Mean first passage times for piecewise deterministic Markov processes and the effects of critical points

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Abstract.
In this paper, we use probabilistic methods to determine the mean first passage time (MFPT) for a two-state piecewise deterministic Markov process (PDMP), also known as a dichotomous noise process, to escape from a finite interval. In particular, we consider the case where the set of functions generating the piecewise deterministic dynamics have one or more critical points. In order to solve this type of problem, we partition the domain into a set of subintervals that contain no critical points and impose conditions at the critical points separating these regions. Our analysis exploits the fact that a PDMP satisfies the *strong Markov property*. We prove that in the absence of common critical points, the MFPT is finite. Through specific examples, we also explore how the MFPT depends on the number of critical points and prove that the MFPT can be infinite if there are common critical points.

1. Introduction

Piecewise deterministic Markov processes (PDMPs) [9] are stochastic hybrid systems that couple a continuous stochastic process $X(t) \in \Sigma \subset \mathbb{R}$ (or $\mathbb{R}^d$) with a discrete Markov chain $N(t) \in \{0, 1, \ldots, N-1\}$ according to piecewise deterministic dynamics: if $N(t) = n$ then $X(t) = x(t)$ with $\dot{x} = F_n(x)$ and $\{F_n(x), n \in \Gamma\}$ a set of continuous functions. Such processes are finding an increasing number of applications in biophysics, ranging from gene networks to ion channels to motor-driven intracellular transport [3]. One important feature of a PDMP is that the existence and singularity structure of a stationary density depends on the presence or otherwise of critical points of the functions $F_n(x)$, that is points where one or more of the functions vanish. This particular issue has been explored extensively in the case of two-state PDMPs ($N = 2$), which are also known as dichotomous noise processes in the physics literature [2].

In this paper, we use probabilistic methods previously developed for analyzing Brownian motion in switching environments [4, 5, 6, 8], to determine the mean first passage time (MFPT) for a two-state PDMP to escape from a finite interval with one or more critical points. In order to solve this type of problem, we partition the domain into a set of subintervals that contain no critical points and impose conditions at the critical points separating these regions. Our analysis exploits the fact that a PDMP satisfies the *strong Markov property* [10], which allows us to calculate various conditional expectations. Recall that a stochastic process is said to have the Markov property if the conditional probability distribution of future states of the process (conditional on both past and present states) depends only upon the present state, not on the sequence of events that preceded it. Similarly, it has the strong Markov property if the same conditions hold, except that the meaning of "present" is defined in terms of a stopping time. A *stopping time* S for a continuous stochastic process X is a time that depends on the path $\{X(t), t \in \mathbb{R}^+\}$, and is thus a random variable. A defining feature of a stopping time is that one knows at time t whether or not $S \leq t$, that is, knowledge of the sample path $\{X(s), s \leq t\}$ is sufficient to determine whether or not $S \leq t$. It immediately follows that a first passage time is a stopping time. Given any stopping time S with respect to X, if the stochastic process $Y(t) = X(t+S) - X(S)$ is independent of $\{X(s), s < S\}$ then X is said to satisfy the strong Markov property.

The structure of the paper is as follows. In section 2 we describe some of the basic features of two-state PDMPs, including the effect of critical points on the existence of a stationary density. We also derive the MFPT equations in the case of escape from a bounded interval in the absence of critical points. In section 3 we use probabilistic methods based on the strong Markov property and conditional expectations to extend the analysis of MFPTs to the case of one or more critical points. An important step in this analysis is a proof that the MFPTs are finite. Some explicit examples are presented in section 4, which illustrate how the MFPT depends on the number of critical points. We also prove that the MFPT can be infinite if there are common critical points (regardless of initial condition).

2. Two-state PDMP: dichotomous noise processes

Consider a system the states of which are described by a pair $(x, n) \in \Sigma \times \{0, 1\}$, where x is a continuous variable in $\Sigma \subset \mathbb{R}$ and n is a discrete stochastic variable taking values in the finite set $\Gamma \equiv \{0, 1\}$. When the internal state is n, the system evolves according to the ordinary differential equation (ODE)

$$\dot{x} = F_n(x), \tag{2.1}$$

where $F_n : \Sigma \to \mathbb{R}$ is a continuous function. The discrete state $N(t) \in \{0,1\}$ evolves according to a two-state Markov chain with x-independent matrix generator

$$\mathbf{A} = \begin{pmatrix} -\beta & \alpha \\ \beta & -\alpha \end{pmatrix}. \tag{2.2}$$

Let $X(t)$ and $N(t)$ denote the stochastic continuous and discrete variables, respectively, at time $t > 0$, given the initial conditions $X(0) = x_0, N(0) = n_0$. Introduce the probability density $p_n(x,t|x_0,n_0,0)$ with

$$\mathbb{P}\{X(t) \in (x, x+dx),\ N(t) = n|x_0,n_0\} = p_n(x,t|x_0,n_0,0)dx.$$

It follows that p evolves according to the forward differential Chapman-Kolmogorov (CK) equation [12, 3] (commonly termed the master equation)

$$\frac{\partial p_n}{\partial t} = \mathbb{L}p_n, \tag{2.3}$$

with the operator $\mathbb{L}$ (dropping the explicit dependence on initial conditions) defined according to

$$\mathbb{L}p_n(x,t) = -\frac{\partial F_n(x)p_n(x,t)}{\partial x} + \sum_{m=0,1} A_{nm}p_m(x,t). \tag{2.4}$$

The first term on the right-hand side represents the probability flow associated with the piecewise deterministic dynamics for a given n, whereas the second term represents jumps in the discrete state n.

Now define the averaged function $\overline{F} : \mathbb{R} \to \mathbb{R}$ by

$$\overline{F}(x) = \sum_{n=0,1} \rho_n F_n(x),$$

where $\rho_{0,1}$ are the two components of the stationary distribution of the generator $\mathbf{A}$:

$$\rho_0 = \frac{\alpha}{\alpha+\beta}, \quad \rho_1 = \frac{\beta}{\alpha+\beta}. \tag{2.5}$$

That is, $\sum_{m=0,1} A_{nm}(x)\rho_m = 0$ for $n = 0,1$ and fixed x. Intuitively speaking, one would expect the stochastic hybrid system (2.1) to reduce to the deterministic dynamical system

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & \overline{F}(X(t)) \\ X(0) & = & x_0 \end{array} \right. \tag{2.6}$$

in the fast switching limit $\alpha, \beta \to \infty$ for all $x \in \Sigma$. The Markov chain then undergoes many jumps over a small time interval Δt during which $\Delta x \approx 0$, and thus the relative frequency of each discrete state n is approximately ρ_n. This can be made precise in terms of a law of large numbers for PDMPs proven in [14, 11].

It is useful to write the CK equation (2.3) in the component form

$$\frac{\partial p_0}{\partial t} = -\frac{\partial}{\partial x}(F_0(x)p_0(x,t)) - \beta p_0(x,t) + \alpha p_1(x,t) \tag{2.7a}$$

$$\frac{\partial p_1}{\partial t} = -\frac{\partial}{\partial x}(F_1(x)p_1(x,t)) + \beta p_0(x,t) - \alpha p_1(x,t). \tag{2.7b}$$

Adding this pair of equations and setting $p = p_0 + p_1$ yields the conservation equation

$$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial J(x,t)}{\partial x} \tag{2.8}$$

with flux

$$J(x,t) = F_0(x)p_0(x,t) + F_1(x)p_1(x,t). \tag{2.9}$$

2.1. Stationary density

The existence of solutions and the form of the boundary conditions depends crucially on the signs of the functions $F_n(x), n = 0, 1$. In order to illustrate this, consider the bounded interval $\Sigma = [0, L]$. First, suppose that $F_0(x) > 0$ and $F_1(x) < 0$ for all $x \in [0, L]$. We can then impose reflecting boundary conditions at both ends by setting $J(0, t) = J(L, t) = 0$. (Since p_n, $n = 0, 1$, are positive functions, the flux J can only vanish for non-zero densities if the functions F_n have opposite signs at the boundaries.) In this case, a unique stationary solution can be constructed as follows. Setting time derivatives to zero and adding the pair of equations (2.7a) and (2.7b) yields

$$\frac{\partial}{\partial x}(F_0(x)p_0(x)) + \frac{\partial}{\partial x}(F_1(x)p_1(x)) = 0, \tag{2.10}$$

that is,

$$F_0(x)p_0(x) + F_1(x)p_1(x) = c,$$

for some constant c. The reflecting boundary conditions imply that $c = 0$. Since $F_n(x)$ is non-zero for all $x \in \Sigma$, we can express $p_1(x)$ in terms of $p_0(x)$:

$$p_1(x) = -\frac{F_0(x)p_0(x)}{F_1(x)}. \tag{2.11}$$

Substituting into Eq. (2.7a) gives

$$0 = \frac{\partial}{\partial x}(F_0(x)p_0(x)) + \left(\frac{\alpha}{F_1(x)} + \frac{\beta}{F_0(x)}\right) F_0(x)p_0(x). \tag{2.12}$$

This yields the solutions

$$p_n(x) = \frac{1}{Z|F_n(x)|} \exp\left(-\int_0^x \left(\frac{\alpha}{F_1(y)} + \frac{\beta}{F_0(y)}\right) dy\right),$$

with the constant Z obtained by imposing the normalization $\int_0^L p(x)dx = 1$, assuming the latter is finite. (A similar analysis would also hold if the rates α, β were x-dependent.)

The analysis is more subtle if the functions have critical points within the domain $[0, L]$, as highlighted in the review by Bena [2]. First, suppose that $F_n(x)$ has a stable

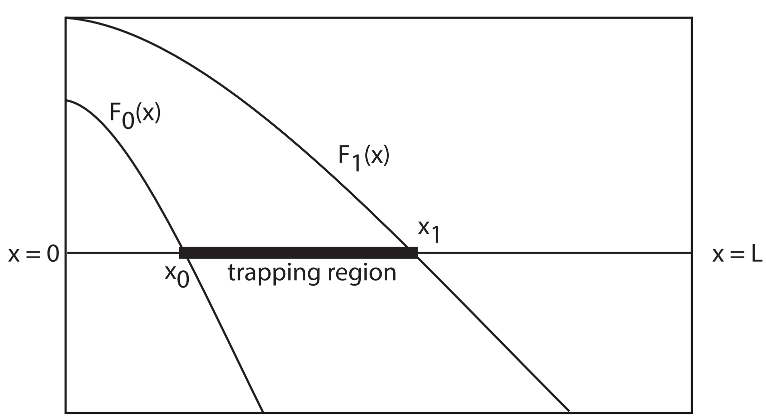

Figure 1: Trapping region of a one-dimensional PDMP when $F_n(x)$ has a single critical point at $x = x_n$ with $F'(x_n) < 0$.

critical point at $x_n \in [0, L]$ with $F'(x_n) < 0$, $n = 0, 1$, and $x_0 < x_1$, see Fig. 1. It follows that $F_0(x) < 0$ and $F_1(x) > 0$ in the open interval (x_0, x_1), which acts as a trapping region. One can extend the above construction of the steady-state density by restricting x to the sub-interval (x_0, x_1) and imposing reflecting boundary conditions at $x = x_0, x_1$. One finds that there are removable singularities in the densities at the boundaries. Note that the function $F_n(x)$ cannot have more than one stable critical point, since this would mean that it also has at least one unstable critical point x_c for which $F_n'(x_c) > 0$. It is then no longer possible to construct a stationary density on the interval $[0, L]$. However, one can consider two-state PDMPs with unstable critical points by taking periodic boundary conditions. One then finds that for certain periodic dichotomous flows $F_{0,1}(x)$, the asymptotic state is characterized by a nonzero stationary flux through the system [2].

2.2. First passage times in the absence of critical points

Now suppose that there is a reflecting boundary condition at $x = 0$ and an absorbing boundary condition at $x = L$:

$$J(0, t) = 0, \quad p_0(L, t) = 0, \tag{2.13}$$

with $J(x, t)$ given by equation (2.9). For the moment, assume that $F_0(x) < 0$ and $F_1(x) > 0$ for all $x \in [0, L]$ so there are no critical points. (These conditions will be relaxed in sections 3 and 4.) A natural quantity of interest for the associated PDMP is the first passage time, defined according to

$$T = \inf\{t > 0 : X(t) > L\}. \tag{2.14}$$

Let $\tau_m(x)$ be the mean first passage time (MFPT) to be absorbed at L given given $X(0) = x$ and $N(0) = m$:

$$\tau_m(x) = \mathbb{E}[T|X(0) = x, N(0) = m] = \mathbb{E}_{x,m}[T], \tag{2.15}$$

where T is the FPT (2.14) and $\mathbb{E}_{x,m}$ denotes expectation conditioned on $X(0) = x, N(0) = m$. (We now use x' to denote the final position $X(t)$ and x to denote the initial position $X(0)$.) There are two alternative methods for calculating the MFPT, one based on Laplace transforming the forward CK equation (2.3), and the other based on solving the corresponding backward equation. We will follow the latter here.

Setting $q_m(x, t) = \sum_{n=0,1} p_n(x', t|x, m, 0)$, the backward CK equation takes the form

$$\frac{\partial q_0(x, t)}{\partial t} = F_0(x)\frac{\partial}{\partial x}q_0(x, t) - \beta q_0(x, t) + \beta q_1(x, t) \tag{2.16a}$$

$$\frac{\partial q_1(x, t)}{\partial t} = F_1(x)\frac{\partial}{\partial x}q_1(x, t) + \alpha q_0(x, t) - \alpha q_1(x, t), \quad x \in (0, L). \tag{2.16b}$$

The associated boundary conditions are

$$q_0(0, t) = q_1(0, t), \quad q_1(L, t) = 0.$$

Let $\mathcal{P}_m(x, t)$ be the survival probability density that the particle hasn't yet been absorbed at $x' = 0$ up to time t, given that it started at x in state m. That is,

$$\mathcal{P}_m(x, t) = \int_0^L \sum_{n=0,1} p_n(x', t, 0|x, m, 0)dx'. \tag{2.17}$$

Differentiating with respect to t, and using the backward CK equations (2.16a) and (2.16b) yields

$$\frac{\partial \mathcal{P}_0(x,t)}{\partial t} = F_0(x)\frac{\partial}{\partial x}\mathcal{P}_0(x,t) - \beta\mathcal{P}_0(x,t) + \beta\mathcal{P}_1(x,t) \tag{2.18a}$$

$$\frac{\partial \mathcal{P}_1(x,t)}{\partial t} = F_1(x)\frac{\partial}{\partial x}\mathcal{P}_1(x,t) + \alpha\mathcal{P}_0(x,t) - \alpha\mathcal{P}_1(x,t), \quad y \in (0,L). \tag{2.18b}$$

The MFPT is related to the survival probability density according to

$$\tau_m(x) = -\int_0^\infty t\partial_t\mathcal{P}_m(x,t)dt = \int_0^\infty \mathcal{P}_m(x,t)dt \tag{2.19}$$

after integration by parts. It follows that τ_m evolves according to the equations

$$-1 = F_0(x)\frac{\partial}{\partial x}\tau_0(y) - \beta\tau_0(x) + \beta\tau_1(x) \tag{2.20a}$$

$$-1 = F_1(x)\frac{\partial}{\partial x}\tau_1(y) + \alpha\tau_0(x) - \alpha\tau_1(x), \quad x \in (0,L), \tag{2.20b}$$

supplemented by the boundary conditions

$$\tau_0(0) = \tau_1(0), \quad \tau_1(L) = 0.$$

Now suppose that there exists an absorbing boundary at both ends, and introduce the FPT

$$\widehat{T} = \inf\{t > 0 : X(t) \notin (0,L)\}. \tag{2.21}$$

The corresponding MFPT, $\widehat{\tau}_n(x) = \mathbb{E}_{x,n}[\widehat{T}]$, still satisfies equations of the form (2.20a) and (2.20b), except the boundary conditions become

$$\widehat{\tau}_0(0) = 0, \quad \widehat{\tau}_1(L) = 0.$$

The splitting probability for which side of $(0,L)$ that $X(t)$ exits can also be determined. In particular, using the backward CK equation one can show that the splitting probability to exit at 0,

$$\pi_n(x) = \mathbb{P}_{x,n}[X(\widehat{T}) = 0],$$

satisfies

$$0 = F_0(x)\frac{\partial}{\partial x}\pi_0(x) - \beta\pi_0(x) + \beta\pi_1(x) \tag{2.22a}$$

$$0 = F_1(x)\frac{\partial}{\partial x}\pi_1(x) + \alpha\pi_0(x) - \alpha\pi_1(x), \quad x \in (0,L), \tag{2.22b}$$

supplemented by the boundary conditions

$$\pi_0(0) = 1, \quad \pi_1(L) = 0.$$

3. MFPT for a two-state PDMP with critical points on an interval

Suppose that we now allow for $F_0(x)$ and $F_1(x)$ to have critical points in $[0,L]$. We first consider the case that both $F_0(0) \geq 0$ and $F_1(0) \geq 0$ so that $X(t)$ can only escape through $x = L$. (We will consider the general case in which escape is possible through both $x = 0$ and $x = L$ in section 3.3 below.) In order to ensure that $X(t)$ eventually reaches $x = L$ we make the following assumption.

Assumption 1. There are no points $x_1 \leq x_2 \in [0,L]$ such that

$$F_0(x_1), F_1(x_1) \geq 0, \quad F_0(x_2), F_1(x_2) \leq 0.$$

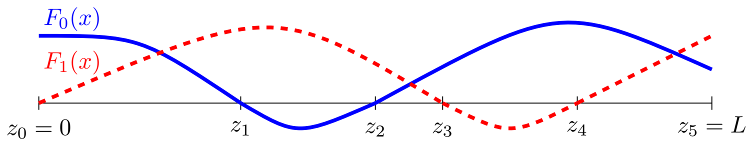

Figure 2: Example illustrating labeling of boundaries and critical points.

If Assumption 1 is violated, then $X(t)$ will never reach L if $x_1 \le X(0) \le x_2$ (nor 0 in the case of escape from either end). We note that this assumption excludes the possibility of a common critical point, $z \in [0, L]$ such that $F_0(z) = F_1(z) = 0$. Of course, if a common critical point $z \in [0, L]$ exists and $X(0) = z$, then $X(t) = z$ for all $t \ge 0$.

3.1. Finite MFPT

Before calculating the MFPT, we first prove in this section that it is finite under Assumption 1. This preliminary step is more than a technicality. Indeed, we later demonstrate in Example 4.2 that the MFPT can be infinite if Assumption 1 is violated, even if the FPT is finite with positive probability. That is, an infinite MFPT can occur in situations other than the trivial case where the particle is confined to a trapping region. Furthermore, when we calculate the MFPT in section 3.2 below, we will derive formulas that are not a priori clear to be nonsingular. Hence, our work in this section relieves us from the need to check that these general formulas are devoid of singularities (though we check this for the examples in section 4).

In addition to Assumption 1, we assume that $F_0(x)$ and $F_1(x)$ are continuous and have finitely many critical points in $[0, L]$. That is, if

$$Z := \big\{z \in [0, L] : \{F_0(z) = 0\} \cup \{F_1(z) = 0\}\big\}, \tag{3.1}$$

then $|Z| < \infty$. Assigning labels to the critical points and $\{0, L\}$, we let

$$0 = z_0 < z_1 < \ldots < z_{K-1} < z_K = L \tag{3.2}$$

be such that $\cup_{k=0}^{K}\{z_k\} = Z \cup \{0, L\}$, see Fig. 2

Let T denote the first time $X(t)$ is larger that $x = L$, see equation (2.14). The following theorem asserts that T has finite expectation.

Theorem 1. *Suppose Assumption 1 holds and* $F_0(0) \ge 0$ *and* $F_1(0) \ge 0$. *If* $x \in [0, L]$ *and* $n \in \{0, 1\}$, *then* $\mathbb{E}_{x,n}[T] < \infty$.

Proof. Fix $n \in \{0, 1\}$. It is immediate that for any $x \in [0, L]$,

$$\mathbb{E}_{x,n}[T] \le \mathbb{E}_{0,0}[T] + \mathbb{E}_{0,1}[T], \tag{3.3}$$

so it suffices to prove $\mathbb{E}_{0,n}[T] < \infty$. Define the first time $X(t)$ reaches $x = z_k$,

$$t_k := \inf\{t > 0 : X(t) > z_k\}, \quad k \in \{0, \ldots, K\}. \tag{3.4}$$

Observe that the time to reach $x = z_2$ starting from $x = 0$ is simply the time to reach $x = z_1$ starting from $x = 0$ plus the time to reach $x = z_2$ starting from $x = z_1$. More

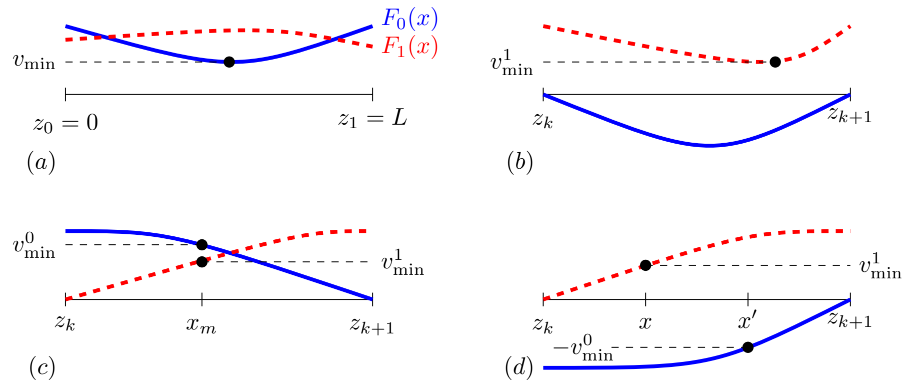

Figure 3: Some possible cases for signs of $F_0(x)$ (blue, solid) and $F_1(x)$ (red, dashed) on (z_k, z_{k+1}).

precisely,

$$\begin{aligned}\mathbb{E}_{0,n}[t_2] &= \mathbb{E}_{0,n}[t_1] + \mathbb{E}_{0,n}\big[\mathbb{E}_{0,n}[t_2 - t_1|\mathcal{F}(t_1)]\big] \\ &= \mathbb{E}_{0,n}[t_1] + \mathbb{E}_{0,n}\big[\mathbb{E}_{X(t_1),N(t_1)}[t_2]\big] \\ &\leq \mathbb{E}_{0,n}[t_1] + \mathbb{E}_{z_1,0}[t_2] + \mathbb{E}_{z_1,1}[t_2]. \end{aligned} \tag{3.5}$$

The first line of equation (3.5) follows from the definition of conditional expectation, the second line uses the strong Markov property and the fact that t_1 is finite almost surely by Lemma 1 below, and the last line holds since $X(t_1) = z_1$ almost surely, $N(t_1) \in \{0, 1\}$ almost surely, and $t_2 \geq 0$ almost surely. Combining the upper bound in (3.5) with Lemma 1 yields that

$$\mathbb{E}_{0,n}[t_2] < \infty. \tag{3.6}$$

Having established (3.6), one can then use the same argument as in equation (3.5) to prove that $\mathbb{E}_{0,n}[t_3] < \infty$. Continuing in this manner, it follows that $E_{0,n}[t_K] < \infty$. Since $T = t_K$ almost surely, the proof is complete. □

Lemma 1. *If $k \in \{0, \dots, K-1\}$ and $n \in \{0, 1\}$, then $E_{z_k,n}[t_{k+1}] < \infty$.*

Proof. The proof proceeds by considering the various possible cases for the signs and critical points of F_0, F_1 on $[z_k, z_{k+1}]$. By Assumption 1, there are three possible cases. *Case (a).* First, suppose that F_0, F_1 are both nonzero on $[z_k, z_{k+1}]$, which is only possible if $[z_k, z_{k+1}] = [0, L]$ and $F_0 > 0, F_1 > 0$ on $[0, L]$. Observe that the minimum velocity of $X(t)$,

$$v_{\min} := \inf_{x\in[0,L],n\in\{0,1\}} F_n(x) > 0,$$

is strictly positive (see Fig. 3(a)). Thus, $X(t)$ must reach $x = L$ if $N(t)$ ever has a holding time larger than $L/v_{\min}$. Defining the fastest possible switching rate

$$\lambda_{\max} := \max\{\alpha, \beta\} < \infty,$$

it follows that the probability that a given holding time of $N(t)$ is larger than $L/v_{\min}$ is bounded below by

$$p := \exp(-\lambda_{\max} L/v_{\min}) > 0.$$

Thus, the expected number of times that $N(t)$ switches before $X(t)$ reaches $x = L$ is bounded above by $1+1/p$, which is one plus the expected value of a geometric random variable with parameter p. Defining the slowest possible switching rate,

$$\lambda_{\min} := \min\{\alpha, \beta\} > 0,$$

it follows that

$$\mathbb{E}_{z_k,n}[t_{k+1}] < (1+1/p)/(p\lambda_{\min}) < \infty.$$

Case (b). Next, suppose that exactly one of the functions, F_0, F_1 has either one or two critical points on $[z_k, z_{k+1}]$. Without loss of generality, suppose that $F_0(z_k)F_0(z_{k+1}) = 0$. In this case, we must have that $F_1 > 0$ on $[z_k, z_{k+1}]$ by Assumption 1. Observe that the minimum velocity of $X(t)$ when $N(t) = 1$,

$$v^1_{\min} := \inf_{x\in[z_k,z_{k+1}]} F_1(x) > 0,$$

is strictly positive (see Fig. 3(b)). Thus, $X(t)$ must reach z_{k+1} if $N(t)$ is ever in state 1 for longer than time $L/v^1_{\min}$. Proceeding along similar lines as in Case (a) above, it follows that $\mathbb{E}_{z_k,n}[t_{k+1}] < \infty$.

Case (c). Finally, suppose that each of the functions F_0, F_1 has exactly one critical point on $[z_k, z_{k+1}]$. By Assumption 1, we must have that $F_0 > 0, F_1 > 0$ on (z_k, z_{k+1}). Without loss of generality, suppose $F_1(z_k) = F_0(z_{k+1}) = 0$. Let $x_m \in (z_k, z_{k+1})$ and define the first time $X(t)$ reaches $x = x_m$,

$$s_{x_m} := \inf\{t > 0 : X(t) = x_m\}.$$

Observe that the minimum velocity of $X(t)$ when $X(t) \in [z_k, x_m]$ and $N(t) = 0$,

$$\inf_{x\in[z_k,x_m]} F_0(x) > 0,$$

is strictly positive (see Fig. 3(c)). Hence, proceeding along similar lines as in Case (a) above, it follows that $\mathbb{E}_{z_k,n}[s_{x_m}] < \infty$. Thus, using the strong Markov property in an argument analogous to that in (3.5), we have

$$\mathbb{E}_{z_k,n}[t_{k+1}] \le \mathbb{E}_{z_k,n}[s_{x_m}] + \mathbb{E}_{x_m,0}[t_{k+1}] + \mathbb{E}_{x_m,1}[t_{k+1}]. \tag{3.7}$$

Next, observe that the minimum velocity of $X(t)$ when $X(t) \in [x_m, z_{k+1}]$ and $N(t) = 1$,

$$\inf_{x\in[x_m,z_{k+1}]} F_1(x) > 0,$$

is strictly positive. Hence, proceeding along similar lines as in Case (a) above, it follows that $\mathbb{E}_{x_m,n'}[t_{k+1}] < \infty$ for $n' \in \{0, 1\}$. By (3.7), the proof is complete.

□

3.2. Calculating the MFPT

Given the FPT T to escape $[0, L]$, see equation (2.14), we define the MFPT conditioned on $X(0) = x$ and $N(0) = n$

$$\tau_n(x) := \mathbb{E}_{x,n}[T].$$

Theorem 1 ensures that $\tau_n(x) < \infty$ for all $x \in [0, L]$ and $n \in \{0, 1\}$. If $\rho_n = \mathbb{P}[N(0) = n]$, then the MFPT conditioned on $X(0) = x$ is given by

$$\tau(x) = \rho_0\tau_0(x) + \rho_1\tau_1(x). \tag{3.8}$$

Away from critical points of F_0, F_1, the MFPT $\tau_n(x)$ satisfies equations (2.20*a*) and (2.20*b*). It is convenient to work with the following sums and differences

$$S := \tau_0 + \tau_1, \quad \Delta := \tau_0 - \tau_1.$$

In these variables, equations (2.20*a*) and (2.20*b*) become

$$\frac{d}{dx}\Delta - \Gamma_+(x)\Delta = -\gamma_-(x), \tag{3.9a}$$

$$\frac{d}{dx}S - \Gamma_-(x)\Delta = -\gamma_+(x), \quad x \in \cup_{k=0}^{K-1}(z_k, z_{k+1}), \tag{3.9b}$$

where

$$\Gamma_\pm(x) := \frac{\beta F_1(x) \pm \alpha F_0(x)}{F_1(x)F_0(x)}, \quad \gamma_\pm(x) := \frac{F_1(x) \pm F_0(x)}{F_1(x)F_0(x)}.$$

Solving equations (3.9*a*) and (3.9*b*) on each subinterval, (z_k, z_{k+1}), we have that

$$\Delta(x) = \Delta(\overline{z}_k)\psi_\Delta(x) + \eta_\Delta(x), \tag{3.10a}$$

$$S(x) = \Delta(\overline{z}_k)\psi_S(x) + \eta_S(x) + S(\overline{z}_k), \tag{3.10b}$$

where

$$\overline{z}_k := \frac{z_k + z_{k+1}}{2},$$

and for $x \in (z_k, z_{k+1})$ we define

$$\psi_\Delta(x) := \exp\Big(\int_{\overline{z}_k}^{x} \Gamma_+(x')\,dx'\Big), \tag{3.11}$$

$$\eta_\Delta(x) := -\int_{\overline{z}_k}^{x} \gamma_-(x')\exp\Big(\int_{x'}^{x} \Gamma_+(x'')\,dx''\Big)\,dx', \tag{3.12}$$

$$\psi_S(x) := \int_{\overline{z}_k}^{x} \Gamma_-(x')\psi_\Delta(x')\,dx', \tag{3.13}$$

$$\eta_S(x) := \int_{\overline{z}_k}^{x} \big(\Gamma_-(x')\eta_\Delta(x') - \gamma_+(x')\big)\,dx'. \tag{3.14}$$

We have chosen $\overline{z}_k$ as the lower limit of integration in (3.11)-(3.14) so that $\psi_\Delta, \eta_\Delta, \psi_S, \eta_S$ are well-defined away from the set of critical points, $Z \subset [0, L]$. It thus remains to determine the constants $\Delta(\overline{z}_k)$ and $S(\overline{z}_k)$ in (3.10*a*)-(3.10*b*) and to determine the values of Δ and S on Z.

The constants are determined by imposing conditions at each $x = z_k$. Fix $n \in \{0, 1\}$. By Assumption 1, there are three cases.

Case 1. If $F_n(z_{k+1}) > 0$, then $X(t)$ will immediately pass $x = z_{k+1}$ if $X(0) = z_{k+1}-$ and $N(0) = n$. Thus, we have the continuity condition

$$\tau_n(z_{k+1}-) = \tau_n(z_{k+1}+), \tag{3.15}$$

where $f(z\pm)$ denotes the limit from the right or left, $\lim_{x\to z\pm} f(x)$.
Case 2. Instead, if $F_n(z_{k+1}) = 0$ and $F_n > 0$ on (z_k, z_{k+1}), and if $X(0) = z_{k+1}-$ and $N(0) = n$, then $X(t)$ will be stationary until $N(t)$ switches. Since $N(t)$ switches at a time that is exponentially distributed with rate α if $n = 1$ and β if $n = 0$, we thus obtain the condition

$$\tau_n(z_{k+1}-) = n\alpha^{-1} + (1-n)\beta^{-1} + \tau_{1-n}(z_{k+1}-). \tag{3.16}$$

Case 3. Finally, if $F_n(z_{k+1}) = 0$ and $F_n < 0$ on (z_k, z_{k+1}), then Assumption 1 ensures that $F_n(z_k) = 0$. Hence, if $X(0) = z_k+$ and $N(0) = n$ then $X(t)$ will be stationary until $N(t)$ switches. We thus obtain the condition

$$\tau_n(z_k+) = n\alpha^{-1} + (1-n)\beta^{-1} + \tau_{1-n}(z_k+). \tag{3.17}$$

It is clear that for fixed $n \in \{0, 1\}$, each of the K points, $\{z_{k+1}\}_{k=0}^{K-1}$, falls into exactly one of these 3 cases. Hence, we obtain $2K$ conditions to determine the $2K$ constants, $\{\Delta(\overline{z}_k), S(\overline{z}_k)\}_{k=0}^{K-1}$.

We now determine values of Δ and S at critical points. By the reasoning of Case 1 above, if $F_n(z_k) \neq 0$, then τ_n is continuous at z_k,

$$\tau_n(z_k) = \tau_n(z_k-) = \tau_n(z_k+).$$

Further, by the reasoning of Case 2 above, if $F_n(z_k) = 0$, then

$$\tau_n(z_k) = n\alpha^{-1} + (1-n)\beta^{-1} + \tau_{1-n}(z_k).$$

3.3. Escape through $x = 0$ or $x = L$

We now relax the assumption that $F_0(0) \geq 0$ and $F_1(0) \geq 0$ so that $X(t)$ may now escape $[0, L]$ through either $x = 0$ or $x = L$. In making this generalization, the first thing to notice is that $X(t)$ can only pass through a given critical point of F_n in one direction, see Fig. 4. That is, recall (3.2) and partition the critical points into the following two sets,

$$Z_l := \big\{k : \{F_0(z_k) + F_1(z_k) < 0\} \cap \{F_0(z_k)F_1(z_k) = 0\}\big\},$$
$$Z_r := \big\{k : \{F_0(z_k) + F_1(z_k) > 0\} \cap \{F_0(z_k)F_1(z_k) = 0\}\big\}.$$

Then, if $k \in Z_l$ and $X(0) \leq z_k$, then $X(t) \leq z_k$ for all $t \geq 0$. Similarly, if $k \in Z_r$ and $X(0) \geq z_k$, then $X(t) \geq z_k$ for all $t \geq 0$. Therefore, Assumption 1 implies that if $k_l \in Z_l$ and $k_r \in Z_r$, then $k_l < k_r$. Thus, defining the index

$$K' := \max\big\{k : \{k \in Z_l\} \cup \{-1\}\big\} \in \{-1, 0, \dots, K\},$$

we have that

$$\begin{aligned} F_0(z_k) + F_1(z_k) &< 0, \quad \text{if } k \leq K' \quad \text{and} \quad F_0(z_k)F_1(z_k) = 0,\\ F_0(z_k) + F_1(z_k) &> 0, \quad \text{if } k > K' \quad \text{and} \quad F_0(z_k)F_1(z_k) = 0. \end{aligned}$$

Therefore, if $X(0) \geq z_{K'+1}$, then $X(t)$ can only exit through $x = L$ and the problem reduces to that considered in section 3.2. Similarly, if $X(0) \leq z_{K'}$, then $X(t)$ can only exit through $x = 0$ and the problem reduces to that considered in section 3.2.

Thus, we need only to consider the MFPT to escape $[0, L]$ when $X(0) \in (z_{K'}, z_{K'+1})$ with $K' \in \{0, 1, \dots, K-1\}$. Before computing this MFPT, we first prove that it is finite.

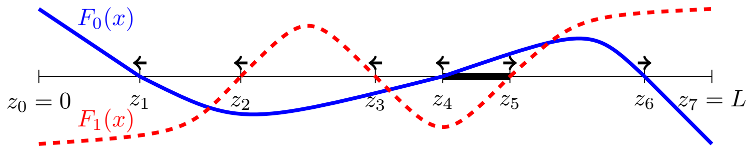

Figure 4: In this example, if $X(0) \leq z_4$, then $X(t)$ will exit through $x = 0$. Similarly, if $X(0) \geq z_5$, then $X(t)$ will exit through $x = L$. Hence, $K' = 4$ and if $X(0) \notin (z_4, z_5)$, then the problem reduces to that considered in section 3.2.

Theorem 2. *Suppose Assumption 1 holds. If $x \in [0, L]$ and $n \in \{0, 1\}$, then $\mathbb{E}_{x,n}[T] < \infty$.*

Proof. As described above, we need only to consider the case that $X(0) = x \in (z_{K'}, z_{K'+1})$ with $K' \in \{0, 1, \ldots, K-1\}$. Define the first time $X(t)$ escapes $(z_{K'}, z_{K'+1})$,

$$\widehat{T} := \inf\{t > 0 : X(t) \notin (z_{K'}, z_{K'+1})\}. \tag{3.18}$$

If $\mathbb{E}_{x,n}[\widehat{T}] < \infty$, then by the strong Markov property and Theorem 1,

$$\mathbb{E}_{x,n}[T] \leq \mathbb{E}_{x,n}[\widehat{T}] + \sum_{y=z_{K'}, z_{K'+1}} \sum_{n=0,1} \mathbb{E}_{y,n}[T] < \infty.$$

Hence, it remains only to show that $\mathbb{E}_{x,n}[\widehat{T}] < \infty$

Without loss of generality, suppose that $F_1(z_{K'}) = F_0(z_{K'+1}) = 0$. The main difficulty in the proof is that we cannot bound $|F_0|$ or $|F_1|$ away from zero on $(z_{K'}, z_{K'+1})$. To ameliorate this problem, fix $x < x'$ satisfying

$$z_{K'} < x < x' < z_{K'+1}.$$

Let $s_0 = 0$ and define the sequence of stopping times

$$s_m := \min\Big\{\widehat{T}, \inf\big\{t > s_{m-1} : \{X(t) \in \mathcal{Y}\} \cap \big\{X(t) \neq X(s_{m-1})\big\}\big\}\Big\},$$

where

$$\mathcal{Y} := \{z_{K'}, x, x', z_{K'+1}\}.$$

Observe that $\{s_m\}_{m=1}^{\infty}$ is the sequence of times before $\widehat{T}$ in which $X(t)$ hits $z_{K'}$, x, x', or $z_{K'+1}$.

Observe that the minimum speed of $X(t)$ when $X(t) \in [z_{K'}, x']$ and $N(t) = 0$,

$$\inf_{x \in [z_{K'}, x']} |F_0(x)| > 0,$$

is strictly positive (see Fig. 3(d)). Hence, proceeding along the same lines as in the proof of Theorem 1, it follows that

$$\mathbb{E}_{x,n}[s_1] < \infty.$$

Continuing in this manner, it follows that

$$\mathbb{E}_{x,n}[s_m] < \infty, \quad m \geq 0.$$

Now, define a discrete time Markov chain, $\{(Y_m, J_m)\}_{m=0}^{\infty}$, by

$$(Y_m, J_m) = (X(s_m), n(s_m)), \quad m \geq 0,$$

on the finite state space $\mathcal{Y} \times \{0,1\}$. The fact that $(X(t), N(t))$ is a strong Markov process ensures that (Y_m, J_m) is indeed a Markov chain. By the assumptions on the signs of F_0, F_1 on $[z_k, z_{k+1}]$, it is immediate that the only absorbing states for (Y_m, J_m) are $(z_{K'}, 0)$ and $(z_{K'+1}, 1)$. Let M be the number of discrete time steps that (Y_m, J_m) takes before reaching one of these absorbing states. Since (Y_m, J_m) has a finite state space, M has finite expectation and is therefore finite almost surely. Hence,

$$\widehat{T} = s_M = \sum_{m=1}^{M} (s_m - s_{m-1}) \quad \text{almost surely.}$$

Taking expectation yields

$$\mathbb{E}_{x,n}[\widehat{T}] = \sum_{i=0}^{\infty} \sum_{m=1}^{i} \mathbb{E}_{x,n}[(s_m - s_{m-1}) 1_{M=i}]. \tag{3.19}$$

Since $s_m - s_{m-1}$ has finite expectation, we have that

$$\begin{aligned} &\mathbb{E}_{x,n}[(s_m - s_{m-1}) 1_{M=i}] \\ &= \mathbb{E}_{x,n}\Big[1_{M=i} \mathbb{E}_{x,n}\big[s_m - s_{m-1} | \sigma(M, Y_m, Y_{m-1}, J_m, J_{m-1}))\big]\Big], \end{aligned} \tag{3.20}$$

where $\sigma(Z_1, \ldots, Z_p)$ denotes the σ-algebra [10] generated by random variables $Z_1, \ldots, Z_p$. Now by the strong Markov property, we have that for $m \in \{1, \ldots, i\}$

$$\begin{aligned} &\mathbb{E}_{x,n}[s_m - s_{m-1} | \sigma(M, Y_m, Y_{m-1}, J_m, J_{m-1})] \\ &= \mathbb{E}_{y,j}[s_1 | Y_1 = y', J_1 = j'], \quad \text{if } Y_{m-1} = y, Y_m = y', J_{m-1} = j, J_m = j. \end{aligned} \tag{3.21}$$

Proceeding along the same lines as in the proof of Theorem 1, it follows that

$$\mathbb{E}_{y,j}[s_1 | Y_1 = y', J_1 = j'] < \infty, \quad \text{if } (y, y') \in \mathcal{Y}^2, \quad (j, j') \in \{0,1\}^2.$$

Since $\mathcal{Y}^2 \times \{0,1\}^2$ is finite, we thus have

$$C := \max_{(y,y') \in \mathcal{Y}^2, (j,j') \in \{0,1\}^2} \mathbb{E}_{y,j}[s_1 | Y_1 = y', J_1 = j'] < \infty. \tag{3.22}$$

Therefore, combining (3.19)-(3.22) yields

$$\mathbb{E}_{x,n}[\widehat{T}] \leq C \sum_{i=0}^{\infty} \sum_{m=1}^{i} \mathbb{P}_{x,n}(M = i) = C \mathbb{E}_{x,n}[M] < \infty.$$

□

We now calculate the MFPT for an initial condition $X(0) = x \in (z_{K'}, z_{K'+1})$ with $K' \in \{0, 1, \ldots, K-1\}$. Without loss of generality, suppose that $F_1(z_{K'}) = F_0(z_{K'+1}) = 0$ and, hence, $F_0(x) < 0, F_1(x) > 0$ in $(z_{K'}, z_{K'+1})$. Observe that by the strong Markov property we have that

$$\tau_n(x) = \widehat{\tau}_n(x) + \pi_n(x)\tau_0(z_{K'}) + (1 - \pi_n(x))\tau_1(z_{K'+1}), \tag{3.23}$$

where $\widehat{\tau}_n(x)$ is the MFPT to escape $(z_{K'}, z_{K'+1})$,

$$\widehat{\tau}_n(x) = \mathbb{E}_{x,n}[\widehat{T}],$$

and $\pi_n(x)$ is the splitting probability for which side of $(z_{K'}, z_{K'+1})$ that $X(t)$ exits through,

$$\pi_n(x) = \mathbb{P}_{x,n}[X(\widehat{T}) = z_{K'}].$$

In (3.23), we have used the fact that $N(\widehat{T}) = 0$ if $X(\widehat{T}) = z_{K'}$, and $N(\widehat{T}) = 1$ if $X(\widehat{T}) = z_{K'+1}$. One can show that $\widehat{\tau}_n$ satisfies equations (2.20a) and (2.20b), and therefore

$$\widehat{\tau}_n = \frac{1}{2}(S + (-1)^n \Delta),$$

where S and Δ are given by (3.10a)-(3.10b) and the constants $S(\overline{z}_{K'}), \Delta(\overline{z}_{K'})$ are chosen to satisfy the boundary conditions

$$\widehat{\tau}_0(z_{K'}+) = \widehat{\tau}_1(z_{K'+1}-) = 0.$$

Similarly, one can show that π_n satisfies equations (2.22a) and (2.22b), and therefore

$$\pi_n = 1/2(S + (-1)^n \Delta),$$

where S and Δ are given by (3.10a)-(3.10b) with $\gamma_\pm \equiv 0$. The constants $S(\overline{z}_{K'}), \Delta(\overline{z}_{K'})$ are chosen to satisfy the boundary conditions

$$\pi_0(z_{K'}+) = 1, \quad \pi_1(z_{K'+1}-) = 0.$$

4. Examples

Having developed the general theory, we now apply it to four examples. The first two examples illustrate the behavior of the MFPT as a function of the starting position for the case of a single critical point, while the last two examples examine the MFPT as a function of the number of critical points.

Example 4.1. Suppose that

$$\begin{aligned}
&F_1(x) > 0, \quad x \in [0, L], \\
&F_0(x) < 0, \quad x \in (0, L], \quad F_0(0) = 0.
\end{aligned}$$

The MFPT for $X(t)$ to reach escape $[0, L]$ given that $X(0) = x$ is determined by the functions $\Delta(x)$ and $S(x)$ which are given by (3.10a)-(3.10b). Further, we have that the constants $S(L/2), \Delta(L/2)$ must satisfy (3.15) with $n = 1$ and (3.17) with $n = 0$, which in this case become

$$\begin{aligned}
&\Delta(L/2)\psi_S(L) + \eta_S(L) + S(L/2) - \Delta(L/2)\psi_\Delta(L) - \eta_\Delta(L) \\
&\qquad = \pi_0^1(L)\big(\Delta(L/2)\psi_S(L) + \eta_S(L) + S(L/2) + \Delta(L/2)\psi_\Delta(L) + \eta_\Delta(L)\big), \\
&\Delta(L/2)\psi_S(0) + \eta_S(0) + S(L/2) + \Delta(L/2)\psi_\Delta(0) + \eta_\Delta(0) \\
&\qquad = 2/\beta + \Delta(L/2)\psi_S(0) + \eta_S(0) + S(L/2) - \Delta(L/2)\psi_\Delta(0) - \eta_\Delta(0).
\end{aligned}$$

Solving this linear system yields

$$\Delta(L/2) = \frac{c_{22} d_1}{c_{11} c_{22}}, \quad S(L/2) = \frac{c_{11} d_2 - c_{21} d_1}{c_{11} c_{22}},$$

where

$$\begin{aligned}
c_{11} &:= 2\psi_\Delta(0), \\
c_{21} &:= (1 - \pi_0^1(L))\psi_S(L) - (1 + \pi_0^1(L))\psi_\Delta(L), \quad c_{22} := 1 - \pi_0^1(L), \\
d_1 &:= 1/\beta - 2\eta_\Delta(0), \\
d_2 &:= -(1 - \pi_0^1(L))\eta_S(L) + (1 + \pi_0^1(L))\eta_\Delta(L).
\end{aligned}$$

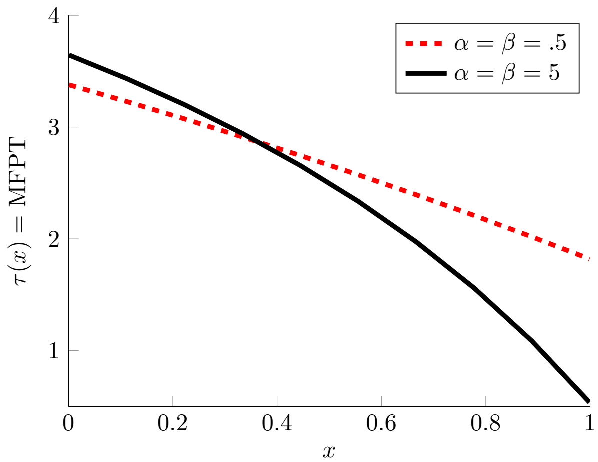

Figure 5: MFPT to escape $[0, 1]$ as a function of starting location for Example 4.1 with $F_0(x) = -x$ and $F_1(x) = 1$.

Fig. 5 plots $\tau(x)$ defined by equation (3.8) as a function of x for the particular case that $F_0(x) = -x$, $F_x(x) = 1$, and $L = 1$, for various choices of switching rates, $\alpha = \beta$. From this plot, one notices that $\tau(x)$ increases as the switching rate increases for x near 0, while $\tau(x)$ decreases as the switching rate increases for x near L. This non-monotonic behavior can be understood as follows. If $X(t)$ starts near the critical point $x = 0$, then a faster switching rate makes it more difficult to get away from $x = 0$ since the holding times of $N(t)$ in state 1 are shorter. By similar reasoning, if $X(t)$ starts near the absorbing boundary $x = L$, then a faster switching rate makes it more difficult to get away from $x = L$, and hence the MFPT to reach $x = L$ decreases.

Example 4.2. In this example, we show that the MFPT can be infinite if Assumption 1 is violated, even when there is a positive probability that the FPT is finite. That is, the MFPT can be infinite even if the particle is not confined to a trapping region. Let

$$F_0(x) = -2x, \quad F_1(x) = x, \quad \alpha = \beta = 2, \quad \text{and } L = 2.$$

This example is very similar to Example 4.1 above, except that $F_1(0) = 0$ here, while $F_1(0) > 0$ in Example 4.1. Observe that Assumption 1 is violated in this example since $x = 0$ is a common critical point, $F_0(0) = F_1(0) = 0$.

For simplicity, we assume that $X(0) = 1$ and $N(0) = 0$. The MFPT to escape $[0, 2]$ has a positive probability of being finite. To see this, let $\{s_k\}_{k=1}^{\infty}$ be the sequence of holding times of $N(t)$ and observe that they are independent and exponentially distributed with rate $\alpha = \beta = 2$. Therefore, $X(t)$ is essentially just products of either

$$\exp(-2s_{2k}) \quad \text{and} \quad \exp(s_{2k-1}).$$

Hence, we see that the probability that the FPT is, for example, less than one is

$$\mathbb{P}_{1,0}(T < 1) \geq \mathbb{P}(s_1 < 0.1 \cap s_2 > 0.9) = (1 - \exp(-0.2)) \exp(-0.9) > 0.07,$$

since $X(s_1 + s_2) > \exp(0.9) \exp(-0.2) > 2$ if $s_1 < 0.1$ and $s_2 > 0.9$.

Though the particle has a positive probability of escaping $[0, L]$ in finite time, we now show that its MFPT to escape is infinite. Define the discrete time process

$\{Y_n\}_{n=0}^{\infty}$ by

$$Y_n := X\Big(\sum_{k=1}^{2n} s_k\Big) = \exp\Big(\sum_{k=1}^{n}(s_{2k} - 2s_{2k-1})\Big).$$

Observe that $\{Y_n\}_{n=0}^{\infty}$ is a martingale since [10]

$$\begin{aligned}\mathbb{E}[Y_{n+1}|\sigma(Y_0,\dots,Y_n)] &= \mathbb{E}[\exp(s_{2(n+1)} - s_{2(n+1)-1})Y_n|\sigma(Y_0,\dots,Y_n)]\\ &= Y_n\mathbb{E}[\exp(s_{2(n+1)} - s_{2(n+1)-1})] = Y_n,\end{aligned}$$

where we have used that

$$\mathbb{E}[\exp(s_{2(n+1)} - s_{2(n+1)-1})] = \frac{\alpha}{\alpha-1}\frac{\alpha}{\alpha+2} = 1, \tag{4.1}$$

since $\alpha = 2$. By Doob's martingale inequality [10], we then have that

$$\mathbb{P}_{1,0}\Big(\sup_{0\le n\le M} Y_n \ge 2\Big) \le \frac{\mathbb{E}_{1,0}[Y_M]}{2} = \frac{1}{2}, \quad M \in \mathbb{N}.$$

Furthermore, observe that if

$$t \in \Big[\sum_{k=1}^{2n} s_k, \sum_{k=1}^{2(n+m)} s_k\Big],$$

then

$$X(t) \le \sup_{n\le k\le n+m} Y_k. \tag{4.2}$$

Taking $M \to \infty$ in (4.1) and using (4.2), we find that

$$\mathbb{P}_{1,0}\Big(\sup_{t\ge 0} X(t) \ge 2\Big) \le \frac{1}{2}.$$

Therefore, the MFPT to escape $[0,2]$ is certainly infinite since the FPT is infinite with positive probability.

Investigating this example further, we note that $\{s_{2k} - 2s_{2k-1}\}_{k=1}^{\infty}$ are independent and identically distributed with a negative mean,

$$\mathbb{E}[s_{2k} - 2s_{2k-1}] = -1/\alpha = -1/2 < 0.$$

Hence, the strong law of large numbers ensures that $Y_n \to 0$ almost surely as $n \to \infty$. Therefore, $X(t) \to 0$ almost surely as $t \to \infty$ by (4.2). Thus, while $X(t)$ has a positive probability of escaping any given finite interval, it will nonetheless converge to 0 with probability one as $t \to \infty$. That is, though $X(t)$ is never confined to the trapping "region", $x = 0$, we have that $X(t)$ approaches this "region" as $t \to \infty$.

Example 4.3. Suppose that

$$F_0(x) = \sin(xK\pi), \quad F_1(x) = v_1 > 0,$$

for $K \in \mathbb{N}$, see Fig. 6. Observe that since $F_0(0) = 0, F_1(0) > 0$, $X(t)$ must exit through $x = L$. Further, observe that F_0 has $K+1$ critical points, $z_k = k/K$, for $k \in \{0, 1, \dots, K\}$.

The MFPT for $X(t)$ to reach L given that $X(0) = x$ is determined by the functions $\Delta(x)$ and $S(x)$ which are given by (3.10a)-(3.10b) on each subinterval (z_k, z_{k+1}). Further, the constants $S(\overline{z}_k), \Delta(\overline{z}_k)$ must satisfy (3.15) with $n = 1$ and either (3.16) or (3.17) with $n = 0$, depending if k is even or odd.

Solving this system, in Fig. 7 we plot the MFPT for $X(t)$ to reach $x = L$ as a function of the number of critical points, $K + 1$. In this example, we see that the MFPT approaches a finite, nonzero limit as the number of critical points increases. To understand this, observe that as the number of critical points grows, $X(t)$ approaches the process that is fixed in space when $N(t) = 0$, and this process clearly has a finite MFPT to reach $x = L$. In particular, using the approximation, $F_0(x) = 0$, we solve (2.20a)-(2.20b) and find the large K approximation,

$$\tau(x) \approx \frac{\alpha/\beta}{\alpha+\beta} + \frac{\alpha+\beta}{\beta}\frac{L-x}{v_1}. \tag{4.3}$$

Fig. 7 verifies that the actual MFPT approaches (4.3) for large K. Furthermore, from Fig. 7 we notice that the MFPT is non-monotonic in the switching rates, $\alpha = \beta$, when the number of critical points is not too large (less than about 50 in this example).

Example 4.4. Suppose that

$$F_0(x) = \sin(xM\pi) + 1/2, \quad F_1(x) = \sin(xM\pi + \pi) + 1/2,$$

for $M \in \mathbb{Z}$, see Fig. 8. Observe that since $F_0(0) > 0, F_1(0) > 0$, $X(t)$ must exit through $x = L$. Further, observe that if M is even, then F_0 and F_1 each have M critical points, while if M is odd, then F_0 has $M - 1$ critical points if M and F_1 has $M + 1$ critical points. These critical points are given by $z_0 = 0$, $z_1 = 1/(6M)$,

$$z_k = z_{k-1} + \frac{2}{3M}, \quad \text{if } k \text{ is even and } k < 2M+1,$$
$$z_k = z_{k-1} + \frac{1}{3M}, \quad \text{if } k \text{ is odd and } k < 2M+1,$$

and $z_{2M+1} = 1$.

The MFPT for $X(t)$ to reach escape $[0, L]$ given that $X(0) = x$ is determined by the functions $\Delta(x)$ and $S(x)$ which are given by (3.10a)-(3.10b) on each subinterval (z_k, z_{k+1}). Further, the constants $S(\overline{z}_k), \Delta(\overline{z}_k)$ must satisfy (3.15) with $n = 0$ and (3.16) with $n = 1$ if $k = 0$. These constants must satisfy (3.15) with $n = 0$ and (3.17) with $n = 1$ if $k = 2$. Continuing in this manner yields a system that these constants must satisfy.

Solving this system, in the left panel of Figure 9 we plot the MFPT for $X(t)$ to reach $x = L$ conditioned on $X(0) = 0$ as a function of the number of critical points,

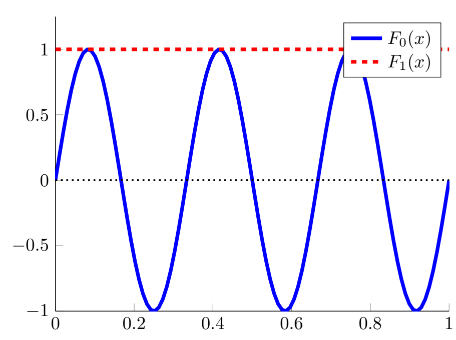

Figure 6: Example 4.3 with $F_0(x) = \sin(6\pi x)$ and $F_1(x) = 1$.

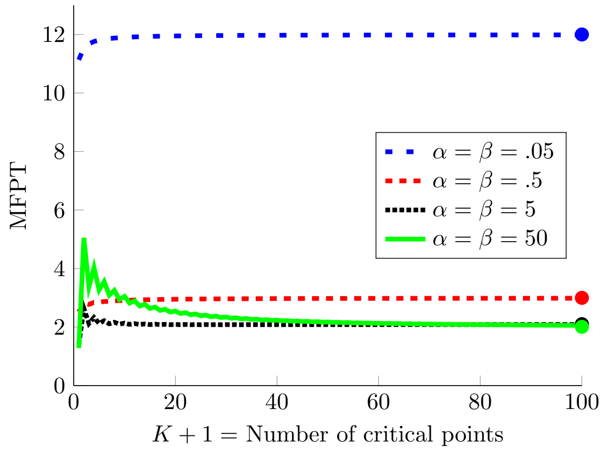

Figure 7: MFPT to escape $[0, 1]$ with $F_0(x) = \sin(K\pi x)$ and $F_1(x) = 1$ (Example 4.3) with $X(0) = 0$, as a function of the number of critical points, $K+1$ for various choices of the switching rates, α, β. The dots at $K + 1 = 100$ are the large K approximation in (4.3).

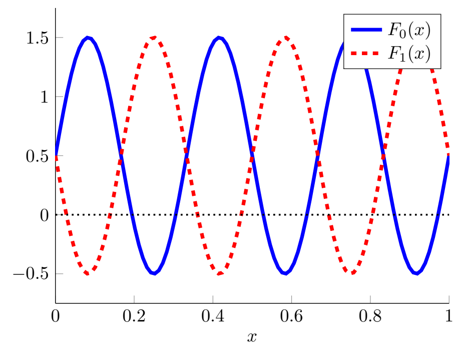

Figure 8: Example 4.4 with $F_0(x) = \sin(6x\pi) + 1/2$ and $F_1(x) = \sin(6x\pi + \pi) + 1/2$.

$2M$. In this example, we see that the MFPT increases linearly in the number of critical points. From the right panel of Fig. 9, we see that this MFPT, $\tau(0)$, is approximately

$$\tau(0) \approx M/\alpha \quad \text{if } M/\alpha \gg 1.$$

One can understand this approximate formula as follows. The distance between critical points is proportional to $1/M$, so the time it takes $X(t)$ to go from one critical point to the next is negligible compared to the time it takes for $N(t)$ to switch if $M/\alpha \gg 1$. Hence, the MFPT will approximately be the number of critical points that $X(t)$ gets "stuck" at (which is approximately M since each F_n has approximately M critical points) multiplied by the time it takes for $N(t)$ to switch (which is $1/\alpha = 1/\beta$).

5. Discussion

In this paper we have used probabilistic methods based on conditional expectations and the strong Markov property to analyze the MFPT for a two-state PDMP to escape

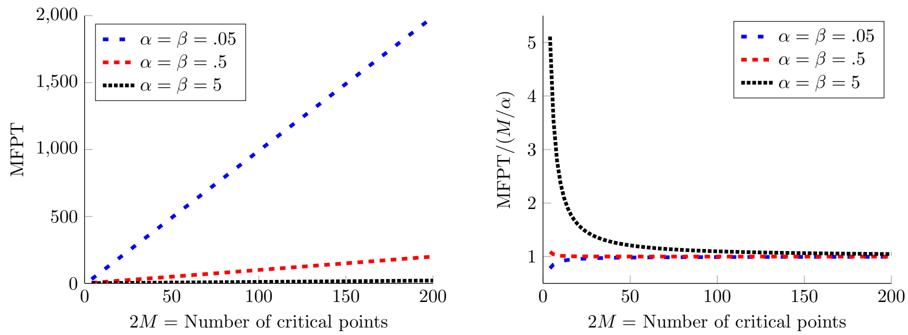

Figure 9: MFPT to escape $[0, 1]$ with $F_0(x) = \sin(Mx\pi) + 1/2$ and $F_1(x) = \sin(Mx\pi + \pi) + 1/2$ (Example 4.4) increases linearly in the number of critical points, $2M$.

from a finite interval. The complicating factor is the presence of critical points of the functions generating the underlying piecewise deterministic dynamics. Although it is well known that such critical points lead to singularities in the stationary density (if it exists), the effect on the MFPT has not previously been explored in any detail. In principle, it would be possible to extend our analysis to multi-state PDMPs ($N > 2$) and higher spatial dimensions. However, it is notoriously difficult to find explicit solutions for the stationary density [1], even in cases where it is known to exist, suggesting that the corresponding MFPT equations will be equally hard to solve.

Another possible extension of our work would be to consider functionals of the continuous process $X(s)$, $0 \in [0, t]$, which take the form

$$\mathcal{T} := \int_0^t U(X(s))ds, \tag{5.1}$$

where $U(x)$ is some prescribed function or distribution such that $\mathcal{T}$ has positive support. One example is the so-called residence or occupation time in an interval $(a, b) \subset \Sigma$:

$$\mathcal{T} := \int_0^t I_{(a,b)}(X(s))ds, \tag{5.2}$$

where $1_{\mathcal{V}}(x)$ denotes the indicator function of the set $\mathcal{V}$, that is, $1_{\mathcal{V}}(x) = 1$ if $x \in \mathcal{V}$ and is zero otherwise. Since $X(t)$, $t \geq 0$, is a stochastic process, it follows that each realization of the PDMP will typically yield a different value of $\mathcal{T}$, which means that $\mathcal{T}$ will be distributed according to some probability density $P(\mathcal{T}, t|x_0, 0)$ for $X(0) = x_0$. If $X(s)$ is taken to be a Brownian motion, rather than the continuous part of a PDMP, then $\mathcal{T}$ in equation (5.1) is known as a Brownian functional. Brownian functionals are finding an increasing number of applications in physics, biology and computing [15]. In particular, the statistical properties of a Brownian functional can be analyzed using path integrals, resulting in the classical *Feynman-Kac* formula [13]. Recently, one of us has derived an analogous Feynman-Kac formula for PDMP functionals of the form (5.1) [7]. Now suppose that the upper limit of the integral (5.1) is taken to be a stopping time such as the FPT to escape some bounded interval. In contrast to Brownian motion, the continuous component of a PDMP is not a Markov process on its own (even in the weak sense), whereas the full system $\{X(t), N(t)\}$ satisfies the strong

Markov property. One way to handle this issue is to use conditional expectations and probabilistic methods along the lines outlined in our paper.

Acknowledgements

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