TRIPLE SOLIDS AND SCROLLS

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dedicated to Enrique Arrondo on the occasion of his 60th birthday

Abstract. Let Y be a smooth projective variety of dimension $n \geq 2$ endowed with a finite morphism $\phi : Y \to \mathbb{P}^n$ of degree 3, and suppose that Y, polarized by some ample line bundle, is a scroll over a smooth variety X of dimension m. Then $n \leq 3$ and either $m = 1$ or 2. When $m = 1$, a complete description of the few varieties Y satisfying these conditions is provided. When $m = 2$, various restrictions are discussed showing that in several instances the possibilities for such a Y reduce to the single case of the Segre product $\mathbb{P}^2 \times \mathbb{P}^1$. This happens, in particular, if Y is a Fano threefold as well as if the base surface X is $\mathbb{P}^2$.

Introduction

As observed by Fujita in his book [12, (10.9.1), (10.11)], and reaffirmed in the supplementary note circulating as a manuscript "Problems on Polarized Varieties", the problem of classifying the triple covers of $\mathbb{P}^n$ is still open, in particular for $n = 2$ and 3, in which cases such coverings are not necessarily of triple section type [13]. Moreover, as far as we know, there has been no recent contribution in this direction, except for [11], where the authors consider a special class of surfaces represented as triple planes. Actually, in spite of several general results on triple covers existing in the literature (e.g. see [23], [29]), nothing seems explicitly aimed at the study of 3-dimensional triple solids and, more specifically, of threefolds which admit at the same time a projective bundle structure. The unsolved case left open in [19, Sec. 4] exactly reflects this lack of knowledge.

More generally, let Y be a projective n-fold and let $\phi : Y \to \mathbb{P}^n$ be a finite morphism of degree d. Suppose that $d = 2$ or 3 and that Y is a scroll for some polarization at the same time. Then, a classical result of Lazarsfeld [22] implies that $n = 2$ if $d = 2$ and that $n = 2$ or 3 if $d = 3$. In this paper we investigate the possible varieties occurring precisely in this setting. In Section 2 we present some concrete examples to illustrate this situation. The hearth of the paper is Section 3 in which we provide their description, which is complete as far as scrolls over a curve are concerned. The crucial remark is that if Y is a scroll with respect to some polarizing line bundle, then it is also a scroll, via the same projection, with respect to the ample and spanned line bundle $H := \phi^*\mathcal{O}_{\mathbb{P}^n}(1)$. This allows us to work with the ample and spanned vector bundle obtained by pushing down H via the scroll projection. According to this, the case of double solids is easily settled by Proposition 3.1, while the more delicate case of triple solids that are scrolls over a curve is dealt with in Theorem 3.2. The last three Sections are devoted to discussing the case of triple solids that are scrolls over a

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surface, which looks even more interesting and intriguing. More specifically, in Section 4 we exhibit several different situations in which the only possibility for Y is given by the Segre product $\mathbb{P}^2 \times \mathbb{P}^1$; in particular, this is the case if we assume in addition that Y is a Fano manifold. Furthermore, we have the opportunity to amend a flaw in a result of Ballico [3, Theorem]. Moreover, in Section 5 we provide further restrictions on Y deriving from the consideration of the triple plane induced by ϕ on the general element $S \in \phi^*|\mathcal{O}_{\mathbb{P}^3}(1)|$ and of its Tschirnhaus bundle. Finally, in Section 6 we focus on scrolls whose base surface is $\mathbb{P}^2$. Proposition 6.1 shows the extremely severe conditions that this hypothesis entails on the various numerical characters involved in the discussion. Comparing the Tschirnhaus bundle of ϕ with that of the triple plane induced on S in a special situation arising from our analysis, we finally succeed to prove that necessarily $Y = \mathbb{P}^2 \times \mathbb{P}^1$, even in this case (Theorem 6.2).

However, from a complementary point of view suggested by this result, we would like to emphasize that also for scrolls Y over $\mathbb{P}^2$ with respect to some ample line bundle L, distinct from the product, it may happen that Y contains a smooth surface S with the structure of a triple plane even as a very ample divisor, but with $\mathcal{O}_Y(S) \neq L$ (see Remark 6.3).

Throughout the whole paper a relevant role is played by Miranda's formulas, which allow to express the invariants of a triple plane by means of the Chern classes of its Tschirnhaus bundle.

1. Background material

We work over the field of complex numbers and we use the standard notation from algebraic geometry. By a little abuse we make no distinction between a line bundle and the corresponding invertible sheaf. Moreover, the tensor products of line bundles are denoted additively. The pullback $i^*\mathcal{E}$ of a vector bundle $\mathcal{E}$ on X by an embedding of projective varieties $i\colon Y \hookrightarrow X$ is denoted by $\mathcal{E}_Y$. We denote by K_X the canonical bundle of a smooth variety X.

A *polarized manifold* is a pair $(X, \mathcal{L})$ consisting of a smooth projective variety X and an ample line bundle $\mathcal{L}$ on X. The sectional genus and the Δ-genus of a polarized manifold $(X, \mathcal{L})$ are defined as $g(X, \mathcal{L}) = 1 + \frac{1}{2}\big(K_X + (\dim X - 1)\mathcal{L}\big) \cdot \mathcal{L}^{\dim X - 1}$ and $\Delta(X, \mathcal{L}) = \dim X + \mathcal{L}^{\dim X} - h^0(X, \mathcal{L})$, respectively. A polarized manifold $(X, \mathcal{L})$ is said to be a *scroll* (over W) if it is a classical scroll, namely if there exist a smooth projective variety W of positive dimension and a surjective morphism $\pi\colon X \to W$ such that $(F, \mathcal{L}_F) \cong \big(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(1)\big)$ with $m = \dim X - \dim W$ for any fiber F of π. This condition is equivalent to saying that $X = \mathbb{P}_W(\mathcal{F})$ for some ample vector bundle $\mathcal{F}$ of rank ≥ 2 on W, and $\mathcal{L}$ is the tautological line bundle. If $\mathcal{L}$ is a line bundle on a projective manifold X, we denote by $\varphi_{\mathcal{L}}$ the rational map $X - - \to \mathbb{P}^{h^0(\mathcal{L})-1}$ associated with the complete linear system $|\mathcal{L}|$.

We will use the symbol $\mathbb{F}_e$ to denote the Segre–Hirzebruch surface $\mathbb{P}\big(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e)\big)$ of invariant $e (\geq 0)$, and σ and f will denote the section of minimal self-intersection $-e$ and a fiber respectively.

2. Some covers of $\mathbb{P}^n$ admitting a scroll structure

Let Y be a smooth projective variety with $\dim Y = n$ and let $d \geq 2$ be an integer: if Y is endowed with a d-uple branched covering of $\mathbb{P}^n$ we will refer to Y as a *d-uple n-solid*. Any smooth projective variety Y of dimension n embedded in some projective space can be regarded as a d-uple n-solid, where $d = \deg Y$, by projecting it onto a $\mathbb{P}^n$ from a suitable linear space. This is true in particular when Y is a scroll over a positive dimensional projective variety. In this case, however, the integers n and d are not completely unrelated, due to the following general fact.

Lemma 2.1. *Let Y be any projective bundle over a smooth positive dimensional projective variety and a d-uple n-solid at the same time. Then $d \geq n \geq 2$.*

Proof. Clearly $n \geq 2$. Suppose that $d \leq n-1$. Since Y has Picard number $\rho(Y) \geq 2$, we get a contradiction by a well known result of Lazarsfeld (see [22, Proposition 3.1]). □

Example 2.2. Let $Y = \mathbb{P}_{\mathbb{P}^1}(\mathcal{V})$, where

$$\mathcal{V} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(\alpha_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(\alpha_{n-1}) \tag{2.2.1}$$

with $0 \leq \alpha_1 \leq \cdots \leq \alpha_{n-1}$ and let $\alpha = \sum_{i=1}^{n-1} \alpha_i = \deg \mathcal{V}$. Let ξ be the tautological line bundle, let F be a fiber of the projection $\pi\colon Y \to \mathbb{P}^1$ and let $L = \xi + bF$ for some integer b. Due to the normalization (2.2.1), we know that L is ample if and only if it is very ample if and only if $b > 0$ ([5, Lemma 3.2.4]). So let $b > 0$; then the morphism φ_L embeds Y in $\mathbb{P}^N$ as a scroll of degree

$$d := L^n = \deg\big(\mathcal{V} \otimes \mathcal{O}_{\mathbb{P}^1}(b)\big) = \alpha + nb \tag{2.2.2}$$

by the Chern–Wu relation, where $N + 1 = h^0(L) = h^0\big(\mathcal{V} \otimes \mathcal{O}_{\mathbb{P}^1}(b)\big) = n + nb + \alpha = n + d$. Let Λ be a general linear subspace of $\mathbb{P}^N$ of dimension $N - 1 - n$. Then, projecting $\varphi_L(Y)$ from Λ onto a $\mathbb{P}^n \subset \mathbb{P}^N$ skew with Λ, we get a map $\phi\colon Y \to \mathbb{P}^n$, which is a finite morphism of degree d and $L = \phi^*\mathcal{O}_{\mathbb{P}^n}(1)$. On the other hand, (Y, L) is a scroll over $\mathbb{P}^1$. Clearly $d \geq n$, according to Lemma 2.1. In this specific case this simply follows from (2.2.2), taking into account that $b > 0$ and $\alpha \geq 0$.

Remark 2.3. If (Y, L) is an n-dimensional scroll as in Example 2.2 and $d = n$, then $(Y, L) = \big(\mathbb{P}^1 \times \mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^{n-1}}(1,1)\big)$. Actually, equality $d = n$ implies $b = 1$ and $\alpha = 0$ by (2.2.2), and the latter in turn implies that $\mathcal{V} = \mathcal{O}_{\mathbb{P}^1}^{\oplus n}$.

In particular, according to Example 2.2, we get: for $(n, d) = (2, 2)$ the smooth quadric surface of $\mathbb{P}^3$ described as a double plane via projection from a general point; for $(n, d) = (2, 3)$ the rational cubic scroll of $\mathbb{P}^4$ described as a triple plane via projection from a general line; for $(n, d) = (3, 3)$ the Segre product $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$ described as a triple solid via projection from a general line. In all these cases, of course, the line bundle L making Y a scroll is very ample.

Example 2.4. Let $B \subset \mathbb{P}^2$ be an irreducible projective curve whose dual $B^\vee \subset \mathbb{P}^{2\vee}$ is a smooth curve of degree d. The following construction is inspired by [8, §3]. In $\mathbb{P}^2 \times \mathbb{P}^{2\vee}$ consider the incidence variety $T = \{(x, \ell) \in \mathbb{P}^2 \times \mathbb{P}^{2\vee} \mid x \in \ell\}$. Then T is a smooth threefold which is endowed with two $\mathbb{P}^1$-bundle structures $p\colon T \to \mathbb{P}^2$, $q\colon T \to \mathbb{P}^{2\vee}$ via the projections of $\mathbb{P}^2 \times \mathbb{P}^{2\vee}$ onto the factors.

Now consider the smooth curve $B^\vee$ and let $S = q^{-1}(B^\vee)$. Clearly S is a smooth surface and $\pi := q|_S\colon S \to B^\vee$ makes S a $\mathbb{P}^1$-bundle over $B^\vee$. On the other hand, since S is a divisor on T

belonging to the linear system $|q^*\mathcal{O}_{\mathbb{P}^{2\vee}}(d)|$, we see that $f := p|_S \colon S \to \mathbb{P}^2$ is a finite morphism of degree $\deg f = p^{-1}(x) \cdot S$ where $x \in \mathbb{P}^2$ is any point. We thus get

$$\deg f = \big(p^*\mathcal{O}_{\mathbb{P}^2}(1)\big)^2 \cdot q^*\mathcal{O}_{\mathbb{P}^{2\vee}}(d) = d.$$

Looking at the construction more closely, we can note that the branch locus of the d-uple plane $f \colon S \to \mathbb{P}^2$ is B. To see this, note that $S = \{(x, \ell) \mid x \in \ell, \ell \in B^\vee\}$, while, $p^{-1}(x) = \{(x, \ell) \mid \ell \ni x\} = \{(x, \ell) \mid \ell \in L_x\}$, for any $x \in \mathbb{P}^2$, where L_x is the line in $\mathbb{P}^{2\vee}$ corresponding to the pencil of lines through x. Now fix $x \in \mathbb{P}^2$: then

$$f^{-1}(x) = p^{-1}(x) \cap S = \{(x, \ell) \mid \ell \ni x, \ell \in B^\vee\} = \{(x, \ell) \mid \ell \in L_x \cap B^\vee\}.$$

This shows that the pre-images of x via $f \colon S \to \mathbb{P}^2$ correspond to the intersections of L_x with $B^\vee$. As $B^\vee$ is smooth those pre-images are not d distinct points if and only if the line L_x is tangent to $B^\vee$. By biduality, this is equivalent to saying that $x \in B$.

Let $L = f^*\mathcal{O}_{\mathbb{P}^2}(1)$. Then L is an ample and spanned line bundle on S. Note that $L = \big(p^*\mathcal{O}_{\mathbb{P}^2}(1)\big)_S$. Moreover, for any fiber $\mathfrak{f}$ of $\pi \colon S \to B^\vee$ we have $L \cdot \mathfrak{f} = p^*\mathcal{O}_{\mathbb{P}^2}(1) \cdot \big(q^*\mathcal{O}_{\mathbb{P}^{2\vee}}(1)\big)^2 = 1$. This says that (S, L) is a scroll over $B^\vee$, a smooth curve of genus $\binom{d-1}{2}$.

Set $\mathcal{L} = \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^{2\vee}}(1,1)$ and recall that $(\mathbb{P}^2 \times \mathbb{P}^{2\vee}, \mathcal{L})$ is the del Pezzo fourfold of degree six. Since $T \in |\mathcal{L}|$, $(T, \mathcal{L}_T)$ is the following del Pezzo threefold of degree six: $T = \mathbb{P}_{\mathbb{P}^{2\vee}}(T_{\mathbb{P}^{2\vee}})$, with $\mathcal{L}_T$ being the tautological line bundle [12, Chapter I, §8 (8.7), (8.8)]. Furthermore,

$$\mathcal{L}_S = (\mathcal{L}_T)_S = \big(p^*\mathcal{O}_{\mathbb{P}^2}(1) + q^*\mathcal{O}_{\mathbb{P}^{2\vee}}(1)\big)_S = f^*\mathcal{O}_{\mathbb{P}^2}(1) + \pi^*\mathcal{O}_{B^\vee}(1) = L + \pi^*\mathcal{O}_{B^\vee}(1).$$

This shows that $S = \mathbb{P}_{B^\vee}\big((T_{\mathbb{P}^{2\vee}}(-1))_{B^\vee}\big)$, with L being the tautological line bundle.

In particular this gives

Example 2.5. Set $d = 3$. Then the previous construction exhibits a smooth surface which is a triple plane and a scroll over an elliptic curve at the same time. We can show that $\big(T_{\mathbb{P}^{2\vee}}(-1)\big)_{B^\vee} = \mathcal{U} \otimes \mathcal{O}_{B^\vee}(z)$, where $z \in B^\vee$, and $\mathcal{U}$ is an indecomposable vector bundle of degree one on the elliptic curve $B^\vee$. Thus, letting $M = L + \mathfrak{f}_0$, where $\mathfrak{f}_0$ is any fiber of π, we see that M is very ample [15, Exerc. 2.12, p. 385] and φ_M embeds S as an elliptic quintic scroll in $\mathbb{P}^4$. Since $L = M - \mathfrak{f}_0$, we can regard the triple plane $f \colon S \to \mathbb{P}^2$ as the projection of the quintic elliptic scroll from its fiber $\mathfrak{f}_0$ onto a plane skew with it. Note that here L is ample and spanned but not very ample.

3. Double and triple solids admitting a scroll structure

In this Section we slightly change the perspective. Let $\phi : Y \to \mathbb{P}^n$ be a d-uple n-solid, where $d = 2$ or 3: we wonder when Y admits an ample line bundle L such that (Y, L) is a scroll over a projective manifold of dimension $m \geq 1$. Let us consider the line bundle $H = \phi^*\mathcal{O}_{\mathbb{P}^n}(1)$, which, of course, is ample and spanned (in principle, it could also be very ample, as some examples in Section 2 show). Let $\pi : Y \to X$ be the projection of the scroll (Y, L): since $\mathrm{Pic}(Y)$ is generated by L and

$\pi^*\mathrm{Pic}(X)$ we can write $H = aL + \pi^*\mathcal{O}_X(D)$, where a is a positive integer and D is a divisor on X. Then

$$
\begin{aligned}
(3.0.1) \qquad d &= H^n = (aL + \pi^*D)^n \\
&= a^nL^n + na^{n-1}L^{n-1}\cdot\pi^*D + \cdots + \binom{n}{m}a^{n-m}L^{n-m}\cdot(\pi^*D)^m \\
&= aK,
\end{aligned}
$$

since $n \geq m+1$, where

$$
(3.0.2) \qquad K = a^{n-1}L^n + na^{n-2}L^{n-1}\cdot\pi^*D + \cdots + \binom{n}{m}a^{n-m-1}L^{n-m}\cdot(\pi^*D)^m.
$$

Now, if d (≥ 2) is prime, we deduce from (3.0.1) that either $a = 1$ (and then $K = d$), or $a = d$, which implies that $1 = K = d^{n-m-1}K'$, where K' is an integer, and this is impossible if $n \geq m+2$. On the other hand, if $a = 1$, then the pair (Y, H) itself is a scroll over X, hence we can suppose that $Y = \mathbb{P}(\mathcal{E})$, where $\mathcal{E} = \pi_*H$ is a vector bundle on X of rank $n-m+1$, which is ample and spanned, so being its tautological line bundle H. In particular, let $d = 2$; then $n = 2$ by Lemma 2.1, hence $m = 1$, i.e. X is a curve. Then necessarily $a = 1$; otherwise we would get $a = 2$, and then $1 = K = 2\big(L^2 + \deg(D)\big)$ by (3.0.2), which is absurd. Similarly, let $d = 3$; then $n = 2$ or 3 by Lemma 2.1, hence either $n = 2$ and $m = 1$, or $n = 3$ and $m = 1, 2$. If $n = 3$, i.e. Y is a threefold, then we obtain that necessarily $a = 1$. Otherwise, we would get $a = 3$ and

$$
1 = K = \begin{cases} 9L^3 + 9L^2\cdot\pi^*D & \text{if } m = 1, \\ 9L^3 + 9L^2\cdot\pi^*D + 3L\cdot(\pi^*D)^2 & \text{if } m = 2, \end{cases}
$$

by (3.0.2), which is clearly impossible. Therefore, $a = 1$ in all these cases, hence the problem becomes determining when (Y, H) itself is a scroll. Note that case $d = 3$ with $n = 2$ is not covered by the previous analysis; it will be discussed in the proof of Theorem 3.2. For $d = 2$ the answer is very easy and is given by the following

Proposition 3.1. *Let $\phi : Y \to \mathbb{P}^n$ be any smooth double n-solid with $n \geq 2$. Then there is no polarization on Y making it a scroll over a smooth projective variety of positive dimension except for $\big(Y, \phi^*\mathcal{O}_{\mathbb{P}^n}(1)\big) = \big(\mathbb{P}^1\times\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)\big)$.*

Proof. Let $\phi\colon Y \to \mathbb{P}^n$ be the finite morphism of degree 2 making Y a double n-solid and consider the ample and spanned line bundle $H := \phi^*\mathcal{O}_{\mathbb{P}^n}(1)$. If $\pi : Y \to X$ is a scroll for some polarization, then $n = 2$, X is a curve and (Y, H) itself is a scroll via π in view of the above discussion. Then $\mathcal{E} := \pi_*H$ is an ample and spanned rank-2 vector bundle on X. Let $B \in |\mathcal{O}_{\mathbb{P}^2}(2b)|$, $b \geq 1$, be the branch locus of ϕ. Comparing the expression of

$$K_Y = -2H + \pi^*(K_X + \det\mathcal{E})$$

with that given by the ramification formula

$$K_Y = \phi^*\mathcal{O}_{\mathbb{P}^2}(b-3) = (b-3)H$$

we conclude that $b=1$ and $K_X+\det\mathcal{E}=\mathcal{O}_X$, because H and $\pi^*\mathcal{O}_X(1)$ are linearly independent in $\mathrm{Pic}(Y)$. This shows that $X=\mathbb{P}^1$ and $\det\mathcal{E}=\mathcal{O}_{\mathbb{P}^1}(2)$, which in turn implies $\mathcal{E}=\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}$. Equivalently, $(Y,H)=\big(\mathbb{P}^1\times\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)\big)$. □

As to case $d=3$ is concerned, here let us start assuming that $m=1$.

Theorem 3.2. *Let $\phi : Y\to\mathbb{P}^n$ be any smooth triple n-solid with $n\geq 2$. Then there is no polarization on Y making it a scroll over a smooth projective curve except for the following three pairs $\big(Y,\phi^*\mathcal{O}_{\mathbb{P}^n}(1)\big)$:*

(1) $\big(\mathbb{P}^2\times\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^1}(1,1)\big)$;

(2) $(\mathbb{F}_1,[\sigma+2f])$, *where σ is the (-1)-section and f is a fiber;*

(3) $\big(\mathbb{P}_C(\mathcal{U}),L\big)$, *where $\mathcal{U}$ is an indecomposable vector bundle of degree one on an elliptic curve C and L is the tautological line bundle of $\mathcal{U}\otimes\mathcal{O}_C(z)$, z being a point of C.*

Proof. Let $\phi\colon Y\to\mathbb{P}^n$ be the finite morphism of degree 3 making Y a triple n-solid and consider the ample and spanned line bundle $H:=\phi^*\mathcal{O}_{\mathbb{P}^n}(1)$ again. Assume that (Y,L) is a scroll over a smooth curve X of genus $q:=g(X)$ for some ample line bundle L and let $\pi\colon Y\to X$ be the scroll projection. From Lemma 2.1 we know that $n=2$ or 3.

First suppose that $n=3$. Then, according to the discussion at the beginning of this Section, (Y,H) itself is a scroll over X. By [22, Theorem 1] we know that ϕ induces an isomorphism $0=H^1(\mathbb{P}^3,\mathbb{C})\cong H^1(Y,\mathbb{C})$. Therefore $h^1(\mathcal{O}_Y)=0$, and then the scroll structure of (Y,H) over X implies that $q=0$. Thus (Y,H) is a scroll over $\mathbb{P}^1$, so $Y=\mathbb{P}_{\mathbb{P}^1}\big(\oplus_{i=1}^3\mathcal{O}_{\mathbb{P}^1}(a_i)\big)$. Since H is ample and $H^3=3$, we derive $Y=\mathbb{P}_{\mathbb{P}^1}\big(\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 3}\big)$, hence $(Y,H)=\big(\mathbb{P}^2\times\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^1}(1,1)\big)$ (see Remark 2.3).

Next, let $n=2$. In this case what we observed at the beginning of this Section implies that $a=1$, unless in the following case:

$$a=3 \qquad \text{and} \qquad K=1. \tag{3.2.1}$$

Claim. Case (3.2.1) cannot occur.

To prove the claim, consider the scroll (Y,L) again, set $\mathcal{E}'=\pi_*L$, so that $L^2=\deg\mathcal{E}'$, and recall that $H=aL+\pi^*\mathcal{O}_X(D)$ for some divisor D on X. If (3.2.1) holds, then (3.0.2) gives

$$1=K=3\deg\mathcal{E}'+2\deg D. \tag{3.2.2}$$

Let's prove that (3.2.2) does not occur. We can write $\phi_*\mathcal{O}_Y=\mathcal{O}_{\mathbb{P}^2}\oplus\mathcal{T}$, where $\mathcal{T}$, the Tschirnhaus bundle of ϕ, is a vector bundle of rank 2 on $\mathbb{P}^2$. Then the branch locus of ϕ is an element of $|2\det\mathcal{T}^\vee|$ [23, Proposition 4.7]. Set $b_i=c_i(\mathcal{T})$. By applying the Riemann–Hurwitz formula to the curve $\phi^{-1}(\ell)$ where $\ell\subset\mathbb{P}^2$ is a general line, we get

$$2g(Y,H)-2=3(-2)+(-2b_1). \tag{3.2.3}$$

On the other hand, since $K_Y=-2L+\pi^*(K_X+\det\mathcal{E}')$, taking into account the expression of H and condition (3.2.2), the genus formula shows that

$$2g(Y,H)-2=(K_Y+H)\cdot H=2\ (3\deg\mathcal{E}'+2\deg D)+6\ (q-1)=6q-4.$$

Combining this with (3.2.3) we get

$$-b_1 = 3q + 1. \tag{3.2.4}$$

Now, since Y is a $\mathbb{P}^1$-bundle over X, we know that $K_Y^2 = 8(1-q)$ and the topological Euler–Poincaré characteristic is $e(Y) = 4(1-q)$. Thus, eliminating b_2 from Miranda's formulas for the triple plane $\phi : Y \to \mathbb{P}^2$ [23, Proposition 10.3]

$$K_Y^2 = 27 + 12b_1 + 2b_1^2 - 3b_2 \qquad \text{and} \qquad e(Y) = 9 + 6b_1 + 4b_1^2 - 9b_2, \tag{3.2.5}$$

and using (3.2.4) we obtain the following equation $9q^2 - 29q + 12 = 0$, which has no integral solution. This proves the claim.

Therefore $a = 1$ even if $n = 2$, hence (Y, H) itself is a scroll over X; so $g(Y, H) = q$ and then the Riemann–Hurwitz formula applied to the curve $\phi^{-1}(\ell)$ now gives

$$-b_1 = q + 2, \tag{3.2.6}$$

Moreover, $K_Y^2 = 8(1-q)$ and $e(Y) = 4(1-q)$ again. In this case, eliminating b_2 from Miranda's formulas, we get $q(q-1) = 0$. If $q = 0$, from Example 2.2 we see that $\alpha = b = 1$, hence $Y = \mathbb{P}\big(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2)\big)$: this gives case (2) in the statement. On the other hand, for $q = 1$ we get case (3). This is a consequence of the following lemma. □

Lemma 3.3. *Let (Y, H) be a surface scroll over a smooth curve C of genus one, for some ample and spanned line bundle H. If $H^2 = 3$, then $Y = \mathbb{P}_C(\mathcal{U})$, $\mathcal{U}$ being the nontrivial extension*

$$0 \to \mathcal{O}_C \to \mathcal{U} \to \mathcal{O}_C(p) \to 0,$$

with $p \in C$, and $H = [\sigma + f]$, where σ denotes the tautological section on Y.

Proof. Write $Y = \mathbb{P}_C(\mathcal{V})$, where $\mathcal{V}$ is a rank-2 vector bundle on C, that we can suppose to be normalized as in [15, p. 373]. Denote by σ and f the tautological section and a fiber, respectively. Then $\sigma^2 = -e$, where $e = -\deg \mathcal{V}$ is the invariant of Y. Since (Y, H) is a scroll, up to numerical equivalence, we can write $H = [\sigma + bf]$ for some integer b. Thus $H^2 = -e + 2b$, and then condition $H^2 = 3$ gives $b = \frac{1}{2}(e+3)$, which implies that e is odd. Moreover, the ampleness conditions say that $b > e$ if $e \geq 0$ and $b \geq 0$ if $e = -1$ [15, Propositions 2.20 and 2.21, p. 382]. This, combined with the above expression of b shows that there are only two possible cases, namely:

$$(e, b) = (-1, 1) \quad \text{or} \quad (1, 2). \tag{3.3.1}$$

In the latter case, $H = [\sigma + 2f]$ is clearly not spanned, since its restriction to the elliptic curve σ has degree $\deg H_\sigma = (\sigma + 2f) \cdot \sigma = 1$. On the contrary, in the former case, $\mathcal{V} = \mathcal{U}$ [15, pp. 376–377] and we can check the spannedness of H by using Reider's theorem [26]. Set $M = H - K_Y$, then $M = 3\sigma$, up to numerical equivalence. In particular $M^2 = 9 > 5$, hence Reider's theorem applies. Suppose, by contradiction, that $H = K_Y + M$ is not spanned; then there exists an effective divisor D on Y such that either

$$D \cdot M = 0 \quad \text{and} \quad D^2 = -1, \tag{3.3.2}$$

or

$$D \cdot M = 1 \quad \text{and} \quad D^2 = 0. \tag{3.3.3}$$

Up to numerical equivalence we can write $D = x\sigma + yf$ for suitable integers x, y, and then we get $D \cdot M = 3(x + y)$, while $D^2 = x(2y + x)$. Clearly, the expression of $D \cdot M$ rules out the possibility in (3.3.3). Suppose (3.3.2) holds. Then $x = 1$ and $y = -1$. So $D = [\sigma - f]$ and therefore $D \cdot \sigma = 0$. However, since $e = -1$, the elliptic curve σ moves in an algebraic family (parameterized by the base curve C itself), sweeping out the whole surface Y. Thus the equality $D \cdot \sigma = 0$ would imply that D cannot be effective, a contradiction. Therefore H is spanned in the former case of (3.3.1). □

4. Scrolls over surfaces

Consider triple n-solids again. According to Lemma 2.1, apart from scrolls over curves, there is only one more possibility for Y being a scroll for some polarization, namely, that $n = 3$ and $\dim X = 2$. In this Section and the following ones we focus precisely on this case, showing that Y must satisfy several restrictions. A further motivation for this study is provided by an unresolved situation in [19, p. 687]. So, let $\phi : Y \to \mathbb{P}^3$ be a triple solid, and suppose that (Y, L) is a scroll over a smooth surface X via $\pi : Y \to X$, for some ample line bundle L. In this case the argument at the beginning of Section 3 says that

$$(Y, H) \text{ itself is scroll over } X \text{ via } \pi, \text{ where } H = \phi^*\mathcal{O}_{\mathbb{P}^3}(1). \tag{4.0.1}$$

We can thus suppose that $\mathcal{E} := \pi_*H$ is an ample and spanned rank-2 vector bundle on X and $Y = \mathbb{P}_X(\mathcal{E})$, with tautological line bundle H. When we refer to (4.0.1), implicitly we also mean that $\mathcal{E}$ is as above. In this case, since $\pi_*\mathcal{O}_Y = \mathcal{O}_X$ [15, Proposition 7.11, p. 162], we have

$$h^i(\mathcal{O}_Y) = h^i(\mathcal{O}_X) \quad i = 0, \dots, 3 \tag{4.0.2}$$

[15, Exerc. 4.1, p. 222]. In particular, $\chi(\mathcal{O}_Y) = \chi(\mathcal{O}_X)$.

Notice that the pair (Y, H) as in case (1) of Theorem 3.2 can also be regarded as a scroll over a surface by taking $(X, \mathcal{E}) = \left(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2}\right)$. We will refer to this case as the *obvious case* in the subsequent discussion. First of all, given any smooth triple solid $\phi : Y \to \mathbb{P}^3$ and $H = \phi^*\mathcal{O}_{\mathbb{P}^3}(1)$, we have $h^0(H) \geq 4$; on the other hand $H^3 = 3$, hence the Δ-genus of (Y, H) is $\Delta(Y, H) = 6 - h^0(H) \leq 2$. In our setting (i.e. taking into account the additional scroll structure of Y), the situation is simpler. In fact we have

Proposition 4.1. *Either $\Delta(Y, H) = 2$, or $\Delta(Y, H) = 0$ and (Y, H) is as in the obvious case; in particular, if H is very ample, then (Y, H) is as in the obvious case.*

Proof. Suppose that $\Delta(Y, H) < 2$; since Y is a scroll over a surface, its Picard number is $\rho(Y) \geq 2$, so combining this with Fujita's classification of polarized manifolds of low Δ-genus [12, Theorem 5.10 and Corollary 6.7] we immediately get what is stated. In particular, if H is very ample then $|H|$ embeds our threefold Y in $\mathbb{P}^N$, with $N = h^0(H) - 1 \geq 4$, hence it cannot be $\Delta(Y, H) = 2$. □

Remark 4.2. In particular, the fact that in the setting (4.0.1) it can be $\Delta(Y,H)=2$ amends a result of Ballico [3, Theorem] (actually, the assertion that $H^3=3$ would imply the obvious case is not proved there). However, assuming in our setting that either Y is Fano or $X=\mathbb{P}^2$, we will see that the obvious case is the only possibility (cf. Proposition 4.10 and Theorem 6.2).

More generally, with an eye to the characterization of projective manifolds admitting a given variety as a hyperplane section, Proposition 4.1 suggests the following.

Proposition 4.3. *Let $\mathcal{X}\subset\mathbb{P}^N$ be a regular projective n-fold, with $n\geq 3$. If a general surface section $\mathcal{Y}$ of $\mathcal{X}$ is a triple plane via the hyperplane bundle map, then either*

(1) *$\mathcal{X}\subset\mathbb{P}^{n+1}$ is a smooth cubic hypersurface, or*
(2) *$n=3$ and $\mathcal{X}\subset\mathbb{P}^5$ is the Segre product $\mathbb{P}^2\times\mathbb{P}^1$.*

Proof. Let Z be a general 3-dimensional linear section of $\mathcal{X}$ and set $\mathcal{H}=\mathcal{O}_{\mathbb{P}^N}(1)|_Z$, so that $\mathcal{Y}\in|\mathcal{H}|$. Clearly, $h^0(Z,\mathcal{H})=1+h^0(\mathcal{Y},\mathcal{H}_{\mathcal{Y}})$ since $h^1(\mathcal{O}_Z)=h^1(\mathcal{O}_{\mathcal{X}})=0$, by the Lefschetz theorem. Hence

$$\Delta(Z,\mathcal{H})=3+\mathcal{H}^3-h^0(Z,\mathcal{H})=3+\mathcal{H}_{\mathcal{Y}}^2-1-h^0(\mathcal{Y},\mathcal{H}_{\mathcal{Y}})\leq 1,$$

since $\mathcal{H}_{\mathcal{Y}}^2=3$ and $h^0(\mathcal{Y},\mathcal{H}_{\mathcal{Y}})\geq 4$, $\mathcal{H}_{\mathcal{Y}}$ being a very ample divisor on $\mathcal{Y}$. If $\Delta(Z,\mathcal{H})$=1, then $Z\subset\mathbb{P}^4$ is a smooth cubic threefold by [12, Corollary 6.7] and then $\mathcal{X}$ is as in (1). On the other hand, if $\Delta(Z,\mathcal{H})=0$, then Z is the Segre product $\mathbb{P}^2\times\mathbb{P}^1\subset\mathbb{P}^5$, which, however, cannot ascend to higher dimensions. Actually, $\mathcal{X}$ is a scroll over $\mathbb{P}^1$ and then (2.2.2) shows that $n=3$, i.e, $\mathcal{X}=Z$, as in (2). □

From now on we will assume that our triple solid Y has the additional structure of a scroll over a smooth surface. So we will always refer to the setting (4.0.1).

The structure of triple solid given by ϕ, combined with the Chern–Wu relation implies:

$$3=H^3=c_1(\mathcal{E})^2-c_2(\mathcal{E}) \tag{4.0.3}$$

$c_i(\mathcal{E})$ denoting the i-th Chern class of $\mathcal{E}$. So we have

Remark 4.4. $\mathcal{E}$ is Bogomolov stable unless (Y,H) is as in the obvious case. Actually, (4.0.3) says that $c_1(\mathcal{E})^2-4c_2(\mathcal{E})=3\big(1-c_2(\mathcal{E})\big)\leq 0$, because $c_2(\mathcal{E})>0$ due to the ampleness of $\mathcal{E}$ [6]; therefore $\mathcal{E}$ is Bogomolov semistable. Moreover it is properly semistable if and only if $c_2(\mathcal{E})=1$ and this occurs only for $(X,\mathcal{E})=\big(\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2}\big)$ by [20]. Hence, apart from the obvious case, $\mathcal{E}$ is Bogomolov stable.

Here we collect some properties of Y.

Proposition 4.5. *We have:*

(a) *$h^1(\mathcal{O}_Y)=0$;*
(b) *X is a regular surface;*
(c) *a general element S in the linear subsystem $\phi^*|\mathcal{O}_{\mathbb{P}^3}(1)|\subseteq|H|$ is a smooth regular surface;*
(d) *the ramification divisor R of ϕ is very ample;*

(e) (Y,R) *is a conic fibration over* X *via* π*, with empty discriminant locus. In particular, letting* $P := \mathbb{P}_X(\mathcal{F})$*, where* $\mathcal{F} = \pi_* R$ *and denoting by* ξ *the tautological line bundle and by* $\widetilde{\pi} : P \to X$ *the bundle projection,* Y *is contained in* P *as a smooth divisor of relative degree* 2*, belonging to the linear system* $|2\xi - 2\widetilde{\pi}^*(K_X + 2\det\mathcal{E})|$ *and* $\xi_Y = R$*.*

Proof. (a) follows from [22, Theorem 1], and then equation (4.0.2) implies (b). As H is ample and $\phi^*|\mathcal{O}_{\mathbb{P}^3}(1)|$ is base-point free, its general element S is a smooth surface by the Bertini theorem: the fact that $h^1(\mathcal{O}_S) = 0$ follows from the Lefschetz theorem [27]. This proves (c). The ramification formula says that

$$K_Y = \phi^* K_{\mathbb{P}^3} + R = -4H + R,$$

hence $R = K_Y + 4H$. Since H is ample and spanned with $H^3 = 3$ it thus follows from [21, Theorem 3.1] that R is a very ample divisor. This gives (d). Finally, by the canonical bundle formula, we have

$$K_Y = -2H + \pi^*(K_X + \det\mathcal{E}),$$

and by comparing the two expressions of K_Y we get the relation

$$R = 2H + \pi^*(K_X + \det\mathcal{E}).$$

The first assertion in (e) follows from the fact that R restricts to every fiber of π as $\mathcal{O}_{\mathbb{P}^1}(2)$: the discriminant is empty since every fiber is irreducible. Furthermore, as a conic fibration over X, Y is contained as a smooth divisor of relative degree 2 inside $P := \mathbb{P}_X(\mathcal{F})$, where $\mathcal{F} = \pi_* R$; more precisely, letting ξ denote the tautological line bundle and $\widetilde{\pi} : P \to X$ the bundle projection extending π, we have that $Y \in |2\xi + \widetilde{\pi}^*\mathcal{B}|$ for some line bundle $\mathcal{B}$ on X and $\xi_Y = R$. Recalling that $\pi_* H = \mathcal{E}$, from the expression of R we get

$$\mathcal{F} = \pi_*\big(2H + \pi^*(K_X + \det\mathcal{E})\big) = S^2\mathcal{E} \otimes (K_X + \det\mathcal{E}),$$

where S^2 stands for the second symmetric power. Since $\mathrm{rk}(\mathcal{F}) = 3$, this gives

$$c_1(\mathcal{F}) = 3c_1(\mathcal{E}) + 3(K_X + \det\mathcal{E}) = 3(K_X + 2\det\mathcal{E}).$$

The condition expressing the fact that the discriminant locus of (Y,R) is empty is given by $2c_1(\mathcal{F}) + 3\mathcal{B} = \mathcal{O}_X$ [7, p. 76]. Therefore we get $\mathcal{B} = -\frac{2}{3}c_1(\mathcal{F}) = -2(K_X + 2\det\mathcal{E})$, and this concludes the proof. □

Proposition 4.6. *Suppose that* (Y,H) *is not as in the obvious case. Then* $K_X + \det\mathcal{E}$ *is ample and spanned.*

Proof. Suppose that $K_X + \det\mathcal{E}$ is not ample. Then, according to [14, Main Theorem], $(X,\mathcal{E})$ is one of the following pairs:

(a) X is a $\mathbb{P}^1$-bundle over a smooth curve C and $\mathcal{E}_f = \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}$, for every fiber of the bundle projection $p : X \to C$;

(b) $\big(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}(1)\big)$;

(c) $(\mathbb{P}^2, T_{\mathbb{P}^2})$ (tangent bundle);

(d) $\left(\mathbb{Q}^2, \mathcal{O}_{\mathbb{Q}^2}(1)^{\oplus 2}\right)$.

Note that the right hand term in equality (4.0.3) is equal to 7 in case (b) and 6 in cases (c) and (d), a contradiction. In case (a) we can set $X = \mathbb{P}_C(\mathcal{V})$ where $\mathcal{V}$ is a rank-2 vector bundle over C of degree $v := \deg \mathcal{V}$, and up to a twist by a line bundle we can suppose that $v = 0$ or -1 according to whether it is even or odd, respectively; moreover, letting ξ denote the tautological line bundle and $p : X \to C$ the projection we have $\mathcal{E} = \xi \otimes \pi^*\mathcal{G}$ for some rank-2 vector bundle $\mathcal{G}$ on C. Set $\gamma := \deg \mathcal{G}$. Then $\xi^2 = v$, $c_1(\mathcal{E})^2 = (2\xi + \gamma f)^2 = 4(v+\gamma)$, and $c_2(\mathcal{E}) = \xi^2 + \gamma = v + \gamma$. Then equality (4.0.3) gives $v + \gamma = 1$. But $c_2(\mathcal{E}) = 1$ implies that $(X, \mathcal{E}) = \left(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2}\right)$ by [20]. Thus (Y, H) is as in the obvious case, a contradiction. Therefore $K_X + \det \mathcal{E}$ is ample. Moreover, it is also spanned in view of [18, Theorem A], since $\mathcal{E} = \pi_* H$ is ample and spanned. $\square$

Recall that the triple cover $\phi : Y \to \mathbb{P}^3$ is said to be of triple section type if Y is contained in the total space of an ample line bundle on $\mathbb{P}^3$ as a triple section [13]. As a consequence of Proposition 4.6 we get the following conclusion (compare with [19, Proposition 4.4]).

Corollary 4.7. *ϕ is not of triple section type. In particular, ϕ is not a cyclic cover.*

Proof. If ϕ is of triple section type, then $K_Y = \phi^*\mathcal{O}_{\mathbb{P}^3}(k) = kH$ for some integer k, [22, Proposition 3.2] (see also [13, Theorem 2.1]). Taking into account the canonical bundle formula we thus get $kH = -2H + \pi^*(K_X + \det \mathcal{E})$. Therefore $k = -2$ and $K_X + \det \mathcal{E} = \mathcal{O}_X$, due to the injectivity of the homomorphism $\pi^* : \mathrm{Pic}(X) \to \mathrm{Pic}(Y)$. This conclusion, however, contradicts Proposition 4.6. Note also that it is not satisfied even when (Y, H) is as in the obvious case. $\square$

Proposition 4.8. *If $\mathcal{E}$ fits into an exact sequence*

$$0 \to M \to \mathcal{E} \to N \to 0,$$

where M and N are ample line bundles, then (Y, H) can only be as in the obvious case. In particular, except for that case, $\mathcal{E}$ is indecomposable.

Proof. Assuming that $\mathcal{E}$ fits into an exact sequence as above, we have that $c_1(\mathcal{E}) = M + N$ and $c_2(\mathcal{E}) = M \cdot N$. Thus (4.0.3) becomes

$$3 = H^3 = c_1(\mathcal{E})^2 - c_2(\mathcal{E}) = M^2 + M \cdot N + N^2,$$

and $M^2 = M \cdot N = N^2 = 1$, because both M and N are ample. But then $(M - N) \cdot M = 0$ and $(M - N)^2 = 0$, hence the Hodge index theorem implies that M and N are numerically equivalent. As $\mathcal{E}$ is spanned, N is spanned too and then (X, N) is a surface polarized by an ample and spanned line bundle with $N^2 = 1$. Therefore $X = \mathbb{P}^2$ and $M = N = \mathcal{O}_{\mathbb{P}^2}(1)$; then $\mathcal{E} = M \oplus N$ since $\mathrm{Ext}^1(N, M) = H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) = 0$. $\square$

Here is a consequence of Proposition 4.8.

Corollary 4.9. *If (Y, H) is not as in the obvious case, then $c_2(\mathcal{E}) \geq 3$.*

Proof. Since $\mathcal{E}$ is ample and spanned we know that $c_2(\mathcal{E}) \geq 1$ with equality occurring only in the obvious case, as already said. Let $c_2(\mathcal{E}) = 2$. Then a result of Noma [24, Theorem 6.1] shows that either $\mathcal{E}$ is decomposable, which is impossible by Proposition 4.8, or X is not a regular surface, which contradicts Proposition 4.5 (b). $\square$

Finally, we can prove

Proposition 4.10. *Suppose that Y is a Fano threefold; then (Y, H) is as in the obvious case.*

Proof. Due to the assumption, $\mathcal{E}$ is a Fano bundle on X [28]. Let $\mathcal{F}$ be another rank-2 vector bundle on X such that $Y = \mathbb{P}_X(\mathcal{F})$. Denoting by ξ its tautological line bundle, from the fact that $\mathcal{E} = \mathcal{F} \otimes \mathcal{O}_X(D)$ for some divisor D on X, we see that $H = \xi + \pi^* D$. Since $c_1(\mathcal{E}) = c_1(\mathcal{F}) + 2D$ and $c_2(\mathcal{E}) = c_2(\mathcal{F}) + c_1(\mathcal{F}) \cdot D + D^2$, we get from (4.0.3) that

$$3 = H^3 = c_1(\mathcal{F})^2 - c_2(\mathcal{F}) + 3D \cdot \big(c_1(\mathcal{F}) + D\big), \tag{4.10.1}$$

hence $\xi^3 = c_1(\mathcal{F})^2 - c_2(\mathcal{F})$ is also divisible by 3. Moreover, the fact that Y is Fano implies that X is a del Pezzo surface [28, Proposition 1.5]. We can therefore assume that $\mathcal{F}$ is normalized in an appropriate way. Suppose that (Y, H) is not as in the obvious case. Then, checking the list of the rank-2 Fano bundles on surfaces [28, Theorem] and taking into account Proposition 4.8 and (4.10.1) we see that if our $(X, \mathcal{F})$ is in that list, then the possibilities for $(X, \mathcal{E})$, if any, restrict to the following:

(1) $X = \mathbb{P}^2$ and $\mathcal{E}$ is a stable spanned bundle fitting in an exact sequence
$0 \to \mathcal{O}_{\mathbb{P}^2}(-2) \to \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \to \mathcal{E} \to 0$ (case 7 in [28, Theorem]; here $\mathcal{E} = \mathcal{F}(1)$);

(2) $X = \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathcal{E}$ is a stable spanned bundle fitting in an exact sequence
$0 \to \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, -1) \to \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}^{\oplus 3} \to \mathcal{E} \to 0$ (case 12 in [28, Theorem]; here $\mathcal{E} = \mathcal{F}(1, 1)$).

However, in these cases the vector bundle $\mathcal{E}$ is not ample. To see this suppose we are in case (1), consider the inclusion of $Y = \mathbb{P}_{\mathbb{P}^2}(\mathcal{E})$ in $\mathbb{P}^2 \times \mathbb{P}^2 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}^{\oplus 3})$ corresponding to the surjection $\mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \to \mathcal{E}$ and call $\rho : Y \to \mathbb{P}^2$ the restriction of the second projection p_2 of $\mathbb{P}^2 \times \mathbb{P}^2$ to Y (note that π is the restriction of the first projection). Then, for the tautological line bundle of $\mathcal{E}$ we have that $H = \rho^* \mathcal{O}_{\mathbb{P}^2}(1)$. Fix a point $x \in \mathbb{P}^2$: then $\gamma := \rho^{-1}(x) = p_2^{-1}(x) \cap Y$ is a curve inside Y and clearly if $\ell \subset \mathbb{P}^2$ is a general line, we get $H \cap \gamma = \rho^{-1}(\ell) \cap \rho^{-1}(x) = \emptyset$. Therefore H, hence $\mathcal{E}$, is not ample. The same argument applies to case (2) and this concludes the proof. $\square$

5. Further constraints on Y deriving from S as triple plane

Let Y, H and $\mathcal{E}$ be as in (4.0.1), and let S be a general element of the linear subsystem $\phi^*|\mathcal{O}_{\mathbb{P}^3}(1)| \subseteq |H|$ (recall that equality holds except when (Y, H) is as in the obvious case). Then S is a smooth regular surface, by Proposition 4.5 (c), and the polarized surface (S, H_S) inherits from (Y, H) the structure of a triple plane $\varphi := \phi|_S : S \to \mathbb{P}^2$, where $H_S = \varphi^* \mathcal{O}_{\mathbb{P}^2}(1)$. Moreover, referring to the scroll structure of (Y, H), by restricting the projection $\pi : Y \to X$ to S we get a birational morphism $r = \pi|_S : S \to X$. More precisely, the pair (S, H_S) has $(X, \det \mathcal{E})$ as its adjunction theoretic minimal reduction, the reduction morphism being r. This means that S is a meromorphic non-holomorphic section of π which contains $s > 0$ fibres $e_1, \dots, e_s$ of $\pi : Y \to X$, and these curves, which are lines of (S, H_S), are contracted by the birational morphism r to a finite subset of X; in addition, X can contain no line with respect to $\det \mathcal{E}$, $\mathcal{E}$ being ample of rank 2. Hence $(X, \det \mathcal{E})$ is the minimal reduction of (S, H_S). In particular, S is not minimal; moreover, $H_S = r^* \det \mathcal{E} - \sum_{i=1}^s e_i$, so that $(\det \mathcal{E})^2 = c_1(\mathcal{E})^2 = 3 + s$, which combined with (4.0.3) shows that

(5.0.1)
$$s = c_2(\mathcal{E}).$$

We have also $K_S = r^* K_X + \sum_{i=1}^s e_i$, hence $K_S + H_S = r^*(K_X + \det \mathcal{E})$, which has the following consequence on the sectional genus:

(5.0.2)
$$g(Y, H) = g(S, H_S) = g(X, \det \mathcal{E}).$$

We set $g := g(Y, H)$. Furthermore, $K_S^2 = K_X^2 - s$ and $e(S) = e(X) + s$. Consider the exact sequence

(5.0.3)
$$0 \to \mathcal{O}_Y \to H \to H_S \to 0.$$

By pushing (5.0.3) down via π we get the sequence

$$0 \to \mathcal{O}_X \to \mathcal{E} \to \det \mathcal{E} \otimes \mathcal{J}_Z \to 0,$$

defined by the multiplication by θ, the section of $\mathcal{E}$ that corresponds to the section of H defining S in the isomorphism $H^0(Y, H) \cong H^0(X, \mathcal{E})$. Here Z stands for the zero locus of θ and $\mathcal{J}_Z$ for its ideal sheaf. Recall that Z consists of $s = c_2(\mathcal{E})$ points of X, by (5.0.1). Clearly,

$$h^0(\det \mathcal{E} \otimes \mathcal{J}_Z) = h^0(\det \mathcal{E}) - t$$

where t is the number of linearly independent linear conditions to be imposed on an element of $|\det \mathcal{E}|$ to contain Z. Of course, $t \leq \mathrm{Card}(Z) = s$. On the other hand, recalling that X is regular by Proposition 4.5 (b), we see from the cohomology of the exact sequence above that

$$h^0(\det \mathcal{E} \otimes \mathcal{J}_Z) = h^0(\mathcal{E}) - 1 = h^0(H) - 1 = 3,$$

provided that (Y, H) is not as in the obvious case. So we have

Proposition 5.1. *Suppose that (Y, H) is not as in the obvious case. Then $\varphi : S \to \mathbb{P}^2$ factors through r and the rational map defined by the linear subsystem of $|\det \mathcal{E}|$ of curves passing through $c_2(\mathcal{E})$ points of X that impose only $h^0(\det \mathcal{E}) - 3$ linearly independent linear conditions on them.*

The following result will have relevant consequences.

Proposition 5.2. *Suppose that (Y, H) is not as in the obvious case. Then $g \geq 3$, equality implying $X = \mathbb{P}^2$ and $\mathcal{E}$ indecomposable of generic splitting type $(2, 2)$, in particular semistable, with $c_2(\mathcal{E}) = 13$.*

Proof. Look at $(X, \det \mathcal{E})$. Polarized surfaces with sectional genus ≤ 1 are well known [12]. By Proposition 4.6 we know that $K_X + \det \mathcal{E}$ is ample and spanned, since (Y, H) is not as in the obvious case. We can thus exclude that $(X, \det \mathcal{E})$ is such a pair. Therefore $g \geq 2$. However, it cannot be $g = 2$, since every ample and spanned rank 2 vector bundle of c_1-sectional genus 2 (i.e.,

$g(X, \det \mathcal{E}) = 2$) on a surface is decomposable [10, proof of the Theorem in the appendix], but this is in contrast with Proposition 4.8. Finally, suppose that $g = 3$. Then a close check of the list in [10, Theorem 2.1, (III)] combined with Proposition 4.8 again confines the possibilities to the following single case: $X = \mathbb{P}^2$ and $\mathcal{E}$ indecomposable with $\det \mathcal{E} = \mathcal{O}_{\mathbb{P}^2}(4)$. Now, let (a_1, a_2), with $a_1 \geq a_2$, be the generic splitting type of $\mathcal{E}$ (i.e., $\mathcal{E}_\ell = \mathcal{O}_\ell(a_1) \oplus \mathcal{O}_\ell(a_2)$ for the general line $\ell \subset \mathbb{P}^2$). Clearly $(a_1, a_2) = (3, 1)$ or $(2, 2)$, due to the ampleness. Suppose that $(a_1, a_2) = (3, 1)$. Then $\mathcal{E}$ has no jumping lines [25, p. 29], so that it is uniform. Thus, according to a theorem of Van de Ven [25, p. 211], $\mathcal{E}$ is either $\mathcal{O}_{\mathbb{P}^2}(3) \oplus \mathcal{O}_{\mathbb{P}^2}(1)$, or a twist of the tangent bundle. Both cases, however, have to be excluded: the former would contradict Proposition 4.8, while in the latter $\det \mathcal{E}$ could not be $\mathcal{O}_{\mathbb{P}^2}(4)$. Therefore $(a_1, a_2) = (2, 2)$. It thus follows from [25, Lemma 2.2.1, p. 209] that $\mathcal{E}$ is semistable. Finally, (4.0.3) implies $c_2(\mathcal{E}) = 13$. □

Remark 5.3. Note that spanned rank-2 vector bundles on $\mathbb{P}^2$ with Chern classes $(c_1, c_2) = (4, 13)$ do exist according to [9, Theorem 0.1]. Anyway, the general stable rank-2 vector bundle with these Chern classes is certainly not spanned, since its invariants do not satisfy the conditions in [16, Theorem 2.6]. As a consequence, [16, Theorem 5.1] is not applicable to establish the ampleness. Moreover, for a vector bundle like $\mathcal{E}$, giving rise to a pair (Y, H) with $g = 3$, if any, we know that $h^0(\mathcal{E}) = 4$ and by the Riemann–Roch theorem combined with the exact cohomology sequence induced by (5.0.3) it follows easily that $h^1(\mathcal{E}) = 1$. Therefore, such an $\mathcal{E}$, if any, would be quite special in moduli by the Weak Brill–Noether theorem for $\mathbb{P}^2$ [16, Theorem 2.4]. In fact, such a vector bundle does not exist, according to what we will prove in Section 6.

Now let us focus on the triple plane $\varphi : S \to \mathbb{P}^2$ induced by ϕ, deriving further restrictions on Y. Let B be the branch locus and let $\mathcal{T}$ be the rank 2 vector bundle on $\mathbb{P}^2$ such that $\varphi_* \mathcal{O}_S = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{T}$, i.e. the Tschirnhaus bundle of φ. Set $b_i = c_i(\mathcal{T})$. Then $B \in |2 \det \mathcal{T}^\vee|$ so that $b := \deg B = -2b_1 > 0$; moreover, if φ is general in the sense of [23, p. 1154], then B is irreducible and has only cusps as singularities, their number being $c = 3b_2$ [23, Lemma 10.1]. Furthermore, the Riemann–Hurwitz theorem applied to $\varphi^{-1}(\ell)$, where $\ell \subset \mathbb{P}^2$ is a general line, gives

$$b = 2g + 4. \tag{5.0.4}$$

As an immediate consequence of Proposition 5.2 we have

Remark 5.4. If (Y, H) is not as in the obvious case, then $b \geq 10$, equality implying $g = 3$.

Consider the equalities $h^i(\mathcal{O}_S) = h^i(\mathcal{O}_{\mathbb{P}^2}) + h^i(\mathcal{T})$ coming from the definition of $\mathcal{T}$. For $i = 1$, since S is regular we get $h^1(\mathcal{T}) = 0$. On the other hand we know that $h^0(\mathcal{T}) = 0$, hence letting $i = 2$ we see that $h^2(\mathcal{O}_S) = h^2(\mathcal{T}) = \chi(\mathcal{T})$, which can be computed with the Riemann–Roch theorem [4, p. 26]. In conclusion, we obtain $p_g(S) = \frac{1}{8} b(b-6) + 2 - \frac{c}{3}$, hence

$$\chi(\mathcal{O}_S) = 1 + p_g(S) = \frac{1}{8} b(b-6) + 3 - \frac{c}{3}.$$

On the other hand, Miranda's formulas (3.2.5), rewritten for S in terms of b and c, provide the following values of K_S^2 and $e(S)$:

$$K_S^2 = 27 - 6b + \frac{1}{2}b^2 - c \qquad \text{and} \qquad e(S) = 9 - 3b + b^2 - 3c. \tag{5.0.5}$$

Recalling that $r : S \to X$ is a birational morphism which factors through s blowing-ups, this immediately gives the corresponding numerical characters of X.

It is useful to recall that for (Y, H) as in the obvious case, the pair (S, H_S) is as in (2) of Theorem 3.2. In particular, we have $g = 0$; moreover, for the triple plane $\varphi : S \to \mathbb{P}^2$ induced by ϕ, the Tschirnhaus bundle is $\mathcal{T} = \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$ [23, Table 10.5]. Then the branch curve B of φ is a quartic, since $b = -2b_1 = 4$, and (5.0.5) shows that $c = 3$. Furthermore, φ maps the only (-1)-line of (S, H_S) (namely the only fiber of $\pi : Y \to X$ that S contains), isomorphically to a line $\ell \subset \mathbb{P}^2$, which is bitangent to B (there is only one bitangent line in this case, by Plücker formulas).

Coming back to the general case, a natural question concerning the Tschirnhaus bundle of φ is what happens when $\mathcal{T}$ is decomposable, namely $\mathcal{T} = \mathcal{O}_{\mathbb{P}^2}(-m) \oplus \mathcal{O}_{\mathbb{P}^2}(-n)$ for some positive integers m, n, as in [23, p. 1156]. As we have seen, $(m, n) = (1, 1)$ corresponds to (Y, H) being as in the obvious case. We have $b = 2(m + n)$, $c = 3mn$ and we can rewrite the invariants of S in terms of m, n as in [23, Corollary 10.4]. In particular, we have $p_g(S) = (\frac{1}{2})(m^2 + n^2 - 3m - 3n) + 2$, $K_S^2 = 2(m + n - 3)^2 - 3(mn - 3)$, $e(S) = 4(m + n)^2 - 6(m + n) - 9(mn - 1)$. Then we immediately obtain the following result.

Proposition 5.5. *If $\mathcal{T}$ is decomposable and X is a surface with $p_g(X) = 0$, then (Y, H) is necessarily as in the obvious case.*

Proof. Since p_g is a birational invariant, we have $p_g(S) = 0$. Hence (m, n) must be an integral point of the curve Γ represented in the (m, n)-plane by the equation

$$m^2 + n^2 - 3m - 3n + 4 = 0.$$

Note that Γ is a circle centered at $(\frac{3}{2}, \frac{3}{2})$ with radius $\frac{1}{\sqrt{2}}$; hence its integral points are $(1, 1)$, $(1, 2)$, $(2, 1)$, $(2, 2)$ only. In view of the symmetry between m and n we can confine to consider the three pairs $(m, n) = (1, 1), (1, 2), (2, 2)$. In all these cases we have $b = 2(m + n) \leq 8$, hence the assertion follows from Remark 5.4. $\square$

Still about the branch curve B, we have

Proposition 5.6. *Let things be as in the setting* (4.0.1)*, let S be a smooth element of $\phi^*|\mathcal{O}_{\mathbb{P}^3}(1)|$, and suppose that $\varphi : S \to \mathbb{P}^2$ is a general triple plane, then*

$$\frac{1}{6}\, b^2 < c \leq \min\left\{\frac{1}{16}\, b(5b - 6) - \frac{s}{2}\ ,\ \frac{3}{8}\, b(b - 6) + 6\right\}.$$

Proof. To prove the lower bound for c, note that the ramification divisor of φ is $R_S := R \cap S$, R being the ramification divisor of ϕ. So, $\varphi(R_S) = B$. As $H_S = \varphi^*\mathcal{O}_{\mathbb{P}^2}(1)$ we have $H_S \cdot R_S = \deg B = b$. Hence the Hodge index theorem gives the inequality $b^2 = (R_S \cdot H_S)^2 \geq H_S^2 R_S^2 = 3R_S^2$. On the other hand $R_S = K_S + 3H_S$ by the ramification formula. Having all the ingredients, recalling (5.0.4) and the expression of K_S^2, we can thus compute

$$R_S^2 = (K_S + 3H_S)^2 = K_S^2 + 6(2g-2) + 3H_S^2 = \frac{1}{2}\, b^2 - c.$$

Then the above inequality says that $c \geq \frac{1}{6}\, b^2$ (compare with [11, Corollary 2.7]) Now suppose that equality holds. Then R_S and H_S are linearly dependent in $\mathrm{NS}(S) \otimes \mathbb{Q}$. But this implies that either $H_S \equiv tK_S$ for some rational t or $K_S \equiv 0$. The latter case cannot occur, since S is not minimal. In the former case, for a (-1)-curve $e \subset S$ we have $0 < H_S e = tK_S e = -t$, hence t is negative. Then $-K_S \equiv \frac{p}{q} H_S$ for some positive $\frac{p}{q} \in \mathbb{Q}$. This means that S is a del Pezzo surface: in particular we have that $-K_S = \frac{p}{q} H_S$. By combining the classification of these surfaces with the fact that S is not minimal we argue that $-K_S$ is not divisible in $\mathrm{NS}(S)$. Note that the same is true for H_S, since $H_S^2 = 3$. Thus the above equality allows us to conclude that $-K_S = H_S$, hence $g = 1$. But this is impossible in view of Proposition 5.2, taking also into account that $g = 0$ if (Y, H) is as in the obvious case. Therefore the inequality we obtained above is strict.

As to the upper bounds for c, the one with respect to the second term in the min derives from the obvious inequality $p_g(S) \geq 0$ combined with the expression of $p_g(S)$. To prove the other bound we use the inequality $K_X^2 \leq 3e(X)$. Recall that if X is a surface of general type, this is just the Bogomolov–Miyaoka–Yau inequality [4, p. 275], while if X has Kodaira dimension ≤ 1 then the above inequality follows immediately from the theory of minimal models, simply recalling that X is regular, due to Proposition 4.5 (b). On the other hand, since S is obtained from X via s blowing-ups we have that $3e(S) - K_S^2 = 4s + \big(3e(X) - K_X^2\big) \geq 4s$, Due to the expression of both K_S^2 and $e(S)$ provided by (5.0.5) we can immediately convert this inequality into the bound with respect to the first term in the min. □

Comments. i) Concerning the upper bound with respect to the first term of the min in Proposition 5.6 one can say a bit more if X is not of general type, since instead of looking at $3e(S) - K_S^2$ one can use better lower bounds for $2e(S) - K_S^2$ in terms of s, according to the Kodaira dimension. In particular, if S is rational, then the upper bound for c in Proposition 5.6 can be improved. Actually, $S \neq \mathbb{P}^2$, hence, there exists a birational morphism $S \to S_0$, where S_0 is a Segre–Hirzebruch surface (either a minimal model or $\mathbb{F}_1$). Then $K_S^2 = 8 - t$, $e(S) = 4 + t$ for some $t \geq 0$ (the number of blow-ups factoring this birational morphism). Thus $2e(S) - K_S^2 = 3t$. Moreover, $s \leq t$ since X is not necessarily minimal, unless $X = \mathbb{P}^2$, in which case $s = t + 1$. So, apart from this case, $s \leq t = \frac{1}{3}\big(2e(S) - K_S^2\big)$ and taking into account (5.0.5), this gives $c \leq \frac{3}{10} b^2 - \frac{3}{5}(s + 3)$.
ii) According to [11, Corollary 2.7] the inequality $c \geq \frac{1}{6} b^2$ holds for any general triple plane. We emphasize that the inequality proved in Proposition 5.6 is strict because it refers only to triple planes deriving from a triple solid as in (4.0.1).

We conclude this Section with a general property that the pair $(X, \mathcal{E})$ has to satisfy if Y is as in our setting. Recall that $\mathcal{E}$ is ample and spanned of rank 2, and $h^0(\mathcal{E}) \geq 4$ (with equality except when (Y, H) in the obvious case); so let V be a 4-dimensional vector subspace of $H^0(X, \mathcal{E})$ spanning $\mathcal{E}$ and let $\mathbb{G} := \mathbb{G}(1, 3)$ be the grassmannian of the codimension 2 vector subspaces of V. According to [1, Remark 2.6], since $\mathcal{E}$ is ample and spanned by V, $\mathcal{E}$ defines a morphism $\psi : X \to \mathbb{G}$, finite to its image $W := \psi(X)$, such that $\mathcal{E} = \psi^*\mathcal{Q}$, where $\mathcal{Q}$ is the universal rank-2 quotient bundle of $\mathbb{G}$.

Proposition 5.7. *Consider the morphism $\psi : X \to \mathbb{G}$ defined by $\mathcal{E}$, and write $W = \alpha\Omega(0,3) + \beta\Omega(1,2)$, as a linear combination of the usual Schubert cycle classes with integral coefficients $\alpha = W \cdot \Omega(0,3)$ and $\beta = W \cdot \Omega(1,2)$ in the cohomology ring of $\mathbb{G}$. Then $s = c_2(\mathcal{E}) = \beta \deg \psi$ and $3 = \alpha \deg \psi$. In particular, if s and 3 are coprime, then ψ is birational and W has bidegree $(3,s)$. Moreover, if ψ is an embedding, then the following relation holds, connecting $c_2(\mathcal{E})$ with the Chern classes of the Tschirnhaus bundle of φ:*

$$(s-2)(s-3) = -2b_1^2 - 2b_1 + 6b_2.$$

Proof. Writing $W = \alpha\Omega(0,3) + \beta\Omega(1,2)$ as we said, and recalling that $c_1(\mathcal{Q}) = \mathcal{O}_{\mathbb{G}}(1)$ and $c_2(\mathcal{Q}) = \Omega(1,2)$, by the functoriality of the Chern classes we get

$$c_2(\mathcal{E}) = \psi^* c_2(\mathcal{Q}_W) = \deg \psi \ \Big(\Omega(1,2) \cdot \big(\alpha\Omega(0,3) + \beta\Omega(1,2)\big)\Big) = \beta \ \deg \psi.$$

and

$$c_1(\mathcal{E})^2 = \psi^*\big(\mathcal{O}_{\mathbb{G}}(1)_W\big)^2 = \deg \psi \ \deg(W) = (\alpha + \beta) \deg \psi.$$

Therefore, (4.0.3) gives $3 = \alpha \deg \psi$ and this, in turn, combined with (5.0.1) proves the assertion on the birationality of ψ and the bidegree of W. Finally, if ψ is an embedding, the "formule clef" applied to the smooth congruence $X \cong W$ in $\mathbb{G}$ [2, Proposition 2.1] implies

$$9 + s^2 = 3(3+s) + 4(2g-2) + 2K_X^2 - 12\chi(\mathcal{O}_X).$$

Taking into account (5.0.4) and the expressions of K_X^2 and $\chi(\mathcal{O}_X)$ deriving from (5.0.5) in view of the birationality between S and X, this proves the final relation. □

Remark 5.8. We emphasize that $\psi : X \to \mathbb{G}$ can be an embedding although $\mathcal{E}$ is not very ample (see [1, Proposition 2.4 and Remarks 2.5 and 2.6]). However, if (Y, H) is as in the obvious case, then $\mathcal{E}$ is very ample, ψ is in fact an embedding, and W is the Veronese surface; in this case both sides of the equality in the last display are equal to 10.

6. Scrolls over $\mathbb{P}^2$

Let (Y, H), $\pi : Y \to X$, and $\mathcal{E}$ be as in our setting (4.0.1) again and let $\varphi : S \to \mathbb{P}^2$ be the triple plane induced by ϕ as in Section 5. When $X = \mathbb{P}^2$, the possibilities for $\mathcal{E}$, b and c are extremely restricted.

Proposition 6.1. *Let things be as above and suppose that $\varphi : S \to \mathbb{P}^2$ is a general triple plane. If $X = \mathbb{P}^2$, then either (Y, H) is as in the obvious case or it has the following characters:*

(6.1.1) $$c_1(\mathcal{E}) = \mathcal{O}_{\mathbb{P}^2}(4), \quad s = 13, \quad b = 10, \quad c = 21.$$

Proof. For $X = \mathbb{P}^2$ we have $3e(S) - K_S^2 = 3e(X) - K_X^2 + 4s = 4s$. Then (5.0.5) combined with the expression $c = \frac{3}{8}\, b(b-6) + 6$ deriving from the condition $p_g(S) = 0$ leads to the equation $b^2 - 30b + 8(12+s) = 0$. Hence $b = 15 \pm \sqrt{D}$, where $D = 129 - 8s$. Imposing that D is non-negative we get the bound $s \leq 16$ and then the list of the admissible values of s follows by requiring that D is the square of an integer. These values, together with the corresponding b and c deriving from the above relations, are summarized in Table 1 below.

case	1	2	3	4	5	6	7	8	9	10	11	12
s	1	6	10	13	15	16	16	15	13	10	6	1
b	4	6	8	10	12	14	16	18	20	22	24	26
c	3	6	12	21	33	48	66	87	111	138	168	201

Table 1.

Clearly case 1 corresponds to (Y, H) being as in the obvious case while case 4 corresponds to the further possibility mentioned in the statement. So, it is enough to show that all remaining cases cannot occur. Clearly cases 2 and 3 are ruled out by Remark 5.4. Consider the remaining cases 5–12 and set $c_1(\mathcal{E}) = \mathcal{O}_{\mathbb{P}^2}(a)$. Recalling (5.0.4), (5.0.2) and the fact that $\big(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(a)\big)$ is the minimal reduction of (S, H_S), Clebsch formula implies that $g = \frac{1}{2}b - 2 = \frac{1}{2}(a-1)(a-2)$. This rules out all cases except cases 7 and 11, in which we get $a = 5$ and 6 respectively. However, computing $c_1(\mathcal{E})^2 - c_2(\mathcal{E}) = a^2 - s$ in these cases we see that condition (4.0.3) is not satisfied. □

Now suppose that (Y, H) is a scroll over $X = \mathbb{P}^2$ with the characters as in (6.1.1). For the description of the triple plane $\varphi : S \to \mathbb{P}^2$ in this case we refer to [11, 2.2 and 3.4]. We have $\mathcal{T} = T_{\mathbb{P}^2}(-4) = \Omega^1_{\mathbb{P}^2}(-1)$, in view of the natural identification $\Omega^1_{\mathbb{P}^2} \cong T_{\mathbb{P}^2} \otimes \det \Omega^1_{\mathbb{P}^2} = T_{\mathbb{P}^2}(-3)$. In particular, $\mathcal{T}$ is stable. Moreover, we can observe that in this case the triple plane $\varphi : S \to \mathbb{P}^2$ is general, regardless of the assumption made in Proposition 6.1. Actually, the vector bundle $S^3\mathcal{T}^\vee \otimes \det \mathcal{T} = S^3\big(T_{\mathbb{P}^2}(-1)\big) \otimes \mathcal{O}_{\mathbb{P}^2}(1)$ is spanned, due to the Euler sequence. Thus by combining [23, Theorem 1.1] and [29, Theorem 2.1 and Theorem 3.2] with the fact that Corollary 4.7 prevents φ from being totally ramified, we conclude that $\varphi : S \to \mathbb{P}^2$ is general. By (5.0.4) we get $g = 3$, since $b = 10$, and then Proposition 5.2 applies. We know that $(X, \det \mathcal{E}) = \big(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(4)\big)$, hence $h^0(\det \mathcal{E}) = 15$ and therefore Proposition 5.1 tells us that the triple plane $\varphi : S \to \mathbb{P}^2$ is defined via the linear system of plane quartics passing through 13 points that impose only 12 independent linear conditions on them (see also [11, Proposition 3.7] and [23, p. 1158]). Clearly, such a triple plane exists. However, it cannot derive from a pair (Y, H) as in our setting. To see this, let $\widetilde{\mathcal{T}}$ be the Tschirnhaus bundle of ϕ, i.e. $\phi_*\mathcal{O}_Y = \mathcal{O}_{\mathbb{P}^3} \oplus \widetilde{\mathcal{T}}$. If $\Pi = \mathbb{P}^2 \subset \mathbb{P}^3$ is the plane such that $S = \phi^{-1}(\Pi)$, then

$$\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{T} = \varphi_*\mathcal{O}_S = \phi_*(\mathcal{O}_Y|_S) = (\mathcal{O}_{\mathbb{P}^3} \oplus \widetilde{\mathcal{T}})|_\Pi = \mathcal{O}_\Pi \oplus \widetilde{\mathcal{T}}|_\Pi,$$

hence $\widetilde{\mathcal{T}}|_{\mathbb{P}^2} = \mathcal{T}$, and therefore $c_i(\mathcal{T}) = c_i(\widetilde{\mathcal{T}})|_{\mathbb{P}^2}$. We know that $b_1 = -\frac{b}{2} = -5, b_2 = \frac{1}{3}c = 7$ by Proposition 6.1. As a consequence, $\widetilde{\mathcal{T}}$ has Chern classes $-5h$ and $7h^2$, respectively, where $h = \mathcal{O}_{\mathbb{P}^3}(1)$. But this contradicts the Schwarzenberger condition $c_1 \cdot c_2 \equiv 0 \pmod 2$, necessary for the existence of a rank 2 vector bundle on $\mathbb{P}^3$ [25, p. 113].

In conclusion, we have the following result.

Theorem 6.2. *Let $\phi : Y \to \mathbb{P}^3$ be a triple solid, $H = \phi^*\mathcal{O}_{\mathbb{P}^3}(1)$ and suppose that $\varphi : S \to \mathbb{P}^2$ is a general triple plane. If Y is a scroll over $\mathbb{P}^2$ for some polarization, then (Y, H) is necessarily as in the obvious case.*

The above result does not mean that the pair (Y, H) as in the obvious case is the only scroll over $\mathbb{P}^2$ containing a smooth surface which is a triple plane. From this perspective we would like to emphasize the following fact.

Remark 6.3. Given a scroll (Y, L) over $\mathbb{P}^2$ for some ample line bundle L, which is not as in the obvious case, it may happen that Y contains a smooth surface S such that: i) S has the structure of a triple plane, ii) $M := \mathcal{O}_Y(S)$ is a very ample line bundle, and iii) $M \neq L$. To give an example, consider $Y := \mathbb{P}(T_{\mathbb{P}^2})$. Recalling that Y is contained in $P := \mathbb{P}^2_1 \times \mathbb{P}^2_2$ as a smooth element of $|\mathcal{O}_P(1,1)|$, we see that Y has two distinct structures of $\mathbb{P}^1$-bundle over $\mathbb{P}^2$, $\pi_i : Y \to \mathbb{P}^2_i$ $(i = 1, 2)$, induced by the projections of P onto the two factors. Set $L := \big(\mathcal{O}_P(1,1)\big)_Y$; then L is very ample, and we can regard (Y, L) as a scroll over $\mathbb{P}^2$, e.g. via π_1. As is well-known, the general element $\Sigma \in |L|$ is a del Pezzo surface of degree 6 and $\pi_1|_\Sigma : \Sigma \to \mathbb{P}^2_1$ is a birational morphism consisting of the blow-up a three general points. Now look at $\mathrm{Pic}(Y)$, which can be generated by L and $h := \pi_1^*\mathcal{O}_{\mathbb{P}^2_1}(1)$. The line bundle $M := L + 2h$ is clearly very ample. Let $S \in |M|$ be a general element: then S is a smooth surface, and $\varphi := \pi_2|_S : S \to \mathbb{P}^2_2$ is a triple plane. Actually, φ is a finite morphism and recalling that $\mathcal{O}_P(1,0)^3 = \mathcal{O}_P(0,1)^3 = 0$ and $\mathcal{O}_P(1,0)^2 \cdot \mathcal{O}_P(0,1)^2 = 1$, we see that its degree is computed by

$$S \cdot \big(\mathcal{O}_P(0,1)_Y\big)^2 = (L + 2h) \cdot \big(\mathcal{O}_P(0,1)_Y\big)^2 = \mathcal{O}_P(3,1) \cdot \mathcal{O}_P(1,1) \cdot \big(\mathcal{O}_P(0,1)\big)^2 = 3.$$

Finally, restricting our attention to triple solids with sectional genus 3, we want to stress that Theorem 6.2 constitutes a significant progress compared with [17, Proposition 3.3].

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References

[1] Enrique Arrondo, *Subvarieties of Grassmannians*, Lect. Notes Series Dipart. di Matematica Univ. Trento **10**, 1996.

[2] Enrique Arrondo and Ignacio Sols, *On congruences of lines in the projective space* (Chapter 6 written in collaboration with M. Pedreira), Mém. Soc. Math. France (2e série) **50** (1992).

[3] Edoardo Ballico, *On ample and spanned vector bundles with zero Δ-genera*, Manuscripta Math. **70** (1991), 153–155.

[4] Wolf P. Barth, Klaus Hulek, Chris A.M. Peters, and Antonius Van de Ven, *Compact complex surfaces* (2nd enlarged edition), Ergebnisse der Math. **4**, Springer, 2004.

[5] Mauro C. Beltrametti and Andrew J. Sommese, *The adjunction theory of complex projective varieties*, Exp. Math. vol. 16, de Gruyter, Berlin, 1995.

[6] Spencer Bloch and David Gieseker *The positivity of the Chern classes of an ample vector bundle*, Invent. Math. **12** (1971), 112–117.

[7] Robert Braun, Giorgio Ottaviani, Michael Schneider, and Frank-Olaf Schreyer, *Classification of conic bundles in $\mathbb{P}_5$*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **23**(1) (1996), 69–97.

[8] Fabrizio Catanese, *On a problem of Chisini*, Duke Math. J. **53** (1986), 33–42.

[9] Philippe Ellia, *Chern classes of rank two globally generated vector bundles on* $\mathbb{P}^2$, Rend. Lincei Mat. Appl. **24** (2013), 147–163.

[10] Yoshiaki Fukuma and Hironobu Ishihara, *Complex manifolds polarized by an ample and spanned line bundle of sectional genus three*, Arch. Math. **71** (1998), 159–168.

[11] Daniele Faenzi, Francesco Polizzi, and Jean Vallès, *Triple planes with* $p_g = q = 0$, Trans. Amer. Math. Soc. **371** (2019), 589–639.

[12] Takao Fujita, *Classification theories of polarized varieties*, London Math. Soc. Lecture Notes Series **155**, Cambridge Univ. Press, Cambridge, 1990.

[13] Takao Fujita, *Triple covers by smooth manifolds*, J. Fac. Sci. Univ. Tokyo, Sect. IA Math. **35**, 1 (1988), 169–175.

[14] Takao Fujita, *On adjoint bundles of ample vector bundles*. In "Proc. of the Conference in Algebraic Geometry", Bayreuth, 1990, Lect. Notes in Math. **1507**, Springer, 1992, pp. 105–112.

[15] Robin Hartshorne, *Algebraic Geometry*, Springer-Verlag, 1977.

[16] Jack Huizenga and John Kopper, *Ample stable vector bundles on rational surfaces*, Comm. Algebra **50**, 9 (2022), 3744–3760.

[17] Antonio Lanteri and Elvira L. Livorni *Triple solids with sectional genaus three*, Forum Math. **2** (1990), 297–306.

[18] Antonio Lanteri and Hidetoshi Maeda, *Adjoint bundles of ample and spanned vector bundles on algebraic surfaces*, J. reine angew. Math. **433** (1992), 181–199.

[19] Antonio Lanteri and Carla Novelli, *Ample vector bundles of small* Δ*-genera*, Journal of Algebra **323** (2010), 671–697.

[20] Antonio Lanteri and Andrew J. Sommese *A vector bundle characterization of* $\mathbb{P}^n$, Abh. Math. Sem. Univ. Hamburg **58** (1988), 89–96.

[21] Antonio Lanteri, Marino Palleschi, and Andrew J. Sommese, *Very ampleness of* $K_X \otimes \mathcal{L}^{\dim X}$ *for ample and spanned line bundles* $\mathcal{L}$, Osaka J. Math. **26** (1989), 647–664.

[22] Robert Lazarsfeld, *A Barth–type theorem for branched coverings of projective space*, Math. Ann. **249** (1980), 153–162.

[23] Rick Miranda, *Triple covers in algebraic geometry*, Amer. J. Math. **107** (1985), 1123–1158.

[24] Atsushi Noma, *Classification of rank-*2 *ample and spanned vector bundles on surfaces whose zero loci consist of general points*, Trans. Amer. Math. Soc. **342** (1994), 867–894.

[25] Christian Okonek, Michael Schneider, and Heinz Spindler, *Vector bundles on complex projective spaces*, Birkhäuser, 1980.

[26] Igor Reider, *Vector bundles of rank 2 and linear systems on algebraic surfaces*, Ann. of Math. **127**, 2 (1988), 309–316.

[27] Andrew J. Sommese, *On manifolds that cannot be ample divisors*, Math. Ann. **221** (1976), 55–72.

[28] Michał Szurek and Jarosław A. Wiśniewski, *Fano bundles of rank* 2 *on surfaces*, Compos. Math. **76** (1990), 295–305.

[29] Sheng-Li Tan, *Triple covers on smooth algebraic varieties*, In "Geometry and nonlinear partial differential equations" (Hangzhou, 2001), AMS/IP Stud. Adv. Math. vol. 29, Amer. Math. Soc., Providence, RI, 2002, pp. 143–164.

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