c}] + \mathbb{E}[\tau 1_A]}{1 - \mathbb{P}(A)} + \frac{\mathbb{P}(A)}{1 - \mathbb{P}(A)}\mathbb{E}[T_e] \quad \textit{as } D \to 0. \tag{5.3}$$

Proof. Since $T_e \to \infty$ in probability as $D \to 0$ by assumption, we have by Lemma 3 in the Appendix that

$$|\mathbb{P}(N \cap A) - \rho_1 \mathbb{P}(A)| \to 0 \quad \text{as } D \to 0. \tag{5.4}$$

Equation (5.4) essentially says that the state of the switching boundary at time $\tau + T_e$ is independent of the history before time τ. The desired approximation quickly follows. □

It turns out that the behavior of the approximation in Theorem 1 depends on the sign of V' in the interval $[L-\delta, L]$. The different cases are illustrated in figures 3 and 4.

Case I: $V' = 0$ on $[L-\delta, L]$

In this case, the condition $T_e \to \infty$ in probability as $D \to 0$ holds by Lemma 1 in the Appendix. This result is not surprising since the mean time for a pure Brownian particle to travel from $x = L - \delta$ to $x = L$ for fixed $\delta > 0$ diverges as $D \to 0$.

Hence, the approximation in Theorem 1 holds. Further, all the terms on the right-hand side of (5.3) are known. First, observe that the numerator of the first term is simply the expected exit time for a pure Brownian particle diffusing in the interval $[L-\delta, L]$, starting at $x = L$, with an absorbing boundary at $x = L - \delta$ and a boundary at $x = L$ that switches between reflecting and absorbing. We can find an explicit expression for this quantity by solving the following boundary value problem.

$$Dw_0''(y) - \beta w_0(y) + \alpha w_1(y) = -\rho_0, \tag{5.5a}$$

$$Dw_1''(y) + \beta w_0(y) - \alpha w_1(y) = -\rho_1, \tag{5.5b}$$

with

$$w_0(L) = 0, \quad \partial_y w_1(L) = 0, \quad w_0(L-\delta) = w_1(L-\delta) = 0.$$

The derivation of this boundary value problem is similar to the derivation of equations (3.17a) and (3.17b). Solving this, we obtain the numerator in the first term of equation (5.3)

$$\mathbb{E}[T1_{A^c}] + \mathbb{E}[\tau 1_A] = \frac{\delta^2}{2D} \frac{1}{1 + \xi\delta(\rho_0/\rho_1)\coth(\xi\delta)} \tag{5.6}$$

where $\xi = \sqrt{(\alpha+\beta)/D}$.

Second, observe that $\mathbb{P}(A)$ is the probability that a pure Brownian particle diffusing in the interval $[L-\delta, L]$ exits at $x = L - \delta$ rather than $x = L$, given that it

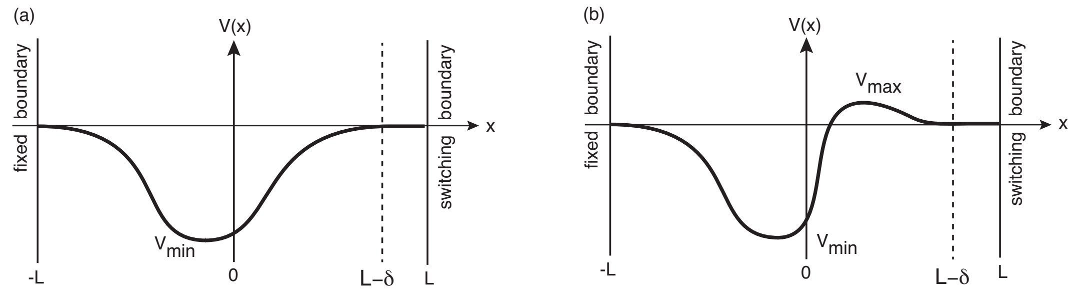

Figure 3. Smooth potential well that is flat in a neighborhood of the switching boundary. (a) $V_{\max} = V(L)$. (b) $V_{\max} > V(L)$.

starts at $x = L$ and the boundary $x = L - \delta$ is absorbing and the boundary $x = L$ switches between reflecting and absorbing. Thus, performing a calculation similar to the one that yielded equation (4.2), we have that

$$\mathbb{P}(A) = 1 - \big(1 + (\rho_1/\rho_0)(\xi\delta)^{-1}\tanh(\xi\delta)\big)^{-1}.$$

Note that we have the simplifications

$$\begin{aligned}\frac{\mathbb{P}(A)}{1-\mathbb{P}(A)} &= (\rho_1/\rho_0)(\xi\delta)^{-1}\tanh(\xi\delta),\\ \frac{\mathbb{E}[T1_{A^c}] + \mathbb{E}[\tau 1_A]}{1-\mathbb{P}(A)} &= \frac{\delta^2}{2D}(\rho_1/\rho_0)(\xi\delta)^{-1}\tanh(\xi\delta).\end{aligned}$$

Finally, $\mathbb{E}[T_e]$ is a classical quantity that we found in equation (2.17). Specifically,

$$\mathbb{E}[T_e] = \frac{1}{D}\int_{L-\delta}^{L}\int_{-L}^{z} \mathrm{e}^{[V(z)-V(x)]/D}dx\,dz. \tag{5.7}$$

Thus, our approximation in equation (5.3) simplifies to

$$\begin{aligned} w_1(L) &\sim (\rho_1/\rho_0)(\xi\delta)^{-1}\tanh(\xi\delta)\Big(\frac{\delta^2}{2D} + \mathbb{E}[T_e]\Big) \qquad (5.8)\\ &= \frac{1}{\sqrt{D}}\frac{\beta/\alpha}{\delta\sqrt{\alpha+\beta}}\tanh(\xi\delta)\Big(\frac{\delta^2}{2} + \int_{L-\delta}^{L}\int_{-L}^{z}\mathrm{e}^{[V(z)-V(x)]/D}dx\,dz\Big). \qquad (5.9)\end{aligned}$$

Since $V'(x) = 0$ for $x \in [L-\delta, L]$, we have that $V(x) = V(L)$ for $x \in [L-\delta, L]$. Hence, our approximation simplifies to

$$w_1(L) \sim \frac{1}{\sqrt{D}}\frac{\beta/\alpha}{\sqrt{\alpha+\beta}}\tanh(\xi\delta)\int_{-L}^{L} e^{[V(L)-V(x)]/D}\,dx, \tag{5.10}$$

as $D \to 0$. Hence, a straightforward application of Laplace's method yields the following result.

Theorem 2. *Assume the potential V is constant in a neighborhood of $x = L$ and assume V is twice differentiable with a unique minimum $x_{\min} \in [-L, L]$ with $V''(x_{\min}) < 0$. Then*

$$w_1(L) \sim \sqrt{2\pi}\frac{\beta/\alpha}{\sqrt{\alpha+\beta}}\frac{e^{[V(L)-V(x_{\min})]/D}}{\sqrt{V''(x_{\min})}}, \quad \textit{as } D \to 0. \tag{5.11}$$

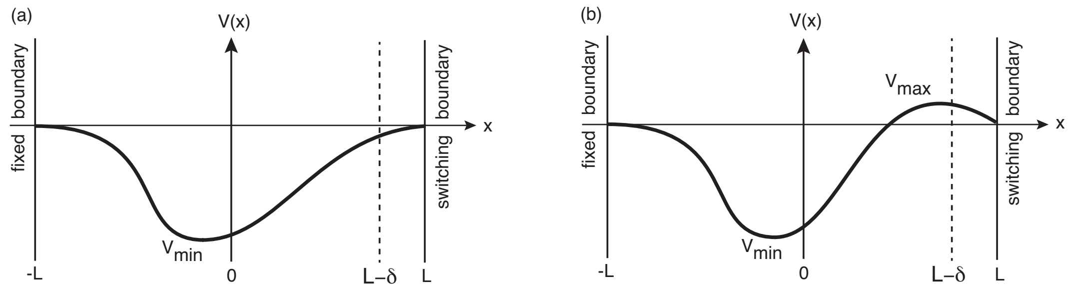

Figure 4. Smooth potential well with (a) $V'(x) > 0$ and (b) $V'(x) < 0$ on $[L, L-\delta]$.

We can now compare the contribution $w_1(L)$ to the MFPT in the case of a non-switching boundary, which is given by equation (2.17), which we will denote by S. Suppose that the particle starts in a neighborhood of the minimum of the potential at $x = x_{\min}$. For small D, we can use Laplace's method to give

$$\begin{aligned} S &= \frac{1}{D}\int_{x_{\min}}^{L}\int_{-L}^{z} \mathrm{e}^{[V(z)-V(x)]/D} dx\, dz \\ &\sim \frac{1}{D}\left[\int_{-L}^{L} \mathrm{e}^{-V(x)/D} dx\right]\left[\int_{-x_{\min}}^{L} \mathrm{e}^{V(z)/D} dz\right] \\ &\sim \sqrt{\frac{2\pi}{V''(x_{\min})}}\mathrm{e}^{-V(x_{\min})/D}\left[\int_{-x_{\min}}^{L} \mathrm{e}^{V(z)/D} dz\right], \quad \text{as } D \to 0. \end{aligned}$$

Evaluation of the final integral depends on whether or not $V(x)$ has a local maximum in the domain $[x_{\min}, L]$ such that $V_{\max} > V(L)$, see Fig. 3. If this is the case then we can apply Laplace's method to obtain the classical Kramers' formula

$$S = 2\pi\sqrt{\frac{1}{V''(x_{\min})|V''(x_{\max})|}}\mathrm{e}^{[V(x_{\max})-V(x_{\min})]/D}. \tag{5.12}$$

In this case, $S \gg w_1(L)$ for small D and the effect of the switching boundary is negligible. On the other hand, if $V(x_{\max}) = V(L)$, then

$$S = \delta\sqrt{\frac{2\pi}{V''(x_{\min})}}\mathrm{e}^{[V(L)-V(x_{\min})]/D}, \tag{5.13}$$

and S is comparable in size to $w_1(L)$.

Case II: $V' > 0$ on $[L-\delta, L]$

In this case, the condition $T_e \to \infty$ in probability as $D \to 0$ holds by Lemma 1. To see intuitively why this must be true, note that $T_e \to \infty$ in probability as $D \to 0$ in the case $V' = 0$ on $[L-\delta, L]$ studied above and T_e can only be larger if $V' > 0$ on $[L-\delta, L]$. Hence, the approximation (5.3) holds.

Theorem 3. *Assume $V' > 0$ on $[L-\delta, L]$. Then $\mathbb{P}(A) \to \rho_1$ as $D \to 0$ and*

$$w_1(L) \sim \frac{\rho_1}{\rho_0}\mathbb{E}[T_e] = \frac{\beta}{\alpha}\mathbb{E}[T_e] \quad \textit{as } D \to 0. \tag{5.14}$$

with $\mathbb{E}[T_e]$ *given by the classical quantity (5.7).*

Before giving the proof of Theorem 3, we first give an intuitive explanation. Since $V' > 0$ on $[L-\delta, L]$ and D is small, if the boundary is initially reflecting, then the drift term will dominate and thus with high probability the particle will hit $L-\delta$ before exiting. Further, once the particle hits $L-\delta$, by the time it reaches L again the state of the boundary will be roughly independent of the particle's last visit to L. Thus, we can think of the time to exit as a series of independent Bernoulli trials with probability of success equal to ρ_0, where we must wait time $\mathbb{E}[T_e]$ between trials. Hence for small D,

$$w_1(L) \approx \rho_0 \mathbb{E}[T_e] \sum_{k=1} k\rho_1^k = \rho_0 \mathbb{E}[T_e] \frac{\rho_1}{(\rho_0)^2} = \frac{\beta}{\alpha}\mathbb{E}[T_e].$$

Proof. Define

$$a := \inf\{V'(x) : x \in [L-\delta, L]\},$$

and note that $a > 0$ since $V' > 0$ on $[L-\delta, L]$ by assumption. Since we are assuming a reflecting boundary condition at $x = L$, there exists a nonnegative, continuous, nondecreasing process $R(t)$ that increases only when $X(t) = L$ such that

$$X(t) = L + \int_0^t (-V'(X(s)))\, ds + \sqrt{2D}W(t) - R(t).$$

Hence,

$$X(t) \leq L - at + \sqrt{2D}W(t). \tag{5.15}$$

For each $k > 0$, define the deterministic time $s_k := 2\delta/(ak)$. For notational ease, define the probability measure $\mathbb{P}_1(B) = \mathbb{P}(B|n(0) = 1)$ for all events B. Then we have that

$$\begin{aligned}\mathbb{P}_1(A^c) = \mathbb{P}_1(T \leq \tau) &= \mathbb{P}_1(T \leq \tau < s_1) + \mathbb{P}_1(\{T \leq \tau\} \cap \{\tau \geq s_1\})\\ &\leq \mathbb{P}_1(T < s_1) + \mathbb{P}(\tau \geq s_1).\end{aligned}$$

Now for each $k > 1$ we have that

$$\begin{aligned}\mathbb{P}_1(T < s_1) &= \mathbb{P}_1(T \in [0, s_k]) + \mathbb{P}_1(T \in (s_k, s_1])\\ &\leq \mathbb{P}_1(n(t) = 0 \text{ for some } t \in [0, s_k])\\ &\quad + \mathbb{P}(L - \delta/k < X(t) \text{ for some } t \in (s_k, s_1]).\end{aligned}$$

By definition of $n(t)$ as a Markov jump process, the first term is simply

$$\mathbb{P}_1(n(t) = 0 \text{ for some } t \in [0, s_k]) = 1 - \exp(-\beta s_k).$$

Thus, we first choose k large to make the first term arbitrarily small. Next, for our fixed large k, we have by equation (5.15) that

$$\begin{aligned}&\mathbb{P}(L - \delta/k < X(t) \text{ for some } t \in (s_k, s_1])\\ &\leq \mathbb{P}(at - \delta/k < \sqrt{2D}W(t) \text{ for some } t \in (s_k, s_1])\\ &\leq \mathbb{P}(\delta/k < \sqrt{2D}W(t) \text{ for some } t \in (s_k, s_1]) \to 0 \quad \text{as } D \to 0.\end{aligned}$$

A similar argument shows that $\mathbb{P}_1(\tau \geq s_1) \to 0$ as $D \to 0$.

Thus, we have shown that $\mathbb{P}_1(A^c) = \mathbb{P}(A^c|\{n(0) = 1\}) \to 0$ as $D \to 0$. Since $\mathbb{P}(A^c|\{n(0) = 0\}) = 1$, it follows that $\mathbb{P}(A) \to \rho_1$ as $D \to 0$. Using Theorem 1, it follows that

$$w_1(L) \sim \frac{\mathbb{E}[T1_{A^c}] + \mathbb{E}[\tau 1_A]}{\rho_0} + \frac{\rho_1}{\rho_0}\mathbb{E}[T_e] \quad \text{as } D \to 0.$$

Now, the numerator of the first term is the expected exit time from the interval $[L-\delta, L]$ starting from L. This is dominated by $\mathbb{E}[T_e]$ for small D, so we obtain equation (5.14). $\square$

<u>*Case III:* $V' < 0$ *on* $[L-\delta, L]$</u>

In this case, we cannot guarantee that $T_e \to \infty$ in probability as $D \to 0$. Thus, we cannot use the approximation in Theorem 1 and we will not derive an exact asymptotic formula for $w_1(L)$ as in cases 1 and 2 above. Nonetheless, the exact equation (5.2) can be used to show that the switching boundary doesn't contribute to the classical Kramers' escape rate from a well in the interior. Using equation (5.2) and Lemma 2 of the appendix, we have the bound

$$\mathbb{E}[T] \leq \frac{1}{1 - \mathbb{P}(A)/\rho_1}\big(\mathbb{E}[T1_{A^c}] + \mathbb{E}[\tau 1_A] + \mathbb{P}(A)\mathbb{E}[T_e]\big). \tag{5.16}$$

Let t_p denote the expected exit time of a pure Brownian particle from the interval $[L-\delta, L]$, starting at L with a reflecting boundary at $L-\delta$ and a switching boundary at L, and let p denote the probability that this pure Brownian particle hits $L-\delta$ before exiting. It's easy to see that

$$\mathbb{P}(A) \leq p < \rho_1 \quad \text{and} \quad \mathbb{E}[T1_{A^c}] \leq t_p.$$

Further, since $\tau \geq 0$, we have that

$$\mathbb{E}[\tau 1_A] \leq \mathbb{E}[\tau].$$

Hence, by equation (5.16), we have the bound

$$\mathbb{E}[T] \leq \frac{1}{1 - p/\rho_1}\big(t_p + \mathbb{E}[\tau] + p\mathbb{E}[T_e]\big). \tag{5.17}$$

Everything on the righthand side of this inequality is known. First, it follows from equations (4.2) and (4.1) that

$$p = 1 - \big(1 + (\rho_1/\rho_0)(\xi\delta)^{-1}\tanh(\xi\delta)\big)^{-1}, \quad \text{and}$$
$$t_p = \frac{2\beta(\delta)\coth\big(2\xi\delta\big)}{\sqrt{D}\alpha\sqrt{\alpha+\beta}},$$

where $\xi = \sqrt{(\alpha+\beta)/D}$. Next, it follows from equation (2.17) that

$$\mathbb{E}[\tau] = \frac{1}{D}\int_{L-\delta}^{L}\int_{z}^{L} \mathrm{e}^{[V(z)-V(x)]/D}dx\,dz.$$

and $\mathbb{E}[T_e]$ is given by equation (5.7).

From this bound, we can show that the switching boundary doesn't contribute to the classical Kramers' escape rate from a well in the interior of the domain. Suppose V has a local minimum at $x_w < L$. Let S be the classical quantity of the MFPT to $x = L$ starting from x_w given in equation (2.17):

$$S = \frac{1}{D}\int_{x_w}^{L}\int_{-L}^{z} \mathrm{e}^{[V(z)-V(x)]/D}dx\,dz. \tag{5.18}$$

Then, we've already shown in equation (3.19) that the expected exit time with a switching boundary starting from x_w is $w(x_w) = S + w_1(L) = S + \mathbb{E}[T]$. We need only show that $\mathbb{E}[T]/S \to 0$ as $D \to 0$. This follows quickly from equation (5.17). First, observe that $p \to 0$ and $t_p/S \to 0$ as $D \to 0$. Next, if we let x_0 be where V achieves a maximum between x_w and L and choose $\delta > 0$ sufficiently small so that $V(L-\delta) - V(L) < V(x_0) - V(x_w)$, then $\mathbb{E}[\tau]/S \to 0$ as $D \to 0$. Finally, since $p \to 0$ as $D \to 0$, we have that $p\mathbb{E}[T_e]/S \to 0$ as $D \to 0$. Thus, we conclude that in this case, the asymptotic expected exit time from a well is unaffected by the switching boundary.

5.2. *Splitting probability approximation*

Similar to what we found above for the MFPT, the behavior of the splitting probability for small D depends on the the sign of V' near $x = L$. Let $\delta > 0$ be sufficiently small so that V' doesn't change signs on $[L - \delta, L]$.

Case A: $V' \leq 0$ on $[L-\delta, L]$

Let q denote the probability that a pure Brownian particle starting at $x = L$ exits at a switching boundary at $x = L$ before hitting $L - \delta$. It's easy to see that

$$\pi_1(L) = \pi(L) - \pi_0(L) = \pi(L) - \rho_0 \geq q - \rho_0.$$

It follows from equation (4.2) that

$$q = \big(1 + (\rho_1/\rho_0)(\xi\delta)^{-1}\tanh(\xi\delta)\big)^{-1}. \tag{5.19}$$

Since $q \to 1$ as $D \to 0$, we have that $\pi_1(L) \to \rho_1$ as $D \to 0$, and thus by equation (3.11),

$$|\pi(y) - h(y)| \to 0 \quad \text{as } D \to 0.$$

Hence, in this case, the switching boundary doesn't affect the splitting probability in the limit that $D \to 0$.

Case B: $V' > 0$ on $[L-\delta, L]$

In this case, the switching boundary does yield a nontrivial contribution to the hitting probability, which is summarized by the following theorem, which follows from Lemma 4 of the appendix:

Theorem 4. *Assume $V' > 0$ on $[L-\delta, L]$. Then*

$$\left|\pi_1(L) - \frac{\rho_0\rho_1 h(L-\delta)}{1 - \rho_1 h(L-\delta)}\right| \to 0 \quad \textit{as } D \to 0.$$

5.3. *Numerical results*

In Fig. 5 we plot the relative errors in the asymptotic formulae for $w_1(L)$ and $\pi_1(L)$ compared to the corresponding expressions obtained by numerically solving the ODEs (3.17*a*), (3.17*b*) and (3.9*a*), (3.9*b*). The particular potentials chosen for the three cases $V' = 0, V' > 0$ and $V' < 0$ on $[L-\delta, L]$ are as follows:

$$V(x) = \begin{cases} -\exp(-1/(1-4x^2)) & \text{if } |x| < 1/2 \\ 0 & \text{if } |x| \geq 1/2 \end{cases},$$

$$V(x) = x^4,$$
$$V(x) = x^6 - 10x^4 + 10x^2.$$

To numerically solve equations (3.17a), (3.17b), (3.9a), and (3.9b), we used the MATLAB built-in function bvp5c. Because the solutions to these equations become singular as $D \to 0$, accurate numerical solution was not available for $D < .01$. Furthermore, Monte Carlo simulations take a prohibitively long amount of time to complete for small D. Nonetheless, it can still be seen from the range of D values considered that in all cases the error vanishes for small D.

For $V' = 0$, the error in the splitting probability vanishes slowly, but this is precisely what our analysis predicts. Since it is shown in section 5.2 that if $V' = 0$ in a neighborhood of $x = L$, then $\pi_1(L) \to \rho_1$ as $D \to 0$, we define the error in the splitting probability for $V' = 0$ as $\rho_1 - \pi_1(L) \geq 0$. In the example of pure brownian motion considered in section 4.1, it follows from equation (4.2) that the error $\rho_1 - \pi_1(L)$ is given by $1 - \big(1 + (\rho_1/\rho_0)(\xi 2L)^{-1}\tanh(\xi 2L)\big)^{-1}$. For small D, this is essentially of the form $1 - 1/(1+\sqrt{D})$. A symmetric potential well in the interior of the domain should not cause the splitting probability to depart much from the case of pure Brownian motion. Thus, we expect the error to be of the form $1 - 1/(1+\sqrt{D})$ for the potential considered in Fig. 5 in the $V' = 0$ case, which is indeed what the graph looks like.

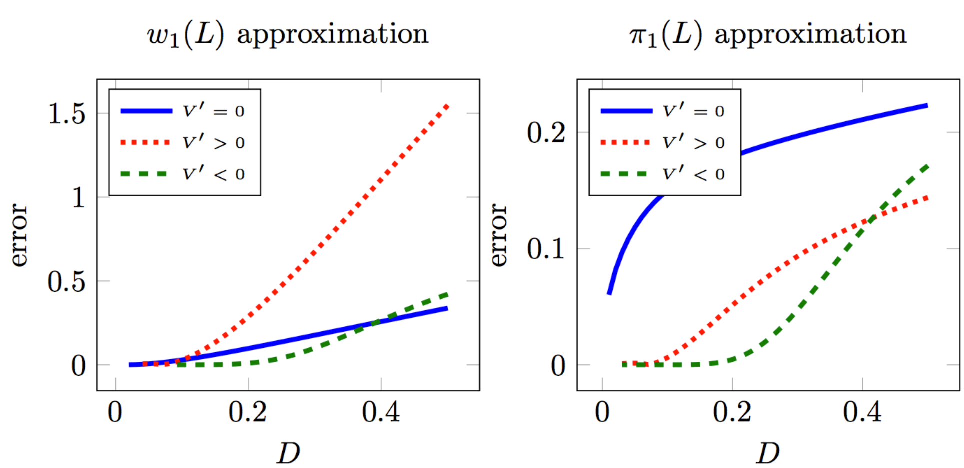

Figure 5. The error for the approximations to $w_1(L)$ and $\pi_1(L)$ as a function of the diffusion coefficient, D, and the sign of V' near the switching boundary. The $V' = 0$ plots correspond to $V(x) = -\exp(-1/(1-4x^2))$ for $|x| < 1/2$ and 0 otherwise. The $V' > 0$ plots correspond to $V(x) = x^4$. The $V' < 0$ plots correspond to $V(x) = x^6 - 10x^4 + 10x^2$. The error is defined differently for the different plots. If $\tilde{w}$ denotes $w_1(L)$ found by solving equations (3.17a) and (3.17b) numerically, then for the $V' = 0$ and $V' > 0$ plots for $w_1(L)$ the error is $|\tilde{w}/w - 1|$ where w is given by equation (5.10) and (5.14), respectively. For the $V' < 0$ plot, the error is defined as $\tilde{w}/S$ where S is given by equation (5.18) to show that the contribution of the switching boundary is negligible. If $\tilde{\pi}$ denotes $\pi_1(L)$ found by solving equations (3.9a) and (3.9b) numerically, then for the $V' = 0$ and $V' < 0$ plots, the error is $|\tilde{\pi} - \rho_1|$. For the $V' > 0$ plot, the error is $|\tilde{\pi} - \pi|$ where π is given by the expression in Theorem 4. In every plot, $L = \alpha = \beta = 1$. For more details on the numerics, see section 5.3.

6. Discussion

In this paper, we considered diffusion in a potential well with a boundary that randomly switches between absorbing and reflecting. We studied the mean first passage time to exit the domain and the probability of exiting out one particular end of the domain. We showed that both of these statistics involve a correction to the classical results obtained in the case of a static boundary. These corrections involve statistics of a particle that starts at the switching boundary in the reflecting state. Using the theory of stochastic hybrid systems, we derived exact formulae for the corrections in terms of the spectral decomposition of the associated Fokker-Planck differential operator. Further, using tools from probability theory, we proved asymptotic formulae for the corrections in the small diffusion limit. These asymptotic formulae show how the classical Kramers' reaction rate theory is affected by a switching boundary.

There are a number of possible extensions of our work. One is to consider two-dimensional or three-dimensional escape problems. Here the boundary of a finite domain is now a closed curve or a closed surface rather than a set of isolated points. One issue would be to specify whether the whole boundary simultaneously switches between an open and a closed state, or different regions of the boundary switch independently. It is likely that the resulting statistics will depend on how the associated potential approaches the switching boundary. Another extension would be to combine the switching boundary problem with a fluctuating potential, which would be relevant to the flow of ions through a gated ion channel.

Appendix

In this appendix, we prove various lemmas used in section 5.

Lemma 1. *Assume $V' \geq 0$ on $[L-\delta, L]$. Then $T_e \to \infty$ in probability as $D \to 0$.*

Proof. Let $Z(t) = \sqrt{2D}W(t)$ and define the stopping times

$$\begin{aligned} \tau_\delta &:= \inf\{t \geq 0 : Z(t) = \delta\} \\ \tau_{-\delta} &:= \inf\{t \geq 0 : Z(t) = -\delta\} \\ S_\delta &:= \inf\{t \geq 0 : |Z(t)| = \delta\} = \min\{\tau_\delta, \tau_{-\delta}\}. \end{aligned}$$

Observe that for each $t > 0$

$$\begin{aligned} &\mathbb{P}(T_e < t) \leq \mathbb{P}(S_\delta < t) \\ &= \mathbb{P}(\{\tau_\delta < t\} \cap \{\tau_\delta < \tau_{-\delta}\}) + \mathbb{P}(\{\tau_{-\delta} < t\} \cap \{\tau_\delta > \tau_{-\delta}\}) \\ &\leq 2\mathbb{P}(\tau_\delta < t). \end{aligned}$$

But, by the reflection principle for Brownian motion, we have that

$$\mathbb{P}(\tau_\delta < t) = 2\mathbb{P}(Z(t) \geq \delta) \to 0 \quad \text{as } D \to 0.$$

$\square$

Lemma 2. *Let N denote the event that $n(\tau + T_e) = 1$. We have that*

$$\mathbb{E}[T_r 1_A] = \mathbb{E}[T]\mathbb{P}(N \cap A)/\rho_1.$$

Proof. Observe that $\mathbb{E}[T_r 1_A] = \mathbb{E}[T_r 1_A 1_N]$ since $T_r = 0$ on N^c. Using the tower property of conditional expectation, we have that

$$\mathbb{E}[T_r 1_A] = \mathbb{E}[T_r 1_A 1_N] = \mathbb{E}\big[1_A 1_N \mathbb{E}[T_r | \sigma(A, N)]\big], \tag{A.1}$$

where $\sigma(X, Y)$ denotes the σ-algebra generated by X and Y. By the strong Markov property, we have that $\mathbb{E}[T_r|\sigma(A,N)] = \mathbb{E}[T_r|N]$ almost surely. Further, by definition, $\mathbb{E}[T_r|N] = \mathbb{E}[T_r 1_N]/\mathbb{P}(N)$ on the event N, so (A.1) becomes

$$\mathbb{E}[T_r 1_A] = \frac{\mathbb{E}[T_r 1_N]}{\mathbb{P}(N)} \mathbb{P}(N \cap A) = \frac{\mathbb{E}[T_r]}{\mathbb{P}(N)} \mathbb{P}(N \cap A).$$

By the strong Markov property, we have that $\mathbb{E}[T_r] = \mathbb{E}[T]$ and $\mathbb{P}(N) = \rho_1$. $\square$

Lemma 3. *Let N denote the event that $n(\tau + T_e) = 1$. If $T_e \to \infty$ in probability as $D \to 0$, then*

$$|\mathbb{P}(N \cap A) - \rho_1 \mathbb{P}(A)| \to 0 \quad \textit{as } D \to 0.$$

Proof. We first show that

$$|\mathbb{P}(\{n(\tau + s) = 1\} \cap A) - \rho_1 \mathbb{P}(A)| \to 0 \quad \text{as } s \to \infty. \tag{A.2}$$

By the tower property of conditional expectation, we have that

$$\mathbb{E}[1_{n(\tau+s)=1} 1_A] = \mathbb{E}\big[1_A \mathbb{E}[1_{n(\tau+s)=1} | \sigma(A, n(\tau))]\big]. \tag{A.3}$$

By the strong Markov property, we have

$$\mathbb{E}[1_{n(\tau+s)=1} | \sigma(A, n(\tau))] = \mathbb{P}(n(\tau + s) = 1 | n(\tau)).$$

But, we know that $\mathbb{P}(n(s) = 1) \to \rho_1$ as $s \to \infty$ for any initial $n(0)$ (see, for example, Theorem 3.4.4 in [9]), so it follows that

$$\mathbb{E}[1_{n(\tau+s)=1} | \sigma(A, n(\tau))] \to \rho_1 \quad \text{almost surely as } s \to \infty.$$

Comparing this with equation (A.3) proves equation (A.2).

Next, let μ denote the distribution of T_e and notice that T_e is independent of τ by the strong Markov property. Thus,

$$\begin{aligned}
&|\mathbb{P}(\{n(\tau + T_e) = 1\} \cap A) - \rho_1 \mathbb{P}(A)| \\
&\leq 2\mathbb{P}(T_e \leq K) + \int_K^\infty \big|\mathbb{P}(\{n(\tau + s) = 1\} \cap A) - \rho_1 \mathbb{P}(A)\big| \, \mu(ds) \\
&\leq 2\mathbb{P}(T_e \leq K) + \sup_{s \geq K} \big|\mathbb{P}(\{n(\tau + s) = 1\} \cap A) - \rho_1 \mathbb{P}(A)\big|.
\end{aligned}$$

To complete the proof, we use equation (A.2) to choose K large to make the second term above arbitrarily small and then take D small to make the first term small since $T_e \to \infty$ in probability as $D \to 0$ by assumption. $\square$

Lemma 4. *Define the event $C = \{n(0) = 1\}$. Similar to Theorem 1, we have the following approximation to $\pi_1(L)$. If $T_e \to \infty$ in probability as $D \to 0$, then as $D \to 0$ we have that*

$$\left|\pi_1(L) - \frac{1}{1 - \mathbb{P}(A)h(L - \delta)} \big(\mathbb{P}(A^c \cap C) + \mathbb{P}(A)h(L - \delta)\rho_0\big)\right| \to 0.$$

Proof. Define the three events

$$\begin{aligned} R &= \Big\{ \inf\{t \geq 0 : (X(t) = L) \wedge (n(t) = 0)\} < \inf\{t \geq 0 : X(t) = -L\}\Big\} \\ H &= \Big\{ \inf\{t \geq 0 : X(\tau + t) = L\} < \inf\{t \geq 0 : X(\tau + t) = -L\}\Big\} \\ R_r &= \Big\{ \inf\{t \geq 0 : (X(\tau + T_e + t) = L) \wedge (n(\tau + T_e + t) = 0)\} \\ &\qquad < \inf\{t \geq 0 : X(\tau + T_e + t) = -L\}\Big\}. \end{aligned}$$

Then

$$\pi_1(L) = \mathbb{P}(R \cap C) = \mathbb{P}(C \cap A^c) + \mathbb{P}(R \cap A).$$

Define the event $N = \{n(\tau + T_e) = 1\}$. By the strong Markov property, we have that

$$\begin{aligned} \mathbb{P}(R \cap A) &= \mathbb{E}[1_R 1_A 1_N] + \mathbb{E}[1_H 1_A 1_{N^c}] \\ &= \mathbb{E}[1_R 1_A 1_N] + \mathbb{P}(H)\mathbb{P}(A \cap N^c). \end{aligned}$$

Again, by the strong Markov property, we have that

$$\mathbb{E}[1_R 1_A 1_N] = \mathbb{E}[1_H 1_{R_r} 1_A 1_N] = \mathbb{P}(H)\mathbb{E}[1_{R_r} 1_A 1_N].$$

Using the tower property of conditional expectation and the strong Markov property, we have that

$$\begin{aligned} \mathbb{E}[1_{R_r} 1_A 1_N] &= \mathbb{E}\big[1_A 1_N \mathbb{E}[1_{R_r} | \sigma(N, A)]\big] = \mathbb{E}\big[1_A 1_N \mathbb{E}[1_{R_r} | N]\big] \\ &= \mathbb{E}[1_A 1_N] \frac{\mathbb{E}[1_{R_r} 1_N]}{\mathbb{P}(N)} = \mathbb{P}(A \cap N)\pi_1(L)/\rho_1. \end{aligned}$$

Putting this all together we have that

$$\pi_1(L) = \mathbb{P}(A^c \cap C) + \mathbb{P}(H)\big(\mathbb{P}(A \cap N^c) + \mathbb{P}(A \cap N)\pi_1(L)/\rho_1\big),$$

and thus

$$\pi_1(L) = \big(1 - \mathbb{P}(H)\mathbb{P}(A \cap N)/\rho_1\big)^{-1}\big(\mathbb{P}(A^c \cap C) + \mathbb{P}(H)\mathbb{P}(A \cap N^c)\big).$$

Since $\mathbb{P}(H) = h(L - \delta)$, with h given by equation (2.16), and $|\mathbb{P}(A \cap N) - \rho_1 \mathbb{P}(A)| \to 0$ as $D \to 0$ by Lemma 3, the proof is complete. □

It is now straightforward to prove Theorem 4 of section 5. By Theorem 3, we have that $\mathbb{P}(A) \to \rho_1$ as $D \to 0$. Hence,

$$\mathbb{P}(A^c \cap C) = \mathbb{P}(A^c) - \mathbb{P}(A^c \cap C^c) = \mathbb{P}(A^c) - \rho_0 \to 0 \quad \text{as } D \to 0,$$

Theorem 4 then follows from Lemma 4.

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