STALLINGS AUTOMATA AND APPLICATIONS

BGSMATH GRADUATE COURSE

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Centre de Recerca Matemàtica
January - February 2023

v1.0

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Free groups

UNIVERSAL PROPERTY

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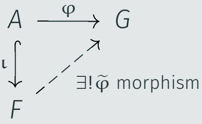

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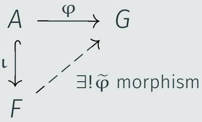

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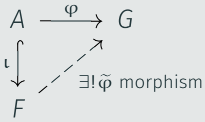

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- $(\mathbb{Z}, +)$ is free over $A = \{1\}$ (i.e., $\{1\}$ is a free basis for $(\mathbb{Z}, +)$);
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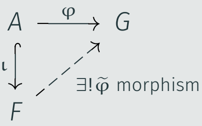

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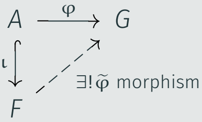

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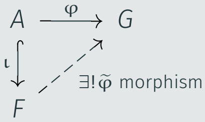

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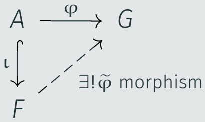

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Question

Which groups are free? Does there exist a free group over any set A?

THE RANK OF A FREE GROUP

Proposition

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Remark

It is clear that $\mathbb{F}_1 \simeq \mathbb{Z}$, but we still do not know whether free groups of higher ranks

$$\mathbb{F}_2, F_3, \ldots, F_{\aleph_0}, F_{\aleph_1}, \ldots$$

do exist. Let us construct them combinatorially ...

Let $A = \{a_1, \ldots, a_r\}$ be a (possibly infinite) set called ***alphabet***.
Then, $\widetilde{A} = \{a_1, \ldots, a_r, a_1^{-1}, \ldots, a_r^{-1}\}$ is an ***involutive alphabet***
($\#\widetilde{A} = 2\#A$). Convention: $(a_i^{-1})^{-1} = a_i$.

CONSTRUCTION OF FREE GROUPS (I)

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A ***word*** on A is a finite sequence of letters from A, $w = a_{i_1} a_{i_2} \cdots a_{i_n}$, $n \geqslant 0$. For $n = 0$ we have the ***empty word***, denoted by $\mathbf{1}$.
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The set $A^* = \{a_{i_1} a_{i_2} \cdots a_{i_n} \mid n \geqslant 0\}$ with the operation of concatenation, $u \cdot v = uv$, is a monoid. Any subset $L \subseteq A^*$ is called a ***language***.

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Free equivalence: For $u, v \in \widetilde{A}^*$, define $\boldsymbol{u \sim^* v} \Leftrightarrow \exists$ a finite chain of elementary reductions/insertions $u = u_1 \sim u_2 \sim \cdots \sim u_n = v$.

CONSTRUCTION OF FREE GROUPS (II)

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The relation $\sim^*$ (or simply $\sim$) is an equivalence in $\widetilde{A}^*$. We denote the quotient by $\mathbb{F}_A = \widetilde{A}^* / \sim = \{[u] \mid u \in \widetilde{A}^*\}$ and $\widetilde{A}^* \twoheadrightarrow \mathbb{F}_A$, $u \mapsto [u]$.

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So, we can think $\mathbb{F}_A$ as $R(A)$ with the operation $u \cdot v = \overline{uv}$, $u, v \in R(A)$.

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Corollary

The map $A \hookrightarrow \mathbb{F}_A$, $a \mapsto [a]$ *is injective.*

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- S is a (free) ***basis*** of $G \Leftrightarrow \pi_S$ is bijective.

THE MEMBERSHIP PROBLEM

(Subgroup) Membership Problem, $\mathrm{MP}(\mathbb{F}_A)$

Given $u, v_1, \ldots, v_n \in \mathbb{F}_A$, decide whether $u \in H = \langle v_1, \ldots, v_n \rangle$;
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Example

Consider $\mathbb{FF}_2 = \langle a, b \rangle$ and the subgroup $H = \langle v_1, v_2, v_3 \rangle \leqslant \mathbb{FF}_2$,
where $v_1 = baba^{-1}$, $v_2 = aba^{-1}$, and $v_3 = aba^2$.
Is it true that $a \in H$?
is it true that $u = b^2aba^{-1}b^7a^{-2}b^{-1}a^2 \in H$?
If yes, express them as a (unique?) word on $\{v_1^{\pm 1}, v_2^{\pm 1}, v_3^{\pm 1}\}$.

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$$\left.\begin{aligned} |v_1|_a &= |baba^{-1}|_a = 0 \\ |v_2|_a &= |aba^{-1}|_a = 0 \\ |v_3|_a &= |aba^2|_a = 3 \end{aligned}\right\} \Rightarrow \quad a \notin H.$$

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But $|u|_a = |b^2aba^{-1}b^7a^{-2}b^{-1}a^2|_a = 1 - 1 - 2 + 2 = 0$; so, $u \in H$?

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So, YES, $u \in H$!!!

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\end{aligned}
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So, YES, $u \in H$!!!

Question

Is this expression unique?

After some calculations ...

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So, YES, $u \in H$!!!

Question

Is this expression unique? How to find it/them systematically?

Subgroup Intersection Problem, SIP($\mathbb{F}_A$)

Given $u_1, \ldots, u_n; v_1, \ldots, v_m \in \mathbb{F}_A$, decide whether the intersection of $H = \langle u_1, \ldots, u_n \rangle$ and $K = \langle v_1, \ldots, v_m \rangle$ is finitely generated; if yes, compute generators for $H \cap K$.

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Example

Consider $\mathbb{F}_2 = \langle a, b \rangle$ and the subgroups

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H = \langle u_1, u_2, u_3 \rangle &\leqslant \mathbb{F}_2, & \text{and} \quad & K = \langle v_1, v_2, v_3 \rangle \leqslant \mathbb{F}_2 \\
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How to find generators (or just elements!) for $H \cap K$?

Clearly, $H \ni u_2 = a^3 = v_2 \in K$. What else?

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Is $H = \langle a^3,\ b^{-1}a^3b,\ a^{-1}ba^3b^{-1}a \rangle$? Do we need more generators?

Digraphs and automata

GOAL AND SEMINAL EXAMPLE

Let $A = \{a_1, \ldots, a_n\}$ and let $\mathbb{F}_n \simeq \mathbb{F}_A = \langle A \mid - \rangle$

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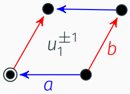

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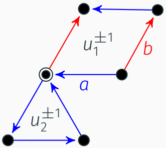

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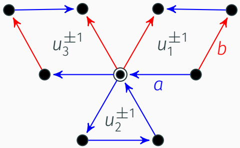

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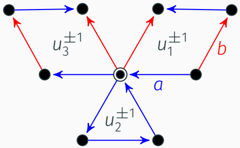

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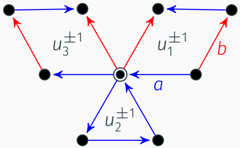

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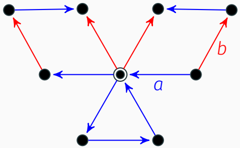

Fact:

H is described by the (reduced) labels of walks ◉ $\rightsquigarrow$ ◉ in $\mathrm{Fl}(u_1, u_2, u_3)$.

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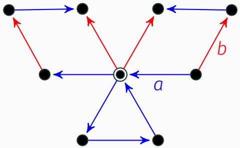

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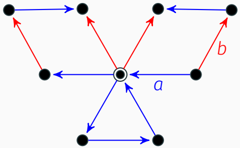

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Flower automata are natural 'drawings' associated to every subgroup of $\mathbb{F}_A$, are they 'nice'?

DIRECTED GRAPHS AND WALKS

A ***directed graph*** (***digraph***) is a tuple $\Delta = (V, E, \iota, \tau)$, where:

- V and E are disjoint sets (of ***vertices*** and ***arcs***, respectively)
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A ***walk*** in a digraph Δ is a finite sequence $\gamma = p_0 e_1 p_1 \cdots e_l p_l$ where $p_i \in V\Delta$, $e_i \in E\Delta$, $\iota e_i = p_{i-1}$ and $\tau e_i = p_i$ for $i = 1, \ldots, l$.

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- The ***length*** of γ is the number of arcs in γ $\qquad (|\gamma| = l)$

We denote by $W\Delta$ the ***set of walks*** in Δ.

LABELLED DIGRAPHS AND LANGUAGES

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Let A be an alphabet. An ***A-digraph*** is a pair $\Gamma = (\Delta, \ell)$, where Δ is a digraph, and $\ell\colon \mathrm{E}\Delta \to A$ is the ***labelling*** of Γ.

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Let Γ be an A-digraph and let $P, Q \in \mathrm{V}\Gamma$. Then,

$$\mathcal{L}_{P,Q}(\Gamma) = \left\{ w \in A^* \, : \, \exists \mathrm{p} \in P, \exists \mathrm{q} \in Q, \ \mathrm{p} \overset{w}{\rightsquigarrow} \mathrm{q} \right\}$$

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AUTOMATA

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An automaton $\Gamma = (\Delta, P, Q)$ is ***pointed*** if it has a unique common initial and terminal state (i.e., if $P = Q = \{\odot\}$).

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An ***A-involutive automaton*** is an $A^{\pm}$-automaton with a labelled involution on its arcs; i.e., to every arc $\mathrm{e} \equiv \mathrm{p} \xrightarrow{a} \mathrm{q}$ we associate a unique arc $\mathrm{e}^{-1} \equiv \mathrm{p} \xleftarrow{a^{-1}} \mathrm{q}$ (the ***inverse*** of e) such that $\mathrm{e}' \neq \mathrm{e}$ and $(\mathrm{e}^{-1})^{-1} = \mathrm{e}$.

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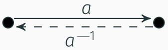

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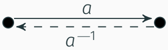

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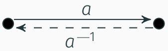

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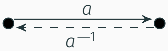

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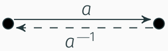

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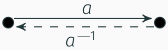

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From now on, automata = pointed involutive automata.

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If Γ is pointed then we say that $\overline{\mathcal{L}}_{\odot}(\Gamma)$ is the ***subgroup recognized by*** Γ, and we write $\overline{\mathcal{L}}_{\odot}(\Gamma) = \langle \Gamma \rangle$.

Remark

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Lemma

If Γ is involutive and deterministic and γ is a walk in Γ, then:

$$\gamma \textit{ is reduced} \quad \Leftrightarrow \quad \ell(\gamma) \textit{ is reduced}$$

and

$$\langle \Gamma \rangle \;=\; \{\ell(\gamma) \,:\, \gamma \equiv \odot \rightsquigarrow \odot \textit{ reduced}\}$$

CORE AND REDUCED AUTOMATA

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Definition

An automaton Γ is ***reduced*** if it is deterministic and core.

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Proposition

Let H be a subgroup of $\mathbb{F}_A$. Then, $\mathrm{Sch}(H, A)$ is deterministic, saturated, connected, and $\langle \mathrm{Sch}(H, A) \rangle = H$.

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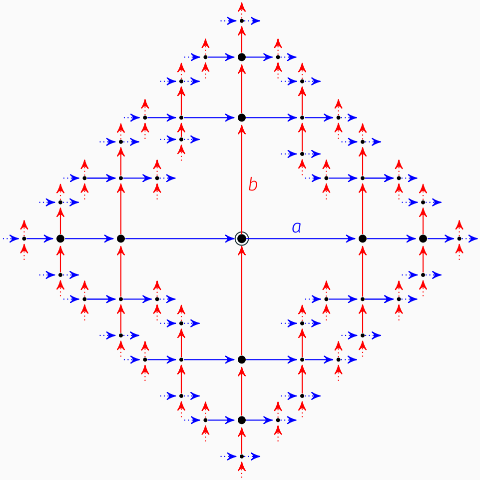
Remark: The Schreier automaton depends on the chosen generating set for G.

The Cayley automaton $\mathrm{Cay}(\mathbb{F}_{\{a,b\}}, \{a, b\})$
(consisting in four ***Cayley branches*** adjacent to the basepoint $\odot$).



STALLINGS AUTOMATON

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Remark: The Stallings automaton $\mathrm{St}(H,A)$ depends on the chosen basis A for the ambient free group.

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Corollary

If $\phi\colon \Gamma \to \Gamma'$ is a homomorphism of automata, then $\mathcal{L}(\Gamma) \subseteq \mathcal{L}(\Gamma')$.

Stallings bijection

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Let Γ, Γ' be reduced (pointed and involutive) A-automata. Then,

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STALLINGS BIJECTION

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Theorem (Stallings, 1983)

Let $\mathbb{F}_A$ be a free group with basis A. Then,

$$\begin{array}{rcl} \{\textit{subgroups of } \mathbb{F}_A\} & \leftrightarrow & \{\textit{(isom. classes of) reduced A-automata}\} \\ H & \mapsto & \mathrm{St}(H, A) \\ \langle \Gamma \rangle & \leftarrow\!\! & \Gamma \end{array}$$

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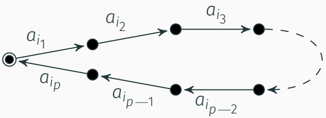

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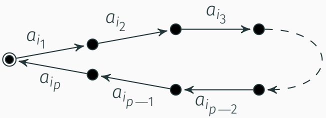

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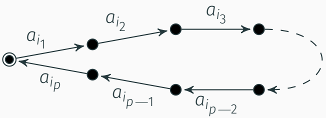

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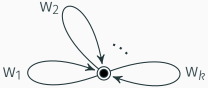

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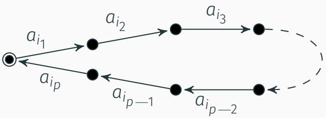

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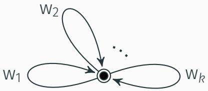

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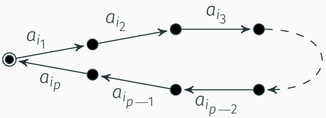

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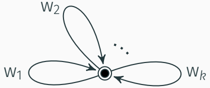

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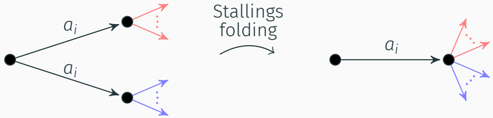

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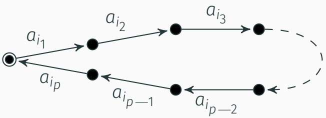

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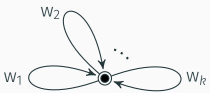

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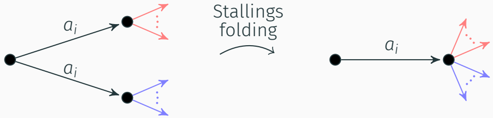

4. Keep folding until (necessarily) reaching $\mathrm{St}(H)$. (why?)

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Remark: the result of the folding process depends neither on the folding sequence *nor on the starting (finite) generating set* for H.

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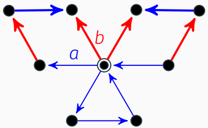

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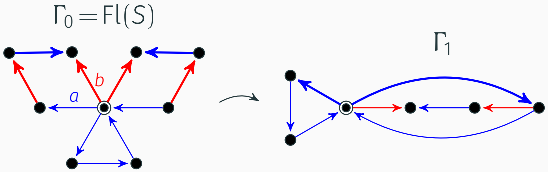

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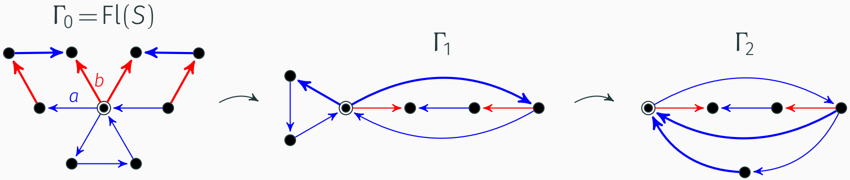

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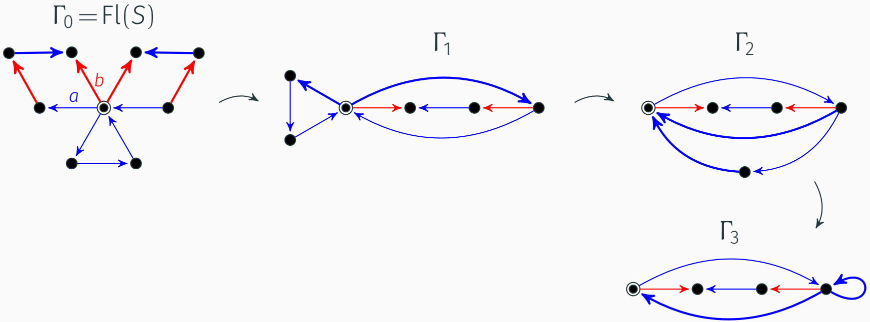

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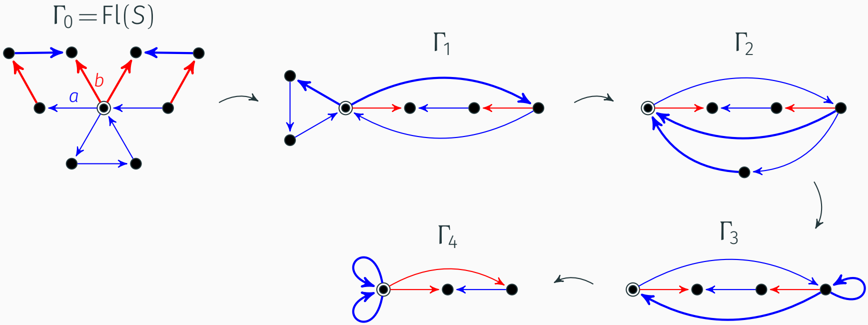

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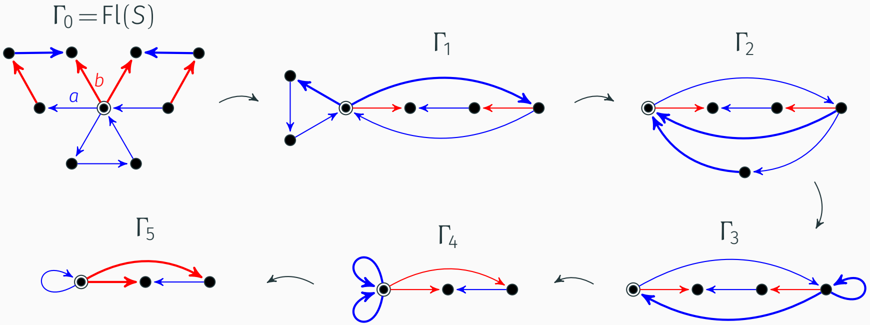

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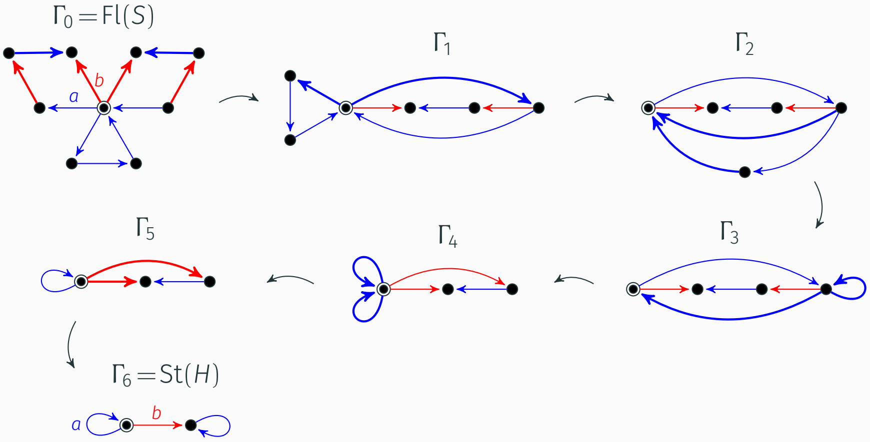

COMPUTABILITY OF GENERATORS ($\longleftarrow$). FREENESS

Theorem

Let Γ be a connected A-automaton, let T be an spanning tree of Γ, and let

$$S_T = \{\, \bar{\ell}(\odot \overset{T}{\rightsquigarrow} \bullet \xrightarrow{\mathrm{e}} \bullet \overset{T}{\rightsquigarrow} \odot) \,:\, \mathrm{e} \in \mathrm{E}^{+}\Gamma \setminus \mathrm{E}T \,\}$$

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COMPUTABILITY OF GENERATORS ($\hookleftarrow$). FREENESS

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Let Γ be a connected A-automaton, let T be an spanning tree of Γ, and let

$$S_T = \{\, \bar{\ell}(\odot \overset{T}{\rightsquigarrow} \bullet \xrightarrow{\mathrm{e}} \bullet \overset{T}{\rightsquigarrow} \odot) \,:\, \mathrm{e} \in \mathrm{E}^{+}\Gamma \setminus \mathrm{E}T \,\}$$

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Sketch of proof. i) Let $w = \overline{\ell}(\gamma) \in \langle \Gamma \rangle$, where γ is reduced. Write:

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where $\mathrm{e}_1, \ldots, \mathrm{e}_l \in \mathrm{E}^{+}\Gamma \smallsetminus \mathrm{E}T$ and $\epsilon_j = \pm 1$.

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It is clear that $w = \overline{\ell}(\gamma) = \overline{\ell}(\gamma') = w_{\mathrm{e}_1}^{\epsilon_1} w_{\mathrm{e}_2}^{\epsilon_2} \cdots w_{\mathrm{e}_l}^{\epsilon_l} \in \langle S_T \rangle$.

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If Γ is finite, then $\mathrm{rk}\langle\Gamma\rangle = \#(E^+ \setminus ET) < \infty$.

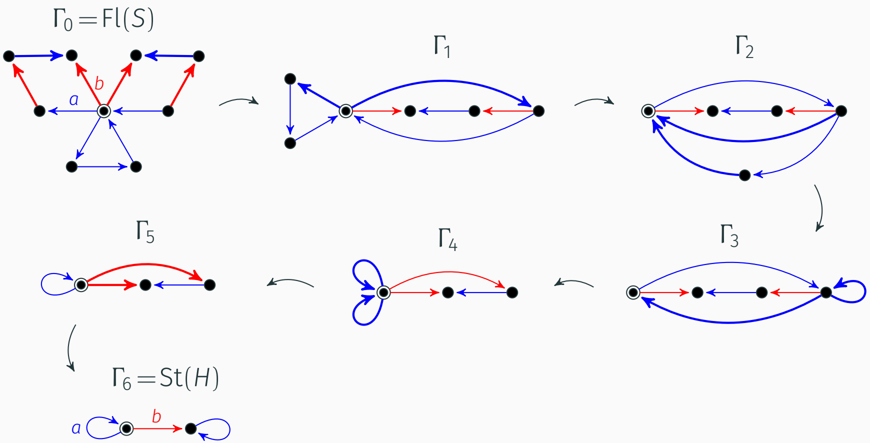
If $\mathrm{rk}\,\Gamma = \mathrm{rk}(\mathrm{core}(\Gamma)) < \infty$ then Γ is finite (why?).

Then, $\mathrm{rk}\langle\Gamma\rangle = \mathrm{rk}\,\Gamma = \#E\Gamma^+ - \#V\Gamma + 1$. □

EXAMPLE

Let $H = \langle \underbrace{a^{-1}bab^{-1}}_{u_1}, \underbrace{a^3}_{u_2}, \underbrace{abab^{-1}}_{u_3} \rangle \leqslant \mathbb{F}_{\{a,b\}}$.

Then, we start by drawing the flower automaton $\mathrm{Fl}(u_1, u_2, u_3)$:



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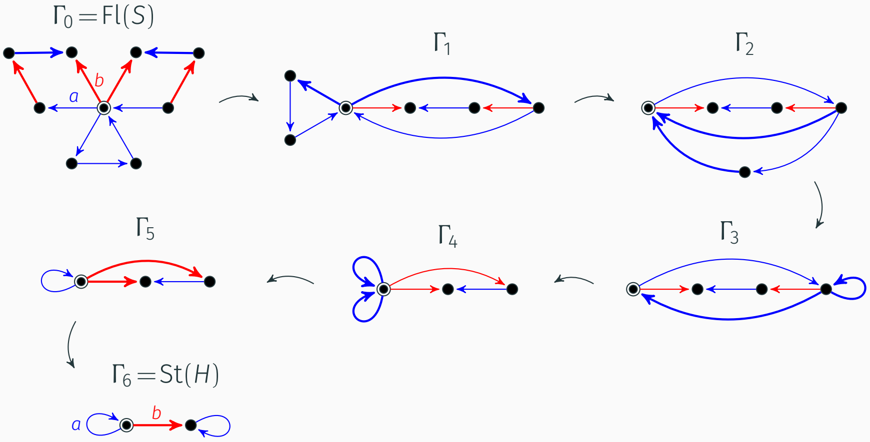

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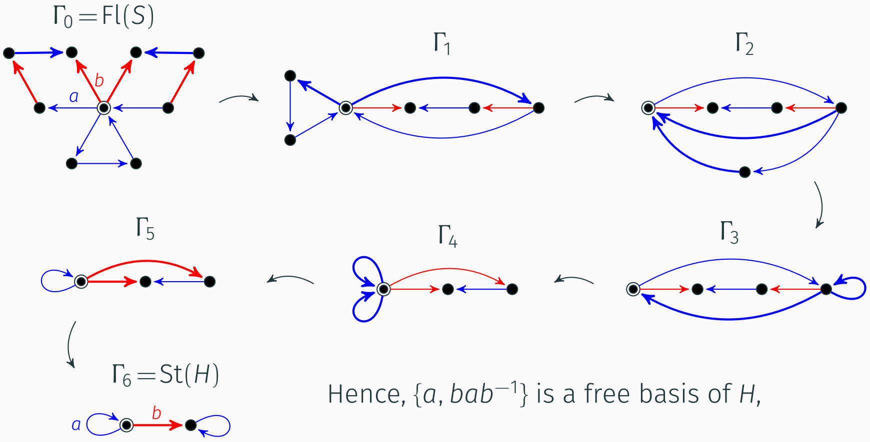

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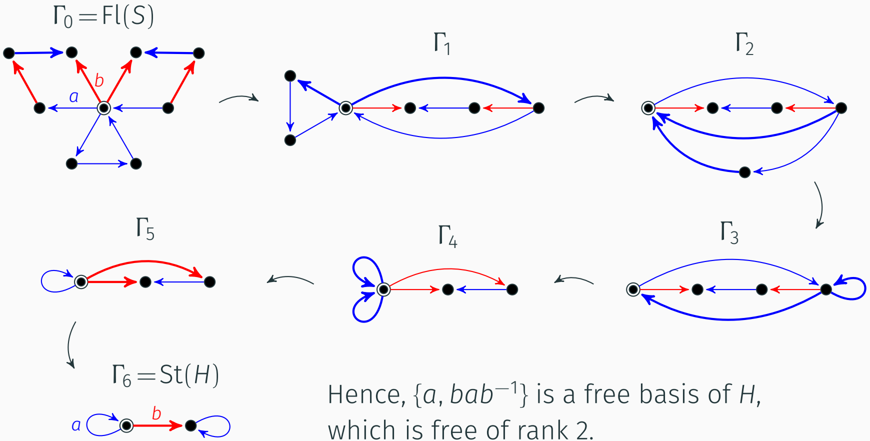

Hence, $\{a, bab^{-1}\}$ is a free basis of H, which is free of rank 2.

Let $\mathbb{F}_A$ be the free group with basis A.

STALLINGS BIJECTION (FULL RESULT)

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is a basis for the subgroup $H = \langle \Gamma \rangle$.

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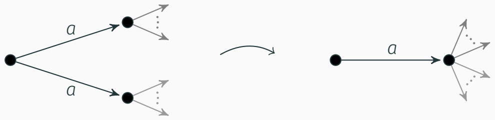

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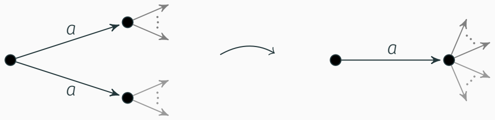

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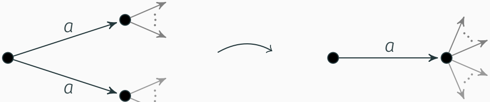

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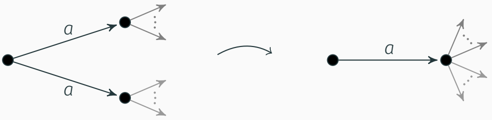

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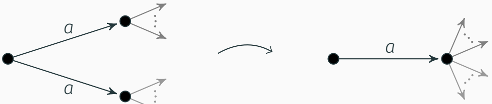

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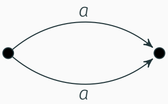

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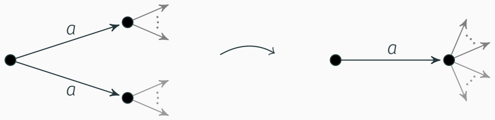

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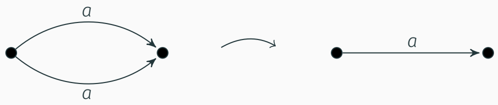

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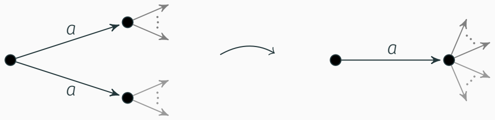

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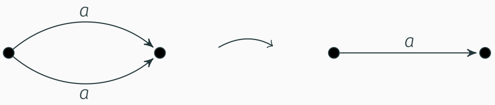

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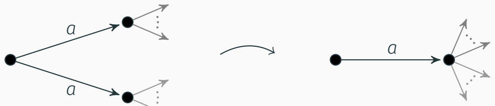

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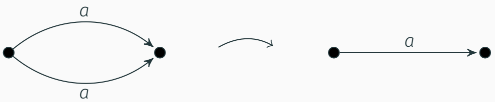

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Remark: If Γ is finite and $\Gamma \curvearrowright \Gamma'$ is a Stallings folding, then:

$$\mathrm{rk}(\Gamma') = \begin{cases} \mathrm{rk}(\Gamma) & \text{if } \Gamma \curvearrowright \Gamma' \text{ is open,} \\ \mathrm{rk}(\Gamma) - 1 & \text{if } \Gamma \curvearrowright \Gamma' \text{ is closed.} \end{cases}$$

FUNDAMENTAL GROUP AND LOSS

Corollary

Let Γ be a connected A-automaton, let T be an spanning tree of Γ, and let S_T be the set of T-petals of Γ. Then,

$$\pi_{\odot}(\widetilde{\Gamma}) \simeq \mathbb{F}_{S_T}$$

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First applications

FIRST APPLICATIONS

Theorem (Nielsen-Schreier)

Subgroups of free groups are again free.

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Proof: Draw it!

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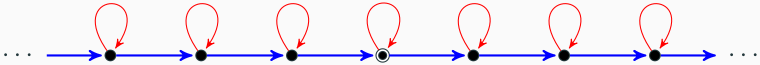

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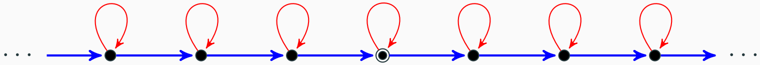

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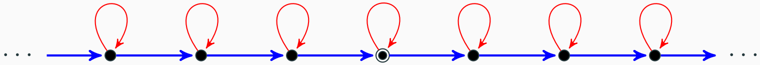

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How many different subgroups of $\mathbb{F}_2$ are there?

Remark

Let $S \subseteq \mathbb{F}_A$. Then,

i) $\langle S \rangle = \mathbb{F}_A \;\Leftrightarrow\; \mathrm{St}(\langle S \rangle) = \mathrm{Fl}(A)$,

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Finitely generated free groups are Hopfian.

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The subgroup membership problem is solvable in $\mathbb{F}_A = \langle A \mid - \rangle$*: given* $v, u_1, \ldots, u_n \in (\widetilde{A})^*$*, it is decidable whether* $v \in H = \langle u_1, \ldots, u_n \rangle$*. In this case, we can compute* v *as a word in* $\{u_1, \ldots, u_n\}$*.*

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When $v \in H$, how to express it as a word in $\{u_1, \ldots, u_n\}$?

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Consider $\mathbb{F}_2 = \langle a, b \rangle$ and the subgroup $H = \langle u_1, u_2, u_3 \rangle \leqslant \mathbb{F}_2$, where

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Let us recover the construction of the Stallings automaton $\mathrm{St}(H)$...

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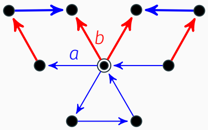

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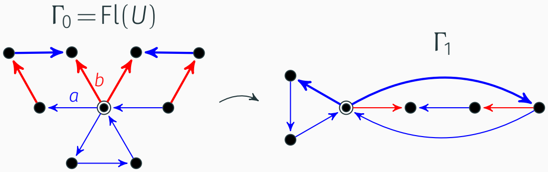

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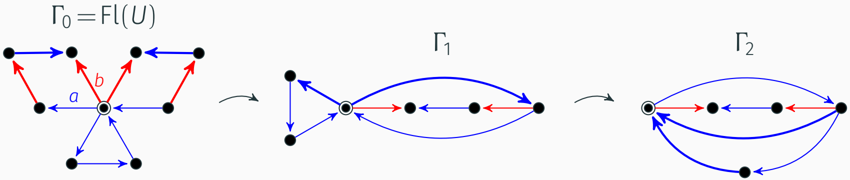

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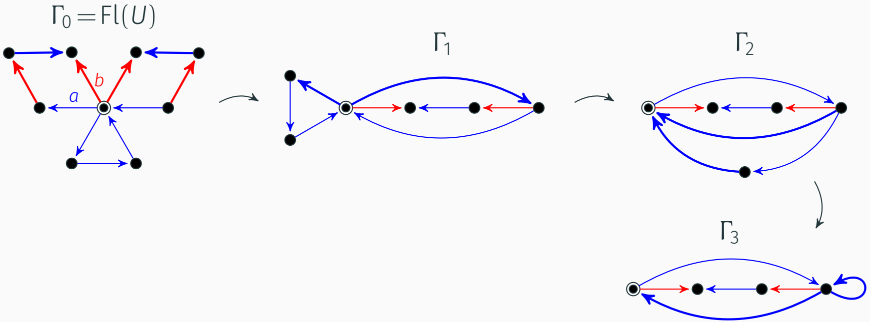

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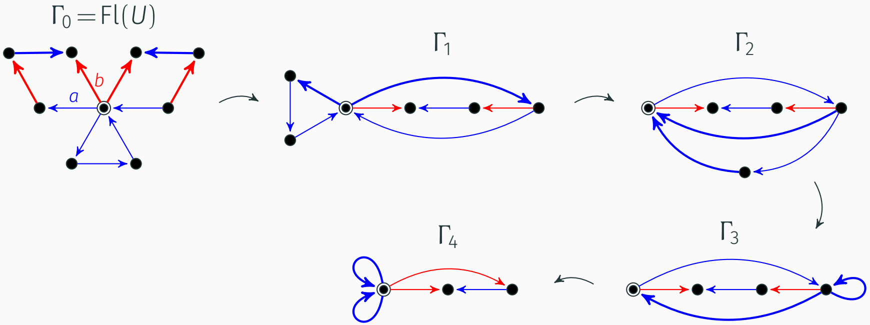

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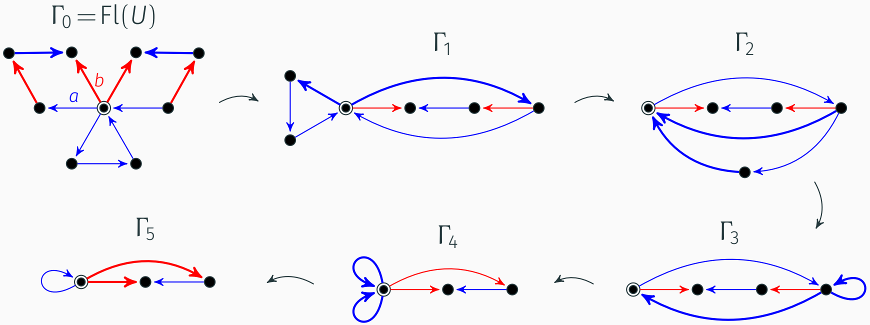

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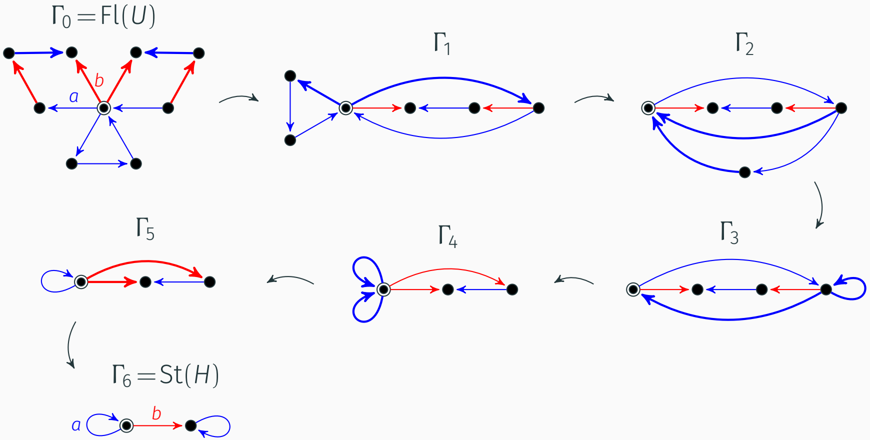

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Let us now express a as a word on $\{u_1, u_2, u_3\}\dots$

When $v \in H$, how to express v as a word in $\{u_1, \ldots, u_n\}$?

(8) Look at the computed tower of foldings

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THE MEMBERSHIP (SEARCH) PROBLEM

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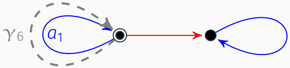
Clearly, $a \in H$ thanks to the walk $\gamma_6 = a_1$:



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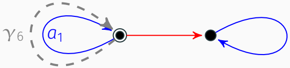
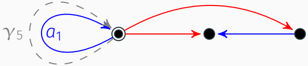
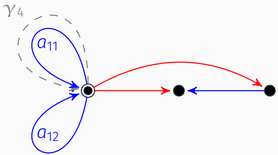


Lifting to Γ_5 (no interaction with the folded arcs), we get $\gamma_5 = a_1$:



Lifting to Γ_4, we have multiple choices (since $\Gamma_4 \rightsquigarrow \Gamma_5$ is a closed folding); we get $\gamma_4 = a_{11}$:



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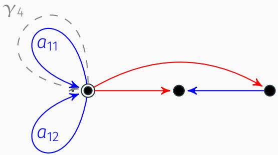

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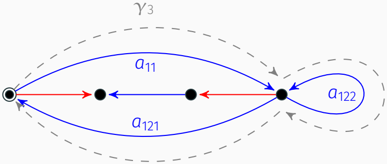

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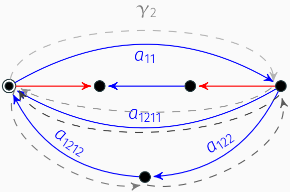

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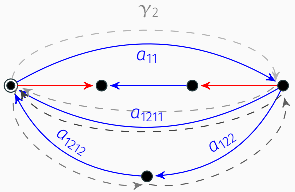
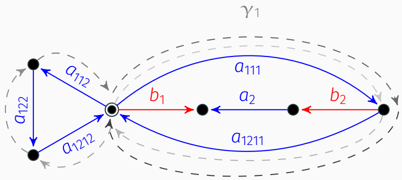
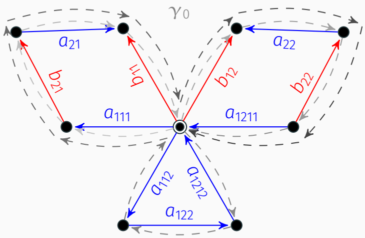


Lifting up to Γ_1, we get $\gamma_1 = a_{111}a_{1211}a_{1212}^{-1}a_{122}^{-1}a_{112}^{-1}a_{111}a_{1211}$:



Finally, lifting to $\Gamma_0 = \mathrm{Fl}(U)$, we get:

$$\gamma_0 = a_{111} b_{21} a_{21} b_{11}^{-1} b_{12} a_{22}^{-1} b_{22}^{-1} a_{1211} a_{1212}^{-1} a_{122}^{-1} a_{112}^{-1} a_{111} b_{21} a_{21} b_{11}^{-1} b_{12} a_{22}^{-1} b_{22}^{-1} a_{1211}$$



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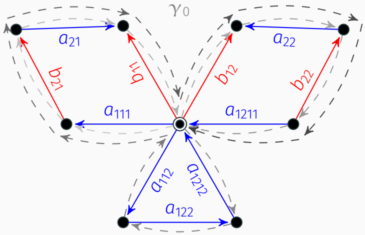

Factorizing through the visits to ◉, we get the desired word:

$$\begin{aligned} a &= (abab^{-1})(ba^{-1}b^{-1}a)(a^{-1}a^{-1}a^{-1})(abab^{-1})(ba^{-1}b^{-1}a) \\ &= u_2u_3^{-1}u_1^{-1}u_2u_3^{-1}. \end{aligned}$$

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Taking $\gamma_4 = a_{12}$ (instead of $\gamma_4 = a_{11}$) at the closed folding, we get the alternative expression:

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The responsible for this is the closed folding …

In general,

At every closed folding $\Gamma_i \curvearrowright \Gamma_{i+1}$, take the reduced non-trivial walk

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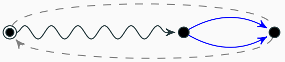

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Proposition

Let $\{u_1, \ldots, u_n\}$ be a set of generators for the (free) subgroup $H = \langle u_1, \ldots, u_n \rangle \leqslant \mathbb{F}_A$. Then,

$$H = \left\langle u_1, \ldots, u_n \mid w_i = 1 \text{ for each closed folding} \right\rangle$$

is a presentation for H with generators $\{u_1, \ldots, u_n\}$.

Definition

Let G be a group, $H \leqslant G$ a subgroup. An ***equation over H*** is an expression of the form $w(X) = h_0 X^{\epsilon_1} h_1 \cdots X^{\epsilon_n} h_n \in H * \langle X \rangle = H * \mathbb{Z}$, where $h_0, \ldots, h_n \in H$, $\epsilon_1, \ldots \epsilon_n = \pm 1$, and $h_i = 1 \Rightarrow \epsilon_i = \epsilon_{i+1}$, for $i = 1, \ldots, n-1$. The ***degree*** is n (for $n = 0$ it is a ***trivial*** equation).

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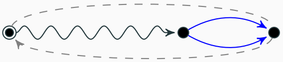

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Theorem

Let $w_1(X), \ldots, w_k(X) \in H * \langle X \rangle$ be the equations computed from the $k \geqslant 0$ closed foldings in the tower. Then,

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Let $w_1(X), \ldots, w_k(X) \in H * \langle X \rangle$ be the equations computed from the $k \geqslant 0$ closed foldings in the tower. Then,

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Ascari. ***Ideals of equations for elements in a free group and Stallings folding***, arXiv.2207.04759v1.

Cosets and index

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Remark: $\mathrm{Sch}(H)$ is a connected, deterministic, and saturated (but not necessarily core) automaton recognizing H.

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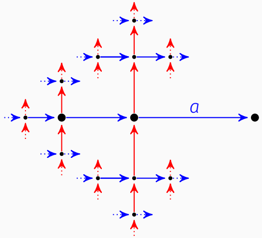

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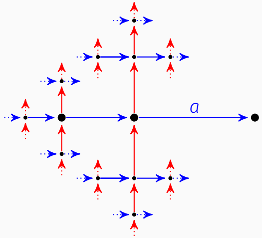

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$\mathrm{Sch}(H)$ is the automaton obtained after adjoining an a-Cayley branch to every a-deficient vertex in $\mathrm{St}(H)$.

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Decide, given words $u_1, \ldots, u_k \in (A^{\pm})^*$, whether $\langle u_1, \ldots, u_k \rangle_G$ has finite index in G.

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Given a finite $S \subseteq \mathbb{F}_A$, we can compute the index of $\langle H \rangle$ in $\mathbb{F}_A$.
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FREE FACTORS AND HANDSHAKING LEMMA

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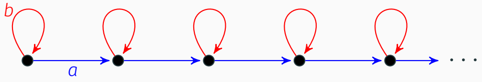

MARSHALL-HALL THEOREM AND RESIDUAL FINITENESS

Theorem (Marshall-Hall Jr.)

If H is a finitely generated subgroup of a free group $\mathbb{F}$, then H is a free factor of a finite-index subgroup of $\mathbb{F}$; i.e.,

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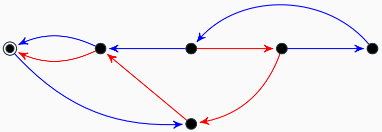

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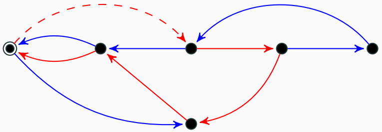

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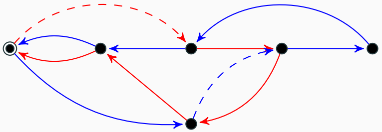

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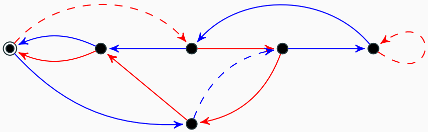

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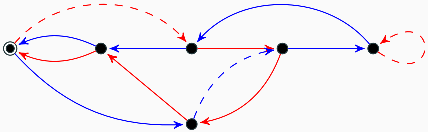

Definition: G is ***residually finite*** if $\forall g \in G \setminus \{1\}$, $\exists N \trianglelefteq_{f.i.} G$ s.t. $g \notin N$.

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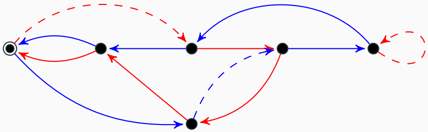

Definition: G is ***residually finite*** if $\forall g \in G \setminus \{1\}$, $\exists N \trianglelefteq_{f.i.} G$ s.t. $g \notin N$.

Theorem

Finitely generated free groups are residually finite.

Theorem (Marshall-Hall Jr.)

If H is a finitely generated subgroup of a free group $\mathbb{F}$, then H is a free factor of a finite-index subgroup of $\mathbb{F}$; i.e.,

$$H \leqslant_{\mathrm{fg}} \mathbb{F} \;\Rightarrow\; \exists K : \; H \leqslant_{\mathrm{ff}} K \leqslant_{\mathrm{fi}} \mathbb{F}.$$

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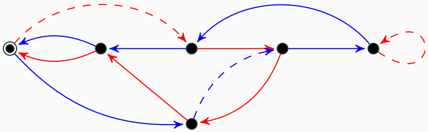

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Prove it using Stallings automata!

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*The **subgroup conjugacy problem** $\mathrm{SCP}(\mathbb{F}_n)$ is decidable.*

$$\mathrm{SCP}(G) \equiv H \sim K\,?_{H,K \leqslant_{\mathrm{fg}} G}$$

Intersections

THE SUBGROUP INTERSECTION PROBLEM

Subgroup Intersection Problem

Given $u_1, \ldots, u_k; v_1, \ldots, v_l \in \mathbb{F}_A$, decide whether the intersection of $H = \langle u_1, \ldots, u_k \rangle$ and $K = \langle v_1, \ldots, v_l \rangle$ is finitely generated; when this is the case, compute generators for $H \cap K$.

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Just playing, we realized that $a^3,\ b^{-1}a^3b,\ a^{-1}ba^3b^{-1}a \in H \cap K$. What else?

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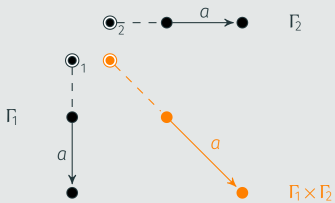

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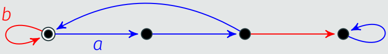

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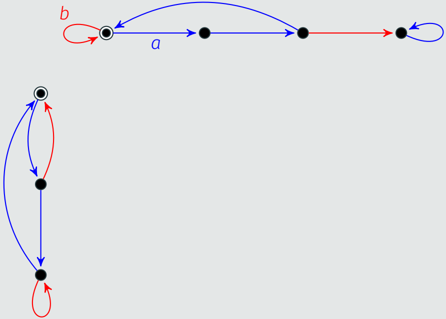

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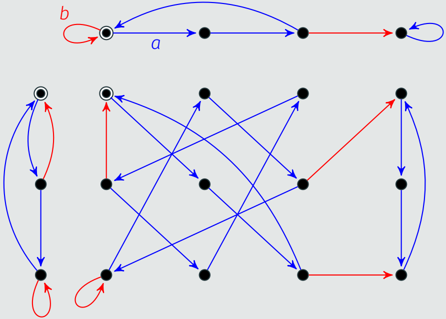

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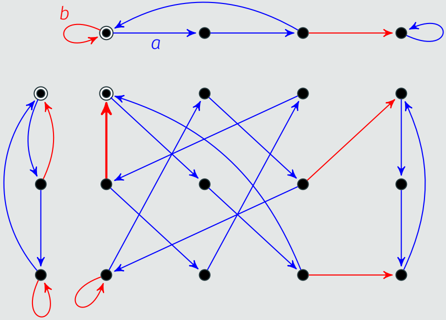

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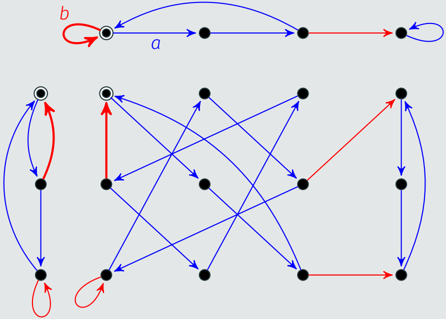

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Two immediate applications follow ...

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(i) Draw the Stallings A-automaton $\mathrm{St}(H)$ for $H = \langle u_1, \ldots, u_k \rangle$;

(ii) draw the Stallings A-automaton $\mathrm{St}(K)$ for $K = \langle v_1, \ldots, v_l \rangle$;

(iii) compute the product $\mathrm{St}(H) \times \mathrm{St}(K)$;

(iv) take the connected component containing $\odot$ and compute its core;

(v) choose a spanning tree and read a free basis for $H \cap K$. $\square$

Example

To compute $H \cap K$ with $H = \langle b, a^3, a^{-1}bab^{-1}a \rangle$, $K = \langle ab, a^3, a^{-1}ba \rangle$ ...

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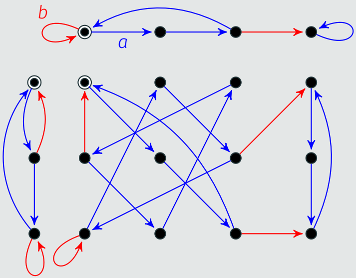

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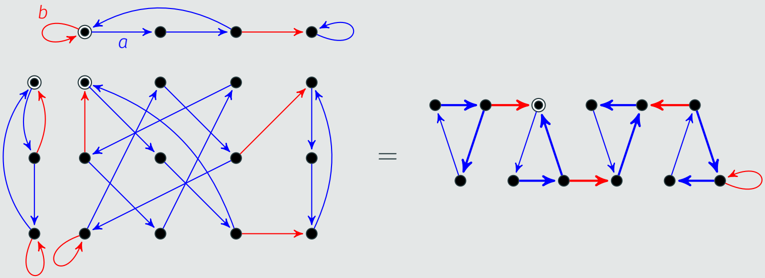

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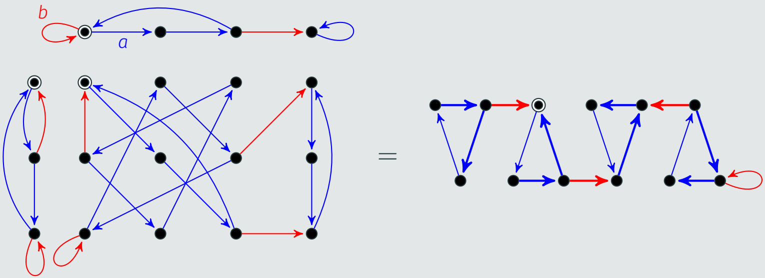

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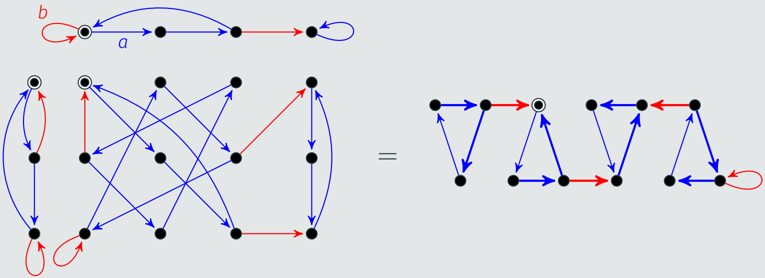

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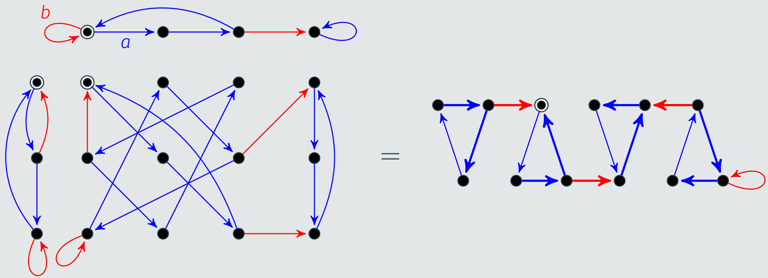

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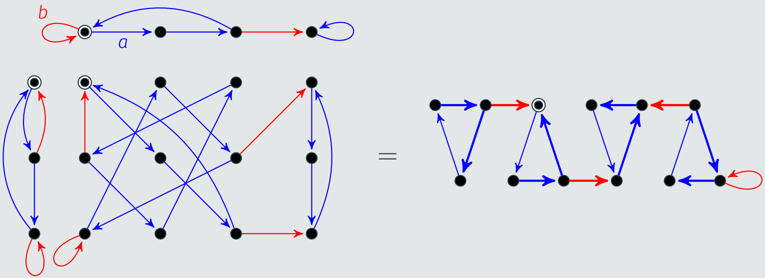

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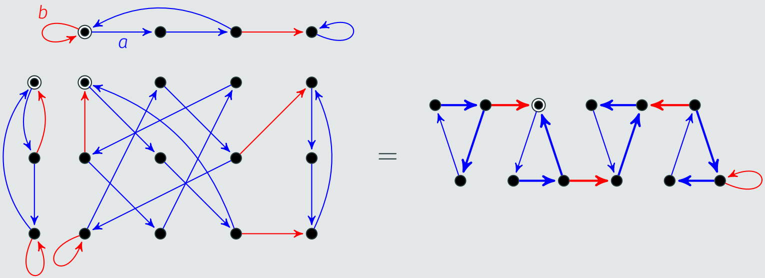

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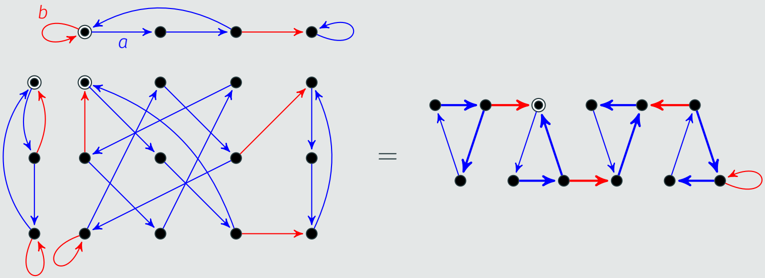

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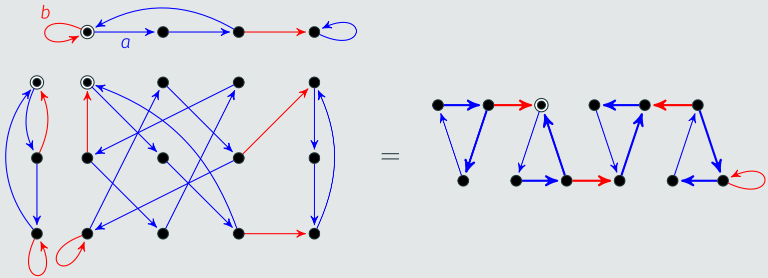

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Hence, the intersection $H \cap K$ has rank equal to 5.

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Given $u, u_1, \ldots, u_k; v, v_1, \ldots, v_l \in \mathbb{F}_A$, decide whether the coset intersection $\langle u_1, \ldots, u_k \rangle u \cap \langle v_1, \ldots, v_l \rangle v$ is empty and, if not, compute a coset representative.

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Observation

If $\Gamma = \mathrm{St}(H)$ and $\gamma = \odot \overset{u}{\rightsquigarrow} p$, then $\overline{\mathcal{L}_{\odot,p}}(\Gamma) = Hu$.

THE COSET INTERSECTION PROBLEM FOR FREE GROUPS

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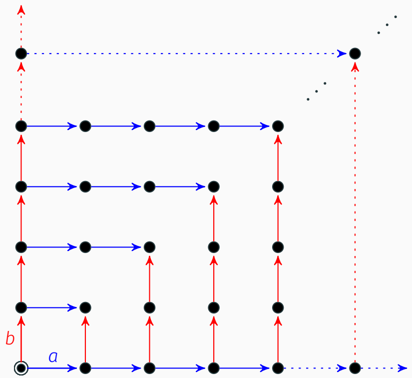

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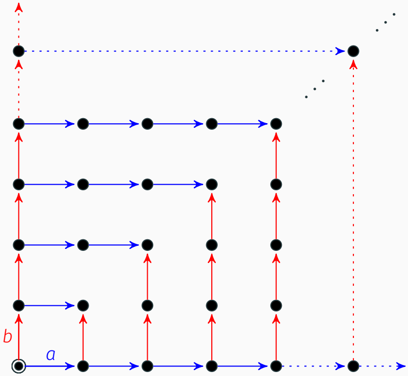

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INTERSECTION AND FREE FACTORS

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- consider $\mathrm{St}(K \cap L)$ and compute $H \cap L = H \cap (K \cap L)$ by looking at the pull-back $\mathrm{St}(H) \times \mathrm{St}(K \cap L)$: it is just the subautomaton of $\mathrm{St}(K \cap L)$ determined by the A-labelled arcs;
- hence, $H \cap L \leqslant_{\text{f.f.}} K \cap L$.

Applying this fact twice, $H \cap H' \leqslant_{\text{f.f.}} K \cap H' \leqslant_{\text{f.f.}} K \cap K'$. $\square$

Definition

The ***reduced rank*** of a group G is $\widetilde{\mathrm{rk}}(G) = \max\{\mathrm{rk}(G) - 1,\ 0\}$, i.e., $\widetilde{\mathrm{rk}}(G) = \mathrm{rk}(G) - 1$ except for the trivial group, for which $\widetilde{\mathrm{rk}}(\{1\}) = 0$.

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BRIEF HISTORY OF THE HANNA NEUMANN INEQUALITY

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Theorem (J. Friedman, 2015; I. Mineyev, 2012)

The factor 2 can be removed in both theorems.

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- conjugating appropriately, we can assume that $\mathrm{St}(H)$ and $\mathrm{St}(K)$ have no vertices of degree 1;
- forget about the double cosets (till the end of proof) and let us show $\widetilde{\mathrm{rk}}(W) \leqslant 2\,\widetilde{\mathrm{rk}}(\mathrm{St}(H))\,\widetilde{\mathrm{rk}}(\mathrm{St}(K))$, where $W = \mathrm{St}(H) \times \mathrm{St}(K)$ and

$$\widetilde{\mathrm{rk}}(W) = \sum_{C \text{ c.c. } W} \widetilde{\mathrm{rk}}(C) = \sum_{C \text{ c.c. } W} \max\{|EC| - |VC|,\ 0\}.$$

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Let X be a finite connected graph. Then,

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(iii) *if* (p,q) *is of degree 1 in W, then* $\big(d(p,q) - 2\big) + 1 \leqslant \big(d(p) - 2\big)\big(d(q) - 2\big)$.

Now,

$$2\,\widetilde{\mathrm{rk}}(W) = \sum_{\substack{C \text{ c.c. } W \\ \text{not tree}}} 2\,\widetilde{\mathrm{rk}}(C)$$

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Finally, let us link the connected components of W with the double cosets $H\backslash\mathbb{F}_A/K$, ...

Lemma

Let $(p, \circledcirc), (p', \circledcirc)$ *be two vertices in* W*, and let* $\circledcirc \overset{x}{\rightsquigarrow} p$ *and* $\circledcirc \overset{x'}{\rightsquigarrow} p'$ *be walks in* $\mathrm{St}(H)$*. Then,*

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$(p, \odot)$ and $(p', \odot)$ belong to the same c.c. of W $\Leftrightarrow$ $HxK = Hx'K$.

Corollary

The following map is a bijection

$$
\begin{array}{rcl}
\alpha\colon H\backslash\mathbb{F}_A/K & \to & \{\text{c.c. of } W\} \\
HxK & \mapsto & \text{the c.c. containing } (p,\odot), \text{ where } \odot \overset{x}{\rightsquigarrow} p \\
H\bar{\ell}(\odot \rightsquigarrow p)K & \leftarrow\!\shortmid & C\,, \text{ where } (p,\odot) \in \mathrm{V}C
\end{array}
$$

further satisfying that, for every $x \in \mathbb{F}_A$, $\langle \alpha(HxK) \rangle_{(p,\odot)} = H^x \cap K$.

Quotients of automata

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- In basic linear algebra:

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Let $H \leqslant K \leqslant \mathbb{F}_A$. We say that $H \leqslant K$ is an ***algebraic extension***, denoted by $\boldsymbol{H \leqslant_{\mathrm{alg}} K}$, if H is not contained in any proper free factor of K, i.e., if

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ALGEBRAIC AND FREE EXTENSIONS

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- if $r(H) \geqslant 2$ and $r(K) \leqslant 2$ then $H \leqslant_{\mathrm{alg}} K$;
- if $H \leqslant_{\mathrm{alg}} K$ and $H \leqslant_{\mathrm{ff}} K$ then $H = K$.

Proposition (Miasnikov–V.–Weil, 2007)

Let $H \leqslant M_i \leqslant K \leqslant \mathbb{F}_A$, for $i = 1, 2$. Then,

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- Additionally, $\mathcal{AE}(H)$ will be computable...

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A morphism of reduced A-automata $f\colon \Gamma_1 \to \Gamma_2$ is called ***onto*** if every edge in Γ_2 is the image of at least one edge from Γ_1. Then, we say that Γ_2 is a ***quotient*** of Γ_1, and write $f\colon \Gamma_1 \twoheadrightarrow \Gamma_2$.

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Let Γ be a finite reduced A-automata, and let $\sim$ be an equivalence relation on $V\Gamma$. We denote by $\Gamma/\sim$ the new reduced A-automata resulting from identifying the vertices according to $\sim$, plus reduction.

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FRINGE OF A SUBGROUP

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For $H \leqslant_{\mathsf{fg}} \mathbb{F}_A$, we have $\mathcal{O}(H) = \{H_0, H_1, \ldots, H_k\}$, all f.g., computable, and with minimum and maximum, $H = H_0 \leqslant H_i \leqslant H_k = \langle A' \rangle \leqslant_{\mathsf{ff}} \mathbb{F}_A$, where $A' \subseteq A$ is the set of letters in use.

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DECIDING FREE-FACTORNESS

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For $H \leqslant K$, $Cl_K(H)$ is the maximal algebraic extension of H contained in K; in particular, it is computable from given generators of H and K.

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Compare with M. Hall's Theorem.

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$\mathcal{V}$ is ***extension-closed*** if $V \trianglelefteq W$ with $V, W/V \in \mathcal{V} \Rightarrow W \in \mathcal{V}$.

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Let G be a group, and $\mathcal{V}$ be a pseudo-variety of finite groups. The ***pro-$\mathcal{V}$ topology on G*** can be defined in several equivalent ways:

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$$d(x, y) = 2^{-r(x,y)},$$

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Observation:

The pro-$\mathcal{V}$ top. is Hausdorff $\Leftrightarrow$ d is a metric $\Leftrightarrow$ G is residually-$\mathcal{V}$.

Proposition (Ribes, Zaleskiĭ)

Let $\mathcal{V}$ be an extension-closed pseudo-variety, and consider $\mathbb{F}_A$ with the pro-$\mathcal{V}$ topology. For a given $H \leqslant_{\mathsf{fg}} \mathbb{F}_A$,

$$H \text{ is } \mathcal{V}\text{-closed} \iff H \text{ is a free factor of a clopen subgroup.}$$

In particular, free factors of $\mathcal{V}$-closed subgroups are $\mathcal{V}$-closed.

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Problem

Find an algorithm to compute the solvable closure $Cl_{sol}(H)$ of a given $H \leqslant_{\mathbf{fg}} \mathbb{F}_A$.

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Theorem (Antolín–Jaikin-Zapirain, 2021)

Let $S \subseteq \mathsf{End}(G)$, where $G = \mathbb{F}_n$ or $G = \mathbb{S}_n$. Then, $\mathsf{Fix}(S)$ is inert.

Theorem (Martino–V. 2003)

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Question

Is the lattice of fixed subgroups of $\mathbb{F}_n$ (by autos or endos) closed under intersections?

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- Hence, $H = M_r \leqslant_{ff} \mathrm{Fix}(\varphi\phi^r)$. $\square$

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- Gromov, Arjantseva, Ol'shanskii, Kapovich, Miasnikov, Schupp, Shpilrain, Ollivier, Jitsukawa, Bassino, Nicaud, W. …

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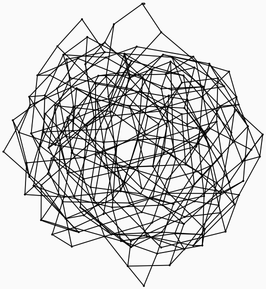

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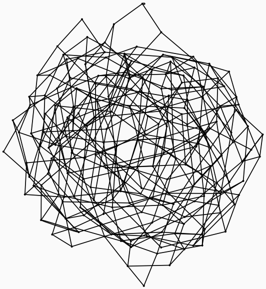

- Work by Bassino, Martino, Nicaud, V., W.

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- Random generation strategy: draw independently, uniformly at random, $|A|$ partial injections, select randomly a base point. This *almost* works...

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- Refer to the Bible: Ph. Flajolet, R. Sedgewick, *Analytic combinatorics*, Cambridge University Press, 2009

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- Example. The EGS of 1 point is z. A permutation is a labeled sequence of points: its EGS is $\frac{1}{1-z} = \sum \frac{n!}{n!} z^n$

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- Also: $\frac{PI_{n-1}}{PI_n} \leqslant \frac{1}{2n}$

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Let $F(z, y)$ is a real function, analytic at $(0, 0)$. Let $J(z) = \sum_{n>0} j_n z^n$, $C(z) = \sum_{n>0} c_n z^n$ and $D(z) = \sum_{n>0} d_n z^n$ with $C(z) = F(z, J(z))$ and $D(z) = \frac{\partial F}{\partial y}(z, J(z))$. If $j_{n-1} = o(j_n)$ and there exists $s \geqslant 1$ such that $\sum_{k=s}^{n-s} |j_k j_{n-k}| = \mathcal{O}(j_{n-s})$, then $c_n = \sum_{k=0}^{s-1} d_k j_{n-k} + \mathcal{O}(j_{n-s})$.

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Proposition

The probability that a size n tuple of partial injections is connected is $1 - \frac{2^r}{n^{r-1}} + o(\frac{1}{n^{r-1}})$: connectedness holds with probability tending to 1

HANDLING CORENESS: COUNT SEQUENCES

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- Let X_n be the random variable which counts the number of sequences in a partial injection of size n.

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Proposition (statistics on the number of sequences)

$\mathbb{E}(X_n) = \sqrt{n}(1 + o(1))$ and $\sigma^2(X_n) = n(1 + o(1))$

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Algorithm

A rejection algorithm to randomly generate a subgroup of $\mathbb{F}_r$:

Draw a random partial injection f_a of $[n]$, independently for each $a \in A$; if the $(f_a)_{a \in A}$ do not induce a Stallings automaton (with base vertex 1), reject and repeat.

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Draw a random partial injection f_a of $[n]$, independently for each $a \in A$; if the $(f_a)_{a \in A}$ do not induce a Stallings automaton (with base vertex 1), reject and repeat.

- The *expected* number of steps is at most 2

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- Still needed: an efficient random generation algorithm for partial injections

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The number of size n subgroups in $\mathbb{F}_r$ is

$$\frac{1}{n!} PI_n^r \left(1 + o(1)\right) \quad \sim \quad n!^{r-1} \frac{n^{1-r/4} e^{2r\sqrt{n}}}{(2\sqrt{e\pi})^r}$$

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DISTRIBUTION OF THE SIZES OF COMPONENTS OF A PARTIAL INJECTION 1/2

- Pointing operator: selecting a vertex in a partial injection. The corresponding EGS is $\Theta \mathrm{PInj}(z) = \sum_n \frac{nPI_n}{n!} z^n = z\frac{d}{dz}\mathrm{PInj}(z)$

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- Now we can randomly generate a partial injection

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- It looks complicated…but it is fast!
- We are dealing with very large numbers: $PI_n \geqslant (n+1)!$ has size $\mathcal{O}(n \log n)$: in the bitcost model, the precomputation is in $\mathcal{O}(n^2 \log n)$ and the cost of one generation is $\mathcal{O}(n^2 \log^2 n)$

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- Comparing the number of size n saturated Stallings automata with the number of general Stallings automata yields the following probability: $\mathcal{O}(n^{r/4}e^{-2r\sqrt{n}}) = o(n^{-k})$

MORE ASYMPTOTIC PROPERTIES

Theorem (Bassino, Martino, Nicaud, V., W.)

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With probablility tending to e^{-r}, H fails to contain a conjugate of a letter.

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- Variant: use the sphere rather than the ball.
- Easy to implement, and questionable (uniqueness).

THE CENTRAL TREE PROPERTY: FREE GENERATION

- The central tree property for $\vec{h} = (h_1, \ldots, h_k)$: small initial cancellation = $\mathrm{St}(H)$ consists of a central tree, and of one loop for each h_i connecting leaves of the tree.

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- If the central tree property holds, then $\vec{h}$ freely generates H.
- Also note: the central tree is usually very small: fix $f(n)$ an unbounded, non-decreasing function. In the few-generator model, generically (only), $\mathrm{lcp}(\vec{h}) < f(n)$.

THE CENTRAL TREE PROPERTY: MALNORMALITY

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- Assume that the central tree property holds. A sufficient condition for *malnormality* can be expressed in terms of common factors occurring in the h_i:
- if $\mathrm{lcp}(\vec{h}) < \frac{1}{4} \min \vec{h}$ and no word of length $\frac{1}{8} \min \vec{h}$ occurs twice as a factor of the elements of $\vec{h}$ and $\vec{h}^{-1}$, then H is malnormal.

- Rigidity: if $\vec{g}$ and $\vec{h}$ have the central tree property and $H(\vec{g}) = H(\vec{h})$, then $\vec{g}$ and $\vec{h}$ coincide up to the order of their elements and replacing a word by its inverse.

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- So: picking a tuple of generators at random is — in practice — a method to randomly generate a subgroup in the sense that collisions are exponentially rare.
- The distribution of subgroups induced is radically different from the distribution based on drawing Stallings automata.
- Malnormality is generic in the word-based model, and negligible in the graph-based model.

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- [Bassino, Nicaud, W.] Whitehead minimality is exponentially generic in the few-generator model (Kapovich, Schupp, Shpilrain for cyclic subgroups)
  and it is also exponentially generic in the graph-based model.

GROUP PRESENTATIONS: AN ODD RESULT

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- Up to density 1/2, $\langle A \mid \vec{h} \rangle$ is generically infinite, hyperbolic (Gromov, Ol'shanskii, Ollivier).
- But the probability that $\mathbb{F}_A / \langle\langle H \rangle\rangle$ is trivial tends to 1 as the size of n grows to infinity.

BEYOND FREE GROUPS: FEW GENERATORS IN HYPERBOLIC GROUPS

- [Gilman, Miasnikov, Osin, 2010] Let G be hyperbolic, A-generated and let $k \geqslant 1$. Exponentially generically, a random k-tuple $\vec{h} = (h_1, \ldots, h_k)$ of elements of G freely generates the subgroup $H(\vec{h}) = \langle \vec{h} \rangle$ of G, and $H(\vec{h})$ is quasi-convex.

- [Kharlampovich, Miasnikov, W., 2017] Let $G = \langle A \mid R \rangle$, finite presentation. Assume that L is a language of representatives. Let $H \leqslant G$ and $\Gamma_L(H)$ be the fragment of the Schreier graph $S(G, H)$ spanned by the loops at H labeled by the L-representatives of the elements of H.

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- Computable if H is L-quasi-convex (semi-algorithm)
- Examples: quasi-convex subgroups of hyperbolic groups, all subgroups of virtually free subgroups.

STALLINGS AUTOMATA

- [Kharlampovich, Miasnikov, W., 2017] Let $G = \langle A \mid R \rangle$, finite presentation. Assume that L is a language of representatives. Let $H \leqslant G$ and $\Gamma_L(H)$ be the fragment of the Schreier graph $S(G, H)$ spanned by the loops at H labeled by the L-representatives of the elements of H.
- A good analogue of Stallings automata: finite if and only if H is L-quasi-convex; membership problem, computation of intersections, decision of finiteness; under reasonable additional hypotheses on G: decision of conjugacy, almost malnormality.
- Computable if H is L-quasi-convex (semi-algorithm)
- Examples: quasi-convex subgroups of hyperbolic groups, all subgroups of virtually free subgroups.
- Generalizes work by Short, Gersten, Kapovich, Gitik, Markus-Epstein, Silva, Soler-Escriva, V.

- [Bassino, Nicaud, W.] The particular case of subgroups of $\mathrm{PSL}_2(\mathbb{Z}) = \mathbb{Z}_2 * \mathbb{Z}_3 = \langle a, b \mid a^2 = b^3 = 1 \rangle$: the Stallings automata are combinatorially nice enough and can be counted: statistics, random generation.

THE MODULAR GROUP

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- E.g., the expected isomorphism type of a subgroup of $\mathrm{PSL}_2(\mathbb{Z})$ of size n is

$$\left(n^{\frac{1}{2}} + o(n^{\frac{1}{2}}), n^{\frac{1}{3}} + o(n^{\frac{1}{3}}), \frac{n}{6} - \frac{1}{3}n^{\frac{2}{3}} + o(n^{\frac{2}{3}})\right),$$

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  and there is strong concentration around these values.
- Also: counting and random generation of finite index subgroups (Stothers, 1970s), free subgroups, subgroups of a fixed isomorphism type.

Enriched Stallings automata

A group is ***free-abelian by free (FABF)*** if it is of the form

$$G_\alpha = \mathbb{F}_n \ltimes_\alpha \mathbb{Z}^m = \left\langle \begin{array}{c} x_1, \ldots, x_n \\ t_1, \ldots, t_m \end{array} \;\middle|\; \begin{array}{ll} t_i t_k = t_k t_i & \forall i, k \in [1, m] \\ x_j^{-1} t_i x_j = t_i \alpha_j & \forall i \in [1, m], \forall j \in [1, n] \end{array} \right\rangle,$$

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FREE-ABELIAN BY FREE GROUPS

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- If $\mathbf{A_1}, \mathbf{A_2}, \ldots, \mathbf{A_n} = \mathbf{I_m}$, then

$G_\alpha = \mathbb{F}_n \times \mathbb{Z}^m$ is a ***free-abelian times free (FATF)*** group.

Let $H \leqslant G_\alpha = \mathbb{F}_n \ltimes_\alpha \mathbb{Z}^m$ and consider the short exact sequence associated to G_α and its restriction to H:

$$\begin{array}{ccccc}
\mathbb{Z}^m & \rightarrowtail & G_\alpha & \underset{\pi}{\overset{\sigma}{\rightleftarrows}} & \mathbb{F}_n \\
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L_H = H \cap \mathbb{Z}^m = \ker(\pi_{|H}) & \rightarrowtail & H & \underset{\sigma_H}{\overset{\pi_{|H}}{\rightleftarrows}} & H\pi
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SUBGROUPS OF FABF GROUPS

Let $H \leqslant G_\alpha = \mathbb{F}_n \ltimes_\alpha \mathbb{Z}^m$ and consider the short exact sequence associated to G_α and its restriction to H:

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Let $H \leqslant G_\alpha = \mathbb{F}_n \ltimes_\alpha \mathbb{Z}^m$. Then,

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Corollary

Subgroups of FABF (resp., FATF) groups are again FABF (resp FATF).

Recall that every subgroup $H \leqslant G_\alpha$ splits as:

$$H = H\pi\sigma \ltimes (H \cap \mathbb{Z}^m), \tag{1}$$

where $\sigma \colon H\pi \to G_\alpha$ is a section of $\pi_H \colon H \to H\pi$

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Definition

A ***'basis'*** of a subgroup $H \leqslant G_\alpha$ is a pair

$$(V\sigma; B) \;=\; (v_1 t^{\mathbf{c}_1}, v_2 t^{\mathbf{c}_2}, \ldots, v_{n'} t^{\mathbf{c}_{n'}}; t^{\mathbf{b}_1}, t^{\mathbf{b}_2}, \ldots, t^{\mathbf{b}_{m'}})$$

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Remark. Note that $V\sigma$ is a free basis of the subgroup $H\pi\sigma$, hence:

- A ***basis*** of H is the result of joining a basis of each factor in (1).

COMPLETION

Let $H \leqslant G_\alpha = \mathbb{F}_n \ltimes \mathbb{Z}^m$ and let $w \in \mathbb{F}_n$.

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If $\{v_1 t^{\mathbf{c}_1}, \ldots, v_{n'} t^{\mathbf{c}_{n'}}; t^{\mathbf{b}_1}, \ldots, t^{\mathbf{b}_{m'}}\}$ is a basis of $\mathbb{F}_n \times \mathbb{Z}^m$ and $w \in \mathbb{F}_n$, then

$$\mathbf{c}_H(w) \;=\; \begin{cases} \varnothing & \text{if } w \notin H\pi \\ w\phi\rho\mathbf{C} + L_H & \text{if } w \in H\pi \,, \end{cases}$$

where $\phi : H\pi \to \mathbb{F}_{n'}$ is the change of basis $x_i \mapsto x_i(v_j)$,

$\rho \colon \mathbb{F}_{n'} \twoheadrightarrow \mathbb{Z}^{n'}$ is the abelianization map,

$\mathbf{C}$ is the $n' \times m$ integer matrix having $\mathbf{c}_i$ as ith row.

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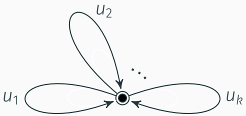

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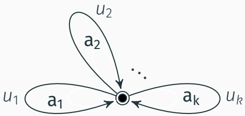

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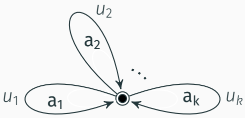

ENRICHED FLOWER AUTOMATA

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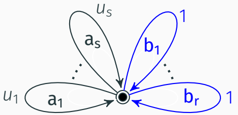

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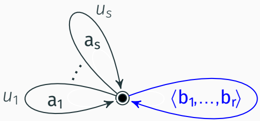

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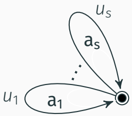

$\langle \mathsf{b}_1, \ldots, \mathsf{b}_r \rangle \leqslant \mathbb{Z}^m$

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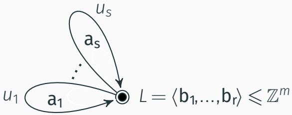
$\mathcal{F}_S \equiv$ 

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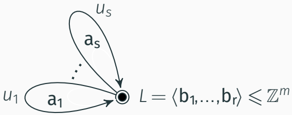

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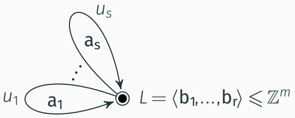
$\mathcal{F}_S \equiv$ 

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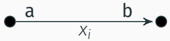

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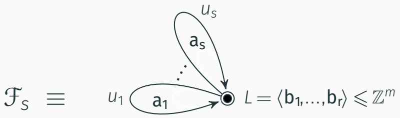

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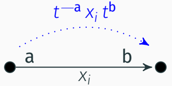

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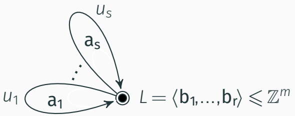

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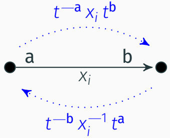

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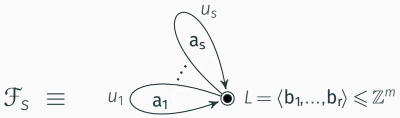

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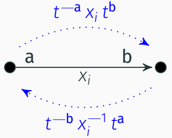

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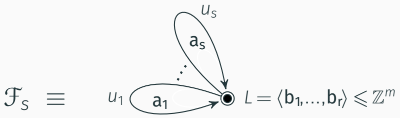

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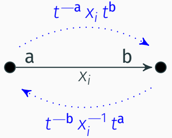

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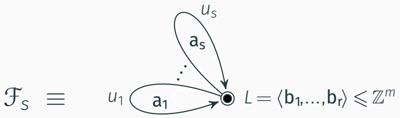

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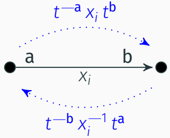

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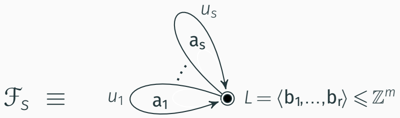

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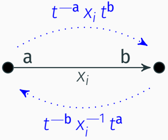

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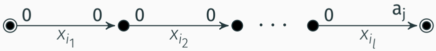

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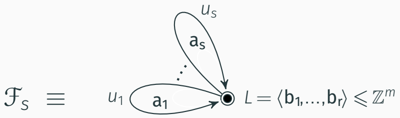

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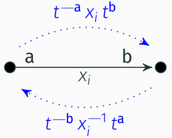

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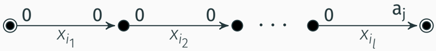

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ENRICHED FLOWER AUTOMATA

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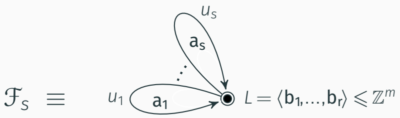

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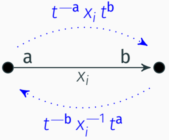

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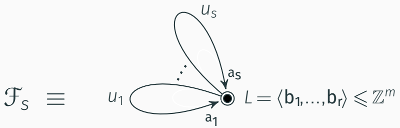

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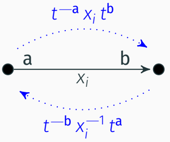

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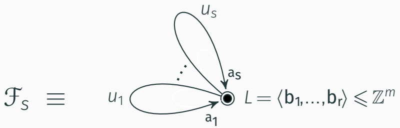

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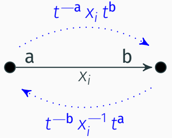

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- $\mathcal{F}_S$ is called the ***(enriched) flower automaton of S***.

ENRICHED AUTOMATA

Definition

A $\mathbb{Z}^m$-***enriched X-automaton*** $\widehat{\Gamma}_L = (\widehat{\Gamma}, L)$ is a pointed involutive automaton $\widehat{\Gamma}$ with:

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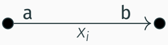

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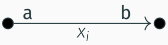

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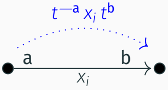

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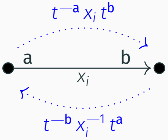

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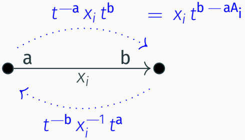

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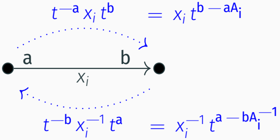

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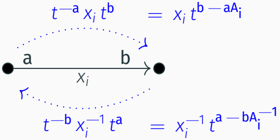

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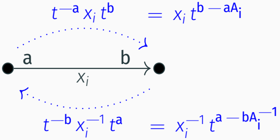
1. the basepoint $\odot$ labelled by a subgroup $L \leqslant \mathbb{Z}^m$.

and every arc having:

2. a middle "free label" $x_i \in X$.
3. two "abelian labels" $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^m$ in the head and tail respectively,

meant to be read (for a given action $\alpha = (\mathbf{A_i})_i \colon \mathbb{F}_X \to \mathsf{GL}_m(\mathbb{Z})$):



Definition.

The ***subgroup recognized*** by $\widehat{\Gamma}_L$ in G_α, denoted by $\langle \widehat{\Gamma}_L \rangle_\alpha$ is the set of α-enriched labels of $\odot$-walks in $\widehat{\Gamma}$.

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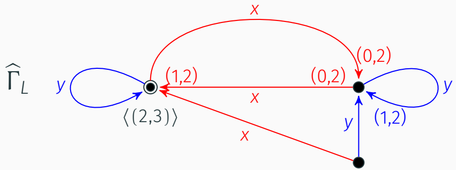
Example: A $\mathbb{Z}^2$-enriched $\{x, y\}$-automaton and its skeleton



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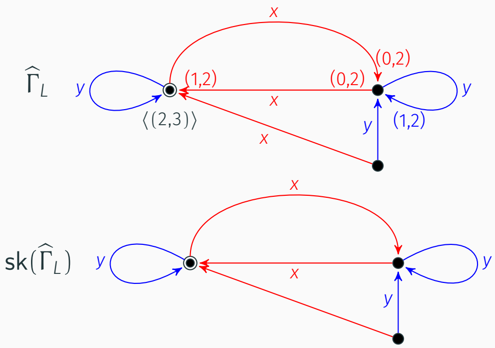

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In order to get rid of these redundancy we introduce different kinds of transformations ...

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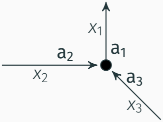

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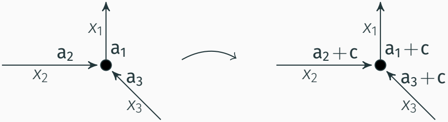

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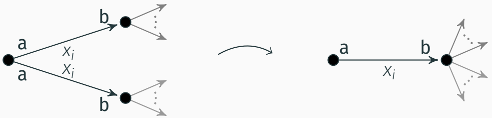

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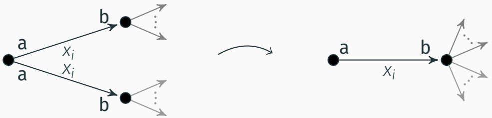

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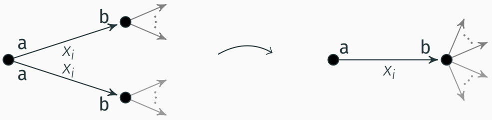

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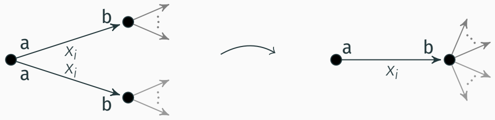

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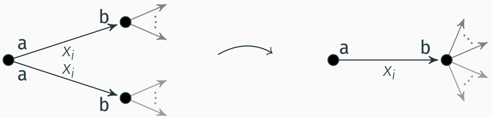

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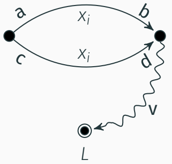

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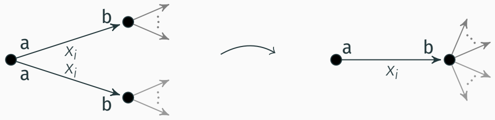

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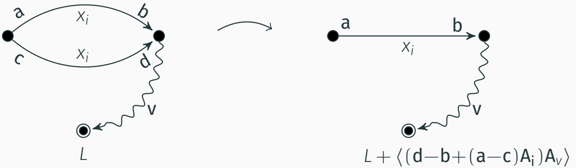

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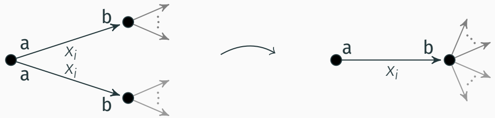

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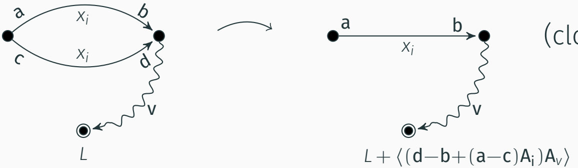

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Proof. Play with abelian transformations. □

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Let $\widehat{\Gamma}_L$ be a reduced automaton recognizing $H \leqslant G_\alpha$. Then,

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NORMALIZATION

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In order to fix redundancy of type II, we choose a spanning tree T in $\widehat{\Gamma}_L$ and we use abelian transformations to obtain an equivalent automaton $\widehat{\boldsymbol{\Delta}}_L$ with zeros at every abelian arc-label except at the end of the arcs outside T.

Lemma

For every reduced automata $\widehat{\Gamma}_L$ and every spanning tree T in $\widehat{\Gamma}_L$ there exists a unique equivalent T-normalized automaton (modulo L).

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Theorem (D.–V.)

There exists a (computable) bijection

$$\begin{array}{rcl} \{\,\textit{(f.g.) subgroups of } \mathbb{F}_n \ltimes \mathbb{Z}^m\,\} & \to & \mathfrak{S} \subseteq \{\,\textit{(finite) enriched automata}\,\} \\ H & \mapsto & \mathsf{St}\,(H) \end{array}$$

FIRST APPLICATIONS: MEMBERSHIP PROBLEM

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A basis for a finitely generated subgroup $H \leqslant G_\alpha$ is computable from any finite set of generators.

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Membership Problem for $G = \langle X \mid R \rangle$, MP($G$)

Given $u, v_1, \ldots, v_k \in \mathbb{F}_X$, decide whether $u \in H = \langle v_1, \ldots, v_k \rangle_G$;
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3. if the final vertex is not $\odot$ return NO;
4. compute the completion $\mathbf{c}_w$ of w in $\widehat{\Gamma}_L$ and check whether $\mathbf{a} - \mathbf{c}_w \in L$. If so return YES, otherwise return NO. □

Intersections in $\mathbb{F}_n \times \mathbb{Z}^m$

FREE-ABELIAN TIMES FREE GROUPS

A group is ***free-abelian times free (FATF)*** if it is of the form

$$\mathbb{F}_n \times \mathbb{Z}^m = \left\langle \begin{array}{l} x_1, \ldots, x_n \\ t_1, \ldots, t_m \end{array} \;\middle|\; \begin{array}{ll} t_i t_k = t_k t_i & \forall i, k \in [1, m] \\ x_j^{-1} t_i x_j = t_i & \forall i \in [1, m], \forall j \in [1, n] \end{array} \right\rangle$$

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Normal form: $w\, t_1^{a_1} \cdots\ t_m^{a_m} = w\, t^{\mathbf{a}} \quad (w \in \mathbb{F}_n, \mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{Z}^m)$.

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Let $H \leqslant \mathbb{F}_n \times \mathbb{Z}^m$. Then,

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$$H \text{ is finitely generated } \Leftrightarrow\; H\pi \text{ is finitely generated}$$

BASES

A basis for $H \leqslant \mathbb{F}_n \times \mathbb{Z}^m$ has the form:

$$v_1 t^{\mathbf{a}_1}, \ldots, v_n t^{\mathbf{a}_{n'}}; t^{\mathbf{b}_1}, \ldots, t^{\mathbf{b}_{m'}}$$

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If $\{v_1 t^{\mathsf{a}_1}, \ldots, v_{n'} t^{\mathsf{a}_{n'}}; t^{\mathsf{b}_1}, \ldots, t^{\mathsf{b}_{m'}}\}$ is a basis of H and $w \in \mathbb{F}_n$, then

$$\mathsf{c}_H(w) = \begin{cases} \varnothing & \text{if } w \notin H\pi \\ w\phi\rho\mathbf{A} + L & \text{if } w \in H\pi \end{cases},$$

where $\phi : H\pi \to \mathbb{F}_{n'}$ is the change of basis $x_i \mapsto x_i(v_j)$
$\rho\colon \mathbb{F}_{n'} \twoheadrightarrow \mathbb{Z}^{n'}$ is the abelianization map,
$\mathbf{A} = (\mathbf{a}_\mathsf{i})_{i \in [1,n']}$ is an integral $n' \times m$ matrix.

SUBGROUP INTERSECTION

Let $H_1, H_2 \leqslant_{\text{fg}} \mathbb{F}_n \times \mathbb{Z}^m$ and respective bases for them, then

$$H_1 = \{wt^{\mathsf{a}} \in \mathbb{F}_n \times \mathbb{Z}^m \mid w \in H_1\pi \text{ and } \mathsf{a} \in w\phi_1\rho_1\mathsf{A}_1 + L_1\},$$

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$$(H_1 \cap H_2)\pi = \left\{ w \in H_1\pi \cap H_2\pi \mid (w\phi_1\rho_1\mathbf{A_1} + L_1) \cap (w\phi_2\rho_2\mathbf{A_2} + L_2) \neq \varnothing \right\}$$

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INTERSECTIONS IN FTFA GROUPS

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So,

$$(H_1 \cap H_2)\pi \leqslant H_1\pi \cap H_2\pi \leqslant \mathbb{F}_n$$

Let $H, H_1, H_2 \leqslant \mathbb{F}_n \times \mathbb{Z}^m$, and $\begin{array}{rl} \pi\colon \mathbb{F}_n \times \mathbb{Z}^m & \to \mathbb{F}_n \\ u\, t^{\mathbf{a}} & \mapsto u \end{array}$

Remarks:

1. $H \simeq H\pi \times (H \cap \mathbb{Z}^m)$.
2. H is f.g. $\Leftrightarrow$ $H\pi$ is f.g.
3. $(H_1 \cap H_2)\pi \leqslant H_1\pi \cap H_2\pi$.

So, if H_1, H_2 are finitely generated:

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INTERSECTIONS IN FTFA GROUPS

Let $H, H_1, H_2 \leqslant \mathbb{F}_n \times \mathbb{Z}^m$, and $\pi \colon \mathbb{F}_n \times \mathbb{Z}^m \to \mathbb{F}_n$
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Example: $\mathbb{F}_2 \times \mathbb{Z}$ is not Howson

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Subgroup Intersection Problem for $G = \langle X \mid R \rangle$, SIP$(G)$

Input: $u_1, \ldots, u_k, v_1, \ldots, v_l \in (X^{\pm})^*$
Decide: $\langle u_1, \ldots, u_k \rangle \cap \langle v_1, \ldots, v_l \rangle$ is f.g.,
and if so, compute generators.

FREE-ABELIAN TIMES FREE GROUPS ARE NOT HOWSON

Lemma

The group $\mathbb{F}_2 \times \mathbb{Z}$ is not Howson.

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Then:

$$\begin{aligned} H \cap K &= \{w(x,y) \mid w \in \mathbb{F}_2\} \cap \{w(xt, y) \mid w \in \mathbb{F}_2\} \\ &= \{w(x,y) \mid w \in \mathbb{F}_2\} \cap \{w(x,y)t^{|w|_x} \mid w \in \mathbb{F}_2\} \\ &= \{w(x,y)t^0 \mid w \in \mathbb{F}_2,\ |w|_x = 0\} \\ &= \langle x^{-k} y x^k,\ k \in \mathbb{Z} \rangle = \langle\langle y \rangle\rangle_{\mathbb{F}_2} \end{aligned}$$

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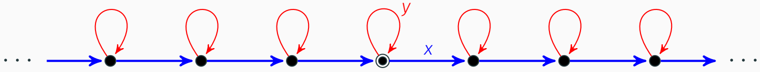
is **not finitely generated**, since its Stallings automaton is **infinite**:



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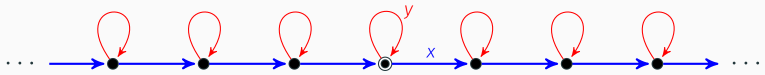

Remark: H and K are free groups with non-f.g. intersection...
doesn't this contradict Howson's property for free groups?

INTERSECTION DIAGRAM

$$
\begin{array}{ccccc}
 & & (H_1 \cap H_2)\pi & & \\
 & & \rotatebox[origin=c]{90}{$\geq$} & & \\
H_1\pi & \xleftarrow{\ \phi_1\ } & H_1\pi \cap H_2\pi & \xrightarrow{\ \phi_2\ } & H_2\pi \\
\Big\downarrow \rho_1 & /// & \Big\downarrow \rho & /// & \Big\downarrow \rho_2 \\
\mathbb{Z}^{r_1} & \xleftarrow{\quad P_1 \quad} & \mathbb{Z}^{r} & \xrightarrow{\quad P_2 \quad} & \mathbb{Z}^{r_2} \\
 & \searrow A_1 & \Big\downarrow R & A_2 \swarrow & \\
 & & \mathbb{Z}^{m} & & \\
 & & \rotatebox[origin=c]{90}{$\leq$} & & \\
 & & L_1 + L_2 & &
\end{array}
$$

INTERSECTION DIAGRAM

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\begin{array}{ccccc}
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 & & \text{\rotatebox{90}{$\leqslant$}} & & \\
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\mathbb{Z}^{r_1} & \xleftarrow{\ \mathbf{P_1}\ } & \mathbb{Z}^{r} & \xrightarrow{\ \mathbf{P_2}\ } & \mathbb{Z}^{r_2} \\
 & \searrow \mathbf{A_1} & \Big\downarrow \mathbf{R} & \mathbf{A_2} \swarrow & \\
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 & & L_1 + L_2 & &
\end{array}
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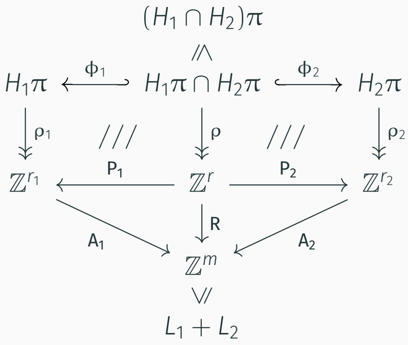
$$
\begin{aligned}
(H_1 \cap H_2)\pi &= \{w \in H_1\pi \cap H_2\pi : \ \mathbf{c}_{H_1}(w) \cap \mathbf{c}_{H_2}(w) \neq \varnothing\} \\
&= \{w \in H_1\pi \cap H_2\pi : \ (w\phi_1\rho_1\mathbf{A_1} + L_1) \cap (w\phi_2\rho_2\mathbf{A_2} + L_2) \neq \varnothing\}
\end{aligned}
$$

INTERSECTION DIAGRAM



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\end{aligned}
$$

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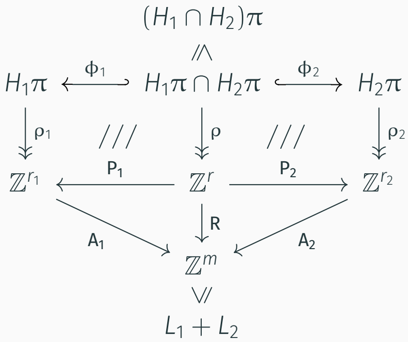

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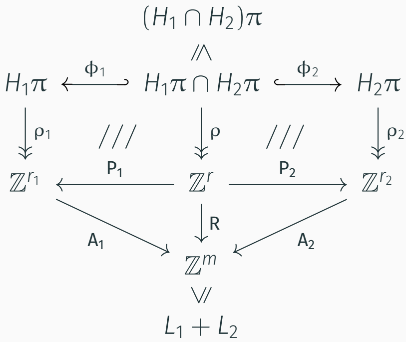
&= \{w \in H_1\pi \cap H_2\pi : w\rho(\mathsf{P}_1\mathsf{A}_1 - \mathsf{P}_2\mathsf{A}_2) \in L_1 + L_2\}
\end{aligned}
$$



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&= (L_1 + L_2)(\mathbf{P_1}\mathbf{A_1} - \mathbf{P_2}\mathbf{A_2})^{\leftarrow}\rho^{\leftarrow}
\end{aligned}
$$

$$
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H_1\pi & \xhookleftarrow{\ \phi_1\ } & H_1\pi \cap H_2\pi & \xhookrightarrow{\ \phi_2\ } & H_2\pi \\
\Big\downarrow{\scriptstyle \rho_1} & /// & \Big\downarrow{\scriptstyle \rho} & /// & \Big\downarrow{\scriptstyle \rho_2} \\
\mathbb{Z}^{r_1} & \xleftarrow{\quad \mathsf{P}_1 \quad} & \mathbb{Z}^{r} & \xrightarrow{\quad \mathsf{P}_2 \quad} & \mathbb{Z}^{r_2} \\
 & \searrow{\scriptstyle \mathsf{A}_1} & \Big\downarrow{\scriptstyle \mathsf{R}} & \swarrow{\scriptstyle \mathsf{A}_2} & \\
 & & \mathbb{Z}^m & & \\
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\end{aligned}
$$

We have:

$$\begin{array}{ccccccc}
\mathbb{F}_n \geqslant H_1\pi \cap H_2\pi \simeq & \mathbb{F}_r & \xrightarrow{\quad\rho\quad} & \mathbb{Z}^r & \xrightarrow{\quad \mathsf{R}\quad} & \mathbb{Z}^m \\
& \triangledown\!\!/ & & \triangledown\!\!/ & & \triangledown\!\!/ \\
(H_1 \cap H_2)\pi \simeq & \underbrace{(L_1+L_2)\mathsf{R}^{\leftarrow}\rho^{\leftarrow}}_{M\rho^{\leftarrow}} & \longleftarrow\!\!\dashv & \underbrace{(L_1+L_2)\mathsf{R}^{\leftarrow}}_{M} & \longleftarrow\!\!\dashv & L_1+L_2
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Theorem

Let $H_1, H_2 \leqslant_{\mathrm{fg}} \mathbb{F}_n \times \mathbb{Z}^m$. Then, TFAE:

1. *the intersection $H_1 \cap H_2$ is finitely generated;*
2. *the projection $(H_1 \cap H_2)\pi$ is finitely generated;*
3. *$(H_1 \cap H_2)\pi$ is either trivial, or has finite index in $H_1\pi \cap H_2\pi$,*
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Corollary

The subgroup intersection problem $\mathrm{SIP}(\mathbb{F}_n \times \mathbb{Z}^m)$ is decidable.

INTERSECTION EXAMPLE

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Let $H_1 = \langle t^{L_1}, x^3\, t^{\mathsf{a}}, yx \rangle$,

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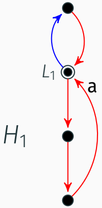

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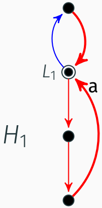

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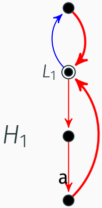

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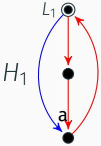

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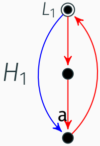

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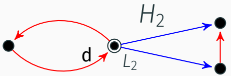

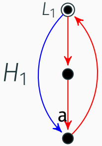

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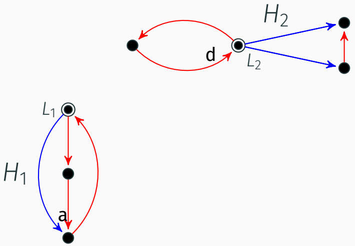

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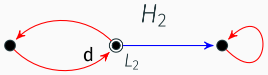

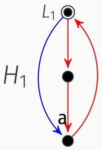

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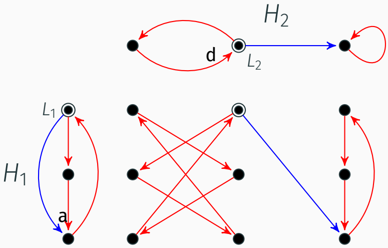

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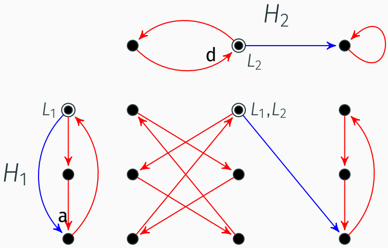

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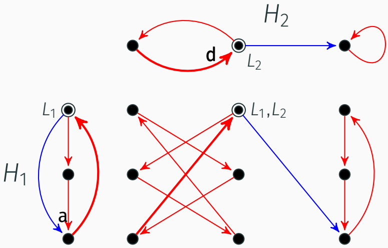

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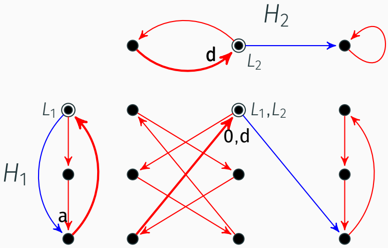

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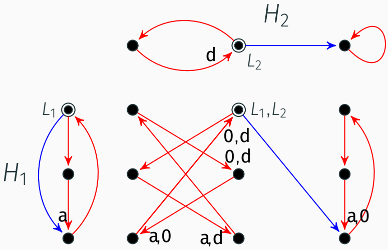

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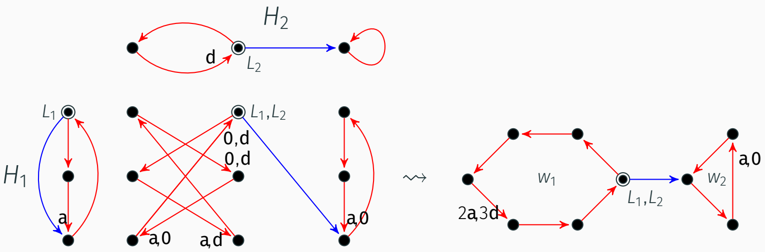

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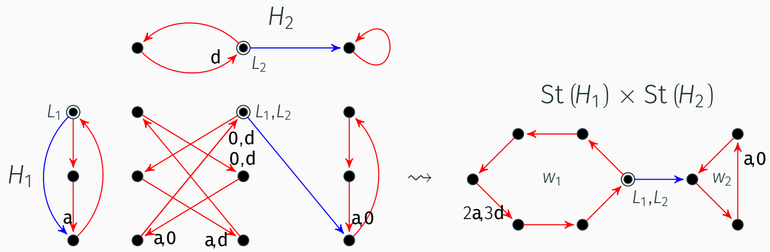

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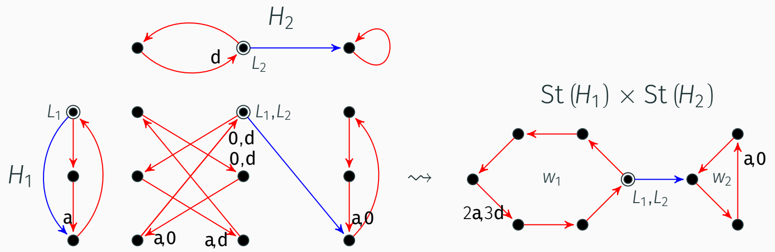

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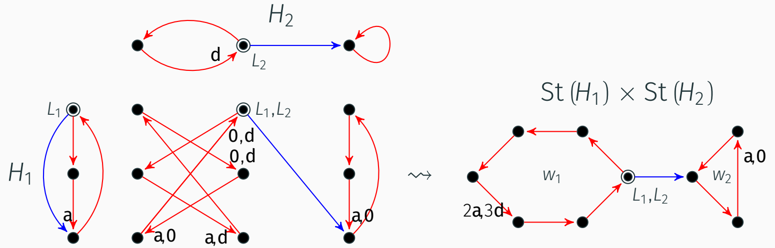

Claim:

$$H_1 \cap H_2 = \{\, u\, t^{\mathbf{a}} : u\, t^{\mathbf{a}} \text{ is componentwise-readable in } \mathrm{St}(H_1) \times \mathrm{St}(H_2) \,\}$$

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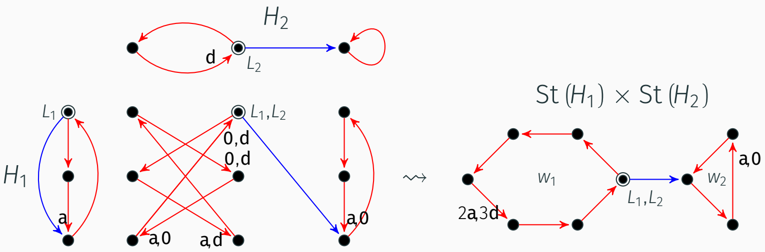

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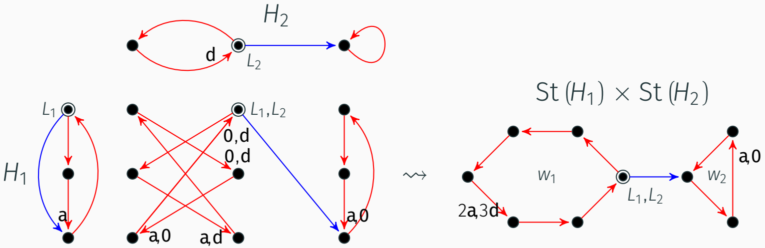

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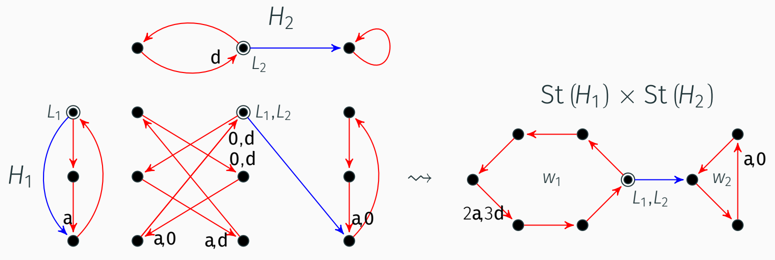

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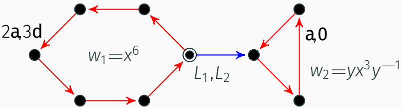
$$= (L_1 + L_2)\mathbf{R}^{\leftarrow} \rho^{\leftarrow}\,, \text{ where } \mathbf{R} = \begin{bmatrix} 2\mathbf{a} - 3\mathbf{d} \\ \mathbf{a} - \mathbf{0} \end{bmatrix} \text{ and } \rho = \mathrm{ab}\,.$$

FROM STALLINGS TO CAYLEY



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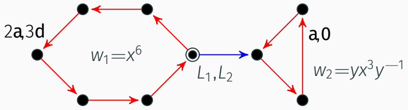

$$\mathsf{R} = \begin{bmatrix} 2a-3d \\ a-0 \end{bmatrix}$$

$$M = (L_1 + L_2)\mathsf{R}^{\leftarrow}$$

We have that $(H_1 \cap H_2)\pi = (L_1 + L_2)\mathsf{R}^{-1}\rho^{-1} = M\rho^{-1}$, i.e.,

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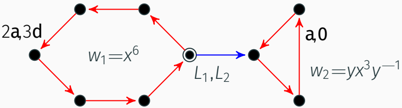

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& \triangledown\!\!/ & & \triangledown\!\!/ & & \triangledown\!\!/ & & \triangledown\!\!/ \\
& (H_1 \cap H_2)\pi & \simeq & M\rho^{-1} & \longleftarrow\!\!\shortmid & M & \longleftarrow\!\!\shortmid & L_1 + L_2
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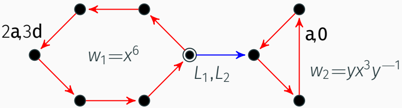

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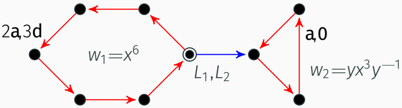

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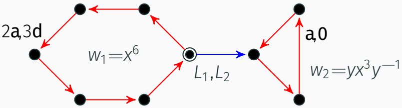

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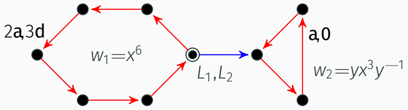

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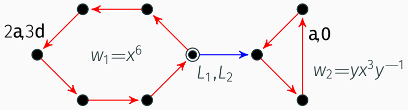
FROM STALLINGS TO CAYLEY



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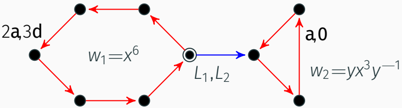

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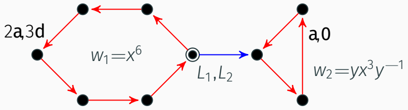

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$$\mathbf{PMQ} = \mathbf{D} = \mathbf{diag}(\delta_1, \delta_2)$$

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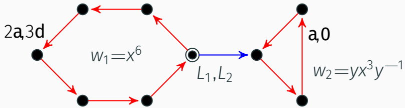
&\simeq \mathrm{Cay}\left(\mathbb{F}_{w_1,w_2}/M\rho^{-1}, \{[w_i]\}_i\right) \\
&\simeq \mathrm{Cay}\left(\mathbb{Z}^2/\langle \mathbf{M} \rangle, \{\mathbf{e}_i\}_i\right) \\
&\simeq \mathrm{Cay}\left(\mathbb{Z}^2/\langle \mathbf{D} \rangle, \{\mathbf{e}_i\mathbf{Q}\}_i\right)
\end{aligned}$$



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&\simeq \operatorname{Cay}\left(\mathbb{Z}^2/\langle \mathbf{D} \rangle, \{\mathbf{e}_i\mathbf{Q}\}_i\right) \\
&\simeq \operatorname{Cay}(\mathbb{Z}/\delta_1\mathbb{Z} \oplus \mathbb{Z}/\delta_2\mathbb{Z}, \{\mathbf{e}_i\mathbf{Q}\}_i)\,.
\end{aligned}$$

Theorem (D.–V.)

Let $H_1, H_2 \leqslant \mathbb{F}_n \times \mathbb{Z}^m$. *Then*

$$\mathrm{St}\left((H_1 \cap H_2)\pi, \{w_i(X)\}_i\right) = \mathrm{Cay}\left(\bigoplus_{i=1}^{r} \mathbb{Z}/\delta_i\mathbb{Z}, \{\mathbf{e}_i\mathbf{Q}\}_i\right),$$

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Theorem (D.–V.)

Let $H_1, H_2 \leqslant \mathbb{F}_n \times \mathbb{Z}^m$. Then,

1. *we can algorithmically decide whether $H_1 \cap H_2$ is f.g.*
2. *if so, $\mathrm{St}(H_1 \cap H_2)$ is computable.*

In particular, $\mathrm{SIP}(\mathbb{F}_n \times \mathbb{Z}^m)$ is solvable.

INTERSECTION SHOWCASE

$$H_1 = \langle t^{L_1}, x^3\, t^{\mathsf{a}}, yx \rangle,\ H_2 = \langle t^{L_2}, x^2\, t^{\mathsf{d}}, yxy^{-1} \rangle \leqslant \mathbb{F}_2 \times \mathbb{Z}^2$$

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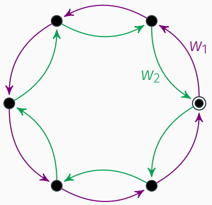

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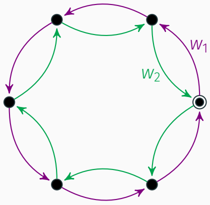

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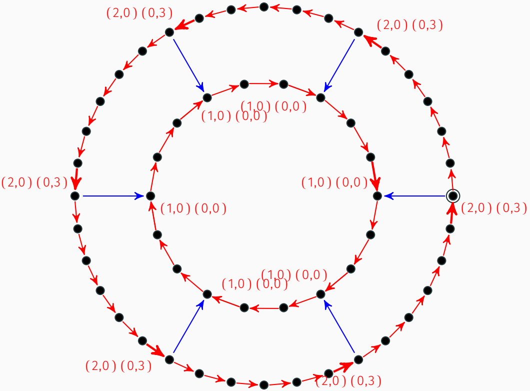

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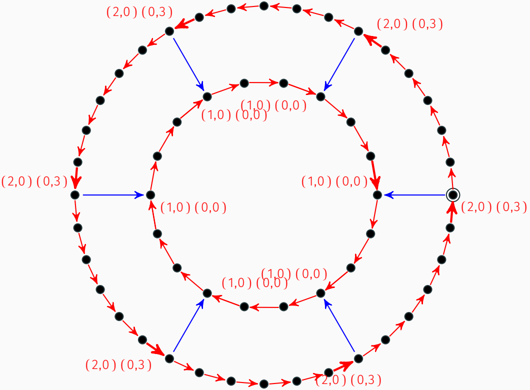

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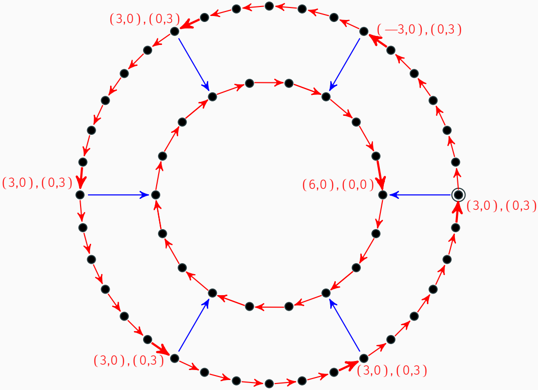

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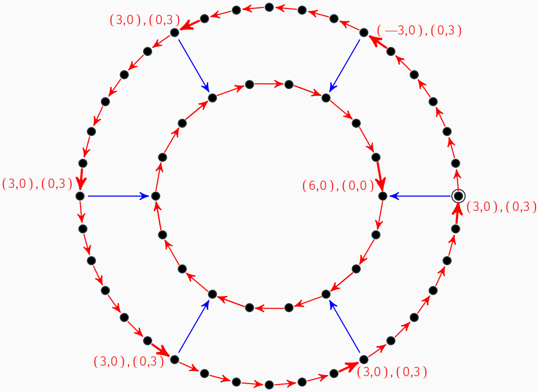

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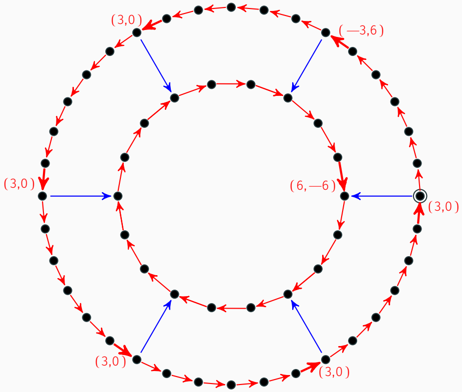

INTERSECTION SHOWCASE

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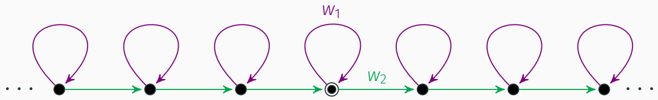

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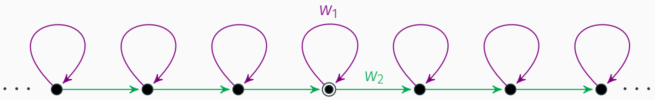

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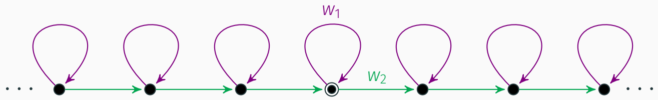

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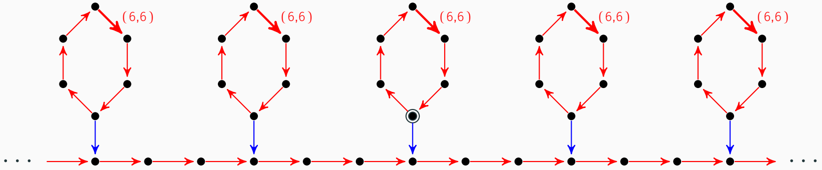

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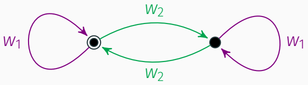

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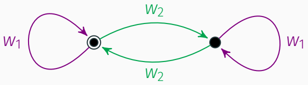

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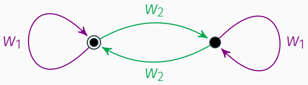

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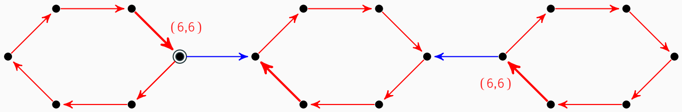

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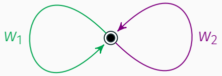

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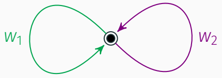

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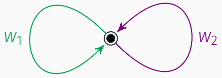

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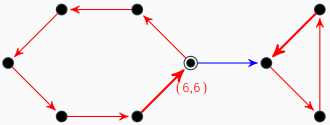

INTERSECTION SHOWCASE

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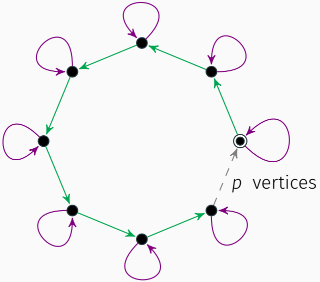

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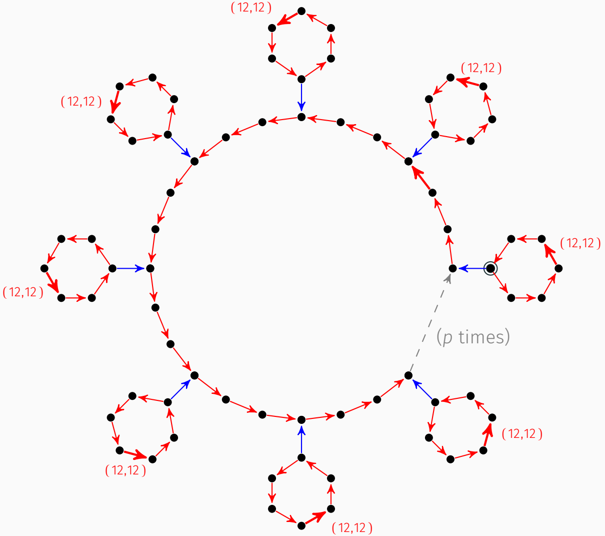

Multiple intersections in $\mathbb{F}_n \times \mathbb{Z}^m$

WHAT ABOUT THE MULTIPLE VERSIONS?

Subgroup Intersection Problem in G, $\mathrm{SIP}(G)$

Given $H_1, H_2 \leqslant_{\mathsf{fg}} G$ (by finite sets of generators), decide whether $H_1 \cap H_2$ is finitely generated; if yes, compute generators for $H_1 \cap H_2$.

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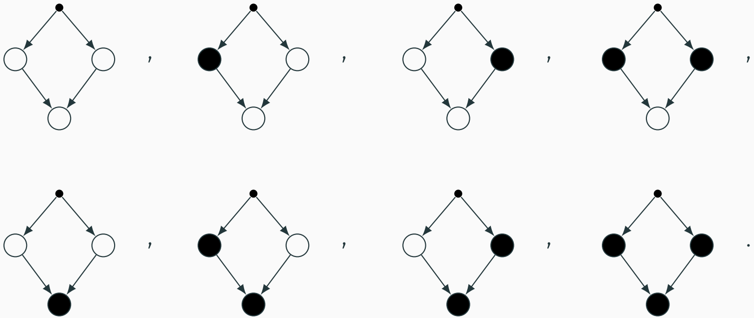
There are subgroups $H_1, H_2, H_3 \leqslant \mathbb{F}_n \times \mathbb{Z}^m$ such that H_1, H_2, H_3 and $H_1 \cap H_2 \cap H_3$ are finitely generated, but $H_1 \cap H_2$, $H_1 \cap H_3$, $H_2 \cap H_3$ are not ...

Let $H_1, H_2 \leqslant G$. There are $2^3 = 8$ possibilities for the finite/infinite generation of H_1, H_2, $H_1 \cap H_2$:



MULTIPLE INTERSECTIONS

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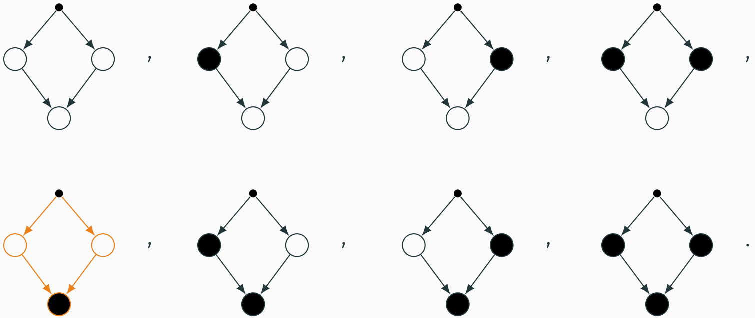

Observation

G is Howson $\Leftrightarrow$ the highlighted 2-configuration is not realizable.

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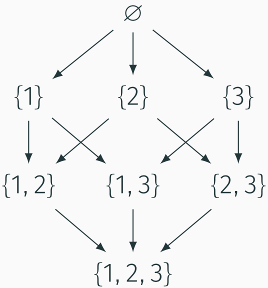
- $\chi_{\{I\}}$ is an ***almost-0*** k-configuration.

INTERSECTION CONFIGURATIONS



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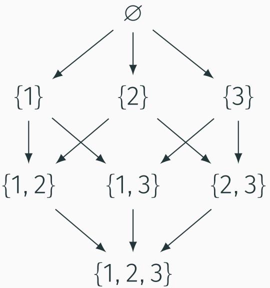

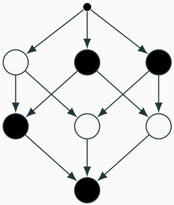

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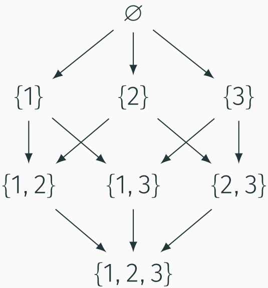

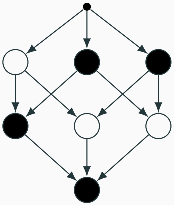

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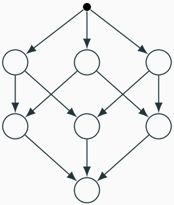

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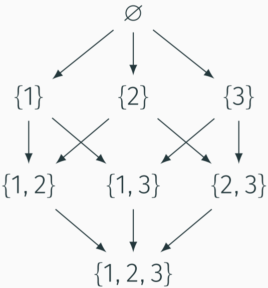

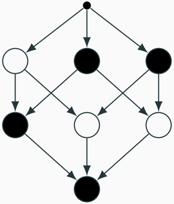

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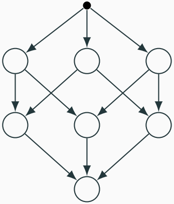

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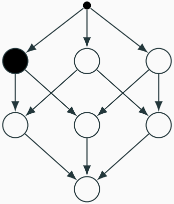

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Let G be a group, and $k \geqslant 1$.

Definition

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$$\forall\, \varnothing \neq I, J \subseteq [k],\ \ (I)\chi = (J)\chi = 0 \ \Rightarrow\ (I \cup J)\chi = 0.$$

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Does there exists a finitely presented intersection-saturated group?

THE MULTIPLE INTERSECTION PROBLEM IS COMPUTABLE

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Let $M', M'' \leqslant \mathbb{F}_n$ *be two subgroups of* $\mathbb{F}_n$ *in free factor position, i.e., such that* $\langle M', M'' \rangle = M' * M''$*. Then, for any* $H_1', \ldots, H_k' \leqslant M' \leqslant \mathbb{F}_n$ *and* $H_1'', \ldots, H_k'' \leqslant M'' \leqslant \mathbb{F}_n$*, then*

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Remark: The same equality is not true, in general, in $\mathbb{F}_n \times \mathbb{Z}^m$.

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Two subgroups $M', M'' \leqslant \mathbb{F}_n \times \mathbb{Z}^m$ are ***strongly complementary***, denoted by $\langle M', M'' \rangle = M' \circledast M''$, if

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Let $M', M'' \leqslant \mathbb{F}_n \times \mathbb{Z}^m$ be strongly complementary. Then, for any $H_1', \ldots, H_k' \leqslant M' \leqslant \mathbb{F}_n \times \mathbb{Z}^m$ satisfying $r' = \mathbf{rk}\left(\cap_{i=1}^k H_i'\pi\right) \geqslant 2$, and any $H_1'', \ldots, H_k'' \leqslant M'' \leqslant \mathbb{F}_n \times \mathbb{Z}^m$ satisfying $r'' = \mathbf{rk}\left(\cap_{i=1}^k H_i''\pi\right) \geqslant 2$,

$$\bigcap\nolimits_{i=1}^{k} \langle H_i', H_i'' \rangle \text{ is f.g.} \;\Leftrightarrow\; \bigcap\nolimits_{i=1}^{k} H_i' \text{ and } \bigcap\nolimits_{i=1}^{k} H_i'' \text{ are both f.g.}$$

Remark: It is not true without the hypotheses.

Lemma

Let $H_1, \dots, H_k \leqslant \mathbb{F}_n \times \mathbb{Z}^m$. *If, for some* $\varnothing \neq I, J \subseteq [k]$, H_I *and* H_J *are f.g. whereas* $H_{I \cup J} = H_I \cap H_J$ *is not, then* $\exists i \in I$, $\exists j \in J$ *s.t. both* $L_i, L_j \leqslant \mathbb{Z}^m$ *have rank strictly smaller than* m.

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Let χ *be a* k*-configuration for which* $\exists\, r \geqslant 2$ *non-empty subsets* $I_1, \ldots, I_r \subseteq [k]$ *s.t.* $\forall j \in \{1, \ldots, r\}$, $(I_1 \cup \cdots \cup \widehat{I_j} \cup \cdots \cup I_r)\chi = 0$ *but* $(I_1 \cup \cdots \cup I_r)\chi = 1$. *Then* χ *is not realizable in* $\mathbb{F}_n \times \mathbb{Z}^{r-2}$.

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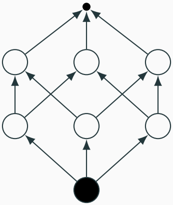
Example: An unrealizable configuration in $\mathbb{F}_2 \times \mathbb{Z}$:



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H_k &= \langle x, yt^{\mathbf{e_1}}; t^{\mathbf{e_2}-\mathbf{e_1}}, \ldots, t^{\mathbf{e_{k-1}}-\mathbf{e_1}} \rangle \\
&= \langle x, yt^{\mathbf{e_1}}, \ldots, yt^{\mathbf{e_{k-1}}} \rangle \leqslant \mathbb{F}_2 \times \mathbb{Z}^{k-1}.
\end{aligned}
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REALIZING k-CONFIGURATIONS

For a given set of indices $\varnothing \neq I \subseteq [k]$, let us compute H_I:

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- *Case 3:* $I = [k]$. In this case,
  $H_I = (H_1 \cap \cdots \cap H_{k-1}) \cap H_k = \langle x, y \rangle \cap \langle x, yt^{\mathbf{e}_1}; t^{\mathbf{e}_2 - \mathbf{e}_1}, \ldots, t^{\mathbf{e}_{k-1} - \mathbf{e}_1} \rangle = \langle\langle x \rangle\rangle_{\mathbb{F}_2}$
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Lemma

Any almost-$\mathbf{0}$ k-configuration $\chi[I_0]$ is realizable in $\mathbb{F}_2 \times \mathbb{Z}^{|I_0|-1}$.

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Theorem (D.–Roy–V.)

For $k \geqslant 1$, every k-configuration χ is realizable in $\mathbb{F}_n \times \mathbb{Z}^m$, for every $n \geqslant 2$ and $m \gg 0$; more precisely, for $m = \sum_{(I)\chi=1}(|I| - 1)$.

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Theorem (D.–Roy–V.)

There exist finitely presented intersection-saturated groups G.

BACK TO THE FREE CASE

Theorem (D.–Roy–V.)

A k-configuration χ is realizable in a free group $\mathbb{F}_n$, $n \geqslant 2$ if and only if χ satisfies the Howson property; i.e., if and only if

$$\forall\, \varnothing \neq I, J \subseteq [k],\ (I)\chi = (J)\chi = 0 \ \Rightarrow\ (I \cup J)\chi = 0.$$

[1] L. Bartholdi and P. Silva. “Rational Subsets of Groups”. In: *Handbook of Automata Theory. Volume II. Automata in Mathematics and Selected Applications*. Berlin: European Mathematical Society (EMS), 2021, pp. 841–869.

[2] F. Bassino, A. Martino, et al. “Statistical Properties of Subgroups of Free Groups”. *Random Structures & Algorithms* 42.3 (May 2013), pp. 349–373.

[3] F. Bassino, C. Nicaud, and P. Weil. “Random Generation of Finitely Generated Subgroups of a Free Group”. *International Journal of Algebra and Computation* 18.02 (Mar. 1, 2008), pp. 375–405.

[4] J. Delgado. “Extensions of Free Groups: Algebraic, Geometric, and Algorithmic Aspects”. PhD thesis. Universitat Politècnica de Catalunya, Sept. 15, 2017.

[5] J. Delgado, M. Roy, and E. Ventura. “Intersection Configurations in Free and Free Times Free-Abelian Groups”. 2022. arXiv: `2107.12426 [math]`.

[6] J. Delgado and E. Ventura. "A List of Applications of Stallings Automata". *Transactions on Combinatorics* 11.3 (June 15, 2022), pp. 181–235.

[7] J. Delgado and E. Ventura. "Stallings Automata for Free-Times-Abelian Groups: Intersections and Index". *Publicacions Matemàtiques* 66.2 (2022), pp. 789–830.

[8] I. Kapovich and A. Myasnikov. "Stallings Foldings and Subgroups of Free Groups". *Journal of Algebra* 248.2 (Feb. 15, 2002), pp. 608–668.

[9] A. Miasnikov, E. Ventura, and P. Weil. "Algebraic Extensions in Free Groups". In: *Geometric Group Theory*. Ed. by G. N. Arzhantseva, J. Burillo, et al. Trends in Mathematics. Birkhäuser Basel, Jan. 1, 2007, pp. 225–253.

[10] P. V. Silva and P. Weil. "On an Algorithm to Decide Whether a Free Group Is a Free Factor of Another". *RAIRO. Theoretical Informatics and Applications* 42.2 (2008), pp. 395–414.

[11] J. R. Stallings. "Topology of Finite Graphs". *Inventiones Mathematicae* 71 (Mar. 1983), pp. 551–565.

Thanks!