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Analysis of a thermoelastic problem with the Moore–Gibson–Thompson microtemperatures

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ABSTRACT

In this paper, we study, from both an analytical and a numerical point of view, a poro-thermoelastic problem with microtemperatures. The so-called Moore–Gibson–Thompson equation is used to model the contribution for the temperature and microtemperatures. An existence and uniqueness result is proved by using the theory of linear semigroups of contractions and, for the one-dimensional case, the exponential energy decay is found under some conditions on the constitutive coefficients. Then, a fully discrete approximation is introduced by using the finite element method and the implicit Euler scheme. We show that the discrete energy decays and we obtain some a priori error estimates from which, under some adequate additional regularity conditions on the continuous solution, we derive the linear convergence of the approximations. Finally, we perform some numerical simulations to demonstrate the accuracy of the approximations and the behavior of the discrete energy and the solution.

1. Introduction

Thought there exists a huge quantity of macroscopic situations which can be described by the classical theory of thermoelasticity, in many physical phenomena the microscopic structure plays a relevant role and must be incorporated in the material model of solids. The Cosserat brothers [1] introduced the micropolar theories at the beginning of the past century, but the studies about microstructure were not developed in a relevant way until the sixties (see the books of Eringen [2] or Ieşan [3]). One of the theories including microstructure is the so-called theory of the materials with voids [4–6] (also known as porous materials). In this case the "bulk density" is the product of the density of the material matrix by the volume fraction. The material points are thus affected by small voids, and this introduces a new degree of freedom. This theory is well-accepted by the scientific community and there exists a substantial volume of research papers dedicated to these models (see, among others, [7–13]). This is because of the large number of applications (building industry, bone repair, etc…) and several interesting comments regarding this line can be found in the book of Straughan [14, p. 307–308].

Between the different aspects concerning the microstructure, we can include the idea of "microtemperatures". Since the materials with microstructure can be understood as composed by microelements, we can suppose them too to be subjected to deformations of the temperature and we can study the variation of the temperature inside the microelement,

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which depends on the direction, and we will call them microtemperatures (see [15–17]). It is worth recalling that this notion comes from the works of Grot [18], Riha [19,20] and Verma et al. [21]; however, the concept has been fully embraced by the scientific community only since the last couple of decades and we can cite several contributions in this sense (see [22–26] among others).

Heat and microheat propagation were proposed within the parabolic theory of partial differential equations, but this fact brings to the instantaneous propagation of thermal (or microthermal) waves which is incompatible with the causality principle. Several people have since tried to propose alternative theories that overcome this unrealistic effect. We can recall the damped hyperbolic heat equation proposed by Cattaneo and Maxwell [27] which has been extended to microtemperatures recently. An alternative equation describing the heat with finite speed propagation of waves is determined by the Moore–Gibson–Thompson (MGT) equation [28]. Although it was originally introduced by Stokes in the mid-nineteenth century [29], this equation is named after the works [30,31], and it plays a paramount role in the description of several physical phenomena (see, e.g., [32–35] and references therein). In fact, this equation has deserved much attention in the last four years in several thermomechanical situations (see, for instance, [36–57]). The present work builds on these previous contributions concerning MGT thermoelastic equation and introduces a theory with porous effects and with heat and microheat effects of the MGT type. Later, we prove the existence and uniqueness of the solutions in the three-dimensional case. We also restrict our attention to the homogeneous one-dimensional case and we show the exponential stability of the solutions (it is known that we cannot expect a similar result in higher dimension). Then, we develop a numerical study of a variational formulation of the problem by using the classical finite element method and the implicit Euler scheme. A discrete version of the energy property and a main a priori error estimates result are proved, from which the linear convergence of the approximations is derived under some regularity conditions. Finally, some one-dimensional numerical simulations are shown, to demonstrate the accuracy of the algorithm and the behavior of the discrete energy. In order to clarify the innovation of this paper, we want to emphasize that, in this work, we propose (for the first time) a Moore–Gibson–Thompson theory for the microtemperatures. Thus, we obtain a new system of equations which has not been previously considered, nor studied in the literature. In view of the techniques we apply here, our work can have several similarities with other previous contributions; however, the aspects concerning MGT microtemperatures are completely new. In this paper, we have tried to see how to extend several previous results to the case when we incorporate these MGT microtemperatures.

The plan of this paper is the following. In the next section we set down the basic equations of the problem. We also provide the basic assumptions and the functional setting as a Cauchy problem in a suitable Hilbert space. The existence and uniqueness of the solutions to this Cauchy problem are proved in Section 3 by means of the theory of semigroup of linear operators. We restrict our attention to the one-dimensional case in Section 4 and we prove the exponential decay of the solutions in this case. Later, we develop the numerical study of the one-dimensional problem (for the sake of simplicity in the writing) in Section 5. By using the finite element method and the implicit Euler scheme, fully discrete approximations are introduced, and a property of the discrete energy some a priori error estimates are proved. Finally, some numerical simulations are presented in Section 6. The Conclusions end the paper.

2. Basic equations and assumptions

We consider a nonhomogeneous porous material occupying a smooth, bounded domain $\Omega \subset \mathbb{R}^3$. First, we consider the evolution equations for the theory of poro-thermoelasticity with microtemperatures for a centrosymmetric material:

$$\begin{aligned}
&\rho \ddot{u}_i = t_{ij,j},\\
&J\ddot{\phi} = h_{j,j} + g,\\
&\rho \dot{\eta} = q_{j,j},\\
&\rho \dot{\epsilon}_i = q_{ji,j} + q_i - Q_i.
\end{aligned}$$

The first two equations represent, respectively, the balances of the linear momentum and of the first stress moment. Here ρ is the mass density, u_i is the displacement vector, t_{ij} is the stress tensor, J is the equilibrated inertia, ϕ is the change in volume fraction with respect to its reference value, h_j is the equilibrated stress and g is the equilibrated body force. Next, we have the balances of the energy and of its first moment, where η is the entropy, q_i is the heat flux vector, ϵ_i is the first moment of the energy vector, q_{ij} is the first heat flux moment tensor and Q_i is the microheat flux average vector.

In order to obtain the final model, we complement the above relations with the constitutive equations (see [58]):

$$\begin{aligned}
&t_{ij} = A_{ijrs}e_{rs} + D_{ij}\phi - a_{ij}\theta,\\
&h_i = A_{ij}\phi_{,j} - N_{ij}T_j,\\
&g = -D_{ij}e_{ij} - \xi\phi + F\theta,\\
&\rho\eta = a_{ij}e_{ij} + F\phi + a\theta,\\
&\rho\epsilon_i = -N_{ji}\phi_{,j} - B_{ij}T_j.
\end{aligned}$$

Here, A_{ijrs} is the elasticity tensor, A_{ij} and ξ characterize the porosity of the material, and B_{ij} plays a similar role to the thermal capacity for the microtemperatures. The tensors D_{ij}, a_{ij}, N_{ij}, as well as the scalars F and a, describe the coupling between the physical quantities appearing in the equations. Furthermore, $e_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i})$ is the linearized strain tensor. Finally, we introduce the constitutive equations for q_i, q_{ij}, Q_i (see again [58]),

$$
\begin{aligned}
q_i(t) &= \int_{-\infty}^{t} g_{ij}(t-s)\theta_{,j}(s)+h_{ij}(t-s)T_j(t-s)ds,\\
q_{ij}(t) &= \int_{-\infty}^{t} P_{ijrs}(t-s)T_{r,s}(s)ds,\\
Q_i(t) &= \int_{-\infty}^{t} [(g_{ij}(t-s)-G_{ij}(t-s))\theta_{,j}(s)+(h_{ij}(t-s)-H_{ij}(t-s))T_j(s)ds,]
\end{aligned}
$$

where θ is the temperature and T_i are the microtemperatures. We shall denote by

$$
\alpha(t)=\alpha(0)+\int_0^t \theta(s)ds, \quad R_i(t)=R_i(0)+\int_0^t T_i(s)ds
$$

the thermal and microthermal displacements, respectively, and, finally, we consider the relaxation functions:

$$
\begin{aligned}
g_{ij}(s) &= \kappa^*_{ij}+\mathrm{e}^{-\frac{s}{\tau}}\left(\frac{1}{\tau}\kappa_{ij}-\kappa^*_{ij}\right),\\
h_{ij}(s) &= H^*_{ij}+\mathrm{e}^{-\frac{s}{\tau}}\left(\frac{1}{\tau}H_{ij}-H^*_{ij}\right),\\
P_{ijrs}(s) &= P^*_{ijrs}+\mathrm{e}^{-\frac{s}{\tau}}\left(\frac{1}{\tau}P_{ijrs}-P^*_{ijrs}\right),\\
G_{ij}(s) &= K^*_{ij}+\mathrm{e}^{-\frac{s}{\tau}}\left(\frac{1}{\tau}K_{ij}-K^*_{ij}\right),\\
H_{ij}(s) &= \Lambda^*_{ij}+\mathrm{e}^{-\frac{s}{\tau}}\left(\frac{1}{\tau}\Lambda_{ij}-\Lambda^*_{ij}\right).
\end{aligned}
$$

The positive constant τ is usually known as *relaxation parameter*. Once again, κ_{ij}, H_{ij}, P_{ijrs}, K_{ij} and Λ_{ij}, as well as their "starred" versions, are tensorial structural coefficients related to the usual constitutive equations for the temperature and microtemperatures (see [15]). It is understood that all the tensors appearing in the above equations might depend on the space variable $\boldsymbol{x}$. Plugging the newly derived constitutive equations into the equations of poro-thermoelasticity, we have the system of field equations:

$$
\begin{aligned}
&\rho\ddot{u}_i=(A_{ijrs}e_{rs}+D_{ij}\phi-a_{ij}\theta)_{,j},\\
&J\ddot{\phi}=(A_{ij}\phi_{,j}-N_{ij}T_j)_{,i}-D_{ij}e_{ij}-\xi\phi+F\theta,\\
&\tau a\dddot{\alpha}+a\ddot{\alpha}=-\tau a_{ij}\ddot{e}_{ij}-a_{ij}\dot{e}_{ij}-\tau F\ddot{\phi}-F\dot{\phi}+(\kappa_{ij}\theta_{,j}+\kappa^*_{ij}\alpha_{,j}+H_{ij}T_j+H^*_{ij}R_j)_{,i},\\
&\tau B_{ij}\dddot{R}_j+B_{ij}\ddot{R}_j=-\tau N_{ji}\ddot{\phi}_{,j}-N_{ji}\dot{\phi}_{,j}+(P_{ijrs}T_{r,s}+P^*_{ijrs}R_{r,s})_{,j}-K_{ij}\theta_{,j}\\
&\qquad -K^*_{ij}\alpha_{,j}-\Lambda_{ij}T_j-\Lambda^*_{ij}R_j.
\end{aligned}
$$

Taking the sum of the derivative of the first equation (multiplied by τ) with the first equation, calling

$$
\hat{u}=\tau\dot{u}+u,
$$

and doing the same for ϕ, we can simplify the system to obtain

$$
\begin{aligned}
&\rho\ddot{u}_i=[A_{ijrs}e_{rs}+D_{ij}\phi-a_{ij}(\tau\dot{\theta}+\theta)]_{,j},\\
&J\ddot{\phi}=[A_{ij}\phi_{,j}-N_{ij}(\tau\dot{T}_j+T_j)]_{,i}-D_{ij}e_{ij}-\xi\phi+F(\tau\dot{\theta}+\theta),\\
&\tau a\dddot{\alpha}+a\ddot{\alpha}=-a_{ij}\dot{e}_{ij}-F\dot{\phi}+(\kappa_{ij}\theta_{,j}+\kappa^*_{ij}\alpha_{,j}+H_{ij}T_j+H^*_{ij}R_j)_{,i},\\
&\tau B_{ij}\dddot{R}_j+B_{ij}\ddot{R}_j=-N_{ji}\dot{\phi}_{,j}+(P_{ijrs}T_{r,s}+P^*_{ijrs}R_{r,s})_{,j}-K_{ij}\theta_{,j}\\
&\qquad -K^*_{ij}\alpha_{,j}-\Lambda_{ij}T_j-\Lambda^*_{ij}R_j,
\end{aligned}
\tag{1}
$$

where we have omitted the hat to simplify the notation. We endow the above system with the Dirichlet boundary conditions:

$$
u_i(\boldsymbol{x},t)_{|\boldsymbol{x}\in\partial\Omega}=\phi(\boldsymbol{x},t)_{|\boldsymbol{x}\in\partial\Omega}=\alpha(\boldsymbol{x},t)_{|\boldsymbol{x}\in\partial\Omega}=R_i(\boldsymbol{x},t)_{|\boldsymbol{x}\in\partial\Omega}=0,
\tag{2}
$$

and the initial conditions, for a.e. $\boldsymbol{x}\in\Omega$,

$$
\begin{aligned}
&u_i(\boldsymbol{x},0)=u_i^0(\boldsymbol{x}), \quad \dot{u}_i(\boldsymbol{x},0)=v_i^0(\boldsymbol{x}), \quad \phi(\boldsymbol{x},0)=\phi^0(\boldsymbol{x}),\\
&\dot{\phi}(\boldsymbol{x},0)=\psi^0(\boldsymbol{x}), \quad \alpha(\boldsymbol{x},0)=\alpha^0(\boldsymbol{x}), \quad \dot{\alpha}(\boldsymbol{x},0)=\theta^0(\boldsymbol{x}),\\
&\ddot{\alpha}(\boldsymbol{x},0)=\vartheta^0(\boldsymbol{x}), \quad R_i(\boldsymbol{x},0)=R_i^0(\boldsymbol{x}), \quad \dot{R}_i(\boldsymbol{x},0)=T_i^0(\boldsymbol{x}),\\
&\ddot{R}_i(\boldsymbol{x},0)=S_i^0(\boldsymbol{x}).
\end{aligned}
\tag{3}
$$

Remark 1. System (1) in full generality can appear quite complicated. However, in order to better understand the thermal and mechanical coupling between the equations, one can look at the isotropic case, where several simplifications occur. Specifically, we have

$$\rho\ddot{u}_i = \mu\Delta u_i + (\lambda+\mu)(\operatorname{div} u)_{,i} + D\phi_{,i} - a^*(\tau\dot{\theta}+\theta)_{,i}, \tag{4}$$

$$J\ddot{\phi} = A\Delta\phi - \xi\phi - D\operatorname{div} u + F(\tau\dot{\theta}+\theta) - N\operatorname{div}(\tau\dot{T}+T), \tag{5}$$

$$\tau a\dddot{\alpha} + a\ddot{\alpha} = \kappa\Delta\theta + \kappa^*\Delta\alpha - a^*\operatorname{div}\dot{u} - F\dot{\phi} + H\operatorname{div} T + H^*\operatorname{div} R, \tag{6}$$

$$\begin{aligned} \tau B\dddot{R}_i + B\ddot{R}_i &= P_6\Delta T_i + P_6^*\Delta R_i + (P_4+P_5)(\operatorname{div} T)_{,i} \\ &\quad + (P_4^*+P_5^*)(\operatorname{div} R)_{,i} - N\dot{\phi}_{,i} - \Lambda T_i - \Lambda^* R_i - K\theta_{,i} - K^*\alpha_{,i}. \end{aligned} \tag{7}$$

Here λ and μ are the Lamè constants, while P_4, P_5 and P_6 are the only coefficients affecting the microtemperatures in the isotropic case, see [16].

Finally, we introduce the phase space associated to our problem

$$\mathcal{H} = V\times H\times V\times H\times V\times V\times H\times V\times V\times H,$$

where V and H are the usual real Sobolev spaces $H_0^1(\Omega)$ and $L^2(\Omega)$, respectively. We note that their complexification is standard. Now, let

$$\begin{aligned} \widetilde{\kappa}_{ij} &= \kappa_{ij} - \tau\kappa_{ij}^*, \\ \widetilde{P}_{ijrs} &= P_{ijrs} - \tau P_{ijrs}^*, \\ \widetilde{\Lambda}_{ij} &= \Lambda_{ij} - \tau\Lambda_{ij}^*, \\ \widetilde{H}_{ij} &= H_{ij} - \tau H_{ij}^*, \\ \widetilde{K}_{ij} &= K_{ij} - \tau K_{ij}^*. \end{aligned} \tag{8}$$

We ask that

(i) There exist positive constants $\rho_0, J_0, \alpha_0, B_0$ such that

$$\rho(\boldsymbol{x}) \geq \rho_0,\ J(\boldsymbol{x}) \geq J_0,\ a(\boldsymbol{x}) \geq a_0,\ B_{ij}(\boldsymbol{x})\xi_i\xi_j \geq B_0\xi_i\xi_i$$

for every $\boldsymbol{\xi} = (\xi_i)$.

(ii) There exists a positive constant C_1 such that

$$A_{ijrs}\xi_{ij}\xi_{rs} + 2D_{ij}\xi_{ij}\eta + \xi\eta^2 \geq C_1(\xi_{ij}\xi_{ij} + \eta^2)$$

for every $\boldsymbol{\xi} = (\xi_{ij})$ and $\eta \in \mathbb{R}$.

(iii) There exists a positive constant C_2 such that

$$P_{ijrs}^*\eta_{ij}\eta_{rs} \geq C_2\eta_{ij}\eta_{ij}, \quad \widetilde{P}_{ijrs}\eta_{ij}\eta_{rs} \geq C_2\eta_{ij}\eta_{ij}$$

for every $\boldsymbol{\eta} = (\eta_{ij})$.

(iv) There exists positive constants C_3, C_3^* such that

$$\kappa_{ij}^*\xi_i\xi_j + (H_{ji}^* + K_{ij}^*)\xi_i\eta_j + \Lambda_{ij}^*\eta_i\eta_j \geq C_3^*(\xi_i\xi_i + \eta_i\eta_i),$$

and

$$\widetilde{\kappa}_{ij}\xi_i\xi_j + (\widetilde{H}_{ji} + \widetilde{K}_{ij})\xi_i\eta_j + \widetilde{\Lambda}_{ij}\eta_i\eta_i \geq C_3(\xi_i\xi_j + \eta_i\eta_i),$$

for every $\boldsymbol{\xi} = (\xi_i)$, $\boldsymbol{\eta} = (\eta_i)$.

(v) We have the following symmetries

$$\begin{aligned} &A_{ijrs} = A_{rsij}, \quad P_{ijrs}^* = P_{jirs}^*, \quad \kappa_{ij}^* = \kappa_{ji}^*, \quad \Lambda_{ij}^* = \Lambda_{ji}^*, \\ &A_{ij} = A_{ji}, \quad \widetilde{P}_{ijrs} = \widetilde{P}_{jirs}, \quad \widetilde{\kappa}_{ij} = \widetilde{\kappa}_{ji}, \quad \widetilde{\Lambda}_{ij} = \widetilde{\Lambda}_{ji}, \end{aligned}$$

as well as the equality[1]

$$H_{ji}^* = K_{ij}^*, \quad \widetilde{H}_{ji} = \widetilde{K}_{ij}.$$

In light of the latter condition, it is understood that, from now on, we will always use K_{ij} in place of H_{ji}.

[1] We recall that this kind of equality is related with the Onsager postulate in the case of the classical theory.

These assumptions are natural in the study of porous thermoelastic materials. The interpretation of the assumption (i) is obvious. Condition (ii) says that the internal mechanical energy is positive. This condition is usual in the studies corresponding to the elastic stability. Conditions (iii) and (iv) are also natural and they are related with the behavior of the heat (and microheat) conduction materials. The symmetries proposed in (v) are usual in the context we consider. We endow $\mathcal{H}$ with the norm

$$\begin{aligned}\|\boldsymbol{U}\|^2_{\mathcal{H}} = \int_\Omega \Big(& A_{ijrs}u_{i,j}u_{r,s} + 2D_{ij}u_{i,j}\phi + \xi|\phi|^2 + A_{ij}\phi_{,i}\phi_{,j} + \rho v_i v_i + J|\psi|^2 + a|\tau\vartheta+\theta|^2 + \kappa^*_{ij}(\tau\theta_{,i}+\alpha_{,i})(\tau\theta_{,j}+\alpha_{,j}) \\ & + \tau\widetilde{\kappa}_{ij}\theta_{,i}\theta_{,j} + B_{ij}(\tau S_i + T_i)(\tau S_j + T_j) + P^*_{ijrs}(\tau T_{i,j} + R_{i,j})(\tau T_{r,s} + R_{r,s}) + \Lambda^*_{ij}(\tau T_j + R_j)(\tau T_i + R_i) \\ & + \tau\widetilde{P}_{ijrs}T_{r,s}T_{i,j} + \tau\widetilde{\Lambda}_{ij}T_jT_i + (H^*_{ji} + K^*_{ij})(\tau\theta_{,j} + \alpha_{,j})(\tau T_i + R_i) + [\tau\widetilde{H}_{ji} + \tau\widetilde{K}_{ij}]T_i\theta_{,j}\Big)\, d\boldsymbol{x}.\end{aligned}$$

Introducing the state vector

$$\boldsymbol{U} = (u_i, v_i, \phi, \psi, \alpha, \theta, \vartheta, R_i, T_i, S_i),$$

we view the system as a Cauchy problem in $\mathcal{H}$ written as follows:

$$\frac{d}{dt}\boldsymbol{U}(t) = \mathbb{A}\boldsymbol{U}(t), \quad \boldsymbol{U}(0) = (u^0_i, v^0_i, \phi^0, \psi^0, \alpha^0, \theta^0, \vartheta^0, R^0_i, T^0_i, S^0_i). \tag{9}$$

Here, $\mathbb{A}$ is the linear operator defined as

$$\mathbb{A}\begin{pmatrix} u_i \\ v_i \\ \phi \\ \psi \\ \alpha \\ \theta \\ \vartheta \\ R_i \\ T_i \\ S_i \end{pmatrix} = \begin{pmatrix} v_i \\ \frac{1}{\rho}[A_{ijrs}e_{rs} + D_{ij}\phi - a_{ij}(\tau\vartheta+\theta)]_{,j}, \\ \psi \\ \frac{1}{J}[A_{ij}\phi_{,j} - N_{ij}(\tau S_j + T_j)]_{,i} - D_{ij}e_{ij} - \xi\phi + F(\tau\vartheta+\theta) \\ \theta \\ \vartheta \\ \frac{1}{\tau a}\mathsf{M} \\ T_i \\ R_i \\ \frac{1}{\tau}C_{ij}\mathsf{N}_j \end{pmatrix}, \tag{10}$$

where

$$\mathsf{M} = -a\vartheta - a_{ij}v_{i,j} - F\psi + (\kappa_{ij}\theta_{,j} + \kappa^*_{ij}\alpha_{,j} + H_{ij}T_j + H^*_{ij}R_j)_{,i}, \tag{11}$$

$$\mathsf{N}_i = -B_{ij}S_j + P_{ijrs}T_{r,sj} + P^*_{ijrs}R_{r,sj} - \Lambda_{ij}T_j - \Lambda^*_{ij}R_j - N_{ji}\psi_{,j} - K_{ij}\theta_{,j} - K^*_{ij}\alpha_{,j}, \tag{12}$$

and C_{ij} is the inverse of the matrix B_{ij}. This operator $\mathbb{A}$ has a (dense) domain:

$$\mathfrak{D}(\mathbb{A}) = \left\{ \boldsymbol{u} \in \mathcal{H} \;\middle|\; \begin{array}{c} v_i, \psi, \vartheta, S_i \in V \\ (A_{ijrs}u_{rs})_{,j} \in H \\ (A_{ij}\phi_{,j})_{,i} \in H \\ (\kappa_{ij}\theta_{,j} + \kappa^*_{ij}\alpha_{,j})_{,i} \in H \\ (P_{ijrs}T_{r,s} + P^*_{ijrs}R_{r,s})_{,j} \in H \end{array} \right\}.$$

3. Existence of solutions

In this section, we obtain an existence and uniqueness theorem for the problem proposed in system (9). We will use the Lumer–Phillips theorem. With standard notation, in what follows $\rho(\mathbb{A})$ denotes the resolvent set of the operator $\mathbb{A}$.

Lemma 1. *The operator $\mathbb{A}$ satisfies*

$$\langle \mathbb{A}\boldsymbol{U}, \boldsymbol{U}\rangle \leq 0,$$

for every $\boldsymbol{U} \in \mathfrak{D}(\mathbb{A})$.

Proof. By the Dirichlet boundary condition and the divergence theorem, a direct computation reveals that

$$\langle \mathbb{A}\boldsymbol{U}, \boldsymbol{U}\rangle = -\int_\Omega \left[\widetilde{\kappa}_{ij}\theta_{,i}\theta_{,j} + 2\widetilde{K}_{ij}\theta_{,i}T_j + \widetilde{\Lambda}_{ij}T_iT_j + \widetilde{P}_{ijrs}T_{r,j}T_{r,s}\right] dv.$$

Assumptions (iii) and (iv) entail the dissipativity of the operator, and the proof is finished.

Lemma 2. *The operator $\mathbb{A}$ is invertible. In particular $0 \in \rho(\mathbb{A})$.*

Proof. For every fixed vector

$$\boldsymbol{f} = (f_i^1, f_i^2, f^3, f^4, f^5, f^6, f^7, f_i^8, f_i^9, f_i^{10}) \in \mathcal{H}$$

we look for a unique solution $\boldsymbol{U} \in \mathfrak{D}(\mathbb{A})$ to the equation

$$\mathbb{A}\boldsymbol{U} = \boldsymbol{f}.$$

In components, equivalently we try to solve in $\mathfrak{D}(\mathbb{A})$ the following system:

$$v_i = f_i^1, \tag{13}$$
$$(A_{ijrs}u_{r,s} + D_{ij}\phi - a_{ij}\theta)_{,j} = \rho f_i^2, \tag{14}$$
$$\psi = f^3, \tag{15}$$
$$[A_{ij}\phi_{,j} - N_{ij}(\tau S_j + T_j)]_{,i} - D_{ij}u_{i,j} - \xi\phi + F(\tau\vartheta + \theta) = Jf^4, \tag{16}$$
$$\theta = f^5, \tag{17}$$
$$\vartheta = f^6, \tag{18}$$
$$-a\vartheta - a_{ij}v_{i,j} - F\psi + (\kappa_{ij}\theta_{,j} + \kappa_{ij}^*\alpha_{,j} + H_{ij}T_j + H_{ij}^*R_j)_{,i} = a\tau f^7, \tag{19}$$
$$T_i = f_i^8, \tag{20}$$
$$S_i = f_i^9, \tag{21}$$
$$\begin{aligned}-B_{ij}S_j + P_{ijrs}T_{r,sj} + P_{ijrs}^*R_{r,sj} - \Lambda_{ij}T_j - \Lambda_{ij}^*R_j - N_{ji}\psi_{,j}&\\ -K_{ij}\theta_{,j} - K_{ij}^*\alpha_{,j} = \tau B_{ij}f_i^{10}.&\end{aligned} \tag{22}$$

We can immediately substitute (13), (15), (17), (18), (20) and (21) into the other four equations to obtain the system:

$$\left.\begin{aligned}&(A_{ijrs}u_{r,s} + D_{ij}\phi)_{,j} = \Psi_i^1,\\ &A_{ij}\phi_{,j} - D_{ij}u_{i,j} - \xi\phi = \Psi^2,\\ &(\kappa_{ij}^*\alpha_{,j} + H_{ij}^*R_j)_{,i} = \Psi^3,\\ &(P_{ijrs}^*R_{r,s})_{,j} - \Lambda_{ij}^*R_j - K_{ij}^*\alpha_{,j} = \Psi_i^4,\end{aligned}\right. \tag{23}$$

where

$$\begin{aligned}\Psi_i^1 &= \rho f_i^2 + (a_{ij}f_i^5)_{,j} + (\tau a_{ij}f_i^6)_{,j},\\ \Psi^2 &= Jf^4 + [N_{ij}(f_i^8 + \tau f_i^9)]_{,j} - F(f^5 + \tau f^6),\\ \Psi^3 &= a\tau f^7 + af^6 + a_{ij}f_{i,j}^1 + Ff^3 + (\kappa_{ij}f_j^5 + K_{ij}f_j^8)_{,i},\\ \Psi_i^4 &= B_{ij}f_j^9 - (P_{ijrs}f_{r,s}^8)_{,j} + A_{ij}f_j^8 + N_{ji}f_{,j}^3 + K_{ij}f_{,j}^5.\end{aligned}$$

Notice that system (23) can be seen as two uncoupled systems: the first system in the variables u_i, ϕ and the second one in the variables α, R_i. Let us focus on the first system. We define the bilinear form:

$$\mathsf{B}_1\left((u_i,\phi),(u_i^*,\phi^*)\right) = \int_\Omega \left[A_{ijrs}u_{i,j}u_{r,s}^* + D_{ij}(u_{i,j}\phi^* + u_{i,j}^*\phi) + \phi\phi^*\right] dv.$$

By assumption (ii) and the Young inequality, it is clear that B_1 is coercive and continuous on the space $V^4 \times V^4$. Since (Ψ_i^1, Ψ^2) is easily seen to belong to $[V^{-1}]^4$, namely, the dual space of V^4, by the Lax–Milgram theorem, we obtain the existence of u_i, ϕ satisfying the first part of (23). By the same token, defining the form:

$$\begin{aligned}\mathsf{B}_2\left((\alpha, R_i),(\alpha^*, R_i^*)\right) = \int_\Omega \Big[\, &k_{ij}^*\alpha_{,i}\alpha_{,j}^* + K_{ij}^*(\alpha_{,i}R_j^* + \alpha_{,i}^*R_j)\\ &+\Lambda_{ij}^*R_iR_j^* + P_{ijrs}^*R_{i,j}R_{r,s}\,\Big] dv,\end{aligned}$$

it is possible to find α, R_i which satisfy the third and fourth equations.

Thus, we have proved the following existence and uniqueness result.

Theorem 3. *The problem (9) admits a unique solution. In fact, the solutions to problem (9) generate a contractive semigroup and, for every $\boldsymbol{U}(0) \in \mathfrak{D}(\mathbb{A})$, there exists a unique solution with the following regularity:*

$$\boldsymbol{U} \in C([0,\infty); \mathfrak{D}(\mathbb{A})) \cap C^1([0,\infty); \mathcal{H}).$$

4. Exponential stability: the one-dimensional system

In this section, we focus on the exponential stability of the system in one space dimension assuming the homogeneous case. In this setting, the system becomes

$$\begin{aligned}
&\rho\ddot{u} = Au_{xx} + D\phi_x - a^*(\tau\ddot{\alpha}_x + \dot{\alpha}_x),\\
&J\ddot{\phi} = A^*\phi_{xx} - N(\tau\ddot{R}_x + \dot{R}_x) - Du_x - \xi\phi + F(\tau\ddot{\alpha} + \dot{\alpha}),\\
&\tau a\dddot{\alpha} + a\ddot{\alpha} = -a^*\dot{u}_x - F\dot{\phi} + k\dot{\alpha}_{xx} + k^*\alpha_{xx} + H\dot{R}_x + H^*R_x,\\
&\tau B\dddot{R} + B\ddot{R} = -N\dot{\phi}_x + P\dot{R}_{xx} + P^*R_{xx} - K\dot{\alpha}_x - K^*\alpha_x - \Lambda\dot{R} - \Lambda^*R.
\end{aligned} \tag{24}$$

To this system of equations, we adjoint the homogeneous null boundary conditions (2) and the corresponding initial conditions (3) to the one-dimensional case.

For the sake of clarity, we write conditions (i)–(iv) in this one-dimensional setting:

(**i***) The coefficients ρ, J, a and B are positive.
(**ii***) $A > 0$ and $A\xi > D^2$.
(**iii***) $P^* > 0$ and $P > \tau P^*$.
(**iv***) $\kappa > 0$, $\kappa\Lambda > K^2$, $\kappa > \tau\kappa^*$, $(\kappa - \tau\kappa^*)(\Lambda - \tau\Lambda^*) > (K - \tau K^*)^2$.

We note that condition (v) is satisfied automatically when we ask $K = H$ and $K^* = H^*$. It is worth imposing a new assumption to guarantee the exponential decay. In this section, we assume that $a^* \neq 0$ and $N \neq 0$.

In the previous section, we have shown an existence and uniqueness theorem which can be applied to this system. It is worth noting that, in order to simplify the notation, we have assumed that the functions took real values. However, we could obtain the same result even if the functions had values in $\mathbb{C}$. That is, if we consider the previous system with complex variables, then we can guarantee that the solutions generate a semigroup of contractions and that the origin of the complex plane is within the resolvent of the operator.

We prove the exponential stability exploiting the classical Prüss result. It is known that, in order to prove the exponential stability for a contractive semigroup, it is enough to show that the imaginary axis is contained in the resolvent set of the operator and that the asymptotic condition

$$\overline{\lim_{|\lambda|\to\infty}} \|(\lambda I - \mathbb{A})^{-1}\| < \infty \tag{25}$$

holds (see [59]). First, we show that the imaginary axis is contained in the resolvent of the operator $\mathbb{A}$. We will proceed by contradiction and so, we assume that there exist two sequences $\lambda_n \in \mathbb{R}$ and

$$\boldsymbol{U}_n = (u_n, v_n, \phi_n, \psi_n, \alpha_n, \theta_n, \vartheta_n, R_n, T_n, S_n) \in \mathfrak{D}(\mathbb{A})$$

such that

$$\|\boldsymbol{U}_{\boldsymbol{n}}\|^2_{\mathcal{H}} = 1, \tag{26}$$

and

$$\|i\lambda_n\boldsymbol{U}_n - \mathbb{A}\boldsymbol{U}_n\|_{\mathcal{H}} \to 0. \tag{27}$$

Since $0 \in \rho(\mathbb{A})$ we can assume that

$$\lambda_n \not\to 0.$$

In components, convergence (27) reads

$$i\lambda_n u_n - v_n \to 0 \quad \text{in } V, \tag{28}$$
$$i\lambda_n\rho v_n - (A\partial_{xx}u_n + D\partial_x\phi_n - a^*\tau\partial_x\vartheta_n - a^*\partial_x\theta_n) \to 0 \quad \text{in } H, \tag{29}$$
$$i\lambda_n\phi_n - \psi_n \to 0 \quad \text{in } V, \tag{30}$$
$$iJ\lambda_n\psi_n - (A^*\partial_{xx}\phi_n - N\tau\partial_x S_n - N\partial_x T_n - D\partial_x u_n - \xi\phi_n + F\tau\vartheta_n + F\theta_n) \to 0 \quad \text{in } H, \tag{31}$$
$$i\lambda_n\alpha_n - \theta_n \to 0 \quad \text{in } V, \tag{32}$$
$$i\lambda_n\theta_n - \vartheta_n \to 0 \quad \text{in } V, \tag{33}$$
$$i\lambda_n\vartheta_n - (-a\vartheta_n - a^*\partial_x v_n - F\psi_n + k\partial_{xx}\theta_n + k^*\partial_{xx}\alpha_n + K\alpha_x T_n + K^*\partial_x R_n) \to 0 \quad \text{in } H,$$
$$i\lambda_n R_n - T_n \to 0 \quad \text{in } V, \tag{34}$$
$$i\lambda_n T_n - S_n \to 0 \quad \text{in } V, \tag{35}$$

$$i\tau B\lambda_n S_n - (-BS_n - N\partial_x\psi_n + P\partial_{xx}T_n + P^*\partial_{xx}R_n - K\partial_x\theta_n - K^*\partial_x\alpha_n - \Lambda T_n - \Lambda^* R_n) \to 0 \quad \text{in } H. \tag{36}$$

By the dissipativity of the operator $\mathbb{A}$ we know that

$$\theta_n, T_n \to 0, \quad \text{in } V.$$

Thanks to (32), (34) and the fact that $\lambda_n \not\to 0$, we infer that

$$\alpha_n, R_n \to 0, \quad \text{in } V.$$

Now, let us multiply by θ_n Eq. (34). Since $\theta_n \to 0$, exploiting Eq. (33) and the fact that $\boldsymbol{U}_n$ is bounded in $\mathcal{H}$, we obtain

$$\vartheta_n \to 0, \quad \text{in } H.$$

A similar multiplication, this time by T_n, of Eq. (36), yields

$$S_n \to 0, \quad \text{in } H.$$

Now, observe that by Eq. (35) and the fact that $T_n \to 0$ in V, we have

$$\frac{S_n}{\lambda_n} \to 0, \quad \text{in } V.$$

Hence, dividing by λ_n Eq. (31), we see at once that $\partial_{xx}\phi_n/\lambda_n$ is bounded in H. This allows us to multiply (36) by $\partial_x\phi_n/\lambda_n$ to obtain

$$\frac{N}{\lambda_n}\langle\psi_n, \phi_n\rangle_V - \frac{1}{\lambda_n}\Big(\partial_x(P^*R_n + PT_n)\cdot\partial_x\phi_n\Big)\Big|_0^L \to 0.$$

Now, we see that, by the Gagliardo-Nirenberg inequality,

$$\left\|\frac{\partial_x(P^*R_n + PT_n)}{\sqrt{|\lambda_n|}}\right\|_{L^\infty} \le C_1\|\partial_x(P^*R_n + PT_n)\|^{\frac{1}{2}}\frac{\|\partial_{xx}(P^*R_n + PT_n)\|^{\frac{1}{2}}}{\sqrt{|\lambda_n|}}$$
$$+C_2\frac{\|\partial_x(P^*R_n + PT_n)\|}{\sqrt{|\lambda_n|}} \to 0.$$

On the other hand, we find that

$$\left\|\frac{\partial_x\phi_n}{\sqrt{|\lambda_n|}}\right\|_{L^\infty} \le C_1\|\partial_x\phi_n\|^{\frac{1}{2}}\frac{\|\partial_{xx}\phi_n\|^{\frac{1}{2}}}{\sqrt{|\lambda_n|}} + C_2\frac{\|\partial_x\phi_n\|}{\sqrt{|\lambda_n|}} \le C,$$

where C is a positive constant independent of n. Therefore, we have

$$\frac{\|\partial_x(P^*R_n + PT_n)\cdot\partial_x\phi_n\|_{L^\infty}}{\lambda_n} \le \frac{\|\partial_x\phi_n\|_{L^\infty}}{\sqrt{|\lambda_n|}}\frac{\|\partial_x(P^*R_n + PT_n)\|_{L^\infty}}{\sqrt{|\lambda_n|}} \to 0.$$

In view of (30), this implies

$$\phi_n \to 0 \quad \text{in } V.$$

A multiplication of (31) by ϕ_n entails also the convergence

$$\psi_n \to 0 \quad \text{in } H.$$

Finally, repeating the argument for u_n and v_n (because $a^* \neq 0$) we reach a contradiction, proving that

$$\boldsymbol{U}_n \to 0 \quad \text{in } \mathcal{H}.$$

Now, we will show that the asymptotic condition (25) is satisfied. Again, we assume that it does not hold and we will arrive to a contradiction. If the condition is not fulfilled, then there will exist a sequence of real numbers $\lambda_n \to \infty$ and a sequence of unit norm vectors at the domain of operator $\mathbb{A}$ such that convergence (27) holds. From this point, we can follow the same argument proposed before since the unique key point is that λ_n does not tend to zero. We arrive again to a contradiction and we have proved the following result.

Theorem 4. *The solutions to system (24) with boundary conditions (2) and initial conditions (3) decay in an exponential form, that is, there exist two constants $M \ge 1$ and $\omega > 0$ such that*

$$\|\boldsymbol{U}(t)\|^2_{\mathcal{H}} \le Me^{-\omega t}\|\boldsymbol{U}(0)\|^2_{\mathcal{H}}.$$

Remark 2. We could also study the decay of solutions in dimension greater than one. Thought we do not give any proof, it is natural to expect that we cannot obtain uniform exponential decay in general. Furthermore, it is possible to show that the isothermal undamped solutions, obtained for the usual thermoelasticity by Dafermos [60], can be also found in our case. Therefore, the study for dimensions greater than one involves several cumbersome issues.

Remark 3. To prove this theorem we have assumed that all the boundary conditions are of Dirichlet homogeneous type. It is clear that we could also consider several other situations, in the sense that we could combine Dirichlet with Neumann homogeneous boundary conditions. The picture regarding exponential stability does not really change in all these other cases; however, these possibilities involve several different problems and the Sobolev space, where the equations would be well posed, changes depending on the particular boundary conditions. For the sake of brevity, we prefer not to dwell on studying all of the possibilities.

5. Numerical analysis of a fully discrete scheme

In this section, we consider a fully discrete approximation of the porous-thermoelastic problem with microtemperatures studied in the previous section, providing some a priori error estimates which lead to the linear convergence under adequate regularity conditions. As usual, we consider the deformation of the body over a finite time interval $[0, T_f]$, with a given final time $T_f > 0$.

For the sake of simplicity in the calculations, we restrict ourselves to the one-dimensional case in a finite spatial interval $(0, \ell)$, $\ell > 0$; however, we note that the extension to the multi-dimensional setting is straightforward and it can be obtained proceeding in a similar way. The numerical analysis presented in this section could be adapted by using the operators introduced in Section 2.

First, we derive the variational formulation of the problem. So, let us denote $v = \dot{u}$, $\psi = \dot{\phi}$, $\theta = \dot{\alpha}$, $\vartheta = \dot{\theta}$, $T = \dot{R}$ and $S = \dot{T}$ and define the variational spaces $Y = L^2(0, \ell)$ and $E = H_0^1(0, \ell)$. In the space Y we represent the inner product and the norm by $(\cdot, \cdot)$ and $\|\cdot\|$, respectively, and, for the space E, we consider the usual inner product and norm defined in $H^1(0, \ell)$, denoted as $(\cdot, \cdot)_E$ and $\|\cdot\|_E$, respectively. Therefore, multiplying the equations of system (24) by adequate test functions in the space E we obtain the following weak problem.

Find the velocity field $v : [0, T_f] \to E$, the porosity speed $\psi : [0, T_f] \to E$, the temperature speed $\vartheta : [0, T_f] \to E$ and the microtemperature speed $S : [0, T_f] \to E$ such that $v(0) = v^0$, $\psi(0) = \psi^0$, $\vartheta(0) = \vartheta^0$, $S(0) = S^0$ and, for a.e. $t \in (0, T)$ and for all $w, m, r, Z \in E$,

$$\rho(\dot{v}(t), w) + A(u_x(t), w_x) = -D(\phi(t), w_x) - a^*(\tau\vartheta_x(t) + \theta_x(t), w), \tag{37}$$

$$\begin{aligned} J(\dot{\psi}(t), m) + A^*(\phi_x(t), m_x) + \xi(\phi(t), m) &= -D(u_x(t), m) + F(\tau\vartheta(t) + \theta(t), m) \\ &\quad -N(\tau S_x(t) + T_x(t), m), \end{aligned} \tag{38}$$

$$\begin{aligned} (\tau a\dot{\vartheta}(t) + a\vartheta(t), r) + \kappa(\theta_x(t), r_x) + \kappa^*(\alpha_x(t), r_x) &= -a^*(v_x(t), r) - F(\psi(t), r) \\ &\quad +H(T_x(t), r) + H^*(R_x(t), r), \end{aligned} \tag{39}$$

$$\begin{aligned} (\tau B\dot{S}(t) + BS(t), Z) + P(T_x(t), Z_x) + P^*(R_x(t), Z_x) + \Lambda(T(t), Z) &= -N(\psi_x(t), Z) - K(\theta_x(t), Z) \\ &\quad -K^*(\alpha_x(t), Z) - \Lambda^*(R(t), Z), \end{aligned} \tag{40}$$

where the displacement, the porosity, the temperature, the microtemperature, the thermal displacement and the microthermal displacement are then recovered from the relations:

$$\begin{aligned} u(t) &= \int_0^t v(s)\,ds + u^0, \quad \phi(t) = \int_0^t \psi(s)\,ds + \phi^0, \\ \theta(t) &= \int_0^t \vartheta(s)\,ds + \theta^0, \quad T(t) = \int_0^t S(s)\,ds + T^0, \\ \alpha(t) &= \int_0^t \theta(s)\,ds + \alpha^0, \quad R(t) = \int_0^t T(s)\,ds + R^0. \end{aligned} \tag{41}$$

Now, we introduce the fully discrete approximation of problem (37)–(41). This is done in two steps. First, we approximate it in the spatial variable. Thus, we construct the finite dimensional space $E^h \subset E$ as follows:

$$E^h = \{z^h \in C([0, \ell]) \cap E \ ; \ z^h_{|[a_i, a_{i+1}]} \in P_1([a_i, a_{i+1}]) \quad \text{for } i = 0, \dots, M-1\}, \tag{42}$$

where we have used a uniform partition of the interval $[0, \ell]$, dividing it into M subintervals denoted by $a_0 = 0 < a_1 < \cdots < a_M = \ell$ with a uniform length $h = a_{i+1} - a_i = \ell/M$. Here, $P_1([a_i, a_{i+1}])$ is the space of polynomials of degree less or equal to 1 for each subinterval $[a_i, a_{i+1}]$, that is, the finite element space E^h is composed of continuous and piecewise affine functions and, as usual, $h > 0$ denotes the spatial discretization parameter. Then, by using the finite element projection operator over E^h denoted by $\mathcal{P}^h$ (see, for instance, the work of Clément [61]) we can define an approximation of the initial conditions given as

$$\begin{aligned} &u^{0h} = \mathcal{P}^h u^0, \quad v^{0h} = \mathcal{P}^h v^0, \quad \phi^{0h} = \mathcal{P}^h \phi^0, \quad \psi^{0h} = \mathcal{P}^h \psi^0, \quad \alpha^{0h} = \mathcal{P}^h \alpha^0, \quad \theta^{0h} = \mathcal{P}^h \theta^0, \\ &\vartheta^{0h} = \mathcal{P}^h \vartheta^0, \quad R^{0h} = \mathcal{P}^h R^0, \quad T^{0h} = \mathcal{P}^h T^0, \quad S^{0h} = \mathcal{P}^h S^0. \end{aligned} \tag{43}$$

Secondly, to obtain the discretization of the time derivatives, we use a uniform partition of the time interval $[0, T_f]$, denoted by $0 = t_0 < t_1 < \cdots < t_N = T_f$, with a time step size $k = T_f/N$ and nodes $t_n = n\,k$ for $n = 0, 1, \dots, N$. Moreover, for a continuous function $f(t)$ let $f_n = f(t_n)$ and, for a sequence $\{w_n\}_{n=0}^N$, let us denote by $\delta w_n = (w_n - w_{n-1})/k$ its divided differences. Therefore, by using the well-known implicit Euler scheme we obtain the fully discrete approximation of problem (37)–(41).

Find the discrete velocity field $\{v_n^{hk}\}_{n=0}^N \subset E^h$*, the discrete porosity speed* $\{\psi_n^{hk}\}_{n=0}^N \subset E^h$*, the discrete temperature speed* $\{\vartheta_n^{hk}\}_{n=0}^N \subset E^h$ *and the discrete microtemperature speed* $\{S_n^{hk}\}_{n=0}^N \subset E^h$ *such that* $v_0^{hk} = v^{0h}$, $\psi_0^{hk} = \psi^{0h}$, $\vartheta_0^{hk} = \vartheta^{0h}$, $S_0^{hk} = S^{0h}$ *and, for* $n = 1, \dots, N$ *and for all* $w^h, m^h, r^h, Z^h \in E^h$,

$$\rho(\delta v_n^{hk}, w^h) + A((u_n^{hk})_x, w_x^h) = -D(\phi_n^{hk}, w_x^h) - a^*(\tau(\vartheta_n^{hk})_x + (\theta_n^{hk})_x, w^h), \tag{44}$$

$$J(\delta\psi_n^{hk}, m^h) + A^*((\phi_n^{hk})_x, m_x^h) + \xi(\phi_n^{hk}, m^h) = -D((u_n^{hk})_x, m^h) + F(\tau\vartheta_n^{hk} + \theta_n^{hk}, m^h) - N(\tau(S_n^{hk})_x + (T_n^{hk})_x, m^h), \tag{45}$$

$$\begin{aligned}(\tau a\delta\vartheta_n^{hk} + a\vartheta_n^{hk}, r^h) + \kappa((\theta_n^{hk})_x, r_x^h) + \kappa^*((\alpha_n^{hk})_x, r_x^h) &= -a^*((v_n^{hk})_x, r^h) - F(\psi_n^{hk}, r^h)\\ &\quad + H((T_n^{hk})_x, r^h) + H^*((R_n^{hk})_x, r^h),\end{aligned} \tag{46}$$

$$\begin{aligned}(\tau B\delta S_n^{hk} + BS_n^{hk}, Z^h) + P((T_n^{hk})_x, Z_x^h) + P^*((R_n^{hk})_x, Z_x^h) + \Lambda(T_n^{hk}, Z^h) &= -N((\psi_n^{hk})_x, Z^h) - K((\theta_n^{hk})_x, Z^h)\\ &\quad - K^*((\alpha_n^{hk})_x, Z^h) - \Lambda^*(R_n^{hk}, Z^h),\end{aligned} \tag{47}$$

where the discrete displacement, the discrete porosity, the discrete temperature, the discrete microtemperature, the discrete thermal displacement and the discrete microthermal displacement are then recovered from the relations:

$$\begin{aligned}&u_n^{hk} = k\sum_{j=1}^n v_j^{hk} + u^{0h}, \quad \phi_n^{hk} = k\sum_{j=1}^n \psi_j^{hk} + \phi^{0h}, \quad \theta_n^{hk} = k\sum_{j=1}^n \vartheta_j^{hk} + \theta^{0h}, \quad T_n^{hk} = k\sum_{j=1}^n S_j^{hk} + T^{0h},\\ &\alpha_n^{hk} = k\sum_{j=1}^n \theta_j^{hk} + \alpha^{0h}, \quad R_n^{hk} = k\sum_{j=1}^n T_j^{hk} + R^{0h}.\end{aligned} \tag{48}$$

By using conditions (i*)-(iv*) on the constitutive coefficients and the well-known Lax–Milgram lemma, it is possible to prove that the fully discrete problem (44)–(48) admits a unique solution.

The aim of this section is to provide an a priori error analysis of this approximation. First, we will obtain a discrete stability result that we will state as follows.

Lemma 5. *Under conditions (i*)-(iv*), we obtain that the sequences* $\{u^{hk}, v^{hk}, \phi^{hk}, \psi^{hk}, \alpha^{hk}, \theta^{hk}, \vartheta^{hk}, R^{hk}, T^{hk}, S^{hk}\}$*, generated by discrete problem (44)–(48), satisfy the stability estimate:*

$$\|v_n^{hk}\|^2 + \|u_n^{hk}\|_E^2 + \|\psi_n^{hk}\|^2 + \|\phi_n^{hk}\|_E^2 + \|\alpha_n^{hk}\|_E^2 + \|\theta_n^{hk}\|_E^2 + \|\vartheta_n^{hk}\|^2 + \|R_n^{hk}\|_E^2 + \|T_n^{hk}\|_E^2 + \|S_n^{hk}\|^2 \le C,$$

where C is a positive constant which is independent of the discretization parameters h and k.

Proof. In order to prove this lemma, we will assume, for simplicity, that $\tau = 1$ and, to simplify the notation, we remove the superscripts in all the variables. If we take as a test function $w^h = v_n^{hk}$ in Eq. (44) we find that

$$\rho(\delta v_n, v_n) + A((u_n)_x, (v_n)_x) = -D((\phi_n, (v_n))_x) - a^*((\vartheta_n)_x + (\theta_n)_x, v_n).$$

Keeping in mind that

$$\begin{aligned}&(\delta v_n, v_n) \ge \frac{1}{2k}\Big\{\|v_n\|^2 - \|v_{n-1}\|^2\Big\},\\ &A((u_n)_x, (u_n)_x) = \frac{A}{2k}\Big\{\|(u_n)_x\|^2 - \|(u_{n-1})_x\|^2 + \|(u_n - u_{n-1})_x\|^2\Big\},\end{aligned}$$

we have

$$\begin{aligned}&\frac{\rho}{2k}\Big\{\|v_n\|^2 - \|v_{n-1}\|^2\Big\} + \frac{A}{2k}\Big\{\|(u_n)_x\|^2 - \|(u_{n-1})_x\|^2 + \|(u_n - u_{n-1})_x\|^2\Big\}\\ &\quad + D((\phi_n, (v_n))_x) \le -a^*((\vartheta_n)_x, v_n) + C(\|(\theta_n)_x\|^2 + \|v_n\|^2).\end{aligned}$$

Now, we proceed in a similar way for the remaining variables. Taking into account that

$$\begin{aligned}&\xi(\phi_n, \psi_n) = \frac{\xi}{2k}\Big\{\|\phi_n\|^2 - \|\phi_{n-1}\|^2 + \|\phi_n - \phi_{n-1}\|^2\Big\},\\ &-a^*((v_n)_x, \vartheta_n) = a^*((\vartheta_n)_x, v_n),\\ &-N((\psi_n)_x, S_n) = N((S_n)_x, \psi_n),\end{aligned}$$

we find that

$$\begin{aligned}&\frac{J}{2k}\Big\{\|\psi_n\|^2 - \|\psi_{n-1}\|^2\Big\} + \frac{\xi}{2k}\Big\{\|\phi_n\|^2 - \|\phi_{n-1}\|^2 + \|\phi_n - \phi_{n-1}\|^2\Big\}\\ &\qquad + \frac{A^*}{2k}\Big\{\|(\phi_n)_x\|^2 - \|(\phi_{n-1})_x\|^2\Big\} + D((u_n)_x, \psi_n)\\ &\quad \le -N((S_n)_x, \psi_n) + C\Big(\|\vartheta_n\|^2 + \|\theta_n\|^2 + \|\psi_n\|^2 + \|(T_n)_x\|^2\Big),\end{aligned}$$

$$\frac{a}{2k}\Big\{\|\vartheta_n\|^2-\|\vartheta_{n-1}\|^2\Big\}+\frac{\kappa}{2k}\Big\{\|(\theta_n)_x\|^2-\|(\theta_{n-1})_x\|^2\Big\}+\kappa^*((\alpha_n)_x,(\vartheta_n)_x)$$
$$+D((\phi_n,(v_n))_x)\le a^*((\vartheta_n)_x,v_n)+C(\|\psi_n\|^2+\|\vartheta_n\|^2+\|(T_n)_x\|^2+\|(R_n)_x\|^2),$$
$$\frac{B}{2k}\Big\{\|S_n\|^2-\|S_{n-1}\|^2\Big\}+\frac{P}{2k}\Big\{\|(T_n)_x\|^2-\|(T_{n-1})_x\|^2\Big\}+\frac{\Lambda}{2k}\Big\{\|T_n\|^2-\|T_{n-1}\|^2\Big\}+P^*((R_n)_x,(S_n)_x)$$
$$\le N((S_n)_x,\psi_n)+C\Big(\|(\alpha_n)_x\|^2+\|(\theta_n)_x\|^2+\|S_n\|^2+\|R_n\|^2\Big).$$

Observing that

$$D((u_n)_x,\psi_n)+D((\phi_n,(v_n))_x)=\frac{D}{k}\Big\{((u_n)_x,\phi_n)-((u_{n-1})_x,\phi_{n-1})$$
$$+((u_n-u_{n-1})_x,\phi_n-\phi_{n-1})\Big\},$$
$$A\|(u_n-u_{n-1})_x\|^2+\xi\|\phi_n-\phi_{n-1}\|^2+2D((u_n-u_{n-1})_x,\phi_n-\phi_{n-1})\ge 0,$$

thanks to the assumptions (i*)-(iv*), combining the previous estimates and summing up to n we obtain

$$\rho\|v_n\|^2+A\|(u_n)_x\|^2+J\|\psi_n\|^2+\xi\|\phi_n\|^2+A^*\|(\phi_n)_x\|^2+2D((u_n)_x,\phi_n)$$
$$+a\|\vartheta_n\|^2+\kappa\|(\theta_n)_x\|^2+2k\kappa^*\sum_{j=1}^n((\alpha_j)_x,(\vartheta_j)_x)+B\|S_n\|^2+P\|(T_n)_x\|^2$$
$$+\Lambda\|T_n\|^2+2kP^*\sum_{j=1}^n((R_j)_x,(S_j)_x)$$
$$\le Ck\sum_{j=1}^n\Big(\|(\alpha_j)_x\|^2+\|(\theta_j)_x\|^2+\|S_j\|^2+\|R_j\|^2+\|\psi_j\|^2+\|\vartheta_j\|^2+\|(T_j)_x\|^2$$
$$+\|(R_j)_x\|^2+\|\theta_j\|^2+\|v_j\|^2\Big)$$
$$+C\Big(\|v^0\|^2+\|u^0\|_E^2+\|\psi^0\|^2+\|\phi^0\|_E^2+\|\vartheta^0\|^2+\|\theta^0\|_E^2+\|S^0\|^2+\|T^0\|_E^2\Big).$$

Thanks again to assumptions (i*)–(iv*) we have

$$A\|(u_n)_x\|^2+\xi\|\phi_n\|^2+2D((u_n)_x,\phi_n)\ge C(\|(u_n)_x\|^2+\|\phi_n\|^2),$$

and, taking into account that

$$\|\alpha_n\|_E^2+\|R_n\|_E^2\le Ck\sum_{j=1}^n\Big(\|\theta_j\|_E^2+\|T_j\|_E^2\Big)+C\Big(\|\alpha^0\|_E^2+\|R^0\|_E^2\Big),$$
$$k\sum_{j=1}^n\Big[((\alpha_j)_x,(\vartheta_j)_x)+((R_j)_x,(S_j)_x)\Big]\le C\Big(\|(\theta_n)_x\|^2+\|\alpha_n\|_E^2+\|\alpha^0\|_E^2+\|\theta^0\|_E^2$$
$$+\|R_n\|_E^2+\|(T_n)_x\|^2+\|T^0\|_E^2+\|R^0\|_E^2+k\sum_{j=1}^n\Big[\|(\theta_j)_x\|^2+\|(T_j)_x\|^2\Big]\Big),$$

using a discrete version of Gronwall's inequality (see, for instance, [7]) we conclude the desired discrete stability property.

Now, we will focus on the derivation of some a priori error estimates. First, we obtain the error estimates on the velocity field. Thus, we subtract variational Eq. (37), at time $t=t_n$ and for a test function $w=w^h\in E^h\subset E$, and discrete variational Eq. (44) to obtain, for all $w^h\in E^h$,

$$\rho(\dot v_n-\delta v_n^{hk},w^h)+A((u_n-u_n^{hk})_x,w_x^h)+a^*((\vartheta_n-\vartheta_n^{hk})_x+(\theta_n-\theta_n^{hk})_x,w^h)+D(\phi_n-\phi_n^{hk},w_x^h)=0.$$

In the previous equation, and also in the rest of this section, again we will assume that $\tau=1$ for the sake of simplicity. We note that it is straightforward to extend the analysis shown below to the general situation.

Taking into account that

$$(\dot v_n-\delta v_n^{hk},v_n-v_n^{hk})=(\dot v_n-\delta v_n,v_n-v_n^{hk})+(\delta v_n-\delta v_n^{hk},v_n-v_n^{hk}),$$
$$\big(\delta v_n-\delta v_n^{hk},v_n-v_n^{hk}\big)\ge\frac{1}{2k}\Big\{\|v_n-v_n^{hk}\|^2-\|v_{n-1}-v_{n-1}^{hk}\|^2\Big\},$$
$$((u_n-u_n^{hk})_x,(v_n-v_n^{hk})_x)\ge((u_n-u_n^{hk})_x,(\dot u_n-\delta u_n)_x)$$
$$+\frac{1}{2k}\Big\{\|(u_n-u_n^{hk})_x\|^2-\|(u_{n-1}-u_{n-1}^{hk})_x\|^2+\|(u_n-u_n^{hk})_x-(u_{n-1}-u_{n-1}^{hk})_x\|^2\Big\},$$
$$((\vartheta_n-\vartheta_n^{hk})_x,v_n-w^h)=-(\vartheta_n-\vartheta_n^{hk},(v_n-w^h)_x),$$
$$((v_n-v_n^{hk})_x,\vartheta_n-r^h)=-(v_n-v_n^{hk},(\vartheta_n-r^h)_x),$$

we find that, for all $w^h \in E^h$,

$$\begin{aligned}
&\frac{\rho}{2k}\left\{\|v_n - v_n^{hk}\|^2 - \|v_{n-1} - v_{n-1}^{hk}\|^2\right\} + D(\phi_n - \phi_n^{hk}, (\delta u_n - \delta u_n^{hk})_x) \\
&\quad + \frac{A}{2k}\left\{\|(u_n - u_n^{hk})_x\|^2 - \|(u_{n-1} - u_{n-1}^{hk})_x\|^2 + \|(u_n - u_n^{hk})_x - (u_{n-1} - u_{n-1}^{hk})_x\|^2\right\} \\
&\quad + a^*((\vartheta_n - \vartheta_n^{hk})_x, v_n - v_n^{hk}) \\
&\leq C\Big(\|\dot{v}_n - \delta v_n\|^2 + \|\dot{u}_n - \delta u_n\|_E^2 + \|v_n - w^h\|_E^2 + \|(u_n - u_n^{hk})_x\|^2 + \|\phi_n - \phi_n^{hk}\|^2 \\
&\quad + \|(\theta_n - \theta_n^{hk})_x\|^2 + \|\vartheta_n - \vartheta_n^{hk}\|^2 + \|v_n - v_n^{hk}\|^2 + (\delta v_n - \delta v_n^{hk}, v_n - w^h)\Big).
\end{aligned}$$

Proceeding in a similar form for the porosity speed, the temperature speed and the microtemperature speed, taking into account that

$$\begin{aligned}
&\xi(\phi_n - \phi_n^{hk}, \psi_n - \psi_n^{hk}) = (\phi_n - \phi_n^{hk}, \dot{\phi}_n - \delta\phi_n) + (\phi_n - \phi_n^{hk}, \delta\phi_n - \delta\phi_n^{hk}), \\
&\xi\left(\phi_n - \phi_n^{hk}, \delta\phi_n - \delta\phi_n^{hk}\right) \geq \frac{\xi}{2k}\left\{\|\phi_n - \phi_n^{hk}\|^2 - \|\phi_{n-1} - \phi_{n-1}^{hk}\|^2 + \|\phi_n - \phi_n^{hk} - (\phi_{n-1} - \phi_{n-1}^{hk})\|^2\right\}, \\
&N((\psi_n - \psi_n^{hk})_x, S_n - Z^h) = -N(\psi_n - \psi_n^{hk}, (S_n - Z^h)_x), \\
&N((\psi_n - \psi_n^{hk})_x, S_n - S_n^{hk}) = -N(\psi_n - \psi_n^{hk}, (S_n - S_n^{hk})_x),
\end{aligned}$$

we have

$$\begin{aligned}
&\frac{J}{2k}\left\{\|\psi_n - \psi_n^{hk}\|^2 - \|\psi_{n-1} - \psi_{n-1}^{hk}\|^2\right\} + D(\delta\phi_n - \delta\phi_n^{hk}, (u_n - u_n^{hk})_x) \\
&\quad + \frac{\xi}{2k}\left\{\|\phi_n - \phi_n^{hk}\|^2 - \|\phi_{n-1} - \phi_{n-1}^{hk}\|^2 + \|\phi_n - \phi_n^{hk} - (\phi_{n-1} - \phi_{n-1}^{hk})\|^2\right\} \\
&\quad + N((S_n - S_n^{hk})_x, \psi_n - \psi_n^{hk}) + \frac{A^*}{2k}\left\{\|(\phi_n - \phi_n^{hk})_x\|^2 - \|(\phi_{n-1} - \phi_{n-1}^{hk})_x\|^2\right\} \\
&\leq C\Big(\|\dot{\psi}_n - \delta\psi_n\|^2 + \|\dot{\phi}_n - \delta\phi_n\|_E^2 + \|\psi_n - m^h\|_E^2 + \|(\phi_n - \phi_n^{hk})_x\|^2 + \|\phi_n - \phi_n^{hk}\|^2 \\
&\quad + \|\theta_n - \theta_n^{hk}\|^2 + \|\vartheta_n - \vartheta_n^{hk}\|^2 + \|(u_n - u_n^{hk})_x\|^2 + \|(T_n - T_n^{hk})_x\|^2 + \|S_n - S_n^{hk}\|^2 \\
&\quad + (\delta\psi_n - \delta\psi_n^{hk}, \psi_n - m^h)\Big) \quad \forall m^h \in E^h, \\
&\frac{a}{2k}\left\{\|\vartheta_n - \vartheta_n^{hk}\|^2 - \|\vartheta_{n-1} - \vartheta_{n-1}^{hk}\|^2\right\} + \kappa^*((\alpha_n - \alpha_n^{hk})_x, (\vartheta_n - \vartheta_n^{hk})_x) \\
&\quad + \frac{\kappa}{2k}\left\{\|(\theta_n - \theta_n^{hk})_x\|^2 - \|(\theta_{n-1} - \theta_{n-1}^{hk})_x\|^2\right\} \\
&\quad + a^*((v_n - v_n^{hk})_x, \vartheta_n - \vartheta_n^{hk}) \\
&\leq C\Big(\|\dot{\vartheta}_n - \delta\vartheta_n\|^2 + \|\dot{\theta}_n - \delta\theta_n\|_E^2 + \|\vartheta_n - r^h\|_E^2 + \|(\theta_n - \theta_n^{hk})_x\|^2 + \|\vartheta_n - \vartheta_n^{hk}\|^2 \\
&\quad + \|(\alpha_n - \alpha_n^{hk})_x\|^2 + \|\psi_n - \psi_n^{hk}\|^2 + \|(T_n - T_n^{hk})_x\|^2 + \|(R_n - R_n^{hk})_x\|^2 + \|v_n - v_n^{hk}\|^2 \\
&\quad + (\delta\vartheta_n - \delta\vartheta_n^{hk}, \vartheta_n - r^h)\Big) \quad \forall r^h \in E^h, \\
&\frac{B}{2k}\left\{\|S_n - S_n^{hk}\|^2 - \|S_{n-1} - S_{n-1}^{hk}\|^2\right\} - N(\psi_n - \psi_n^{hk}, (S_n - S_n^{hk})_x) \\
&\quad + \frac{P}{2k}\left\{\|(T_n - T_n^{hk})_x\|^2 - \|(T_{n-1} - T_{n-1}^{hk})_x\|^2\right\} \\
&\quad + P^*((R_n - R_n^{hk})_x, (S_n - S_n^{hk})_x) + \frac{\Lambda}{2k}\left\{\|T_n - T_n^{hk}\|^2 - \|T_{n-1} - T_{n-1}^{hk}\|^2\right\} \\
&\leq C\Big(\|\dot{S}_n - \delta S_n\|^2 + \|\dot{T}_n - \delta T_n\|_E^2 + \|S_n - Z^h\|_E^2 + \|(T_n - T_n^{hk})_x\|^2 + \|S_n - S_n^{hk}\|^2 \\
&\quad + \|(\alpha_n - \alpha_n^{hk})_x\|^2 + \|(\theta_n - \theta_n^{hk})_x\|^2 + \|(R_n - R_n^{hk})_x\|^2 + \|R_n - R_n^{hk}\|^2 + \|\psi_n - \psi_n^{hk}\|^2 \\
&\quad + (\delta S_n - \delta S_n^{hk}, S_n - Z^h)\Big) \quad \forall Z^h \in E^h.
\end{aligned}$$

Since

$$\begin{aligned}
&D(\phi_n - \phi_n^{hk}, (\delta u_n - \delta u_n^{hk})_x) + D(\delta\phi_n - \delta\phi_n^{hk}, (u_n - u_n^{hk})_x) \\
&= \frac{D}{k}\Big\{(\phi_n - \phi_n^{hk}, (u_n - u_n^{hk})_x) - (\phi_{n-1} - \phi_{n-1}^{hk}, (u_{n-1} - u_{n-1}^{hk})_x) \\
&\quad + (\phi_n - \phi_n^{hk} - (\phi_{n-1} - \phi_{n-1}^{hk}), (u_n - u_n^{hk} - (u_{n-1} - u_{n-1}^{hk}))_x)\Big\},
\end{aligned}$$

and observing that, thanks again to assumptions (i*)-(iv*),

$$\begin{aligned}
&A\|(u_n - u_n^{hk})_x - (u_{n-1} - u_{n-1}^{hk})_x\|^2 + \xi\|\phi_n - \phi_n^{hk} - (\phi_{n-1} - \phi_{n-1}^{hk})\|^2 \\
&\quad + 2D(\phi_n - \phi_n^{hk} - (\phi_{n-1} - \phi_{n-1}^{hk})), (u_n - u_n^{hk} - (u_{n-1} - u_{n-1}^{hk}))_x \geq 0,
\end{aligned}$$

combining the previous estimates, multiplying them by $2k$ and summing up to n, it leads, for all $w^h, m^h, r^h, Z^h \in E^h$,

$$\begin{aligned}
&\rho\|v_n - v_n^{hk}\|^2 + 2D(\phi_n - \phi_n^{hk}, (u_n - u_n^{hk})_x) + A\|(u_n - u_n^{hk})_x\|^2 + J\|\psi_n - \psi_n^{hk}\|^2 \\
&\quad + A^*\|(\phi_n - \phi_n^{hk})_x\|^2 + a\|\vartheta_n - \vartheta_n^{hk}\|^2 + 2\kappa^* k\sum_{j=1}^n ((\alpha_j - \alpha_j^{hk})_x, (\delta\theta_j - \delta\theta_j^{hk})_x) \\
&\quad + \kappa\|(\theta_n - \theta_n^{hk})_x\|^2 + B\|S_n - S_n^{hk}\|^2 + P\|(T_n - T_n^{hk})_x\|^2 + \Lambda\|T_n - T_n^{hk}\|^2 \\
&\quad + 2P^* k\sum_{j=1}^n ((R_j - R_j^{hk})_x, (\delta T_j - \delta T_j^{hk})_x) + \xi\|\phi_n - \phi_n^{hk}\|^2 \\
&\leq Ck\sum_{j=1}^n \Big(\|\dot v_j - \delta v_j\|^2 + \|\dot u_j - \delta u_j\|_E^2 + \|v_j - w_j^h\|_E^2 + \|(u_j - u_j^{hk})_x\|^2 + \|\phi_j - \phi_j^{hk}\|^2 \\
&\quad + \|(\theta_j - \theta_j^{hk})_x\|^2 + \|\vartheta_j - \vartheta_j^{hk}\|^2 + \|v_j - v_j^{hk}\|^2 + (\delta v_j - \delta v_j^{hk}, v_j - w_j^h) \\
&\quad + \|\dot S_j - \delta S_j\|^2 + \|\dot T_j - \delta T_j\|_E^2 + \|S_j - Z_j^h\|_E^2 + \|(T_j - T_j^{hk})_x\|^2 + \|S_j - S_j^{hk}\|^2 \\
&\quad + \|(\alpha_j - \alpha_j^{hk})_x\|^2 + \|(R_j - R_j^{hk})_x\|^2 + \|R_j - R_j^{hk}\|^2 + \|\psi_j - \psi_j^{hk}\|^2 + \|\dot\psi_j - \delta\psi_j\|^2 \\
&\quad + (\delta S_j - \delta S_j^{hk}, S_j - Z_j^h) + \|\dot\phi_j - \delta\phi_j\|_E^2 + \|\psi_j - m_j^h\|_E^2 + \|(\phi_j - \phi_j^{hk})_x\|^2 \\
&\quad + \|\phi_j - \phi_j^{hk}\|^2 + \|\theta_j - \theta_j^{hk}\|^2 + (\delta\psi_j - \delta\psi_j^{hk}, \psi_j - m_j^h) + \|\dot\vartheta_j - \delta\vartheta_j\|^2 + \|\dot\theta_j - \delta\theta_j\|_E^2 \\
&\quad + \|\vartheta_j - r_j^h\|_E^2 + (\delta\vartheta_j - \delta\vartheta_j^{hk}, \vartheta_j - r_j^h) \Big) + C\Big(\|v^0 - v^{0h}\|^2 + \|u^0 - u^{0h}\|_E^2 + \|\psi^0 - \psi^{0h}\|^2 \\
&\quad + \|\phi^0 - \phi^{0h}\|_E^2 + \|\alpha^0 - \alpha^{0h}\|_E^2 + \|R^0 - R^{0h}\|_E^2 + \|\vartheta^0 - \vartheta^{0h}\|^2 + \|\theta^0 - \theta^{0h}\|_E^2 \\
&\quad + \|S^0 - S^{0h}\|^2 + \|T^0 - T^{0h}\|_E^2 \Big).
\end{aligned}$$

Thanks again to assumptions (i*)-(iv*) we find that

$$2D(\phi_n - \phi_n^{hk}, (u_n - u_n^{hk})_x) + A\|(u_n - u_n^{hk})_x\|^2 + \xi\|\phi_n - \phi_n^{hk}\|^2 \geq C(\|(u_n - u_n^{hk})_x\|^2 + \|\phi_n - \phi_n^{hk}\|^2).$$

Recalling that

$$\begin{aligned}
&k\sum_{j=1}^n ((\alpha_j - \alpha_j^{hk})_x, (\delta\theta_j - \delta\theta_j^{hk})_x) \leq ((\alpha_n - \alpha_n^{hk})_x, (\theta_n - \theta_n^{hk})_x) + ((\theta^{0h} - \theta^0)_x, (\alpha_1 - \alpha_1^{hk})_x) \\
&\qquad + Ck\sum_{j=1}^n \|(\theta_j - \theta_j^{hk})_x\|^2 + Ck\sum_{j=1}^n \|\dot\alpha_j - \delta\alpha_j\|_E^2, \\
&k\sum_{j=1}^n ((R_j - R_j^{hk})_x, (\delta T_j - \delta T_j^{hk})_x) \leq ((T_n - T_n^{hk})_x, (R_n - R_n^{hk})_x) + ((T^{0h} - T^0)_x, (R_1 - R_1^{hk})_x) \\
&\qquad + Ck\sum_{j=1}^n \|(T_j - T_j^{hk})_x\|^2 + Ck\sum_{j=1}^n \|\dot R_j - \delta R_j\|_E^2, \\
&\|\alpha_n - \alpha_n^{hk}\|_E^2 \leq C\Big(I_n + k\sum_{j=1}^n \|\theta_j - \theta_j^{hk}\|_E^2 + \|\alpha^0 - \alpha^{0h}\|_E^2\Big), \\
&\|R_n - R_n^{hk}\|_E^2 \leq C\Big(J_n + k\sum_{j=1}^n \|T_j - T_j^{hk}\|_E^2 + \|R^0 - R^{0h}\|_E^2\Big), \\
&k\sum_{j=1}^n (\delta v_j - \delta v_j^{hk}, v_j - w_j^h) = (v_n - v_n^{hk}, v_n - w_n^h) + (v^{0h} - v^0, v_1 - w_1^h) \\
&\qquad + \sum_{j=1}^{n-1} (v_j - v_j^{hk}, v_j - w_j^h - (v_{j+1} - w_{j+1}^h)),
\end{aligned}$$

where similar estimates can be obtained for the differences $\delta S_j - \delta S_j^{hk}$, $\delta\vartheta_j - \delta\vartheta_j^{hk}$ and $\delta\psi_j - \delta\psi_j^{hk}$, and I_n and J_n are the integration errors defined as

$$I_n = \left\| \int_0^{t_n} \theta(s)\,ds - k\sum_{j=1}^n \theta_j \right\|_E^2, \tag{49}$$

$$J_n = \left\| \int_0^{t_n} T(s)\,ds - k\sum_{j=1}^{n} T_j \right\|_E^2, \tag{50}$$

using again a discrete version of Gronwall's inequality (see [7]) we conclude the following a priori error estimates result.

Theorem 6. *Let the assumptions (i*)-(iv*) still hold. If we denote by (v,ψ,ϑ,S) the solution to problem (37)–(41) and by $(v^{hk},\psi^{hk},\vartheta^{hk},S^{hk})$ the solution to problem (44)–(48), then we have the following a priori error estimates, for all $w^h=\{w_j^h\}_{j=0}^N$, $m^h=\{m_j^h\}_{j=0}^N$, $r^h=\{r_j^h\}_{j=0}^N$, $Z^h=\{Z_j^h\}_{j=0}^N\subset E^h$,*

$$\begin{aligned}
&\max_{0\le n\le N}\Big\{ \|v_n-v_n^{hk}\|^2+\|u_n-u_n^{hk}\|_E^2+\|\psi_n-\psi_n^{hk}\|^2+\|\phi_n-\phi_n^{hk}\|_E^2+\|\alpha_n-\alpha_n^{hk}\|_E^2\\
&+\|\theta_n-\theta_n^{hk}\|_E^2+\|\vartheta_n-\vartheta_n^{hk}\|^2+\|R_n-R_n^{hk}\|_E^2+\|T_n-T_n^{hk}\|_E^2+\|S_n-S_n^{hk}\|^2\Big\}\\
&\quad\le Ck\sum_{j=1}^{N}\Big(\|\dot v_j-\delta v_j\|^2+\|\dot u_j-\delta u_j\|_E^2+\|v_j-w_j^h\|_E^2\\
&\quad+\|\dot S_j-\delta S_j\|^2+\|\dot T_j-\delta T_j\|_E^2+\|S_j-Z_j^h\|_E^2+\|\dot\psi_j-\delta\psi_j\|^2\\
&\quad+\|\dot\phi_j-\delta\phi_j\|_E^2+\|\psi_j-m_j^h\|_E^2+\|\dot\vartheta_j-\delta\vartheta_j\|^2+\|\dot\theta_j-\delta\theta_j\|_E^2+\|\vartheta_j-r_j^h\|_E^2\Big)\\
&\quad+C\max_{0\le n\le N}\Big\{\|v_n-w_n^h\|^2+\|\psi_n-m_n^h\|^2+\|\vartheta_n-r_n^h\|^2+\|S_n-Z_n^h\|^2\Big\}\\
&\quad+\frac{C}{k}\sum_{j=1}^{N-1}\Big(\|v_j-w_j^h-(v_{j+1}-w_{j+1}^h)\|^2+\|\psi_j-m_j^h-(\psi_{j+1}-m_{j+1}^h)\|^2\\
&\quad+\|\vartheta_j-r_j^h-(\vartheta_{j+1}-r_{j+1}^h)\|^2+\|S_j-Z_j^h-(S_{j+1}-Z_{j+1}^h)\|^2\Big)\\
&\quad+C\Big(\|v^0-v^{0h}\|^2+\|u^0-u^{0h}\|_E^2+\|\psi^0-\psi^{0h}\|^2+\|\phi^0-\phi^{0h}\|_E^2+\|R^0-R^{0h}\|_E^2\\
&\quad+\|\alpha^0-\alpha^{0h}\|_E^2+\|\vartheta^0-\vartheta^{0h}\|^2+\|\theta^0-\theta^{0h}\|_E^2+\|S^0-S^{0h}\|^2+\|T^0-T^{0h}\|_E^2\Big),
\end{aligned}$$

where C is again a positive constant which does not depend on parameters h and k, and I_j and J_j are the integration errors given in (49) and (50), respectively.

We note that the estimates provided in Theorem 6 can be used to derive the convergence order of the approximations obtained from the fully discrete problem (44)–(48). Thus, as an example, if we assume that the solution to problem (37)–(41) has the additional regularity:

$$\begin{aligned}
&u,\phi\in C^1([0,T_f];H^2(0,\ell))\cap H^2(0,T_f;H^1([0,\ell]))\cap H^3(0,T_f;Y),\\
&\alpha,R\in C^2([0,T_f];H^2(0,\ell))\cap H^3(0,T_f;H^1([0,\ell]))\cap H^4(0,T_f;Y),
\end{aligned}$$

then there exists a positive constant C, assumed to be independent of the discretization parameters h and k, such that

$$\begin{aligned}
&\max_{0\le n\le N}\Big\{ \|v_n-v_n^{hk}\|+\|u_n-u_n^{hk}\|_E+\|\psi_n-\psi_n^{hk}\|+\|\phi_n-\phi_n^{hk}\|_E+\|\alpha_n-\alpha_n^{hk}\|_E+\|\theta_n-\theta_n^{hk}\|_E\\
&\quad+\|\vartheta_n-\vartheta_n^{hk}\|+\|R_n-R_n^{hk}\|_E+\|T_n-T_n^{hk}\|_E+\|S_n-S_n^{hk}\|\Big\}\le C(h+k),
\end{aligned}$$

that is, the linear convergence of the approximations is achieved under this assumed regularity.

6. Numerical results

In this final section, we present the numerical scheme implemented in MATLAB for solving problem (44)–(48), and we describe two numerical examples to demonstrate the accuracy of the approximations and the behavior of the discrete energy in a one-dimensional problem, and the behavior of the solution in a two-dimensional example.

6.1. A one-dimensional example: numerical convergence and discrete energy decay

First, we briefly describe the numerical scheme and then, we show the numerical example.

Given the solution u_{n-1}^{hk}, v_{n-1}^{hk}, ϕ_{n-1}^{hk}, ψ_{n-1}^{hk}, α_{n-1}^{hk}, θ_{n-1}^{hk}, ϑ_{n-1}^{hk}, R_{n-1}^{hk}, T_{n-1}^{hk} and S_{n-1}^{hk} at time t_{n-1}, the variables v_n^{hk}, ψ_n^{hk}, ϑ_n^{hk} and S_n^{hk} are obtained by solving the discrete linear system, for all $w^h,m^h,r^h,Z^h\in E^h$,

$$\rho(v_n^{hk},w^h)+Ak^2((v_n^{hk})_x,w^h)=\rho(v_{n-1}^{hk},w^h)-Ak((u_{n-1}^{hk})_x,w^h)$$

$$
\begin{aligned}
&-Dk(\phi_n^{hk}, w_x^h) - a^*k(\tau(\vartheta_n^{hk})_x + (\theta_n^{hk})_x, w^h),\\
&J(\psi_n^{hk}, m^h) + A^*k(k(\psi_n^{hk})_x, m_x^h) + \xi k^2(\psi_n^{hk}, m^h) = J(\psi_{n-1}^{hk}, m^h)\\
&\quad +A^*k((\phi_{n-1}^{hk})_x, m_x^h) - \xi(\phi_{n-1}^{hk}, m^h) - Nk(\tau(S_n^{hk})_x + (T_n^{hk})_x, m^h)\\
&\quad -Dk((u_n^{hk})_x, m^h) + Fk(\tau\vartheta_n^{hk} + \theta_n^{hk}, m^h),\\
&a(\tau\vartheta_n^{hk} + k\vartheta_n^{hk}, r^h) + \kappa k^2((\vartheta_n^{hk})_x, r_x^h) - \kappa k^3((\vartheta_n^{hk})_x, r_x^h) = a(\tau\vartheta_{n-1}^{hk}, r^h)\\
&\quad -a^*k((v_n^{hk})_x, r^h) - Fk((\psi_n^{hk})_x, r^h) - \kappa k((\theta_{n-1}^{hk})_x, r_x^h)\\
&\quad -\kappa^*k((\alpha_{n-1}^{hk})_x + k(\theta_{n-1}^{hk})_x, r_x^h) + Hk((S_n^{hk})_x, r^h) + H^*k((R_n^{hk})_x, m^h),\\
&B(\tau S_n^{hk} + kS_n^{hk}, Z^h) + Pk((S_n^{hk})_x, Z_x^h) + P^*k^3((S_n^{hk})_x, Z_x^h)\\
&\quad +\Lambda k^2(S_n^{hk}, Z^h) + \Lambda^*k^3(S^{hk,n}, Z^h) = B(\tau S_{n-1}^{hk}, Z^h) - Nk((\psi_n^{hk})_x, Z^h)\\
&\quad -Pk((T_{n-1}^{hk})_x, Z_x^h) - P^*k((R_{n-1}^{hk})_x + k(T_{n-1}^{hk})_x, Z_x^h) - Kk((\theta_n^{hk})_x, Z^h)\\
&\quad -K^*k((\alpha_n^{hk})_x, Z^h) - \Lambda k(T_{n-1}^{hk}, Z^h) - \Lambda^*k(R_{n-1}^{hk} + kT_{n-1}^{hk}, Z^h).
\end{aligned}
$$

This numerical scheme was implemented on a 3.2 GHz PC using MATLAB, and a typical run ($h = k = 0.001$) took about 0.33 s of CPU time.

As a simple example, in order to show the accuracy of the approximations the following problem is considered:

$$
\begin{aligned}
&\rho\ddot{u} = Au_{xx} + D\phi_x - a^*(\tau\ddot{\alpha}_x + \dot{\alpha}_x) + F_1,\\
&J\ddot{\phi} = A^*\phi_{xx} - N(\tau\ddot{R}_x + \dot{R}_x) - Du_x - \xi\phi + F(\tau\ddot{\alpha} + \dot{\alpha}) + F_2,\\
&a(\tau\dddot{\alpha} + \ddot{\alpha}) = -a^*\dot{u}_x - F\dot{\phi} + \kappa\dot{\alpha}_{xx} + \kappa^*\alpha_{xx} + H\dot{R}_x + H^*R_x + F_3,\\
&B(\tau\dddot{R} + \ddot{R}) = -N\dot{\phi}_x + P\dot{R}_{xx} + P^*R_{xx} - K\dot{\alpha}_x - K^*\alpha_x - \Lambda\dot{R} - \Lambda^*R + F_4,
\end{aligned}
$$

with the following initial conditions, for all $x \in (0, 1)$,

$$
\begin{aligned}
&u^0(x) = v^0(x) = \phi^0(x) = \psi^0(x) = \alpha^0(x) = \theta^0(x) = \vartheta^0(x) = x(x-1),\\
&R^0(x) = T^0(x) = S^0(x) = x(x-1),
\end{aligned}
$$

and homogeneous Dirichlet boundary conditions. In the above equations, the (artificial) supply terms F_i ($i = 1, 2, 3, 4$) are given as, for all $(x, t) \in (0, 1) \times (0, 1)$,

$$
\begin{aligned}
&F_1(x,t) = e^t(10x + x(x-1) - 9), \quad F_2(x,t) = -e^t(6x(x-1) - 8x + 8),\\
&F_3(x,t) = -e^t(2x - 9x(x-1) + 5), \quad F_4(x,t) = e^t(12x + 6x(x-1) - 12),
\end{aligned}
$$

and we have used the following data:

$$
\begin{aligned}
&\ell = 1, \quad T_f = 1, \quad \rho = 1, \quad A = 2, \quad D = 1, \quad a^* = 2, \quad J = 1, \quad A^* = 2, \quad \xi = 2,\\
&F = 3, \quad N = 1, \quad \tau = 2, \quad a = 2, \quad \kappa = 2, \quad \kappa^* = 1, \quad H = 1, \quad H^* = 2,\\
&B = 1, \quad P = 1, \quad P^* = 2, \quad K = 2, \quad K^* = 3, \quad \Lambda = 2, \quad \Lambda^* = 1.
\end{aligned}
$$

The exact solution to the above problem can be easily calculated and it has the form, for $(x, t) \in [0, 1] \times [0, 1]$:

$$
u(x,t) = \phi(x,t) = \alpha(x,t) = R(x,t) = e^t x(x-1).
$$

Thus, the approximation errors estimated by

$$
\begin{aligned}
\max_{0\le n\le N} \Big\{ & \|v_n - v_n^{hk}\| + \|u_n - u_n^{hk}\|_E + \|\psi_n - \psi_n^{hk}\| + \|\phi_n - \phi_n^{hk}\|_E + \|\alpha_n - \alpha_n^{hk}\|_E + \|\theta_n - \theta_n^{hk}\|_E\\
& + \|\vartheta_n - \vartheta_n^{hk}\| + \|R_n - R_n^{hk}\|_E + \|T_n - T_n^{hk}\|_E + \|S_n - S_n^{hk}\| \Big\}
\end{aligned}
$$

are presented in Table 1 for several values of the discretization parameters h and k. Moreover, the evolution of the error depending on the parameter $h + k$ is plotted in Fig. 1 by using the diagonal of the previous table. We notice that the convergence of the algorithm is clearly observed, and the linear convergence, stated in the previous section, seems to be achieved.

If we assume now that there are not supply terms, and we use the final time $T_f = 30$, the data

$$
\begin{aligned}
&\rho = 0.1, \quad \ell = 1, \quad A = 4, \quad D = 2, \quad a^* = 0.5, \quad J = 1, \quad A^* = 1, \quad \xi = 3,\\
&F = 1, \quad N = 1, \quad \tau = 1, \quad a = 1, \quad \kappa = 2, \quad \kappa^* = 1, \quad H = 2, \quad H^* = 1,\\
&B = 1, \quad P = 2, \quad P^* = 1, \quad K = 1, \quad K^* = 2, \quad \Lambda = 3, \quad \Lambda^* = 1,
\end{aligned}
$$

and the initial conditions, for all $x \in (0, 1)$,

$$
\begin{aligned}
&u^0(x) = v^0(x) = x(x-1), \quad \phi^0(x) = \psi^0(x) = 0, \quad \alpha^0(x) = \theta^0(x) = \vartheta^0(x) = 0,\\
&R^0(x) = T^0(x) = S^0(x) = 0,
\end{aligned}
$$

Table 1
Example 1D: Numerical errors for some h and k.

$h \downarrow k \rightarrow$	0.01	0.005	0.002	0.001	0.0005	0.0002	0.0001
$1/2^3$	0.605352	0.586122	0.574727	0.570953	0.569071	0.567944	0.567568
$1/2^4$	0.337170	0.309878	0.293825	0.288531	0.285894	0.284316	0.283790
$1/2^5$	0.218431	0.179337	0.156516	0.149024	0.145300	0.143073	0.142332
$1/2^6$	0.180697	0.124535	0.091947	0.081309	0.076035	0.072884	0.071837
$1/2^7$	0.192231	0.112055	0.065443	0.050301	0.042817	0.038356	0.036874
$1/2^8$	0.239866	0.126827	0.060429	0.038845	0.028211	0.021888	0.019790
$1/2^9$	0.321173	0.163364	0.069586	0.038877	0.023752	0.014781	0.011810
$1/2^{10}$	0.441580	0.221993	0.090491	0.047044	0.025562	0.012825	0.008615
$1/2^{11}$	0.613714	0.307612	0.123712	0.062571	0.032182	0.014114	0.008144
$1/2^{12}$	0.857719	0.429621	0.172181	0.086357	0.043533	0.017968	0.009506

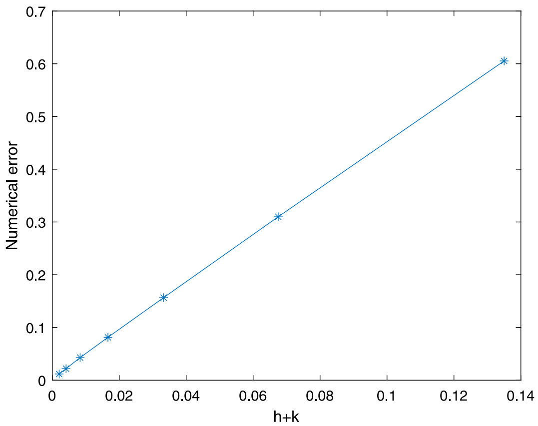

Fig. 1. Example 1D: Asymptotic constant error.

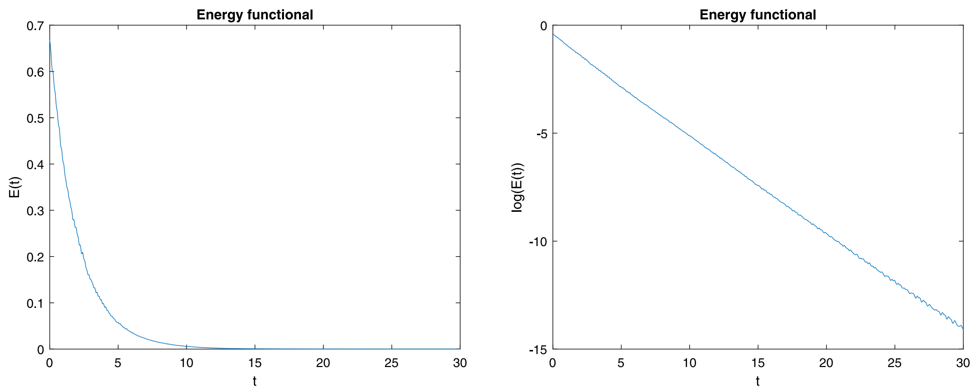

Fig. 2. Example 1D: Evolution in time of the discrete energy (natural and semi-log scales).

taking the discretization parameters $h = k = 0.001$, the evolution in time of the discrete energy defined as

$$E_n^{hk} = \frac{1}{2}\Big(\rho\|v_n^{hk}\|^2 + A\|u_n^{hk}\|_E^2 + J\|\psi_n^{hk}\|^2 + A^*\|\phi_n^{hk}\|_E^2 + \xi\|\phi_n^{hk}\|^2 + a\|\vartheta_n^{hk}\|^2$$
$$+\kappa\|\theta_n^{hk}\|_E^2 + B\|S_n^{hk}\|^2 + P\|T_n^{hk}\|_E^2 + \Lambda\|T_n^{hk}\|^2 \Big),$$

is plotted in Fig. 2 (in both natural and semi-log scales). As can be seen, it converges to zero and an exponential decay seems to be achieved.

6.2. A two-dimensional example

In this section, we will consider the two-dimensional problem for the isotropic case which was defined by system (4)–(7). The domain Ω is assumed to be the quadrangle $(0, 1) \times (0, 1)$, where homogeneous Dirichlet boundary conditions

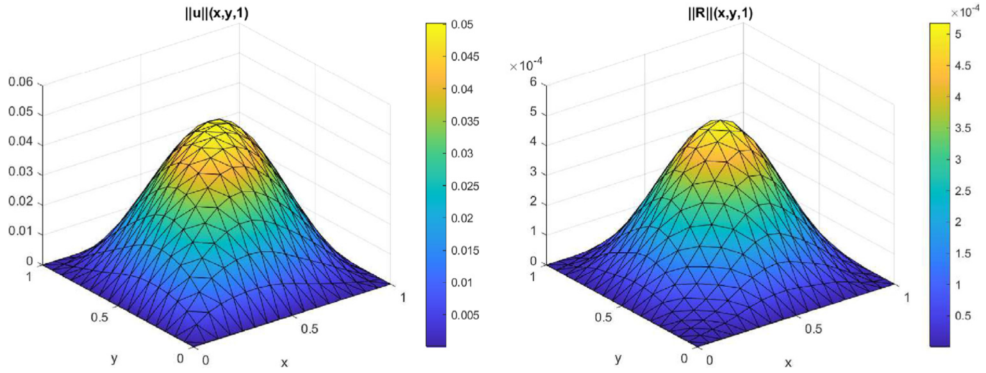

Fig. 3. Example 2D: Norm of the displacement and microthermal displacements at final time.

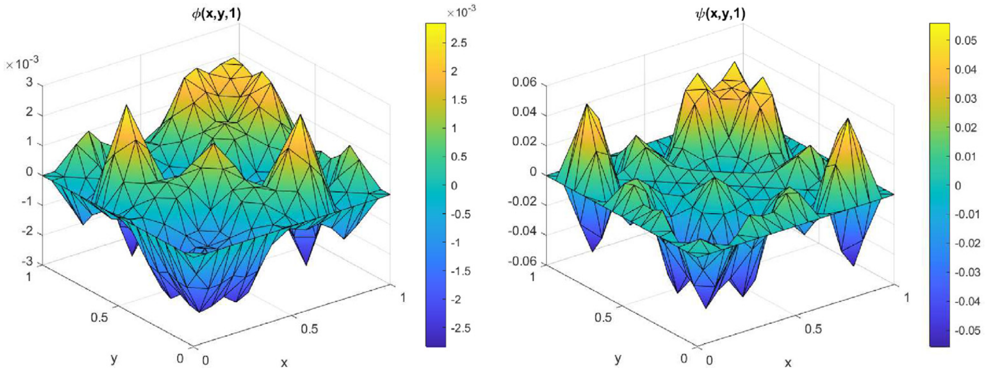

Fig. 4. Example 2D: Porosity and porosity speed at final time.

are prescribed for all the variables. Moreover, we have used the following data:

$$
\begin{aligned}
&T_f = 1, \quad \rho = 1, \quad \mu = 1/2, \quad \lambda = 1/2, \quad D = 1, \quad a^* = 2, \quad J = 1, \quad A = 2, \quad \xi = 2,\\
&F = 3, \quad N = 1, \quad \tau = 2, \quad a = 2, \quad \kappa = 2, \quad \kappa^* = 1, \quad H = 1, \quad H^* = 2,\\
&B = 1, \quad P_6 = 1, \quad P_6^* = 2, \quad P_4 + P_5 = 2, \quad P_4^* + P_5^* = 3, \quad \Lambda = 2, \quad \Lambda^* = 1,\\
&K = 1, \quad K^* = 2,
\end{aligned}
$$

and the initial conditions, for a.e. $(x, y) \in \partial\Omega$ and $i = 1, 2$,

$$u_i^0(x, y) = v_i^0(x, y) = x(x-1)y(y-1), \quad \phi^0 = \psi^0 = \alpha^0 = \theta^0 = \vartheta^0 = R_i^0 = T_i^0 = S_i^0 = 0.$$

Taking as a time discretization parameter $k = 10^{-3}$ and a fixed finite element mesh (with a mesh size h less than 10^{-1}), in Fig. 3 we plot the norm of the displacements and microthermal displacements at final time. We can observe that both have a quadratic shape. In light of the chosen initial data, this is expected for the displacements; however, it is interesting to see that the microthermal displacements, which initially appear only thanks to the coupling, follow the same pattern. The porosity and porosity speed are shown in Fig. 4 at final time. Some oscillations can be found maybe because this function depends on the divergence of the displacements. Finally, in Fig. 5 the thermal displacements and the temperatures are plotted at final time. Again, we note that they are produced only due to the coupling with the other variables. Moreover, there is a skew symmetric behavior (quadratic for each part) for the thermal displacements, meanwhile some oscillations appear in the temperatures.

Taking the final time $T_f = 100$ and parameters $\rho = 10^{-3}$ and $B = 10$, the evolution in time of the discrete energy defined now as

$$
\begin{aligned}
E_n^{hk} = \frac{1}{2}\Big(& \rho\|\boldsymbol{v}_n^{hk}\|^2_{[L^2(\Omega)]^2} + A\|\boldsymbol{u}_n^{hk}\|^2_{[H^1(\Omega)]^2} + J\|\psi_n^{hk}\|^2 + A^*\|\phi_n^{hk}\|^2_E + \xi\|\phi_n^{hk}\|^2 + a\|\vartheta_n^{hk}\|^2 + \kappa\|\theta_n^{hk}\|^2_E\\
& + B\|\boldsymbol{S}_n^{hk}\|^2_{[L^2(\Omega)]^2} + P\|\boldsymbol{T}_n^{hk}\|^2_{[H^1(\Omega)]^2} + \Lambda\|\boldsymbol{T}_n^{hk}\|^2_{[L^2(\Omega)]^2} \Big),
\end{aligned}
$$

is plotted in Fig. 6 (in both natural and semi-log scales). As in the one-dimensional example, it converges to zero and an exponential decay seems to be achieved.

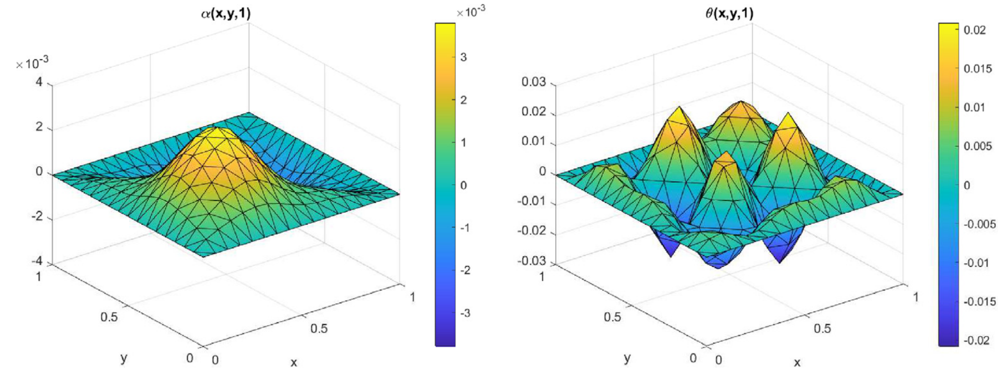

Fig. 5. Example 2D: Thermal displacements and temperatures at final time.

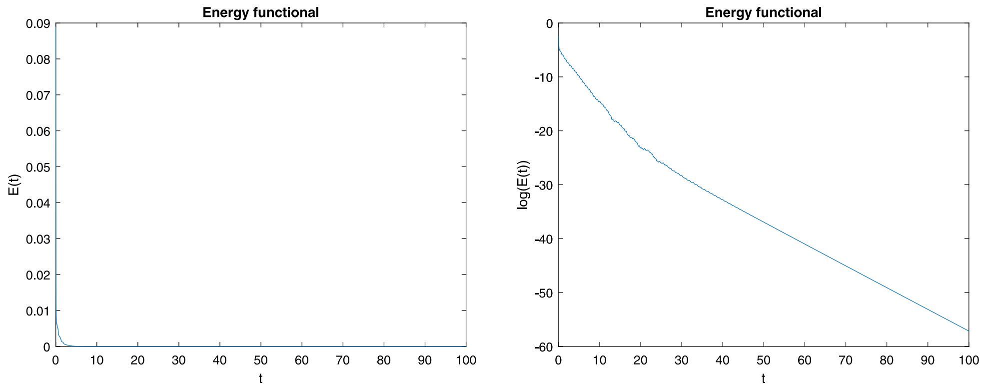

Fig. 6. Example 2D: Evolution in time of the discrete energy (natural and semi-log scales).

7. Conclusions

In this paper, we have studied a problem within the theory of poro-thermoelastic materials with microtemperatures, where the heat and microheat are determined by expression of the MGT type. In particular, the following objectives have been obtained:

1. We have derived the equations corresponding to this theory from a similar theory when the heat and microheat depend on the histories of the temperature and microtemperature respectively.
2. We have shown the existence and uniqueness of solutions by means of the theory of semigroups. This result has been proved in a three-dimensional setting.
3. We have also proved, in the one-dimensional case, the exponential decay of the solutions by using the classical arguments of the theory of semigroups; however, it is not possible to extend this result to a dimension higher than one.
4. Applying the classical finite element method and the implicit Euler scheme, we have introduced a fully discrete approximation, for which we have obtained a discrete stability property and a priori error estimates. It has allowed us to derive the linear convergence of the approximations under suitable regularity conditions.
5. We have implemented a numerical algorithm by using MATLAB and we have performed some examples to show the accuracy of the algorithm and the discrete energy decay.

Data availability

No data was used for the research described in the article

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