Bernstein–Sato Theory for Determinantal Ideals in Positive Characteristic

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Abstract

Let R a polynomial ring over a field k. An algebraic variety over k is a set of points given as the zero loci of polynomials in R. In order to study these varieties and their singularities, a common practice is to construct algebraic invariants to quantify how singular the variety is. This is precisely one of the goals of Bernstein-Sato theory. In characteristic $p > 0$, the theory has seen unparalleled growth and development for the last twenty years. In this project we compute algebraic invariants of this theory for determinantal ideals, that is, ideals generated by the determinants of submatrices of generic matrices of indeterminates.

Keywords: commutative algebra, algebraic geometry, characteristic $p > 0$, singularity theory, Bernstein-Sato theory, determinantal ideals.

MSC2020: 14B05, 14F10, 13C40.

Resumen

Sea R un anillo de polinomios sobre un cuerpo k. Una variedad algebraica sobre k es un conjunto de puntos donde se anulan algunos polinomios de R. A fin de estudiar estas variedades y sus singularidades, una práctica común es construir invariantes algebraicos para cuantificar cómo de singular es la variedad. Este es precisamente uno de los objetivos de la teoría de Bernstein-Sato. Durante los últimos veinte años, la teoría en característica $p > 0$ ha visto un crecimiento y desarrollo sin parangón. En este proyecto calculamos invariantes de la teoría para ideales determinantales, a saber, ideales generados por los determinantes de submatrices de matrices genéricas de indeterminadas.

Palabras clave: álgebra conmutativa, geometría algebraica, característica $p > 0$, teoría de singularidades, teoría de Bernstein-Sato, ideales determinantales.

Resum

Sigui R un anell de polinomis sobre un cos k. Una varietat algebraica sobre k és un conjunt de punts on s'anul·len alguns polinomis de R. Amb la finalitat d'estudiar aquestes varietats i les seves singularitats, una pràctica comuna és construir invariants algebraics per a quantificar com de singular és la varietat. Aquest és precisament un dels objectius de la teoria de Bernstein-Sato. Durant els últims vint anys, la teoria en característica $p > 0$ ha vist un desenvolupament i creixement sense parangó. En aquest projecte calculem invariants de la teoria per ideals determinantals, és a dir, ideals generats pels determinants de submatrius de matrius genèriques d'indeterminades.

Paraules clau: àlgebra commutativa, geometria algebraica, característica $p > 0$, teoria de singularitats, teoria de Bernstein-Sato, ideals determinantals.

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Chapter 1
Introduction

Among the open questions that motivate and drive mathematics, one finds classification problems, and algebra and geometry are no strangers to this.

Fix an algebraically closed field k and let $R = k[x_1, \dots, x_n]$ be a polynomial ring over k. An algebraic variety over k is set of points given as the solution to a set of polynomials $f_1, \dots, f_r \in R$. In this context, the ultimate goal is to classify, up to isomorphism, all the algebraic varieties.

A first distinction that can be made is to discern between non-singular varieties and singular varieties. Roughly speaking, a non-singular algebraic variety does not change its direction suddenly and does not intersect itself, whereas singular algebraic varieties do, therefore they are said to have singularities. At the computational level, if a variety X is determined by polynomials $f_1, \dots, f_r \in R$, then X is said to have a singularity at a point $x \in k^n$ if the differential of the function $(f_1, \dots, f_r)\colon k^n \to k^r$ does not have maximal rank at x. In order to classify singular varieties, one may first consider their singularities and, if they "look" different, then the varieties cannot be isomorphic. In consequence, the problem of classifying varieties has now become the problem of classifying singularities of algebraic varieties. To do so, one constructs algebraic invariants from singularities and compares them.

For the moment, let $k = \mathbb{C}$ be the field of complex numbers and set $R = \mathbb{C}[x_1, \dots, x_n]$. The first way to quantify singularities is the multiplicity or order of vanishing, defined as follows: let $f \in R$ be a polynomial and $x \in \mathbb{C}^n$ a point where $f(x) = 0$. The multiplicity of f at x is the least integer $d \geq 0$ such that $(\partial f)(x) = 0$ for all differential operators ∂ of order $< d$. Although useful at first, the multiplicity is too coarse to measure singularities. Instead, a well-known measure of singularity is the following:

Definition 1.1. The *log-canonical threshold* or *complex singularity exponent* of a polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$ at a point $x \in \mathbb{C}^n$ where $f(x) = 0$ is

$$\operatorname{lct}_x(f) := \sup \left\{ \lambda \in \mathbb{R}_{>0} \;\middle|\; \int_B \frac{1}{|f|^{2\lambda}} < \infty \text{ for some neighborhood } B \text{ of } x \right\}.$$

Example 1.2. The polynomials x^2 and $y^2 + x^3$ in the ring $\mathbb{C}[x, y]$ both have singularities of multiplicity 2 at the origin. In contrast, the log-canonical thresholds are 1 and 5/6, respectively.

In this setting, one can show that $0 < \operatorname{lct}_x(f) < 1$. The example above suggests that the log-canonical threshold is capable of distinguishing between singularities.

Several other algebraic invariants may be constructed from singularities. Denote by $\mathcal{D}_{R|\mathbb{C}}$ the set of $\mathbb{C}$-linear differential operators on $R = \mathbb{C}[x_1, \dots, x_n]$, which has a non-commutative ring structure. Given a polynomial $f \in R$, Bernstein in the context of zeta functions [B72], and Sato in the context of prehomogeneous vector spaces [SS90], discovered the following fact: there exists a differential operator $P(s) \in \mathcal{D}_{R|\mathbb{C}}[s]$ and a polynomial $b(s) \in \mathbb{C}[s]$ such that $P(s) \cdot f^{s+1} = b(s) f^s$ for all integers $s \geq 0$. This motivates the following:

Definition 1.3. The *Bernstein-Sato polynomial* of $f \in \mathbb{C}[x_1, \dots, x_n]$, denoted by $b_f(s) \in \mathbb{C}[s]$, is the minimal monic generator of the ideal generated by the polynomials $b(s) \in \mathbb{C}[s]$ satisfying equations as the one above.

Since its inception, a wealth of connections have been discovered between the log-canonical threshold, the Bernstein-Sato polynomial and other invariants, giving rise to Bernstein-Sato theory. Furthermore, in [BMS06], Budur, Mustaţă and Saito generalized the constructions described so far to non-principal ideals in the polynomial ring $\mathbb{C}[x_1, \dots, x_n]$.

An analogous theory of singularities can be developed when R is a ring of characteristic $p > 0$, that is, R contains the finite field $\mathbb{F}_p$. Every such ring is equipped with the *Frobenius endomorphism* $F\colon R \to R$, which sends $f \mapsto f^p$. In this context, one often starts by asking the ambient space R to have no singularities. This is given by the following celebrated result:

Theorem 1.4 (Kunz's theorem, [K69, Theorem 2.1])**.** A ring R of characteristic $p > 0$ is regular if and only if the Frobenius endomorphism $F\colon R \to R$ is flat.

For a down-to-earth outline of the theory, let $R = \mathbb{F}[x_1, \dots, x_n]$ be a polynomial ring over a field $\mathbb{F}$ of characteristic $p > 0$ and fix a homogeneous polynomial f, i.e. f lies in the homogeneous maximal ideal $\mathfrak{m} = (x_1, \dots, x_n)$ of R. For each positive integer $e \geq 0$, let $\mathfrak{m}^{[p^e]} := (x_1^{p^e}, \dots, x_n^{p^e})$. To define how singular is f at the origin, one needs a way around analysis techniques, since these do not work in characteristic $p > 0$. To begin with, note that the ideal $\mathfrak{m}$ corresponds to the origin in affine n-space over $\mathbb{F}$. The naive approach, which consists in computing how long it takes to the function $1/f^n$ to blow up at the origin, turns out the be right one:

Definition 1.5. The *F-pure threshold* of a homogeneous polynomial $f \in \mathbb{F}[x_1, \dots, x_n]$ at the maximal ideal $\mathfrak{m}$ (i.e. at the origin) is

$$\operatorname{fpt}_0(f) := \sup \left\{ \frac{n}{p^e} \;\middle|\; f^n \notin \mathfrak{m}^{[p^e]},\ \text{for } e, n \in \mathbb{Z}_{\geq 0} \right\}.$$

Via a change of coordinates, the F-pure threshold can be defined at any point where f vanishes. When the point where the F-pure threshold is not mentioned, it is understood to be the origin.

Example 1.6. The F-pure threshold of $f = x^2 + y^3 \in \mathbb{F}_p[x, y]$ at the origin is:

$$\operatorname{fpt}(f) := \begin{cases} 1/2 & \text{if } p = 2, \\ 2/3 & \text{if } p = 3, \\ 5/6 & \text{if } p \equiv 1 \pmod 6, \\ 5/6 - \dfrac{1}{6p} & \text{if } p \equiv 5 \pmod 6. \end{cases}$$

Observe that as the characteristic $p > 0$ grows, the F-pure threshold approaches the log-canonical threshold of the "same" polynomial.

Similarly, differential operators are extremely valuable for the study of singularities in positive characteristic. Furthermore, one can construct the b-function of a polynomial $f \in R$, which is the analogue of the "Bernstein-Sato polynomial". In this case, however, the b-function is an ideal in the algebra $C(\mathbb{Z}_p, \mathbb{F}_p)$ of continuous functions $\mathbb{Z}_p \to \mathbb{F}_p$; its "zeros" are the *Bernstein-Sato roots*, which are p-adic integers $\alpha \in \mathbb{Z}_p$. As before, the constructions described so far for polynomials can be generalized to non-principal ideals $\mathfrak{a}$ in a polynomial ring.

Surprisingly enough, Bernstein-Sato theories in characteristic zero and in positive characteristic are closely related. Let $A = \mathbb{Z}[a^{-1}]$, where $a > 0$ is an integer, and fix a non-zero ideal $\mathfrak{a} \subseteq A[x_1, \ldots, x_n]$. Denote by $\mathfrak{a}_{\mathbb{C}}$ the expansion of $\mathfrak{a}$ to $\mathbb{C}[x_1, \ldots, x_n]$, and by $\mathfrak{a}_p$ the reduction modulo p to $\mathbb{F}_p[x_1, \ldots, x_n]$, where p does not divide a. In this setting, one has the following beautiful theorems:

Theorem 1.7 ([MTW05, Theorem 3.4])**.** One has that

$$\operatorname{lct}(\mathfrak{a}_{\mathbb{C}}) = \lim_{p \to \infty} \operatorname{fpt}(\mathfrak{a}).$$

Theorem 1.8 ([QG21b, Theorem VI.3])**.** Suppose that $\alpha \in \mathbb{Q}$ is a Bernstein-Sato root of $\mathfrak{a}_p$ for infinitely many $p > 0$. Then α is a root of the Bernstein-Sato polynomial $b_{\mathfrak{a}}(s) \in \mathbb{C}[s]$ of $\mathfrak{a}_{\mathbb{C}}$.

Tempting as it may be to study arbitrary algebraic varieties or an arbitrary ideals, it is often advantageous to restrict one's attention to a particular class of ideals. In our case, we are primarily interested in determinantal ideals:

Definition 1.9. Let $X = (x_{ij})$ be a generic matrix of indeterminates of size $m \times n$, with $m \leq n$, and consider the polynomial ring $R = k[X] := k[x_{11}, \ldots, x_{1n}, \ldots, x_{m1}, \ldots, x_{mn}]$ over a field k.

(i) Given an integer $1 \leq t \leq m$, the *ideal I_t of t-minors* of X is the ideal of R generated by the determinants of all the $t \times t$ submatrices of X.

(ii) When $t = m$, I_m is the *ideal of maximal minors.*

In the positive characteristic case, the F-pure threshold of I_t was computed by Miller, Singh and Varbaro in [MSV14]:

Theorem 1.10 ([MSV14, Theorem 1.2])**.** Let $k = \mathbb{F}$ be a field of characteristic $p > 0$ and $R = \mathbb{F}[X]$ a polynomial ring. Then the F-pure threshold of the ideal I_t of t-minors is

$$\operatorname{fpt}(I_t) = \min\left\{ \frac{(m-k)(n-k)}{t-k} \;\middle|\; k = 0, 1, \ldots, t-1 \right\}.$$

Note that $\operatorname{fpt}(I_t)$ does not depend on p. On the other hand, when $k = \mathbb{C}$, Lőrincz, Raicu, Walther and Weyman computed the Bernstein-Sato polynomial in [LRWW17]:

Theorem 1.11 ([LRWW17, Theorem 4.1])**.** Let $k = \mathbb{C}$ and $R = \mathbb{C}[X]$ a polynomial ring. The Bernstein-Sato polynomial of the ideal of maximal minors is

$$b_{I_m}(s) = \prod_{i=n-m+1}^{n} (s+i).$$

The goal of this thesis is to compute algebraic invariants relevant to Bernstein-Sato theory in positive characteristic for determinantal ideals, namely, Frobenius roots of powers of the ideal, ν-invariants and Bernstein-Sato roots. In particular, we aim to give an analogous result to Theorem 1.11 in positive characteristic:

Theorem 1.12 (Consequence of Theorem 5.47)**.** Let $X = (x_{ij})$ be a matrix of indeterminates of size $m \times n$, $m \leq n$. Fix an integer $e \geq 0$ and let I_m be the ideal of maximal minors of X in the polynomial ring $R = \mathbb{F}[X]$. The only Bernstein-Sato root of I_m is $\alpha = -(n-m+1)$.

A by-product of the previous theorem is the computation of Bernstein-Sato roots for a large class of polynomials, to which we refer as determinantal-type polynomials:

Definition 1.13. Let $R = B[x_1, \ldots, x_n]$ be a polynomial ring over a commutative ring B.

(1) A *square-free monomial* is non-trivial monomial $x_1^{a_1} \cdots x_n^{a_n} \in R$, i.e. not a unit of R, such that $0 \leq a_1, \ldots, a_n \leq 1$.

(2) A *determinantal-type polynomial* is a non-zero polynomial whose monomials are square-free.

In this context, we prove the following result:

Theorem 1.14 (Consequence of Theorem 5.58)**.** Let $R = \mathbb{F}[x_1, \ldots, x_n]$ be a polynomial ring over a perfect field $\mathbb{F}$ of characteristic $p > 0$ and let $f \in R$ be a determinantal-type polynomial. The only Bernstein-Sato root of f is $\alpha = -1$.

This thesis is structured as follows:

- Chapter 2 collects background, namely, commutative algebra in characteristic $p > 0$.
- Chapter 3 deals with the construction of differential operators, as well as with the peculiarities in characteristic $p > 0$.
- Chapter 4 introduces the basics of Bernstein-Sato theory in characteristic zero and characteristic $p > 0$, as well as the construction of invariants key to us.

- Chapter 5 is devoted to determinantal ideals and the computations of algebraic invariants for the case of maximal minors.
- Appendix A contains algorithms written in Python and Macaulay2, which have proven useful for the development of this project.

Throughout the text, we assume that all the rings are unitary and, unless otherwise stated, are commutative. Given any ideal $\mathfrak{a}$ in a ring R, we take $\mathfrak{a}^0 = R$.

Chapter 2
Commutative Algebra in Prime Characteristic

Commutative algebra in prime characteristic has some peculiarities that make the theory richer. In this chapter, we begin by discussing the Frobenius endomorphism, which is exclusive to the characteristic $p > 0$ world, and allows one to construct subrings of p-th powers and overrings of p-th roots. Afterwards, we study F-finite rings and polynomial rings over perfect fields of prime characteristic. These include some finiteness conditions that allow for a deeper study of rings and its modules.

2.1. The Frobenius endomorphism

For each prime number $p \in \mathbb{Z}$, the quotient ring $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$ has a natural field structure. It is the *finite field in p elements* and is unique up to isomorphism.

Definition 2.1. A commutative ring R has *characteristic* $p > 0$ if it contains the field $\mathbb{F}_p$, that is to say, R is an $\mathbb{F}_p$-algebra.

We will use the terms "R has characteristic $p > 0$", "R has prime characteristic" and "R has positive characteristic" interchangeably. Denote by $\mathbb{Z}$ the ring of integers with the usual addition and multiplication. For each commutative ring R, there exists a unique ring homomorphism $\mathbb{Z} \to R$ sending $1 \mapsto 1$, that is, $\mathbb{Z}$ is an initial object in the category **CRing** of commutative rings. In turn, this induces a unique $\mathbb{Z}$-algebra structure on R given by $n \cdot x = \sum_{i=1}^{n} x$, with $n \in \mathbb{Z}$ and $x \in R$. This allows one to give an characterize rings having prime characteristic:

Proposition 2.2. Let R be a non-zero commutative ring. Then R has characteristic p if and only if $\ker(\mathbb{Z} \to R) = p\mathbb{Z}$.

Proof. Let $f\colon \mathbb{Z} \to R$ be the unique ring homomorphism from $\mathbb{Z}$ to R. If R has characteristic p, then $p\mathbb{Z} \subseteq \ker f$ since $p \mapsto 0$. As the ideal $p\mathbb{Z} \subseteq \mathbb{Z}$ is maximal and R is non-zero, it follows that $\ker f = p\mathbb{Z}$. Conversely, f defines a surjection $f\colon \mathbb{Z} \to f(\mathbb{Z}) \subseteq R$ and by assumption, $f(\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p$, therefore R contains $\mathbb{F}_p$. □

Every ring of prime characteristic comes equipped with a p-th power map, the Frobenius endomorphism, whose study is essential for singularities.

Definition 2.3. Let R be a ring of characteristic $p > 0$. The *Frobenius endomorphism* $F\colon R \to R$ is the map defined by $r \mapsto r^p$. For each integer $e \geq 0$, the e-th iterated Frobenius is the map $F^e\colon R \to R$ given by $r \mapsto r^{p^e}$. By convention, the 0-th iterated Frobenius is the identity.

In order for the definition to make sense, one needs to prove that the Frobenius endomorphism is, in fact, a ring endomorphism.

Lemma 2.4. If $p \in \mathbb{Z}$ is prime, then for each $1 \leq k \leq p-1$,

$$\binom{p}{k} \equiv 0 \pmod{p}.$$

Proof. The expansion of the binomial coefficient gives

$$\binom{p}{k} = \frac{p(p-1)\cdots(p-k+1)}{k!}.$$

Since p is prime and $1 \leq k \leq p-1$, p does not appear in the prime factorization of $k!$, thereby p divides the binomial coefficient. □

Proposition 2.5. Let R be a ring of characteristic $p > 0$. For each integer $e \geq 0$, the e-th iterated Frobenius $F^e\colon R \to R$ is a ring homomorphism. In particular, for each $x, y \in R$,

$$(x+y)^{p^e} = x^{p^e} + y^{p^e}.$$

Proof. As the composition of ring homomorphisms is again a ring homomorphism, it suffices to prove that $F\colon R \to R$ is, in fact, a homomorphism. It is clear that $F(1) = 1$ and $F(xy) = x^p y^p = F(x)F(y)$ for all $x, y \in R$. Furthermore, since $\binom{p}{0} = \binom{p}{p} = 1$, it follows from Lemma 2.4 that

$$F(x+y) = (x+y)^p = \sum_{k=0}^{p} \binom{p}{k} x^k y^{p-k} = x^p + y^p = F(x) + F(y).$$ □

Proposition 2.6. For each integer $e \geq 1$, the Frobenius endomorphism $F^e\colon \mathbb{F}_{p^e} \to \mathbb{F}_{p^e}$ is the identity of $\mathbb{F}_{p^e}$. More generally, if $d \geq 1$ divides e, then $F^e\colon \mathbb{F}_{p^d} \to \mathbb{F}_{p^d}$ is the identity of $\mathbb{F}_{p^d}$.

Proof. The multiplicative group $\mathbb{F}_{p^e}^\times$ of the finite field $\mathbb{F}_{p^e}$ has order $p^e - 1$. Since the order of an element $a \in \mathbb{F}_{p^e}^\times$ divides the order of the group, it follows that $a^{p^e-1} = 1$, thus $F^e(a) = a^{p^e} = a$. Moreover $F^e(0) = 0$, hence F^e leaves $\mathbb{F}_{p^e}$ fixed. The second statement follows from iterating e/d times the endomorphism F^d. □

Every commutative ring R is naturally an R-module over itself. However, in the positive characteristic setting, one can restrict scalars along the Frobenius endomorphism in order to define an alternative R-module structure on an R-module:

Definition 2.7. Let R be a ring of characteristic $p > 0$, M an R-module and $e \geq 0$ an integer. Define the R-module $F^e_* M$ as follows: its elements are denoted by $F^e_* m$, where $m \in M$, and $F^e_* M$ is isomorphic to M as an abelian group with respect to addition, that is

$$F^e_* m + F^e_* n \coloneqq F^e_*(m + n)$$

for $F^e_* m, F^e_* n \in F^e_* M$. The action of R on $F^e_* M$ is given via restriction of scalars along the e-th iteration of the Frobenius endomorphism, i.e.

$$r \cdot F^e_* m \coloneqq F^e_* r^{p^e} m$$

for $r \in R$ and $F^e_* m \in F^e_* M$.

The elements of the R-modules M and $F^e_* M$ are "the same", but the R-module structures are different. This construction yields a covariant additive functor $F^e_* \colon \mathbf{Mod(R)} \to \mathbf{Mod(R)}$ in the category of modules over a commutative ring R, sending a module $M \mapsto F^e_* M$, and an R-module homomorphism $\varphi \in \operatorname{Hom}(M, N)$ to the R-module homomorphism $F^e_* \varphi \in \operatorname{Hom}(F^e_* M, F^e_* N)$ defined by

$$(F^e_* \varphi)(F^e_* m) \coloneqq F^e_* \varphi(m),$$

where $F^e_* m \in F^e_* M$. We shall be mostly interested in the case when M is a ring.

Definition 2.8. Let R be a ring of characteristic $p > 0$ and $e \geq 0$ an integer. We define the ring $F^e_* R$ as follows: its elements are denoted by $F^e_* x$ for $x \in R$, and $F^e_* R$ is isomorphic to R as a ring, that is

$$F^e_* x + F^e_* y \coloneqq F^e_*(x + y),$$
$$F^e_* x \cdot F^e_* y \coloneqq F^e_*(xy),$$

where $F^e_* x, F^e_* y \in F^e_* R$. The action of R on $F^e_* R$ is given via restriction of scalars along the e-th iterated Frobenius, i.e.

$$r \cdot F^e_* x \coloneqq F^e_* r^{p^e} x$$

for $r \in R$ and $F^e_* x \in F^e_* R$.

It is clear from the definition that $F^e_* R$ is a ring of characteristic p, although we will be primarily interested in its R-module structure. As before, the elements of the rings R and $F^e_* R$ are "the same", but the R-module structures are different. Note that in the case $e = 0$, one has $F^0_* M \cong M$ and $F^0_* R \cong R$.

2.2. Rings of p^e-th powers and rings of p^e-th roots

Definition 2.9. Let R be a ring of characteristic $p > 0$ and let $e \geq 0$ be an integer. The *ring of p^e-th powers* of R is

$$R^{p^e} \coloneqq F^e(R) = \{x^{p^e} \mid x \in R\}.$$

By convention, $R^{p^0} = R$. Note that each R^{p^e} is a subring of R. In addition, for every $e \geq 0$ one has that $R^{p^{e+1}}$ is a subring of R^{p^e} since $F(R^{p^e}) = F^{e+1}(R) = R^{p^{e+1}}$. This results in the following chain of rings:

$$\cdots \subseteq R^{p^e} \subseteq \cdots \subseteq R^{p^2} \subseteq R^p \subseteq R.$$

Example 2.10. Let $R = \mathbb{F}_p[x]$ be a polynomial ring. By Proposition 2.6, F^e leaves $\mathbb{F}_p$ fixed and sends $x \mapsto x^{p^e}$, whence $R^{p^e} = \mathbb{F}_p[x^{p^e}]$. More generally, if $R = \mathbb{F}_p[x_1, \ldots, x_n]$ is a polynomial ring in n variables, then $R^{p^e} = \mathbb{F}_p[x_1^{p^e}, \ldots, x_n^{p^e}]$, and the descending chain of rings reads

$$\cdots \subseteq \mathbb{F}_p[x_1^{p^e}, \ldots, x_n^{p^e}] \subseteq \cdots \subseteq \mathbb{F}_p[x_1^p, \ldots, x_n^p] \subseteq \mathbb{F}_p[x_1, \ldots, x_n].$$

Proposition 2.11. Let R be a reduced ring (i.e. with no nilpotents) of characteristic $p > 0$. Then for each $e \geq 0$, the e-th iterated Frobenius defines an isomorphism $F^e \colon R \to R^{p^e}$.

Proof. It suffices to prove that for every $e \geq 0$, $F \colon R^{p^e} \to R^{p^{e+1}}$ is an isomorphism of rings. If $x^{p^e} \in R^{p^e}$ is such that $x^{p^{e+1}} = 0$, it follows that $x = 0$ because R is reduced, thus F is injective. An element $x^{p^{e+1}} \in R^{p^{e+1}}$ is the image of $x^{p^e} \in R^{p^e}$ under F, hence F is surjective and an isomorphism of rings. □

Observe that the ring of powers R^{p^e} may be used to give an alternative definition of the R-module $F^e_* M$. Indeed, given $r \in R$ and $m \in M$, one has that

$$r \cdot F^e_* m = F^e_* r^{p^e} m \text{ in } F^e_* M \iff r^{p^e} \cdot m = r^{p^e} m \text{ in } M,$$

therefore viewing $F^e_* M$ an R-module is equivalent to view M as an R^{p^e}-module. This characterization will become useful later on when discussing F-finite rings.

From the point of view of R^{p^e}, the ring R can be interpreted as the ring of p^e-th roots. Indeed, since the p^e-th root of $x^{p^e} \in R^{p^e}$ is the element $x \in R$. A natural question arises: can one find p^e-th roots for elements in R? The answer is yes, although it requires more work.

Recall that given a ring R, the total ring of fractions of R, denoted by $K(R)$, is the localization $U^{-1}R$ where $U \subseteq R$ is the set of non-zero-divisors of R. It is the "biggest" localization such that the localization map $R \to U^{-1}R$ is injective.

Theorem 2.12. Let R be a reduced Noetherian ring. Then the total ring of fractions K is isomorphic to a finite product of fields.

Proof. As R is reduced, the nilradical of R coincides with the zero ideal $(0) \subseteq R$, thus it may be expressed as the intersection of all the primes of R. However, note that in such intersection only the minimal primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ of R are required, thus $(0) = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_n$, and this is a minimal primary decomposition. Furthermore, these are the associated primes of (0), thus $U = R - (\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_n)$ is the set of non-zero-divisors of R. It follows that in the localization

$K(R) = U^{-1}R$, the only prime ideals are $U^{-1}\mathfrak{p}_i$, which are different and maximal. By the Chinese Remainder Theorem,

$$K(R) = U^{-1}R = \frac{U^{-1}R}{(0)} = \frac{U^{-1}R}{\bigcap_{i=1}^{n} U^{-1}\mathfrak{p}_i} \cong \prod_{i=1}^{n} \frac{U^{-1}R}{U^{-1}\mathfrak{p}_i},$$

where each $(U^{-1}R)/(U^{-1}\mathfrak{p}_i)$ is a field, from where the result follows. □

Let R be a reduced Noetherian ring and write $K := K(R) = \prod_{i=1}^{n} \mathbb{K}_i$. For every field $\mathbb{K}_i$ fix an algebraic closure $\overline{\mathbb{K}}_i$ and let $\overline{K} = \prod_{i=1}^{n} \overline{\mathbb{K}}_i$. Note that $\overline{K}$ is not a domain, as the direct product of domains is never a domain. Nonetheless, there is an inclusion morphism $R \hookrightarrow K \hookrightarrow \overline{K}$, thus each $x \in R$ may be seen as $(x_1, \dots, x_n) \in \overline{K}$. For each coordinate $x_i \in \overline{\mathbb{K}}_i$ choose a p^e-th root x_i^{1/p^e} and define the p^e-th root of x as

$$x^{1/p^e} := (x_1^{1/p^e}, \dots, x_n^{1/p^e}).$$

Definition 2.13. Let R be a reduced Noetherian ring of characteristic $p > 0$ and let $e \geq 0$ be an integer. The *ring of p^e-th roots* of R is

$$R^{1/p^e} := \{x \in \overline{K} \mid x^{p^e} \in R\}.$$

By convention, we set $R^{1/p^0} = R$.

The construction of the ring of p^e-th of R has a several hiccups: to begin with, the primary decomposition used in Theorem 2.12 is not unique in general, nor is the product of fields in which the total ring of fractions of R decomposes. Furthermore, $\overline{K}$ depends on the choice of algebraic closures, although any two algebraic closures of the same field are isomorphic. These setbacks are fixed by the following proposition:

Proposition 2.14. Let R be a reduced Noetherian ring of characteristic $p > 0$ and $e \geq 0$ an integer. Then the ring R^{1/p^e} of p^e-th roots is a ring and $R \subseteq R^{1/p^e}$. Furthermore, the e-th iterated Frobenius defines an isomorphism $F^e \colon R^{1/p^e} \to R$.

Proof. For each $x \in R$ we have that $x^{p^e} \in R$, hence $R \subseteq R^{1/p^e}$, thus R^{1/p^e} has characteristic $p > 0$. For each pair of elements $\alpha, \beta \in R^{1/p^e}$, $(\alpha - \beta)^{p^e} = \alpha^{p^e} + (-1)^{p^e} \beta^{p^e} \in R$, thus proving that R^{1/p^e} is closed under addition. Furthermore, both addition and multiplication are commutative, and the distributive property holds in R^{1/p^e} because it is a subset of $\overline{K}$, therefore it is a ring.

It is clear that the image of R^{p^e} under F^e is R by construction, hence $F^e \colon R^{1/p^e} \to R$ is a well-defined surjective ring homomorphism. As R is reduced, by Theorem 2.12 the total ring of fractions K of R can be written as $K = \prod_{i=1}^{n} \mathbb{K}_i$, where the $\mathbb{K}_i$ are fields. As before, let $\overline{K} = \prod_{i=1}^{n} \overline{\mathbb{K}}_i$. If $F^e(\alpha) = 0$ for some $\alpha \in R^{1/p^e}$, it follows that $\alpha_i = 0 \in \overline{\mathbb{K}}_i$ for all $i = 1, \dots, n$, whence $\alpha = 0$. This shows that F^e is injective and thus gives an isomorphism $R \cong R^{1/p^e}$. □

It readily follows from Proposition 2.14 that any two rings S and T of p^e-th roots of R are isomorphic, hence the ring of R^{1/p^e} is well-defined up to isomorphism. Furthermore, one readily verifies that $R^{1/p^e} \subseteq R^{1/p^{e+1}}$ for each integer $e \geq 0$. In consequence, a reduced Noetherian ring R of characteristic $p > 0$ gives rise to a chain of rings

$$\cdots \subseteq R^{p^e} \subseteq \cdots \subseteq R^p \subseteq R \subseteq R^{1/p} \subseteq \cdots \subseteq R^{1/p^e} \subseteq \cdots ,$$

which are pairwise isomorphic via an iteration of the Frobenius endomorphism. Moreover, as the following proposition shows, there exists a canonical isomorphism between $F^e_* R$ and R^{1/p^e}.

Proposition 2.15. Let R be a reduced Noetherian ring of characteristic $p > 0$ and $e \geq 0$ an integer. Then the map

$$\begin{aligned} \varphi\colon R^{1/p^e} &\longrightarrow F^e_* R \\ x^{1/p^e} &\longmapsto F^e_* x \end{aligned}$$

is a ring isomorphism.

Proof. The map is well defined since $x = (x^{1/p^e})^{p^e} \in R$, and clearly gives a ring homomorphism. Surjectivity of φ is due to the fact that each $x \in R$ has a p^e-th root in R^{1/p^e}, and injectivity follows from R being reduced. As a result, φ is a ring isomorphism. □

2.3. F-finite rings

Definition 2.16. A Noetherian ring R of characteristic $p > 0$ is an *F-finite ring* if $F_* R$ is a finitely generated R-module.

As previously noted, viewing $F_* R$ as an R-module is equivalent to viewing R as an R^p-module. Since the image of the Frobenius endomorphism $F\colon R \to R$ is R^p, the fact that $F_* R$ is a finitely generated R-module implies that the Frobenius endomorphism is finite. This allows us to characterize F-finite fields:

Example 2.17. Let $\mathbb{K}$ be a field of characteristic $p > 0$. By Proposition 2.11, the Frobenius map defines an isomorphism $\mathbb{K} \cong \mathbb{K}^p$, hence there is an extension of fields $\mathbb{K}^p \subseteq \mathbb{K}$.

Saying that $\mathbb{K}$ is F-finite is equivalent to $[\mathbb{K} : \mathbb{K}^p] < \infty$. Indeed, if we assume F-finiteness then $\mathbb{K}$ is finitely generated over $\mathbb{K}^p$, thus $\mathbb{K} \cong (\mathbb{K}^p)^{\oplus n}/N$ for some integer $n \geq 1$ and submodule $N \subseteq (\mathbb{K}^p)^{\oplus n}$. As a result, the dimension of $\mathbb{K}$ as a $\mathbb{K}^p$-vector space is at most n. The converse is immediate.

In the definition of F-finite ring, one only asks $F_* R$ to be finitely generated over R. This is equivalent, however, to $F^e_* R$ to be finitely generated for some (equivalently, for all) $e \geq 1$.

Proposition 2.18. Let R be a Noetherian ring of characteristic $p > 0$. The following are equivalent:

(1) R is F-finite.
(2) F_*R is a finitely generated R-module.
(3) F^e_*R is a finitely generated R-module for all $e \geq 1$.
(4) F^e_*R is a finitely generated R-module for some $e \geq 1$.

Proof. (1) and (2) are equivalent by definition of F-finite ring, and (3) clearly implies (4).

(2) $\Rightarrow$ (3). By assumption the Frobenius endomorphism $F\colon R \to R$ is finite. Since the composition of finite morphisms is finite, one has that $F^e\colon R \to R$ is finite for all $e \geq 1$, thereby R is a finitely generated R^{p^e}-module, i.e. F^e_*R is a finitely generated R-module.

(4) $\Rightarrow$ (2). Suppose that F^e_*R is a finitely generated R-module, that is R is a finitely generated module over R^{p^e}, and let $\{x_1, \ldots, x_n\} \subset R$ be a set of generators. Then for each $a \in R$ we have $a = a_1^{p^e} x_1 + \cdots + a_n^{p^e} x_n$ for some $a_1, \ldots, a_n \in R$. Define $\alpha_i = a_i^{p^{e-1}} \in R^{p^{e-1}} \subset R$ so that $a = \alpha_1^p x_1 + \cdots + \alpha_n^p x_n$. From the point of view of F_*R, we have

$$F_*a = F_*(\alpha_1^p x_1 + \cdots + \alpha_n^p x_n) = \alpha_1 F_*x_1 + \cdots + \alpha_n F_*x_n,$$

thus F_*R is a finitely generated R-module. □

In the following propositions we present a few examples of F-finite rings, namely, that if R is F-finite, then quotients and localizations of R, as well as polynomial and power series rings over R, are F-finite. This may be summarized by saying that doing algebraic geometry preserves F-finiteness.

Proposition 2.19. Let R be an F-finite ring of characteristic $p > 0$ and $I \subseteq R$ a proper ideal. Then the quotient ring R/I is F-finite.

Proof. The quotient ring R/I is Noetherian because R is Noetherian. Furthermore, $I \cap \mathbb{F}_p = \varnothing$ because I is a proper ideal, hence the quotient ring R/I contains $\mathbb{F}_p$.

Let $F^e_*x_1, \ldots, F^e_*x_n$ be a generating set for F^e_*R over R. We claim that the set $F^e_*\overline{x}_1, \ldots, F^e_*\overline{x}_n$ generates $F^e_*(R/I)$ as an (R/I)-module. Indeed, given $f \in R$ there exist $f_1, \ldots, f_n \in R$ such that $f = \sum_i f_i^{p^e} x_i$. Projecting to the quotient ring gives $\overline{f} = \sum_i \overline{f_i}^{p^e} \overline{x}_i$, hence

$$F^e_*\overline{f} = \sum_i \overline{f}_i F^e_*\overline{x}_i,$$

which shows that $F^e_*(R/I)$ is finitely generated over R/I. □

Proposition 2.20. Let R be an F-finite ring and $W \subseteq R$ a multiplicative subset not containing zero. Then the localization $W^{-1}R$ is F-finite.

Proof. A localization of a Noetherian ring is Noetherian. Since $0 \notin W$, it follows that $W^{-1}R$ has characteristic $p > 0$. Indeed, we have a composition of morphisms $\mathbb{F}_p \hookrightarrow R \to W^{-1}R$ which can be either the zero morphism or injective. In the former case, this would imply that

$W^{-1}R = 0$, but this is a contradiction since the multiplicative subset W does not contain zero. Therefore the composition is injective, which shows that $W^{-1}R$ contains $\mathbb{F}_p$.

Suppose that the set $F_*^e x_1, \dots, F_*^e x_n$ generates $F_*^e R$ as an R-module. We claim that the same set generates $F_*^e(W^{-1}R)$ over $W^{-1}R$. Indeed, pick elements $f \in R$ and $u \in W$. Then there exist $g_1, \dots, g_n \in R$ such that $u^{p^e-1}f = \sum_i g_i^{p^e} x_i$, thus

$$\frac{f}{u} = \frac{1}{u^{p^e}} u^{p^e-1} f = \sum_i \frac{g_i^{p^e}}{u^{p^e}} x_i,$$

and

$$F_*^e \frac{f}{u} = \sum_i \frac{g_i}{u} F_*^e x_i,$$

thus proving that $F_*^e(W^{-1}R)$ is a finitely generated $(W^{-1}R)$-module. □

Proposition 2.21. Let R be an F-finite ring of characteristic $p > 0$. Then a polynomial ring $R[x_1, \dots, x_n]$ in finitely many variables over R is F-finite.

Proof. The polynomial ring $R[x_1, \dots, x_n]$ is Noetherian by Hilbert's basis theorem, and it has characteristic $p > 0$ because the composition $\mathbb{F}_p \hookrightarrow R \hookrightarrow R[x_1, \dots, x_n]$ injects $\mathbb{F}_p$ in $R[x_1, \dots, x_n]$.

Let $F_*r_1, \dots, F_*r_m$ be a set of generators for F_*R as an R-module. In order to show that $F_*R[x_1, \dots, x_n]$ is finitely generated over $R[x_1, \dots, x_n]$, we proceed by induction on the number of variables. For $n = 1$, let $f = a_t x^t + \dots + a_1 x + a_0 \in R[x]$ and write $t = qp + s$, where $q \in \mathbb{N}$ and $0 \le s \le p-1$. Then f can be rewritten as

$$f = \sum_{i=0}^{t} a_i x^i = \sum_{j=0}^{p-1} \sum_{k=0}^{q} a_{kp+j} x^{kp+j} = \sum_{j=0}^{p-1} x^j \sum_{k=0}^{q} a_{kp+j} x^{kp},$$

where we set $a_{qp+j} = 0$ for $s+1 \le j \le p-1$. Since R is finitely generated over R^p, for each $a_{kp+j} \in R$ one can write $a_{kq+j} = \sum_{i=1}^m a_{ijk}^p r_i$ for some $a_{ijk} \in R$. As a result,

$$f = \sum_{j=0}^{p-1} x^j \sum_{k=0}^{q} \sum_{i=0}^{m} a_{ijk}^p r_i x^{kp} = \sum_{i=0}^{m} \sum_{j=0}^{p-1} \left(\sum_{k=0}^{q} a_{ijk} x^k \right)^p r_i x^j,$$

or equivalently,

$$F_*f = \sum_{i=0}^{m} \sum_{j=0}^{p-1} \left(\sum_{k=0}^{q} a_{ijk} x^k \right) F_* r_i x^j,$$

which proves that the set $\{F_* r_i x^j \mid 1 \le i \le m, 0 \le j < p\}$ generates $F_*R[x]$ as module over $R[x]$.

For the inductive step, let $S \coloneqq R[x_1, \dots, x_n]$ and assume that F_*S is a finitely generated S-module. Then $R[x_1, \dots, x_n, x_{n+1}] = S[x_{n+1}]$, and the base case shows that $F_*^e S[x_{n+1}]$ is finitely generated over $S[x_{n+1}]$. □

Proposition 2.22. Let R be an F-finite ring of characteristic $p > 0$. Then a power series ring $R[\![x_1, \dots, x_n]\!]$ in finitely many variables over R is F-finite.

Proof. Analogous to the proof of Proposition 2.21. □

Definition 2.23. A ring homomorphism $R \to S$ is *essentially of finite type* if S is the localization of an R-algebra of finite type.

Proposition 2.24. Let R be an F-finite ring of characteristic $p > 0$. Then any R-algebra S that is essentially of finite type over R is F-finite.

Proof. By assumption S is of the form $S = W^{-1}(R[x_1, \dots, x_n]/I)$, where $I \subseteq R[x_1, \dots, x_n]$ is an ideal and $W \subseteq R[x_1, \dots, x_n]/I$ is a multiplicative set. It follows from Propositions 2.19 to 2.21 that S is F-finite. □

2.4. Polynomial rings over perfect fields

Next we study a particularly well-behaved case of F-finite rings, namely, polynomial rings over perfect fields of positive characteristic. In this case, not only is $F^e_* R$ finitely generated over R, but also free. We begin recalling some facts about perfect fields.

Definition 2.25. A field $\mathbb{K}$ is *perfect* if every irreducible polynomial is separable, i.e. its roots are distinct in a fixed algebraic closure of $\mathbb{K}$.

Definition 2.26. Let $R[x]$ be a polynomial ring and $f = a_n x^n + \cdots + a_1 x + a_0 \in R[x]$ a polynomial. The *formal derivative* of f is defined to be the polynomial

$$f' = n a_n x^{n-1} + (n-1) a_{n-1} x^{n-1} + \cdots + 2a_2 x + a_1.$$

Lemma 2.27. Let $\mathbb{K}$ be a field and $f \in \mathbb{K}[x]$ an irreducible polynomial. Then f is separable if and only if f and f' are coprime.

Proof. If f is separable, it has distinct roots in an algebraic closure of $\mathbb{K}$, thus f and f' do not share common roots. Indeed, in a fixed algebraic closure $\overline{\mathbb{K}}$, f factors as a product of distinct linear factors $f = a\prod_i (x - \alpha_i)$, and its formal derivative reads

$$f' = a \sum_i \prod_{j \neq i} (x - \alpha_j).$$

Choose a root α_k of f. Note that all summands except $\prod_{j \neq k}(x - \alpha_j)$ have $x - \alpha_k$. Thereby, by evaluating f' at α_k, one gets $f'(\alpha_k) = a\prod_{j \neq k}(\alpha_k - \alpha_j)$, i.e. f and f' do not have common roots. As f is irreducible, the ideal (f) is maximal in $\mathbb{K}[x]$ and $f' \notin (f)$, hence $(f) + (f') = \mathbb{K}[x]$, thus f and f' are coprime.

Conversely, suppose that f and f' are coprime. Then there exist $\lambda, \mu \in \mathbb{K}[x]$ such that $\lambda f + \mu f' = 1$. If f was not separable, f and f' would share a common root $\alpha \in \overline{\mathbb{K}}$ in a fixed algebraic closure. Evaluating $\lambda f + \mu f'$ at α gives a contradiction, hence f is separable. □

Lemma 2.28. Let $\mathbb{K}$ be a field of characteristic $p > 0$ and $f \in \mathbb{K}[x]$ a non-constant polynomial. Then $f' = 0$ if and only if $f - \beta \in (x^p) \subseteq \mathbb{K}[x]$ for some $\beta \in \mathbb{K}$.

Proof. Write $f = a_n x^n + \cdots + a_1 x + a_0$ and suppose that

$$f' = \sum_{k=1}^{n} k a_k x^{k-1} = 0.$$

It follows that $a_k = 0$ when k is not a multiple of p, therefore $f - a_0 \in (x^p)$. The converse is clear. □

Theorem 2.29. Let $\mathbb{K}$ be a field of characteristic $p > 0$. Then $\mathbb{K}$ is perfect if and only if the Frobenius endomorphism $F\colon \mathbb{K} \to \mathbb{K}$ is an automorphism of $\mathbb{K}$.

Proof. Suppose that $\mathbb{K}$ is a perfect field. It is clear that $F\colon \mathbb{K} \to \mathbb{K}$ is injective, thus we only need to show that it is surjective, i.e. that $f = x^p - a \in \mathbb{K}[x]$ has a root in $\mathbb{K}$. Suppose it does not. If $\alpha \in \overline{\mathbb{K}}$ is a root of f in a fixed algebraic closure of $\mathbb{K}$, then $\alpha^p = a$, hence $f = (x - \alpha)^p$. Moreover f factors as a product of irreducibles since $\mathbb{K}[x]$ is a UFD. Let $g = (x - \alpha)^n$ be one of the irreducibles. Then

$$g = \sum_{k=0}^{n} \binom{n}{k} x^{n-k}(-\alpha)^k = x^n - n\alpha x^{n-1} + \cdots + (-\alpha)^{n-1}x + (-\alpha)^n,$$

which implies that $n\alpha \in \mathbb{K}$. By assumption $\alpha \notin \mathbb{K}$, so $n = p$. As a result $f = g$, thus f is irreducible but inseparable, contradicting the fact that $\mathbb{K}$ is perfect.

Conversely, suppose that F is an automorphism of $\mathbb{K}$ and that $\mathbb{K}$ is not perfect. Then there exists an irreducible polynomial f in $\mathbb{K}[x]$ with a multiple root α in an algebraic closure of $\mathbb{K}$. We may assume that f is the minimal polynomial of α, which is irreducible. By Lemma 2.27, the polynomials f and f' are not coprime, thus $f' \in (f)$, but by degree considerations one has $f' = 0$. It follows from Lemma 2.28 that $f = a_n x^{pn} + \cdots + a_1 x^p + a_0$. Furthermore each a_k has a p-th root in $\mathbb{K}$, thereby

$$f = (a_n^{1/p} x^n + \cdots + a_1^{1/p} x + a_0)^p,$$

but this contradicts the irreducibility of f. □

A major example of F-finite rings are perfect fields of positive characteristic. This is an immediate consequence of Theorem 2.29.

Example 2.30. If $\mathbb{K}$ is a perfect field of characteristic $p > 0$ and $e \geq 1$ is an integer, then $F^e_* \mathbb{K}$ is a $\mathbb{K}$-vector space of dimension 1. Indeed, each $\alpha \in \mathbb{K}$ has a p^e-th root $\alpha^{1/p^e} \in \mathbb{K}$.

As a result, $F^e_* \alpha = \alpha^{1/p^e} F^e_* 1 \in F^e_* \mathbb{K}$. This induces a field homomorphism $F^e_* \mathbb{K} \to \mathbb{K}$ sending $F^e_* \alpha \mapsto \alpha^{1/p^e}$, which is clearly injective. Furthermore, given $\beta \in \mathbb{K}$ one has that $F^e_* \beta^{p^e} \mapsto \beta$, thus the homomorphism is surjective. In consequence, we have an isomorphism $F^e_* \mathbb{K} \cong \mathbb{K} F^e_* 1 \cong \mathbb{K}$.

Corollary 2.31. Let $\mathbb{K}$ be a perfect field of characteristic $p > 0$. Then for all integers $e \geq 0$,

$$F^e_* \mathbb{K} \cong \mathbb{K}.$$

Proof. It is clear from the construction of $F^e_* \mathbb{K}$ that it is a field. By Theorem 2.29 every $\alpha \in \mathbb{K}$ has a p^e-th root, namely α^{1/p^e}. As a result, for each element in $F^e_* \mathbb{K}$ one can write $F^e_* \alpha = \alpha^{1/p^e} F^e_* 1$. This yields an injective map of fields $\varphi \colon F^e_* \mathbb{K} \to \mathbb{K}$ sending $F^e_* \alpha \mapsto \alpha^{1/p^e}$. Furthermore, given $\beta \in \mathbb{K}$ one has that $F^e_* \beta^{p^e} \mapsto \beta$, thus the map is surjective and an isomorphism. □

Proposition 2.32. Let $\{M_i\}_i$ be a collection of modules over an F-finite ring R of characteristic $p > 0$ and let $e \geq 1$ be an integer. Then there is an isomorphism

$$\bigoplus_i F^e_* M_i \cong F^e_* \bigoplus_i M_i.$$

Proof. The set $F^e_* \bigoplus_i M_i$ has a natural R-module structure, with action defined by $r F^e_*(m_i)_i \coloneqq F^e_*(r^{p^e} m_i)_i$, where $r \in R$ and $(m_i)_i \in \bigoplus_i M_i$. For each k define an R-module homomorphism $F^e_* M_k \to F^e_* \bigoplus_i M_i$ by sending $F^e_* m \mapsto F^e_*(m_i)_i$, where $m_i = m$ if $i = k$ and $m_i = 0$ otherwise. By the universal property of the direct sum, this induces a unique homomorphism

$$\begin{aligned} \bigoplus_i F^e_* M_i &\longrightarrow F^e_* \bigoplus_i M_i \\ (F^e_* m_i)_i &\longmapsto F^e_*(m_i)_i \end{aligned}$$

It follows at once that the map is injective, as $(F^e_* m_i)_i \mapsto 0$ if and only if $m_i = 0$ for all i. In addition, given $F^e_*(m_i)_i \in F^e_* \bigoplus_i M_i$, one has that $(F^e_* m_i)_i \mapsto F^e_*(m_i)_i$, thus it is surjective as well, and therefore an isomorphism. □

Proposition 2.33. Let $\mathbb{K}$ be a perfect field of characteristic $p > 0$, $R = \mathbb{K}[x_1, \ldots, x_n]$ a polynomial ring and $e \geq 1$ an integer. Then

$$F^e_* R \cong \bigoplus_{0 \leq i_1, \ldots, i_n < p^e} R\, F^e_* x_1^{i_1} \cdots x_n^{i_n},$$

thus $\{F^e_* x_1^{i_1} \cdots x_n^{i_n} \mid 0 \leq i_1, \ldots, i_n < p^e\}$ is a basis for $F^e_* R$ as an R-module.

Proof. The ring R can be viewed a $\mathbb{K}$-vector space, with direct sum decomposition

$$R \cong \bigoplus_{k_1, \ldots, k_n \geq 0} \mathbb{K}\, x_1^{k_1} \cdots x_n^{k_n} \cong \bigoplus_{0 \leq i_1, \ldots, i_n < p^e} \bigoplus_{j_1, \ldots, j_n \geq 0} \mathbb{K}\, x_1^{j_1 p^e + i_1} \cdots x_n^{j_n p^e + i_n},$$

where the second isomorphism follows from the fact that each integer $k \geq 0$ can be written uniquely as $k = jp^e + i$ with $j \geq 0$ and $0 \leq i < p^e$. By Example 2.30 one has that $\mathbb{K} \cong F^e_*\mathbb{K}$, and the functor F^e_* commutes with direct sums by Proposition 2.32, whence

$$\begin{aligned} F^e_*R &\cong F^e_* \bigoplus_{0\leq i_1,\ldots,i_n<p^e} \bigoplus_{j_1,\ldots,j_n\geq 0} \mathbb{K}\, x_1^{j_1p^e+i_n} \cdots x_n^{j_np^e+i_n} \\ &\cong \bigoplus_{0\leq i_1,\ldots,i_n<p^e} \bigoplus_{j_1,\ldots,j_n\geq 0} F^e_*(\mathbb{K}\, x_1^{j_1p^e+i_n} \cdots x_n^{j_np^e+i_n}) \\ &\cong \bigoplus_{0\leq i_1,\ldots,i_n<p^e} \bigoplus_{j_1,\ldots,j_n\geq 0} \mathbb{K}\, x_1^{j_1} \cdots x_n^{j_n}\, F^e_* x_1^{i_1} \cdots x_n^{i_n} \\ &\cong \bigoplus_{0\leq i_1,\ldots,i_n<p^e} R\, F^e_* x_1^{i_1} \cdots x_n^{i_n}. \end{aligned}$$

□

Definition 2.34. Let $R = \mathbb{K}[x_1, \ldots, x_n]$ be a polynomial ring over a perfect field $\mathbb{K}$ of characteristic $p > 0$ and let $e \geq 1$ be an integer. We let

$$\mathcal{B}^e(R) \coloneqq \{F^e_* x_1^{i_1} \cdots x_n^{i_n} \mid 0 \leq i_1, \ldots, i_n < p^e\}$$

be the *standard basis* for F^e_*R as a free R-module.

We shall be mostly interested in polynomial rings, however we note that the same fact, with some tweaks, is true for power series rings over perfect fields of prime characteristic.

Proposition 2.35. Let $\{M_i\}_i$ be a collection of modules over an F-finite ring R of characteristic $p > 0$ and let $e \geq 1$ be an integer. Then there is an isomorphism

$$\prod_i F^e_* M_i \cong F^e_* \prod_i M_i.$$

Proof. The set $F^e_* \prod_i M_i$ has a natural R-module structure given by $rF^e_*(m_i)_i \coloneqq F^e_*(r^{p^e} m_i)_i$, where $r \in R$ and $(m_i)_i \in \prod_i M_i$. Note that for each k there is an R-module homomorphism $F^e_* \prod_i M_i \to F^e_* M_k$ sending $F^e_*(m_i)_i \mapsto F^e_* m_k$, thereby inducing a unique R-module homomorphism

$$\begin{aligned} F^e_* \prod_k M_k &\longrightarrow \prod_k F^e_* M_k \\ F^e_*(m_i)_i &\longmapsto (F^e_* m_i)_i. \end{aligned}$$

The same argument given in the proof of Proposition 2.32 shows that this map is, in fact, an isomorphism. □

Proposition 2.36. Let $\mathbb{K}$ be a perfect field of characteristic $p > 0$, $R = \mathbb{K}[\![x_1, \ldots, x_n]\!]$ a power series ring and $e \geq 1$ an integer. Then

$$F^e_* R \cong \bigoplus_{0\leq i_i,\ldots,i_n<p^e} R\, F^e_* x_1^{i_1} \cdots x_n^{i_n},$$

thus $\{F^e_* x_1^{i_1} \cdots x_n^{i_n} \mid 0 \leq i_1, \ldots, i_n < p^e\}$ is a basis for F^e_*R as an R-module.

Proof. The power series ring R admits a decomposition,

$$\begin{aligned} R \cong \prod_{k_1,\dots,k_n \geq 0} \mathbb{K} x_1^{k_1} \cdots x_n^{k_n} &\cong \prod_{0 \leq i_1,\dots,i_n < p^e} \prod_{j_1,\dots,j_n \geq 0} \mathbb{K}\, x_1^{j_1 p^e + i_1} \cdots x_n^{j_n p^e + i_n} \\ &\cong \bigoplus_{0 \leq i_1,\dots,i_n < p^e} \prod_{j_1,\dots,j_n \geq 0} \mathbb{K}\, x_1^{j_1 p^e + i_1} \cdots x_n^{j_n p^e + i_n} \end{aligned}$$

where the second isomorphism follows from the direct product being taken over a finite set, thus it is equivalent to a direct sum over the same set. By Propositions 2.32 and 2.35, the functor F_*^e commutes with direct sums and direct products, hence

$$\begin{aligned} F_*^e R \cong F_*^e &\bigoplus_{0 \leq i_1,\dots,i_n < p^e} \prod_{j_1,\dots,j_n \geq 0} \mathbb{K}\, x_1^{j_1 p^e + i_1} \cdots x_n^{j_n p^e + i_n} \\ \cong &\bigoplus_{0 \leq i_1,\dots,i_n < p^e} \prod_{j_1,\dots,j_n \geq 0} F_*^e (\mathbb{K}\, x_1^{j_1 p^e + i_1} \cdots x_n^{j_n p^e + i_n}) \\ \cong &\bigoplus_{0 \leq i_1,\dots,i_n < p^e} \prod_{j_1,\dots,j_n \geq 0} \mathbb{K}\, x_1^{j_1} \cdots x_n^{j_n}\, F_*^e x_1^{i_1} \cdots x_n^{i_n} \\ \cong &\bigoplus_{0 \leq i_1,\dots,i_n < p^e} R\, F_*^e x_1^{i_1} \cdots x_n^{i_n}. \end{aligned}$$

□

Chapter 3
Rings of Differential Operators

Let X be an algebraic variety over an algebraically closed field $\mathbb{K}$, and let $\mathcal{O}(X)$ be the ring of regular functions on X. Choose a point $x \in X$ and let $\mathfrak{m}_x \subseteq \mathcal{O}(X)$ be the ideal of functions vanishing at x. It is a widely known fact that the localization of $\mathcal{O}(X)$ at the maximal ideal $\mathfrak{m}_x$, or the completion with respect to $\mathfrak{m}_x$, both convey a wealth of information about the behavior of the variety around x; for instance, see [H77, Sections 1.3 and 1.5]. Similarly, among the applications of rings of differential operators one finds the study of the local behavior of algebraic varieties.

In this chapter we describe the construction of rings of differential operators, which was originally given by Grothendieck [G66]. These operators can be defined in great generality: given a scheme X, differential operators arise as morphisms $\mathcal{F} \to \mathcal{G}$ of $\mathcal{O}_X$-modules. Nonetheless, for our purposes it will suffice to study the construction in the affine case, where differential operators admit a down-to-earth description. Afterwards, we restrict ourselves to the positive characteristic world, where these rings admit an even more pleasant characterization. To conclude the chapter, we shall prove that in positive characteristic, differential operators can be factored as the composition of two particular morphisms.

3.1. Construction of the ring of differential operators

Throughout this section, let B denote a commutative ring.

Proposition 3.1 ([AM69, Chapter 2])**.** Let R and S be B-algebras. Then $R \otimes_B S$ has a natural B-algebra structure with multiplication given by

$$(r \otimes s) \cdot (r' \otimes s') = rr' \otimes ss'.$$

Proof. Define the map $\phi\colon R \times S \times R \times S \to R \otimes_B S$ that sends $(r, s, r', s') \mapsto rr' \otimes ss'$. It is B-linear in each coordinate, that is,

$$\phi(r+u, s, r', s') = \phi(r, s, r', s') + \phi(u, s, r', s') = rs \otimes r's' + ur' \otimes ss', \cdots$$
$$\phi(br, s, r', s') = \phi(r, bs, r', s') = \cdots = brr' \otimes ss',$$

where $b \in B$, $r, r', u \in R$ and $s, s' \in S$. As a result, ϕ induces a unique B-module homomorphism $\varphi\colon (R \otimes_B S) \otimes_B (R \otimes_B S) \to R \otimes_B S$ sending $(r \otimes s) \otimes (r' \otimes s') \mapsto rr' \otimes ss'$. By defining multiplication on $R \otimes_B S$ via φ, the B-module $R \otimes_B S$ becomes a commutative ring with identity $1 \otimes 1$, and therefore a B-algebra. □

Let R be a B-algebra. Then the diagonal map $\mu\colon R \otimes_B R \to R$ sending $r \otimes s \mapsto rs$ is well-defined. Indeed, it is immediate to check that the map $R \times R \to R$ sending $(r, s) \mapsto rs$ is B-linear, and thus factors uniquely through μ. We denote its kernel by $J_{R|B} \subseteq R \otimes_B R$, or simply by J when B and R are understood from the context. For each $r \in R$ we let

$$dr := 1 \otimes r - r \otimes 1 \in R \otimes_B R.$$

Proposition 3.2. Let R be a commutative B-algebra, then $J = (dr \mid r \in R)$.

Proof. The inclusion $(dr \mid r \in R) \subseteq J$ is clear. For the converse, suppose that $\mu(\sum_i r_i \otimes s_i) = \sum_i r_i s_i = 0$. For each $r_i \otimes s_i$ one can write $r_i \otimes s_i = (r_i \otimes 1)ds_i + r_i s_i \otimes 1$, hence summing over all i gives

$$\sum_i r_i \otimes s_i = \sum_i (r_i \otimes 1)ds_i + \Big(\sum_i r_i s_i\Big) \otimes 1 = \sum_i (r_i \otimes 1)ds_i,$$

which proves that $J \subseteq (dr \mid r \in R)$. □

Definition 3.3. Let B be a commutative ring and let M and N be R-modules. A *B-linear homomorphism* from M to N is a map $\varphi\colon M \to N$ such that

$$\begin{aligned} \varphi(x + y) &= \varphi(x) + \varphi(y), \\ \varphi(bx) &= \varphi(bx), \end{aligned}$$

for all $b \in B$ and $x, y \in M$. The set of B-linear homomorphisms $M \to N$ is denoted by $\mathrm{Hom}_B(M, N)$. When $M = N$, a B-linear homomorphism $\varphi\colon M \to M$ is said to be a *B-linear endomorphism* of M. The set of B-linear endomorphisms of M is denoted by $\mathrm{End}_B(M)$.

Proposition 3.4. Let B be a commutative ring and let M and N be B-modules. The set $\mathrm{Hom}_B(M, N)$ has a natural B-module structure with addition and action defined as

$$\begin{aligned} (\varphi + \psi)(x) &= \varphi(x) + \psi(x), \\ (b \cdot \varphi)(x) &= b\varphi(x), \end{aligned}$$

where $\varphi, \psi \in \mathrm{Hom}_B(M, N)$, $b \in B$ and $x \in M$.

Proof. We begin by showing that $\mathrm{Hom}_B(M, N)$ is an abelian group with respect to addition. The zero morphism $0\colon R \to R$ is the neutral element, as for any $\varphi \in \mathrm{End}_B(R)$ one has that $(\varphi+0(x) = (0+\varphi)(x) = \varphi(x)$. Given $\varphi \in \mathrm{End}_B(R)$, the map $-\varphi$ defined by $(-\varphi)(x) = -\varphi(x)$ is again in $\mathrm{End}_B(R)$, and is the opposite of φ. Furthermore, addition of morphisms is associative and commutative because it is performed point-wise in N, which is an abelian group with respect to addition. To finish, given a homomorphism $\varphi \in \mathrm{Hom}_B(M,N)$, one has

$$(b \cdot \varphi)(x+y) = b\varphi(x+y) = b\varphi(x) + b\varphi(y) = (b \cdot \varphi)(x) + (b \cdot \varphi)(y)$$
$$(b \cdot \varphi)(\lambda x) = b\varphi(\lambda x) = \lambda b\varphi(x) = \lambda(b \cdot \varphi)(x)$$

for all $b, \lambda \in B$ and $x, y \in M$, which shows that $b \cdot \varphi \in \mathrm{Hom}_B(M,N)$. It is immediate to see that this defines an action, thus $\mathrm{Hom}_B(M,N)$ is a B-module. □

Proposition 3.5. Let R be a commutative algebra over B. Then the set $\mathrm{End}_B(R)$ of B-linear endomorphisms of R has a non-commutative ring structure with respect to point-wise addition and multiplication of endomorphisms, i.e.

$$(\varphi + \psi)(x) \coloneqq \varphi(x) + \psi(x),$$
$$(\varphi\psi)(x) \coloneqq \varphi(\psi(x)),$$

with $\varphi, \psi \in \mathrm{End}_B(R)$ and $x \in R$.

Proof. By Proposition 3.4, the set $\mathrm{End}_B(R) = \mathrm{Hom}_B(R,R)$ is a B-module. It is clear that the neutral element with respect to point-wise composition is the identity $\mathrm{Id}\colon R \to R$. Composition of morphisms in $\mathrm{End}_B(R)$ is again a morphisms of $\mathrm{End}_B(R)$. Indeed, given $\varphi, \psi \in \mathrm{End}_B(R)$, $b \in B$ and $x, y \in R$, one has

$$(\varphi\psi)(x+y) = \varphi(\psi(x) + \psi(y)) = (\varphi\psi)(x) + (\varphi\psi)(y),$$
$$(\varphi\psi)(bx) = \varphi(\psi(bx)) = \varphi(b\psi(x)) = b(\varphi\psi)(x),$$

and the associativity of composition is clear. Non-commutativity follows from the fact that, in general, $\varphi(\psi(x)) \neq \psi(\varphi(x))$. To conclude, given $\varphi, \psi, \theta \in \mathrm{End}_B(R)$,

$$(\varphi(\psi + \theta))(x) = \varphi((\psi + \theta)(x)) = \varphi(\psi(x)) + \varphi(\theta(x))$$

for all $x \in R$, hence $\varphi(\psi + \theta) = \varphi\psi + \varphi\theta$. This shows, composition distributes over addition, which proves that $\mathrm{End}_B(R)$ is a non-commutative ring. □

Proposition 3.6. Let R be a commutative algebra over a ring B. Then the map

$$\begin{aligned} \phi\colon R &\longrightarrow \mathrm{End}_B(R) \\ f &\longmapsto [\phi_f\colon x \longmapsto fx] \end{aligned}$$

is an injective homomorphism of rings and of B-algebras. In particular, it identifies R with a commutative subring of $\mathrm{End}_B(R)$.

Proof. The unit $1 \in R$ is sent to the identity morphism of R, which is R-linear and thus B-linear. Given $f, g \in R$, one has $\phi_{f+g} = \phi_f + \phi_g$ and $\phi_{fg} = \phi_f \phi_g$ since

$$\phi_{f+g}(x) = fx + gx = \phi_f(x) + \phi_g(x),$$
$$\phi_{fg}(x) = fgx = (\phi_f \phi_g)(x),$$

for all $x \in R$, hence ϕ is a ring homomorphism. If $f \in R$ is sent to the zero morphism, then $\phi_f(1) = f = 0$, which proves that ϕ is injective. As the image of a commutative ring under a ring homomorphism is again a commutative ring, it follows that ϕ maps R to a commutative subring of $\operatorname{End}_B(R)$. □

The ring $R \otimes_B R$ has a natural action on the underlying abelian group of $\operatorname{End}_B(R)$ given by

$$((r \otimes s) \cdot \varphi)(x) := r\varphi(sx),$$

where $r \otimes s \in R \otimes_B R$, thus endowing $\operatorname{End}_B(R)$ with an $(R \otimes_B R)$-module structure. Note that for an integer $n \geq 0$, the n-th power of the ideal J is

$$J^n = (dr_1 \cdots dr_n \mid r_1, \dots, r_n \in R).$$

Definition 3.7. Let R be a commutative algebra over a ring B, let $\varphi \in \operatorname{End}_B(R)$ and fix an integer $n \geq 0$.

(1) The endomorphism φ is a *B-linear differential operator of order* $\leq n$ if $J^{n+1} \cdot \varphi = 0$, that is to say, for all $r_1, \dots, r_{n+1}$ and $x \in R$,

$$((dr_1 \cdots dr_{n+1}) \cdot \varphi)(x) = 0.$$

The set of B-linear differential operators of order $\leq n$ is denoted by $\mathcal{D}^n_{R|B}$.

(2) The endomorphism φ is a *B-linear differential operator* if it is a B-linear differential differential operator of order $\leq n$ for some integer $n \geq 0$. The set of B-linear differential operators is denoted by $\mathcal{D}_{R|B}$.

When B is understood from the context, we shall write these sets as $\mathcal{D}^n_R$ and $\mathcal{D}_R$.

Some examples of differential operators on familiar rings are convenient to better understand the abstract definition:

Example 3.8. Let R be a commutative algebra over a ring B and let $\phi \colon R \to \operatorname{End}_B(R)$ be the map defined in Proposition 3.6. Then each ϕ_f is a B-linear differential operator of order ≤ 0. Indeed, for all $r, g \in R$ one has that

$$(dr \cdot \phi_f)(g) = ((1 \otimes r) \cdot f)(g) - ((r \otimes 1) \cdot f)(g) = frg - rfg = 0.$$

Furthermore, as $\phi \colon R \to \operatorname{End}_B(R)$ is an injective ring homomorphism, R is identified with a subring of $\mathcal{D}^0_R$. In fact, as we will show later on in Proposition 3.17, $\mathcal{D}^0_R = \operatorname{End}_R(R) \cong R$.

Example 3.9. Let $B = \mathbb{K}$ be a field and $R = \mathbb{K}[x_1, \ldots, x_n]$ a polynomial ring. The first-order partial derivatives

$$\partial_1 \coloneqq \frac{\partial}{\partial x_1}, \ \ldots, \ \partial_n \coloneqq \frac{\partial}{\partial x_n},$$

are $\mathbb{K}$-linear differential operators on R of order ≤ 1. Indeed, the action of $dr \in J$ on ∂_i reads

$$(dr \cdot \partial_i)(f) = ((1 \otimes r) \cdot \partial_i)(f) - ((r \otimes 1) \cdot \partial_i)(f) = f\partial(r) = \phi_{\partial_i(r)}(f),$$

whence $dr \cdot \partial_i = \phi_{\partial_i(r)}$, and Example 3.8 shows that $dr \cdot \partial_i$ is a differential operator of order ≤ 0.

Example 3.10. Following the previous example, the second-order partial derivatives

$$\partial_{ij} \coloneqq \partial_i \partial_j = \frac{\partial^2}{\partial x_i \, \partial x_j},$$

are $\mathbb{K}$-linear differential operators on R of order ≤ 2. Let $f \in R$, then

$$(dr \cdot \partial_i \partial_j)(f) = (\partial_i r)(\partial_j f) + (\partial_i f)(\partial_j r) + f \partial_i \partial_j(r)$$

for all $dr \in J$, thus $dr \cdot \partial_i \partial_j = \phi_{\partial_i(r)} \partial_j + \phi_{\partial_j(r)} \partial_i + \phi_{\partial_i \partial_j(r)}$. Note that given $u \in R$, the $\mathbb{K}$-linear endomorphism $\phi_u \partial_i$ satisfies

$$(ds \cdot \phi_u \partial_i)(f) = \phi_u(\partial_i(sf)) - s\phi_u(\partial_i(f)) = \phi_u(\partial_i(sf) - s\partial_i(f)) = \phi_u(f\partial_i(s)) = \phi_{u\partial_i(s)}(f),$$

hence $ds \cdot \phi_u \partial_i = \phi_{u\partial_i(s)}$ for all $ds \in J$. Because $\phi_{\partial_i \partial_j(r)}$ is a $\mathbb{K}$-linear differential operator of order ≤ 0, one has that

$$ds \, dr \cdot \partial_i \partial_j = ds \cdot \phi_{\partial_i(r)} \partial_j + ds \cdot \phi_{\partial_j(r)} \partial_i + ds \cdot \phi_{\partial_i \partial_j(r)} = \phi_{\partial_i(r)\partial_j(s)} + \phi_{\partial_i(s)\partial_j(r)},$$

Since $\phi_{\partial_i(r)\partial_j(s)}$ and $\phi_{\partial_i(s)\partial_j(r)}$ are differential operators of order ≤ 0, it follows that $dt \, ds \, dr \cdot \partial_i \partial_j = 0$ for all $dt \in J$, thus $\partial_i \partial_j$ is a differential operator of order ≤ 2.

Example 3.11. The first-order partial derivatives in the previous example commute with each other, that is

$$\partial_{ij} = \partial_i \partial_j = \partial_j \partial_i = \partial_{ji}$$

for all $1 \leq i, j \leq n$. Since the partial derivatives are $\mathbb{K}$-linear endomorphisms of R, it suffices to show this fact for monomials $x_1^{a_1} \cdots x_n^{a_n} \in R$. Furthermore, we may assume that $i = 1$ and $j = 2$, otherwise one can rename the variables and reduce to this case. If $a_1 = 0$ then $\partial_1 x_1^{a_1} \cdots x_n^{a_n} = 0$, thus $\partial_1 \partial_2 x_1^{a_1} \cdots x_n^{a_n} = \partial_2 \partial_1 x_1^{a_1} \cdots x_n^{a_n} = 0$. The same is true when $a_2 = 0$, thus suppose $a_1, a_2 \geq 1$. In this case,

$$\begin{aligned}
\partial_1 \partial_2 (x_1^{a_1} \cdots x_n^{a_n}) &= \partial_1 (a_2 x_1^{a_1} x_2^{a_2 - 1} x_3 \cdots x_n) \\
&= a_1 a_2 x_1^{a_1 - 1} x_2^{a_2 - 1} x_3 \cdots x_n \\
&= \partial_2 (a_1 x_1^{a_1 - 1} x_2^{a_2} x_3 \cdots x_n) \\
&= \partial_2 \partial_1 (x_1^{a_1} \cdots x_n^{a_n}).
\end{aligned}$$

A word of caution on the previous example is in order. Although first-order partial derivatives commute with each other, and, more generally, partial derivatives of any order commute with each other, this is not always the case with differential operators, as the example below shows:

Example 3.12. Let $R = \mathbb{K}[x_1, \ldots, x_n]$ be a polynomial ring over a field $\mathbb{K}$ and let $\partial_i = \partial/\partial x_i$ be the partial derivative with respect to x_i. In view of Proposition 3.6, we may identify the indeterminate x_i with the map ϕ_{x_i}. On the one hand, for all $f \in R$,

$$(\partial_i \, x_i) \, f = \partial_i(x_i f) = f + (x_i \partial_i) \, f.$$

On the other hand, whenever $i \neq j$,

$$(\partial_i \, x_j) \, f = \partial_i(x_j f) = (x_j \partial_i) \, f.$$

This computation shows that in the subring $\mathbb{K}[x_1, \ldots, x_n, \partial_1, \ldots, \partial_n]$ of $\mathrm{End}_{\mathbb{K}}(R)$,

$$\partial_i x_j - x_j \partial_i = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

As shown in Example 3.10, the second-order partial derivatives ∂_{ij} are differential operators of order ≤ 2. Note that these arise as the composition of two differential operators of order ≤ 1, that is, $\partial_{ij} = \partial_i \, \partial_j$. This fact is generalized as follows:

Proposition 3.13. Let R be a commutative algebra over a ring B. Then for each pair of integers $m, n \geq 0$, one has that

$$\mathcal{D}_R^m \circ \mathcal{D}_R^n := \{\varphi \circ \psi \mid \varphi \in \mathcal{D}_R^m, \, \psi \in \mathcal{D}_R^n\} \subseteq \mathcal{D}_R^{m+n}.$$

Proof. For a proof of the fact, see [QG21b, Proposition II.15]. □

Proposition 3.14. Let R be a commutative algebra over a ring B.

(1) The sets of B-linear differential operators give an ascending filtration

$$\mathcal{D}_R^0 \subseteq \mathcal{D}_R^1 \subseteq \mathcal{D}_R^2 \subseteq \cdots.$$

(2) The set $\mathcal{D}_R$ of B-linear differential operators is a non-commutative ring.

Proof. (1) Let $n \geq 0$ be an integer and fix $\varphi \in \mathcal{D}_R^n$, that is, $J^{n+1} \cdot \varphi = 0$. Since $J^{n+2} \subseteq J^{n+1}$ it follows that $J^{n+2} \cdot \varphi = 0$, thus $\varphi \in \mathcal{D}_R^{n+1}$, which shows that $\mathcal{D}_R^n \subseteq \mathcal{D}_R^{n+1}$.

(2) It is clear that the zero and the identity endomorphisms of R are differential operators of order ≤ 0. It follows from Proposition 3.5 that the zero morphism is the neutral element with respect to addition, and the identity is the neutral element with respect to composition. Given $\varphi, \psi \in \mathcal{D}_R$, there exist integers $m, n \geq 0$ such that $\varphi \in \mathcal{D}_R^m$ and $\psi \in \mathcal{D}_R^n$, and Proposition 3.13 shows that $\varphi \circ \psi$ is a differential operator of order $\leq m+n$, hence $\mathcal{D}_R$ is closed under composition. Furthermore, composition distributes over addition because the operations are performed in $\mathrm{End}_B(R)$, thus proving that $\mathcal{D}_R$ is a non-commutative ring. □

It is possible to give an alternative construction of the ring of differential operators on a commutative B-algebra R in terms of the commutator:

Definition 3.15. Let R be a ring (not necessarily commutative). The *commutator* of two elements $r, s \in R$ is defined by

$$[r, s] \coloneqq rs - sr.$$

Example 3.16. Let $R = \mathbb{K}[x_1, \dots, x_n]$ be a polynomial ring over a field $\mathbb{K}$. In Example 3.12, we showed that

$$\partial_i x_j - x_j \partial_i = [\partial_i, x_j] = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 3.17. Let R be a commutative algebra over a ring B. Then $\mathcal{D}^0_{R|B} = \operatorname{End}_R(R)$.

Proof. Let $\varphi \in D^0_{R|B}$ be a B-linear differential operator of order ≤ 0. Then $(dr \cdot \varphi)(1) = \varphi(r) - r\varphi(1) = 0$, which yields $\varphi(r) = r\varphi(1)$ for all $r \in R$, thus φ is R-linear. Conversely, given $\varphi \in \operatorname{End}_R(R)$, one has $(dr \cdot \varphi)(x) = \varphi(rx) - r\varphi(x) = 0$ for all $r, x \in R$, so $J \cdot \varphi = 0$. □

Every R-linear endomorphism of R is in particular B-linear since $\varphi(br) = br\varphi(1) = b\varphi(r)$, therefore $\mathcal{D}^0_{R|B} = \operatorname{End}_R(R) \subseteq \operatorname{End}_B(R)$.

Proposition 3.18. Let R be a commutative B-algebra and let $n \geq 1$ be an integer. The following are equivalent:

(1) One has that $\varphi \in \mathcal{D}^n_{R|B}$.
(2) One has that $\varphi \circ \delta - \delta \circ \varphi \in \mathcal{D}^{n-1}_{R|B}$ for all $\delta \in \mathcal{D}^0_{R|B}$.

Proof. Let $\varphi \in \mathcal{D}^n_{R|B}$ and $\delta = \phi_f$ for some $f \in R$. We have $\varphi \circ \delta - \delta \circ \varphi = df \cdot \varphi$ since, for each $x \in R$, $(\varphi \circ \delta - \delta \circ \varphi)(x) = \varphi(fx) - f\varphi(x) = (df \cdot \varphi)(x)$. For any $r_1, \dots, r_n \in R$, it follows that $dr_1 \cdots dr_n \, df \in J^{n+1}$ and $(dr_1 \cdots dr_n) \cdot (df \cdot \varphi) = (dr_1 \cdots dr_n \, df) \cdot \varphi = 0$, whence $df \cdot \varphi \in D^{n-1}_{R|B}$.

Conversely, take an arbitrary element $f \in R$ and let $\delta = \phi_f$. Suppose that $\varphi \circ \delta - \delta \circ \varphi = df \cdot \varphi \in D^{n-1}_{R|B}$, that is, $J^n \cdot (df \cdot \varphi) = 0$. Then for any $r_1, \dots, r_n \in R$ we have $dr_1 \cdots dr_n \, df \cdot \varphi = 0$ and, since $f \in R$ can take any value, we deduce $\varphi \in D^0_{R|B}$. □

Observation 3.19. It follows from Propositions 3.17 and 3.18 that one may define the ring of B-linear differential operators on a commutative B-algebra R as follows:

(a) The *ring of B-linear differential operators of order* ≤ 0 is given by

$$\mathcal{D}^0_{R|B} \coloneqq \operatorname{Hom}_R(R, R) \subseteq \operatorname{End}_B(R).$$

(b) For each $n \geq 1$, define the *set of B-linear differential operators of order* $\leq n$ as

$$\mathcal{D}^n_{R|B} \coloneqq \left\{ \varphi \in \operatorname{End}_B(R) \,\middle|\, [\varphi, \delta] = \varphi \circ \delta - \delta \circ \varphi \in \mathcal{D}^{n-1}_{R|B} \text{ for all } \delta \in \mathcal{D}^0_{R|B} \right\}.$$

(c) The *ring of B-linear differential operators on R* is

$$\mathcal{D}_{R|B} \coloneqq \bigcup_{n \geq 0} \mathcal{D}^n_{R|B}.$$

3.2. Differential operators in positive characteristic

In the positive characteristic setting, the ring of differential operators admits a particularly nice description due to Yekutieli [Y92]. Throughout this section we let $B = \mathbb{F}$ be a perfect field of characteristic $p > 0$ and R a commutative $\mathbb{F}$-algebra, thus we shall study $\mathbb{F}$-linear endomorphisms and $\mathbb{F}$-linear differential operators on R. As we shall prove, there is no loss of generality by taking $B = \mathbb{F}_p$ rather than any other field of characteristic $p > 0$ contained in R.

Definition 3.20. Let R be a commutative ring of characteristic $p > 0$ and let $\mathfrak{b} \subseteq R$ be an ideal. For each integer $e \geq 0$, the *e-th Frobenius power* of $\mathfrak{b}$ is the ideal

$$\mathfrak{b}^{[p^e]} := F^e(\mathfrak{b})\, R = (f^{p^e} \mid \mathfrak{b} \in I).$$

Proposition 3.21. Let R be a commutative ring of characteristic $p > 0$. Then the e-th Frobenius power of an ideal $\mathfrak{b} \subseteq R$ is a well-defined operation for each integer $e \geq 0$, that is, it does not depend on the choice of generators of $\mathfrak{b}$. Furthermore, if $\mathfrak{b} = (f_i \mid i \in I)$ for some indexing set I, then $\mathfrak{b}^{[p^e]} = (f_i^{p^e} \mid i \in I)$.

Proof. Let $\mathfrak{b} = (f_i \mid i \in I) = (g_j \mid j \in J)$ be two generating sets for $\mathfrak{b}$, where I and J are arbitrary indexing sets. Each generator f_i can be expressed as a finite sum of the form $f_i = \sum_j f_{ij} g_j$ for some $f_{ij} \in R$. By Proposition 2.5, $f_i^{p^e} = \sum_j f_{ij}^{p^e} g_j^{p^e}$ thus $(f_i^{p^e} \mid i \in I) \subseteq (g_j^{p^e} \mid j \in J)$. By symmetry, the reverse inclusion holds as well, thereby the operation is well-defined. □

Proposition 3.22. Let R be a Noetherian ring of characteristic $p > 0$ and let $I \subseteq R$ be an ideal. Then the chains of ideals $\{I^n\}_{n \geq 0}$ and $\{I^{[p^e]}\}_{e \geq 0}$ are cofinal.

Proof. Let $I = (f_1, \ldots, f_r)$ be a set of generators for I. As $I^{[p^e]} = (f_1^{p^e}, \ldots, f_r^{p^e})$ and each generator is in I^{p^e}, it follows that $I^{[p^e]} \subseteq I^{p^e}$.

For the converse fix an integer $e \geq 0$. Note that the power I^n is generated by elements of the form $f_1^{n_1} \cdots f_r^{n_r}$ with $n_1 + \cdots + n_r = n$. Set $n = r(p^e - 1) + 1$. Then for any choice of exponents with sum equal to n, there is some $n_i \geq p^e$. Indeed, for if all $n_i \leq p^e - 1$ then $n_1 + \cdots + n_r \leq r(p^e - 1)$. As a result, for any choice one has $f_1^{n_1} \cdots f_r^{n_r} \in I^{[p^e]}$, hence $I^n \subseteq I^{[p^e]}$. □

Example 3.23. The Noetherian hypothesis cannot be removed in Proposition 3.22. For a counterexample, consider the polynomial ring $R = \mathbb{F}_p[x_1, x_2, \ldots]$ in infinitely many variables and let $\mathfrak{m} = (x_1, x_2, \ldots)$ be the homogeneous maximal ideal. While it is always true that $\mathfrak{m}^{[p^e]} \subseteq \mathfrak{m}^{p^e}$, given an integer $n \geq 0$, there is no $e \geq 0$ satisfying $\mathfrak{m}^n \subseteq \mathfrak{m}^{[p^e]}$. Indeed, choose integers $1 \leq i_1 < i_2 < \cdots < i_n$. Then the monomial $x_{i_1} x_{i_2} \cdots x_{i_n}$ is in $\mathfrak{m}^n$ but not in $\mathfrak{m}^{[p^e]}$.

We emphasize that when no relations among the generators $f_1, \ldots, f_r$ of I are known, the bound $n = r(p^e - 1) + 1$ is optimal. If one sets $n = r(p^e - 1)$ instead, the element $(f_1 \cdots f_r)^{p^e - 1}$

is a generator of I^n, but a priori it is not known whether or not it lies in $I^{[p^e]}$. As we shall see later on in Chapters 4 and 5, for some classes of ideals it is possible to explicitly compute the optimal bounds. In turn, these bounds are extremely useful to compute certain algebraic invariants of the ideals.

Theorem 3.24 ([Y92, Theorem 1.4.9])**.** Let R be an F-finite ring of characteristic $p > 0$. Then the ring of $\mathbb{F}$-linear differential operators on R is

$$\mathcal{D}_{R|\mathbb{F}} = \bigcup_{n=0}^{\infty} \operatorname{End}_{R^{p^e}}(R).$$

In particular, the ring of differential operators on R does not depend on the choice of perfect field $\mathbb{F}$, thus we shall denote it simply by $\mathcal{D}_R$.

Proof. To show ($\subseteq$) let $\xi \in \mathcal{D}_R$ be a differential operator, so that $J^{n+1} \cdot \xi = 0$ for some integer $n \geq 0$. As the ideal $J \subseteq R \otimes_{\mathbb{F}} R$ is finitely generated because R is an F-finite ring, it follows from Proposition 3.22 that there exists an integer $e \geq 0$ such that $J^{[p^e]} \subseteq J^{n+1}$, whence $J^{[p^e]} \cdot \xi = 0$. This means that for all $r, f \in R$,

$$(dr^{p^e} \cdot \xi)(f) = \xi(r^{p^e} f) - r^{p^e} \xi(f) = 0,$$

thus ξ is an R^{p^e}-linear endomorphism of R.

As for ($\supseteq$), given $\xi \in \operatorname{End}_{R^{p^e}}(R)$ it follows that $dr^{p^e} \cdot \xi = 0$ for each $r \in R$, hence $J^{[p^e]} \cdot \xi = 0$. Again by Proposition 3.22, there is an integer $n \geq 0$ such that $J^{n+1} \subseteq J^{[p^e]}$, so ξ is a differential operator of order $\leq n$. ☐

Definition 3.25. Let R be an F-finite ring of characteristic $p > 0$. For each integer $e \geq 0$, we define the set of *differential operators of level e on R* by

$$\mathcal{D}_R^{(e)} \coloneqq \operatorname{End}_{R^{p^e}}(R).$$

To conclude this section, we give an alternative characterization of the set $\mathcal{D}_R^{(e)}$ as the R-module of R-linear endomorphisms of $F_*^e R$. This will prove particularly useful towards the end of the chapter.

Proposition 3.26. Let R be an F-finite ring of characteristic $p > 0$ and $e \geq 0$ an integer.

(1) The set $\mathcal{D}_R^{(e)}$ of differential operators of level e is an R-module with action $(r \cdot \xi)(f) \coloneqq r^{p^e} \xi(f)$, where $\xi \in \mathcal{D}_R^{(e)}$ and $r, f \in R$.
(2) There is an isomorphism $\mathcal{D}_R^{(e)} \cong \operatorname{End}_R(F_*^e R)$.

Proof. (1) As differential operators of level e are additive endomorphisms of R, the set $\mathcal{D}_R^{(e)}$ is an abelian group with respect to point-wise addition of morphisms. Furthermore, $r \cdot \xi$ is additive by definition of the action, and R^{p^e}-linear as $(r \cdot \xi)(s^{p^e} f) = r^{p^e} s^{p^e} \xi(f) = s^{p^e} (r \cdot \xi)(f)$ for all $s \in R$.

(2) Consider the map $\mathcal{D}\colon \mathcal{D}_R^{(e)} \to \operatorname{End}_R(F_*^e R)$ given by $\xi \mapsto [\mathcal{D}\xi\colon F_*^e f \mapsto F_*^e \xi(f)]$. The morphism $\mathcal{D}\xi$ is additive because ξ is, and for all $r \in R$, $F_*^e f \in F_*^e R$, one has that

$$\mathcal{D}\xi(rF_*^e f) = \mathcal{D}\xi(F_*^e r^{p^e} f) = F_*^e \xi(r^{p^e} f) = F_*^e r^{p^e} \xi(f) = rF_*^e \xi(f) = r\mathcal{D}\xi(F_*^e f),$$

hence $\mathcal{D}\xi$ is R-linear. Given $\xi, \xi' \in \operatorname{End}_{R^{p^e}}(R)$ and

$$\mathcal{D}(\xi + \xi')(F_*^e f) = F_*^e(\xi + \xi')(f) = F_*^e \xi(f) + F_*^e \xi'(f) = \mathcal{D}\xi(F_*^e f) + \mathcal{D}\xi'(F_*^e f),$$
$$\mathcal{D}(r \cdot \xi)(F_*^e f) = F_*^e r^{p^e} \xi(f) = rF_*^e \xi(f) = r(\mathcal{D}\xi)(F_*^e f),$$

therefore $\mathcal{D}$ is an R-module homomorphism. In order to show that $\mathcal{D}$ is an isomorphism, let $\xi \in \ker \mathcal{D}$. Then $\mathcal{D}\xi(F_*^e f) = F_*^e \xi(f) = 0$ for all $f \in R$, but this means that $\xi = 0$, which proves that $\mathcal{D}$ is injective. To show surjectivity, pick a morphism $\boldsymbol{\xi} \in \operatorname{End}_R(F_*^e R)$ and define $\xi \in \mathcal{D}_R^{(e)}$ point-wise as follows: if $\boldsymbol{\xi}(F_*^e f) = F_*^e y$, let $\xi(f) \coloneqq y$. Then one can write $\boldsymbol{\xi}$ as $\boldsymbol{\xi}(F_*^e f) = F_*^e \xi(f)$. By construction, it follows that ξ is additive because $\boldsymbol{\xi}$ is and, for all $r, f \in R$ one has

$$F_*^e \xi(r^{p^e} f) = \boldsymbol{\xi}(F_*^e r^{p^e} f) = \boldsymbol{\xi}(rF_*^e f) = r\boldsymbol{\xi}(F_*^e f) = rF_*^e \xi(f) = F_*^e r^{p^e} \xi(f),$$

thus ξ is an R^{p^e}-linear endomorphism of R. This implies that $\mathcal{D}$ is surjective, and thus defines an isomorphism of R-modules. □

3.3. The p^e-linear and p^{-e}-linear maps

Next we introduce two classes of maps $R \to R$ for an F-finite ring R, namely, p^e-linear and p^{-e}-linear maps, which will allow for a better understanding of the ring of differential operators in positive characteristic.

Definition 3.27 ([B13, Definition 2.1])**.** Let R be an F-finite ring of characteristic $p > 0$ and $e \geq 0$ an integer.

(1) An additive map $\psi\colon R \to R$ is a p^e*-linear map* if $\psi(rf) = r^{p^e}\psi(f)$ for every $r, f \in R$. The set of p^e-linear maps on R is denoted by $\mathcal{F}_R^e$.
(2) An additive map $\varphi\colon R \to R$ is a p^{-e}*-linear map* if $\varphi(r^{p^e} f) = r\varphi(f)$ for every $r, f \in R$. The set of p^{-e}-linear maps on R is denoted by $\mathcal{C}_R^e$.

Proposition 3.28. Let R be an F-finite ring of characteristic $p > 0$. A p^e-linear map ψ on R is of the form $\psi = rF^e$ for some $r \in R$, that is to say, $\psi(f) = rf^{p^e}$.

Proof. A p^e-linear operator ψ is determined by its image on 1, since $\psi(f) = f^{p^e}\psi(1) = rf^{p^e} = rF^e(f)$ for each $f \in R$. □

Proposition 3.29. Let $\psi\colon R \to F_*^e R$ be an R-linear map. Then $\psi = F_*^e f \circ F^e$ for some $F_*^e f \in F_*^e R$, that is to say, $\psi(f) = F_*^e(rf^{p^e})$ for $f \in R$. In particular, $R \cong \operatorname{Hom}_R(R, F_*^e R)$.

Proof. Let $F^e_*r = \psi(1)$; by assumption ψ is R-linear, thus for each $f \in R$, $\psi(f) = f\psi(1) = fF^e_*r = F^e_*(rf^{p^e}) = (F^e_*r \circ F^e)(f)$, whence $\psi = F^e_*r \circ F^e$ as desired.

To show the last statement, define $\mathcal{G}\colon R \to \operatorname{Hom}_R(R, F^e_*R)$ by $r \mapsto F^e_*r \circ F^e$. If $\mathcal{G}(r) = 0$ then $\mathcal{G}(r)(1) = F^e_*r = 0$, which shows $\mathcal{G}$ is injective because $r = 0$. Surjectivity is due to the fact every $\psi \in \operatorname{Hom}_R(R, F^e_*R)$ is of the form $\psi = F^e_*r \circ F^e$. Consequently $\mathcal{G}$ defines an isomorphism $R \cong \operatorname{Hom}_R(R, F^e_*R)$. □

Proposition 3.30. Let R be an F-finite ring of characteristic $p > 0$ and fix an integer $e \geq 0$.

(1) The set $\mathcal{F}^e_R$ of p^e-linear maps is an R-module with action $(r \cdot \psi)(f) := r^{p^e}\psi(f)$, for $r, f \in R$ and $\psi \in \mathcal{F}^e_R$.

(2) There is an isomorphism $\mathcal{F}^e_R \cong \operatorname{Hom}_R(R, F^e_*R)$.

Proof. (1) It is clear that $\mathcal{F}^e_R$ is an abelian group with respect to addition of morphisms. Moreover it is closed under the action defined above since the map $r \cdot \psi\colon R \to R$ is additive and $(r \cdot \psi)(sf) = r^{p^e}\psi(sf) = r^{p^e}s^{p^e}\psi(f)$ for each $f \in R$.

(2) Define the morphism $\mathcal{F}\colon \mathcal{F}^e_R \to \operatorname{Hom}_R(R, F^e_*R)$ sending $\psi \mapsto [\mathcal{F}(\psi)\colon f \mapsto F^e_*\psi(f)]$. As ψ is an additive map, so is $\mathcal{F}(\psi)\colon R \to F^e_*R$. Moreover, $\mathcal{F}(\psi)(rf) = F^e_*\psi(rf) = F^e_*r^{p^e}\psi(f) = rF^e_*\psi(f) = r\mathcal{F}(\psi)(f)$ thus $\mathcal{F}(\psi)$ is R-linear. To verify that $\mathcal{F}$ is an R-module map, fix morphisms $\psi, \psi' \in \mathcal{F}^e_R$ and let $r \in R$. Then for all $f \in R$ we find

$$\begin{gathered}\mathcal{F}(\psi + \psi')(f) = F^e_*(\psi + \psi')(f) = F^e_*\psi(f) + F^e_*\psi'(f) = \mathcal{F}(\psi)(f) + \mathcal{F}(\psi')(f),\\ \mathcal{F}(r \cdot \psi)(f) = F^e_*(r \cdot \psi)(f) = F^e_*r^{p^e}\psi(f) = rF^e_*\psi(f) = r\mathcal{F}(\psi)(f),\end{gathered}$$

thus $\mathcal{F}(\psi + \psi') = \mathcal{F}(\psi) + \mathcal{F}(\psi')$ and $\mathcal{F}(r \cdot \psi) = r\mathcal{F}(\psi)$. It remains to show that $\mathcal{F}$ is an isomorphism. If $\psi \in \mathcal{F}^e_R$ is such that $\mathcal{F}(\psi) = 0$ is the zero map, then $F^e_*\psi(1) = 0$, thus $\psi = 0$, which shows that $\mathcal{F}$ is injective. As for surjectivity, fix a morphism $\Psi \in \operatorname{Hom}_R(R, F^e_*R)$. By Proposition 3.29 there exists $r \in R$ such that $\Psi = F^e_*r \circ F^e$. By defining $\psi\colon R \to R$ by $\psi(f) = rf^{p^e}$. It is clear that ψ is p^e-linear and, by construction, $\mathcal{F}(\psi) = \Psi$, which concludes the proof. □

Proposition 3.31. Let R be an F-finite ring of characteristic $p > 0$ and fix an integer $e \geq 0$.

(1) The set $\mathcal{C}^e_R$ of p^{-e}-linear maps is an R-module with action $(r \cdot \varphi)(f) := r\varphi(f)$, where $r, f \in R$ and $\varphi \in \mathcal{C}^e_R$.

(2) There is an isomorphism $\mathcal{C}^e_R \cong \operatorname{Hom}_R(F^e_*R, R)$.

Proof. (1) With respect to point-wise addition of morphisms, it is clear that $\mathcal{C}^e_R$ is an abelian group. Furthermore, the action defined above yields a p^{-e}-linear map since $(r \cdot \varphi)(s^{p^e}f) = r\varphi(r^{p^e}s^{p^e}f) = s\varphi(r^{p^e}f) = s(r \cdot \varphi)(f)$ for all $f \in R$, thus proving that $\mathcal{C}^e_R$ is an R-module.

(2) Let $\mathcal{C}\colon \mathcal{C}^e_R \to \operatorname{Hom}_R(F^e_*R, R)$ be the morphism that sends $\varphi \mapsto [\mathcal{C}(\varphi)\colon F^e_*f \mapsto \varphi(f)]$. Then $\mathcal{C}(\varphi)$ is R-linear since for all $r \in R$ and $F^e_*f \in F^e_*R$, we find $\mathcal{C}(\varphi)(rF^e_*f) = \mathcal{C}(\varphi)(F^e_*r^{p^e}f) =$

$\varphi(r^{p^e}f) = r\varphi(f) = r\mathcal{C}(\varphi)(f)$, and additivity follows from φ being additive. To show that $\mathcal{C}$ is an R-module map, fix morphisms $\varphi, \varphi' \in \mathcal{C}_R^e$ and $r \in R$. The following equalities hold for all $F_*^e f \in F_*^e R$ and $r \in R$:

$$\mathcal{C}(\varphi+\varphi')(F_*^e f) = (\varphi+\varphi')(f) = \varphi(f)+\varphi'(f) = \mathcal{C}(\varphi)(F_*^e f) + \mathcal{C}(\varphi')(F_*^e f),$$
$$\mathcal{C}(r\cdot\varphi)(F_*^e f) = (r\cdot\varphi)(f) = \varphi(r^{p^e}f) = r\varphi(f) = r\mathcal{C}(\varphi)(F_*^e f).$$

As a result $\mathcal{C}(\varphi+\varphi') = \mathcal{C}(\varphi)+\mathcal{C}(\varphi')$ and $\mathcal{C}(r\cdot\varphi) = r\mathcal{C}(\varphi)$. To conclude, we show that $\mathcal{C}$ is an isomorphism. It is injective since if $\mathcal{C}(\varphi) = 0$, then $\mathcal{C}(\varphi)(F_*^e f) = 0$ for all $F_*^e f \in F_*^e R$, but this means $\varphi = 0$. Given $\Phi \in \operatorname{Hom}_R(F_*^e R, R)$ define $\varphi\colon R \to R$ by $\varphi(f) = \Phi(F_*^e f)$. As Φ is additive, so is φ, and for all $r, f \in R$ one has that $\varphi(r^{p^e}f) = \Phi(F_*^e r^{p^e} f) = \Phi(rF_*^e f) = r\Phi(F_*^e f) = r\varphi(f)$, thus proving that $\mathcal{C}$ is surjective and an isomorphism of R-modules. □

The following theorem shows that, in characteristic $p > 0$, a differential operator of level $e \geq 0$ can be expressed as the composition of a p^e-linear and a p^{-e}-linear map. In view of Propositions 3.30 and 3.31, and in order to make the proof more straightforward, we shall identify $\mathcal{F}_R^e = \operatorname{Hom}_R(R, F_*^e R)$ and $\mathcal{C}_R^e = \operatorname{Hom}_R(F_*^e R, R)$.

Theorem 3.32 ([AMJNB21, Remark 2.17])**.** Let R be regular F-finite ring and let $e \geq 0$ be an integer. Then the map

$$\begin{aligned}\Lambda\colon \mathcal{F}_R^e \otimes_R \mathcal{C}_R^e &\longrightarrow \mathcal{D}_R^{(e)}\\ \psi\otimes\varphi &\longmapsto \psi\varphi\end{aligned}$$

defines an isomorphism of R-modules.

Proof. We start by verifying that Λ is well-defined. The morphism $\Lambda'\colon \mathcal{F}_R^e \times \mathcal{C}_R^e \to \mathcal{D}_R^{(e)}$ that sends $(\psi, \varphi) \mapsto \psi\varphi$ is well-defined because the composition is an R-linear homomorphism $F_*^e R \to F_*^e R$. As the category $\mathbf{Mod(R)}$ of modules over a commutative ring R is an abelian category, composition of morphisms distributes over addition, hence Λ' is a additive in each coordinate. Now given $\psi \in \mathcal{F}_R^e$ and $\varphi \in \mathcal{C}_R^e$, for all $r \in R$ and $F_*^e f \in F_*^e R$ one has that

$$\Lambda'(r\cdot\psi, \varphi)(F_*^e f) = ((r\cdot\psi)\varphi)(F_*^e f) = r\psi(\varphi(F_*^e f)) = (\psi(r\cdot\varphi))(F_*^e f) = \Lambda'(\psi, r\cdot\varphi)(F_*^e f),$$

hence $\Lambda'(r\cdot\psi, \varphi) = \Lambda'(\psi, r\cdot\varphi)$, thus proving that Λ' is R-bilinear. It follows that Λ' factors uniquely through the morphism Λ.

The fact that Λ is an isomorphism can be checked locally, i.e. Λ is an isomorphism if and only if for each prime ideal $\mathfrak{p} \in \operatorname{Spec} R$, the map

$$\Lambda_\mathfrak{p}\colon \operatorname{Hom}_{R_\mathfrak{p}}(R_\mathfrak{p}, F_*^e R_\mathfrak{p}) \otimes_{R_\mathfrak{p}} \operatorname{Hom}_{R_\mathfrak{p}}(F_*^e R_\mathfrak{p}, R_\mathfrak{p}) \longrightarrow \operatorname{Hom}_{R_\mathfrak{p}}(F_*^e R_\mathfrak{p}, F_*^e R_\mathfrak{p})$$

induced on the localization at $\mathfrak{p}$ is an isomorphism. As a result, we may assume that R is a local ring. Since $F_*^e R$ is a flat R-module by Kunz's theorem, and finitely generated by assumption, it follows that F^e_*R is projective. Altogether, this implies that F^e_*R is locally free, that is, $F^e_*R \cong R^{\oplus n}$ for some $n \geq 1$.

We construct a two-sided inverse for Λ. For each integer $1 \leq i \leq n$, let $\alpha_i \colon R \to R^{\oplus n}$and $\beta_i \colon R^{\oplus n} \to R$ be the injection and projection on the i-th coordinate. Then $\alpha_i\beta_i \colon R^{\oplus n} \to R^{\oplus n}$ is an R-linear map and $\sum_i \alpha_i\beta_i$ is the identity of $R^{\oplus n}$. Define the following homomorphism:

$$\begin{aligned} \Omega \colon \operatorname{Hom}_R(R^{\oplus n}, R^{\oplus n}) &\longrightarrow \operatorname{Hom}_R(R, R^{\oplus n}) \otimes_R \operatorname{Hom}_R(R^{\oplus n}, R), \\ \xi &\longmapsto \sum_{i=1}^{n} \xi\alpha_i \otimes \beta_i. \end{aligned}$$

Given a differential operator $\xi \in \operatorname{Hom}_R(R^{\oplus n}, R^{\oplus n})$, one has

$$(\Lambda\Omega)(\xi) = \Lambda\left(\sum_{i=1}^{n} \xi\alpha_i \otimes \beta_i\right) = \sum_{i=1}^{n} \xi\alpha_i\beta_i = \xi \sum_{i=1}^{n} \alpha_i\beta_i = \xi,$$

thus Ω is a right-inverse for Λ. Conversely, take $\psi \in \operatorname{Hom}_R(R, R^{\oplus n})$ and $\varphi \in \operatorname{Hom}_R(R^{\oplus n}, R)$. Note that $\varphi\alpha_i$ is an R-linear endomorphism of R, thus $\varphi\alpha_i \in \operatorname{End}_R(R) \cong R$. As a result,

$$(\Omega\Lambda)(\psi \otimes \varphi) = \Omega(\psi\varphi) = \sum_{i=1}^{n} \psi\varphi\alpha_i \otimes \beta_i = \sum_{i=1}^{n} \psi \otimes \varphi\alpha_i\beta_i = \psi \otimes \varphi\left(\sum_{i=1}^{n} \alpha\beta_i\right) = \psi \otimes \varphi.$$

Since the elements of the form $\psi \otimes \varphi$ generate the tensor product, this shows that Ω is a left-inverse for Λ, therefore Λ is an isomorphism. □

Chapter 4

Bernstein-Sato Theory in Positive Characteristic

One of the guiding problems in algebraic geometry is the classification of algebraic varieties. Consider the simplest case: fix an algebraically closed field $\mathbb{K}$, let $R = \mathbb{K}[x_1, \ldots, x_n]$ be a polynomial ring and let $X = V(I) \subseteq \mathbb{K}^n$ be an affine algebraic variety, where $I \subseteq R$ is an ideal. If I is generated by polynomials $f_1, \ldots, f_r \in R$, we say that X is singular at a point $x \in X$ if the differential of the function $(f_1, \ldots, f_r)\colon \mathbb{K}^n \to \mathbb{K}^r$ does not have maximal rank at x. If $Y = V(J) \subseteq \mathbb{K}^n$ is another affine algebraic variety, in order to tell if X and Y are isomorphic, one may look at their singularities.

One of the goals of Bernstein-Sato theory is to construct algebraic invariants from varieties and their singularities, in order to distinguish and classify them.

In this chapter we start by giving the basic definitions and results of the theory over the complex numbers. Then in the positive characteristic setting, we describe in greater depth the most relevant algebraic invariants that will be of interest to us in Chapter 5.

4.1. Bernstein-Sato theory in characteristic zero

Bernstein-Sato theory in characteristic zero can be developed in great generality, for instance, for any regular algebra over a field $\mathbb{K}$ of characteristic zero. For the sake of simplicity and since considering an arbitrary field of characteristic zero other than the complex numbers does not provide further insight, we shall take $\mathbb{K} = \mathbb{C}$. We refer the reader interested in the general case to [AMJNB21, Sections 3 and 5].

Let $R = \mathbb{C}[x_1, \ldots, x_n]$, fix a polynomial $f \in R$ and let $x \in \mathbb{C}^n$ be a point where $f(x) = 0$. A manner to measure how singular is f at x is to study how fast the function $1/f$ blows up at x. This motivates the following definition:

Definition 4.1. The *log-canonical threshold* of f at x is

$$\operatorname{lct}_x(f) \coloneqq \sup \left\{ \lambda \in \mathbb{R}_{>0} \;\middle|\; \int_{B(x)} \frac{1}{|f|^{2\lambda}} < \infty \text{ for some neighborhood } B(x) \text{ of } x \right\}$$

One can show that $0 < \operatorname{lct}_x(f) \leq 1$. Roughly speaking, the smaller the log-canonical threshold of f at x, the faster the function $1/|f|$ blows up at x, hence the more singular it is. In particular, when f is smooth at x, the log-canonical threshold is 1.

Example 4.2 ([BFS13, Example 2.3]). Let $f = x_1^{a_1} \cdots x_n^{a_n} \in R$ be a monomial and let $B \subseteq \mathbb{C}^n$ be an open neighborhood of the origin. In order to compute the supremum of $\lambda \geq 0$ such that the integral

$$\int_B \frac{1}{|x_1^{a_1} \cdots x_n^{a_n}|^{2\lambda}}$$

converges, one may use the change of variables theorem. Altogether, the convergence of the previous integral is equivalent to the convergence of

$$\int \frac{r_1 \cdots r_n}{r_1^{2\lambda a_1} \cdots r_n^{2\lambda a_n}} = \int \frac{1}{r_1^{2\lambda a_1 - 1} \cdots r_n^{2\lambda a_n - 1}}$$

in a neighborhood of the origin. This occurs if and only if $2\lambda a_i - 1 < 1$ for all $i = 1, \ldots, n$, therefore

$$\operatorname{lct}_0(f) = \min\left\{\frac{1}{a_1}, \ldots, \frac{1}{a_n}\right\}.$$

Rings of differential operators are helpful as well to construct algebraic invariants. For example, set $f = x_1 \in R$ and let ∂_{x_1} be the partial derivative with respect to x_1, which is a $\mathbb{C}$-linear differential operator on R. Then one has that $\partial_{x_1} x_1^{s+1} = (s+1)x_1^s$ for each integer $s \geq 0$, which suggests that the polynomial $b(s) = s + 1$ is an invariant associated to f. This same construction can be repeated for any polynomial.

Let $\mathcal{D}_{R|\mathbb{C}}$ be the ring of $\mathbb{C}$-linear differential operators on R and $f \in R$ a polynomial. Bernstein [B72] and Sato [SS90], independently and in different contexts, discovered the following fact: there exists a differential operator $P(s) \in \mathcal{D}_{R|\mathbb{C}}[s]$ and a polynomial $b(s) \in \mathbb{C}[s]$ such that

$$P(s) \cdot f^{s+1} = b(s) f^s.$$

This results in the following definition:

Definition 4.3. The *Bernstein-Sato polynomial* of a polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$, denoted by $b_f(s)$, is the minimal monic generator of the ideal

$$\left\{ b(s) \in \mathbb{C}[s] \;\middle|\; \exists P(s) \in \mathcal{D}_R[s] \text{ such that } P(s) \cdot f^{s+1} = b(s) f^s \text{ for all } s \in \mathbb{Z}_{\geq 0} \right\} \subseteq \mathbb{C}[s].$$

This polynomial is also known as the b-function of f.

Example 4.4. Let $f = x^2 + y^3 \in \mathbb{C}[x, y]$, which exhibits a singularity at the origin since the differential has rank zero. One can verify that

$$\left(\frac{1}{12} y \partial_x^2 \partial_y + \frac{1}{27}\partial_y^3 + \frac{1}{4}\partial_x s + \frac{3}{8}\partial_x^2\right) \cdot f^{s+1} = \left(s + \frac{5}{6}\right)(s+1)\left(s + \frac{7}{6}\right) f^s.$$

Furthermore, in this case, $b_f(s) = (s + \frac{5}{6})(s+1)(s + \frac{7}{6})$.

The Bernstein-Sato polynomial exhibits a number of particularly nice properties. For instance, Kashiwara proved that the roots of the Bernstein-Sato polynomial are rational and negative [K76], and Kollár showed that the largest among these is equal to the negative log-canonical threshold [K97].

A closely related invariant to the log-canonical threshold is the multiplier ideal of a polynomial, defined as follows:

Definition 4.5. Let $f \in \mathbb{C}[x_1, \dots, x_n]$. For each real number $\lambda \geq 0$, the *multiplier ideal* of f with exponent λ is

$$\mathcal{J}(f^\lambda) = \left\{ g \in \mathbb{C}[x_1, \dots, x_n] \;\middle|\; \int_{B(x)} \frac{|g|^2}{|f|^{2\lambda}} < \infty \text{ for some neighborhood } B(x) \text{ of } x \right\}.$$

Roughly speaking, the ideal $\mathcal{J}(f^\lambda)$ consists of polynomials that dampen the growth of $1/|f|^{2\lambda}$ on a neighborhood of the chosen point. Some well-known properties of these ideals are listed below:

Proposition 4.6 ([BFS13, Proposition 2.23])**.** Let $f \in \mathbb{C}[x_1, \dots, x_n]$, then:

(1) For a real number $\lambda > 0$ sufficiently small, $\mathcal{J}(f^\lambda) = R$.
(2) If $\lambda' > \lambda$, then $\mathcal{J}(f^{\lambda'}) \subseteq \mathcal{J}(f^\lambda)$.
(3) The log-canonical threshold of f is $\operatorname{lct}(f) = \sup\{\lambda \in \mathbb{R}_{\geq 0} \mid \mathcal{J}(f^\lambda) = R\}$.
(4) For each real number $\lambda > 0$, there exists $\varepsilon = \varepsilon(\lambda) > 0$ such that $\mathcal{J}(f^\lambda) = \mathcal{J}(f^{\lambda+\varepsilon})$.
(5) There exist $\lambda \in \mathbb{R}_{>0}$ such that $\mathcal{J}(f^{\lambda-\varepsilon}) \supsetneq \mathcal{J}(f^\lambda)$ for all $\varepsilon > 0$.

The proposition above shows that the multiplier ideals of a polynomial are right semicontinuous, and that there are real numbers where these ideals jump, thus motivating the following:

Definition 4.7. The *jumping numbers* of $f \in \mathbb{C}[x_1, \dots, x_n]$ are real numbers $c \geq 0$ such that $\mathcal{J}(f^c) \subsetneq \mathcal{J}(f^{\lambda-\varepsilon})$ for all $\varepsilon > 0$.

In addition, as the theorem below states, in order to know the jumping numbers of a polynomial, it suffices to compute those in the interval $[0, 1] \subseteq \mathbb{R}$:

Theorem 4.8. Let $f \in \mathbb{C}[x_1, \dots, x_n]$, then for each real number $\lambda \geq 1$,

$$\mathcal{J}(f^{\lambda+1}) = (f)\mathcal{J}(f^\lambda).$$

Consider the more general setting of an ideal $\mathfrak{a} = (f_1, \dots, f_r) \subseteq \mathbb{C}[x_1, \dots, x_n]$. It was shown by Budur, Mustaţă and Saito in [BMS06] that one can construct the corresponding Bernstein-Sato polynomial $b_{\mathfrak{a}}(s) \in \mathbb{C}[s]$, and that it satisfies the same properties as the Bernstein-Sato polynomial of a principal ideal. Likewise, for each real number $\lambda \geq 0$, it is possible to define the multiplier ideal $\mathcal{J}(\mathfrak{a}^\lambda)$.

It is worth mentioning that we have presented the definition of these ideals using tools from analysis. Instead, one can define these using algebraic or algebro-geometric tools, thus opening the theory to fields of characteristic zero other than the complex numbers. See, for instance, the survey by Benito, Faber and Smith [BFS13], or the survey by Àlvarez-Montaner, Jeffries and Núñez-Betancourt [AMJNB21].

4.2. Frobenius powers and Frobenius roots

In this section we introduce two well-known operations on the ideals of a ring of characteristic $p > 0$ which are essential for the study of singularities. The first of such operations is the Frobenius power, which we have already introduced in Section 3.2, but we repeat it here for the sake of completeness.

Definition 4.9. Let R be a ring of characteristic $p > 0$ and let $I \subseteq R$ be an ideal. For an integer $e \geq 0$, the e-th *Frobenius power* of I is the ideal

$$I^{[p^e]} := F^e(I)R = (f^{p^e} \mid f \in I).$$

By convention, we set $I^{[p^0]} = I$.

Proposition 4.10. Let R be a ring of characteristic $p > 0$ and let $I = (f_1, \dots, f_n) \subseteq R$ be an ideal generated by n elements.

(1) The Frobenius power $I^{[p^e]}$ is generated by the elements $f_1^{p^e}, \dots, f_n^{p^e}$.
(2) One has that $I^{n(p^e-1)+1} \subseteq I^{[p^e]}$ for each integer $e \geq 0$.
(3) The families of ideals $\{I^s\}_{s \geq 0}$ and $\{I^{[p^e]}\}_{e \geq 0}$ are cofinal.

Proof. (1) The ideal $I^{[p^e]}$ is generated by the elements g^{p^e} where $g \in I$, thus $g = g_1 f_1 + \cdots + g_n f_n$ for some $g_1, \dots, g_n \in R$ and $g^{p^e} = g_1^{p^e} f_1^{p^e} + \cdots + g_n^{p^e} f_n^{p^e}$, which proves $\mathfrak{a}^{[p^e]} \subseteq (f_1^{p^e}, \dots, f_n^{p^e})$. The converse inclusion is clear.

(2) The ideal $I^{n(p^e-1)+1}$ is generated by elements of the form $f_1^{s_1} \cdots f_n^{s_n}$, where $s_1, \dots, s_n \geq 0$ and $s_1 + \cdots + s_n = n(p^e - 1) + 1$. If all the s_i are $\leq p^e - 1$, one reaches a contradiction since $s_1 + \cdots + s_n \leq n(p^e - 1) < n(p^e - 1) + 1$. Therefore there is some $s_i \geq p^e$, which proves that $\mathfrak{a}^{n(p^e-1)+1} \subseteq \mathfrak{a}^{[p^e]}$.

(3) By part (2) one has $I^{n(p^e-1)+1} \subseteq I^{[p^e]}$ for all integers $e \geq 0$ and, from the definition of Frobenius power, it follows that $I^{[p^e]} \subseteq I^{p^e}$. □

As we showed in Example 3.23, the assumption that I is a finitely generated ideal of R cannot be removed.

Proposition 4.11. Let R be a ring of characteristic $p > 0$. Let $I, J \subseteq R$ be ideals and let $e \geq 0$ be an integer.

(1) For each integer $s \geq 0$, $I^{s[p^e]} = I^{[p^e]s}$.
(2) One has that $(I + J)^{[p^e]} = I^{[p^e]} + J^{[p^e]}$.

(3) One has that $(I \cdot J)^{[p^e]} = I^{[p^e]} \cdot J^{[p^e]}$.
(4) If $d \geq 0$ is an integer, $I^{[p^{d+e}]} = (I^{[p^d]})^{[p^e]}$.

Proof. Since the e-th iteration of the Frobenius is a ring endomorphism of R,

(1) $I^{s[p^e]} = F^e(I^s)R = F^e(I)^s R = I^{[p^e]s}$
(2) $(I+J)^{[p^e]} = F^e(I+J)R = F^e(I)R + F^e(J)R = I^{[p^e]} + J^{[p^e]}$
(3) $(I \cdot J)^{[p^e]} = F^e(I \cdot J)R = F^e(I)R \cdot F^e(J)R = I^{[p^e]} \cdot J^{[p^e]}$
(4) $I^{[p^{d+e}]} = F^{d+e}(I)R = F^e(F^d(I)R)R = (I^{[p^d]})^{[p^e]}$. □

The second operation that we define on an ideal I of a regular F-finite ring is the Frobenius root, denoted by $I^{[1/p^e]}$. This class of ideals was first defined in [AMBL05] for principal ideals, and afterwards generalized to arbitrary ideals in [BMS08]

Definition 4.12 ([AMBL05], [BMS08, Definition 2.2])**.** Let R be a regular F-finite ring of characteristic $p > 0$ and let $I \subseteq R$ be an ideal. For an integer $e \geq 0$, the e-th *Frobenius root* of I, denoted by $I^{[1/p^e]}$, is the smallest ideal of R in the sense of inclusion such that $I \subseteq (I^{[1/p^e]})^{[p^e]}$. By convention, we set $I^{[1/p^0]} = \mathfrak{a}$.

Our next task is to show that Frobenius roots are well-defined:

Lemma 4.13. Let M be a finitely generated projective module over a ring R.

(1) Let $\{N_\lambda\}_\lambda$ be a collection of R-modules. Then there is an isomorphism $(\prod_\lambda R/J_\lambda) \otimes_R M \cong \prod_\lambda (M/J_\lambda M)$.
(2) Let $\{J_\lambda\}_\lambda$ be a collection of ideals of R. Then there is an equality $\bigcap_\lambda (J_\lambda M) = \left(\bigcap_\lambda J_\lambda\right) M$.

Proof. (1) As M is a finitely generated projective module, it is finitely presented (for instance, see [Stacks, Lemma 10.78.2]), thus there exist integers $m, n \geq 0$ such that $R^{\oplus m} \to R^{\oplus n} \to M \to 0$. Tensoring with $\prod_\lambda N_\lambda$ is a right-exact functor, thereby one obtains the following diagram with exact rows:

$$\begin{array}{ccccccc}
\left(\prod_\lambda N_\lambda\right) \otimes_R R^{\oplus m} & \longrightarrow & \left(\prod_\lambda N_\lambda\right) \otimes_R R^{\oplus n} & \longrightarrow & \left(\prod_\lambda N_\lambda\right) \otimes_R M & \longrightarrow & 0 \\
\downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow & & \\
\prod_\lambda (N_\lambda \otimes_R R^{\oplus m}) & \longrightarrow & \prod_\lambda (N_\lambda \otimes_R R^{\oplus n}) & \longrightarrow & \prod_\lambda (N_\lambda \otimes_R M) & \longrightarrow & 0.
\end{array}$$

The map $u\colon (\prod_\lambda N_\lambda) \otimes_R R^{\oplus m} \to \prod_\lambda (N_\lambda \otimes_R R^{\oplus m})$ is easily seen to be an isomorphism as it sends $(n_\lambda)_\lambda \otimes_R (r_1, \dots, r_m) \mapsto (n_\lambda \otimes_R (r_1, \dots, r_m))_\lambda$, and similarly for v. Extending on the right with zeros and applying the five lemma, it follows that $(\prod_\lambda N_\lambda) \otimes_R M \cong \prod_\lambda (N_\lambda \otimes_R M)$.

(2) Consider the map $\varphi\colon R \to \prod_\lambda R/J_\lambda$ sending $f \mapsto (\overline{f}_\lambda)_\lambda$, where $\overline{f}_\lambda$ denotes the equivalence class of f in R/J_λ. An element $f \in R$ is sent to zero by φ if and only if f is in each J_λ, thus

one has the following short exact sequence:

$$0 \longrightarrow \bigcap_\lambda J_\lambda \longrightarrow R \xrightarrow{\varphi} \prod_\lambda \frac{R}{J_\lambda} \longrightarrow 0.$$

Because M is, in particular, a flat R-module, the functor $- \otimes_R M$ is exact, thus one obtains the following diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \left(\bigcap_\lambda J_\lambda\right) \otimes_R M & \longrightarrow & M & \xrightarrow{\varphi\otimes M} & \left(\prod_\lambda \frac{R}{J_\lambda}\right) \otimes_R M & \longrightarrow & 0 \\ & & \downarrow\wr & & \| & & \downarrow\wr & & \\ 0 & \longrightarrow & \left(\bigcap_\lambda J_\lambda\right) M & \longrightarrow & M & \xrightarrow{\psi} & \left(\prod_\lambda \frac{M}{J_\lambda M}\right) & \longrightarrow & 0. \end{array}$$

The map $\psi\colon M \to \prod_\lambda (M/J_\lambda M)$ sends $m \mapsto (\overline{m}_\lambda)_\lambda$, thus its kernel is $\ker\psi = \bigcap_\lambda (J_\lambda M)$. By exactness, it follows that $\left(\bigcap_\lambda J_\lambda\right) M = \bigcap_\lambda (J_\lambda M)$. □

Proposition 4.14. Let R be a regular F-finite ring of characteristic $p > 0$ and let $I \subseteq R$ be an ideal. For each integer $e \geq 0$, the e-th Frobenius root of I exists.

Proof. Consider the collection of ideals $\mathcal{A} = \{J \mid J \text{ is an ideal of } R \text{ and } I \subseteq J^{[p^e]}\}$. We claim that:

$$\Big(\bigcap_{J\in\mathcal{A}} J\Big)^{[p^e]} = \bigcap_{J\in\mathcal{A}} J^{[p^e]}.$$

The R-module $F^e_* R$ is flat by Kunz's theorem and, in particular, it is projective. Then by Lemma 4.13,

$$F^e_*\Big(\bigcap_{J\in\mathcal{A}} J\Big)^{[p^e]} = \Big(\bigcap_{J\in\mathcal{A}} J\Big) F^e_* R = \bigcap_{J\in\mathcal{A}} (J F^e_* R) = \bigcap_{J\in\mathcal{A}} (F^e_* J^{[p^e]}),$$

which proves the equality. By construction of the collection $\mathcal{A}$, it follows that $I \subseteq \left(\bigcap_{J\in\mathcal{A}} J\right)^{[p^e]}$ and that $\bigcap_{J\in\mathcal{A}} J$ is the smallest ideal containing I in its Frobenius power. As a result, the Frobenius root of I exists and is equal to $\bigcap_{J\in\mathcal{A}} J$. □

In Section 3.3 we introduced p^{-e}-linear maps, namely, additive maps $\varphi\colon R \to R$ such that $\varphi(r^{p^e} x) = r\varphi(x)$ for all $r, x \in R$. Furthermore, we showed that the set of p^{-e}-linear maps is isomorphic to $\operatorname{Hom}_R(F^e_* R, R)$. We give these maps a name of their own:

Definition 4.15 ([B13, Definition 2.1])**.** Let R be an F-finite ring of characteristic $p > 0$. For an integer $e \geq 0$, the set of *Cartier operators of level* e *on* R is

$$\mathcal{C}^e_R \coloneqq \operatorname{Hom}_R(F^e_* R, R).$$

In particular, when $e = 0$ one has $F^0_* R \cong R$, hence $\mathcal{C}^0_R \cong \operatorname{End}_R(R) \cong R$.

As proven in Proposition 3.31, for each integer $e \geq 0$ the set of Cartier operators of level e has an R-module structure given by $(r \cdot \varphi)(F^e_* f) := r\varphi(F^e_* f)$, where $r, f \in R$, and $\varphi \in \mathcal{C}^e_R$.

Next we define the algebra of Cartier operators on R. Although we will not make use of this algebra, it is interesting in its own right.

Definition 4.16 ([B13, Definition 2.2]). Let R be an F-finite ring of characteristic $p > 0$. The *algebra of Cartier operators on* R is

$$\mathcal{C}_R := \bigoplus_{e=0}^{\infty} \mathcal{C}^e_R.$$

By the proposition below, this algebra admits, in fact, an R-algebra structure.

Proposition 4.17. Let R be a regular F-finite ring of characteristic $p > 0$.

(1) Let $\alpha \in \mathcal{C}^d_R$ and $\beta \in \mathcal{C}^e_R$ be Cartier operators of levels d and e, respectively. Then the composition $\alpha \circ \beta$ is a Cartier operator of level $d + e$, defined by

$$(\alpha \circ \beta)(F^{d+e}_* f) = \alpha(F^d_* \beta(F^e_* f)).$$

In particular, $\mathcal{C}^d_R \circ \mathcal{C}^e_R = \{\alpha \circ \beta \mid \alpha \in \mathcal{C}^d_R, \beta \in \mathcal{C}^e_R\} \subseteq \mathcal{C}^{d+e}_R$.

(2) The algebra of Cartier operators on R is a non-commutative R-algebra.

Proof. (1) Use the characterization of Cartier operators as p^{-e}-linear maps. The composition $\alpha \circ \beta$ is an additive map $R \to R$, as is every composition of additive maps, and for each $r, f \in R$, one has that

$$(\alpha \circ \beta)(r^{p^{d+e}} f) = \alpha(\beta(r^{p^{d+e}} f)) = \alpha(r^{p^d} \beta(f)) = r(\alpha \circ \beta)(f),$$

thus $\alpha \circ \beta$ is a $p^{-(d+e)}$-linear map.

(2) As previously noted, $\mathcal{C}^0_R \cong R$, therefore the structure map $R \to \mathcal{C}_R$ is the injection $R \xrightarrow{\sim} \mathcal{C}^0_R \hookrightarrow \mathcal{C}_R$. It follows from the definition of $\mathcal{C}_R$ that it is an abelian group with respect to point-wise addition of morphisms, with the zero map as the neutral element. Multiplication on $\mathcal{C}_R$ is defined by composition of morphisms as given in part (1), where the neutral element is the identity map of R, which is a Cartier operator of level 0. As composition of morphisms is in general not commutative, $\mathcal{C}_R$ is a non-commutative R-algebra. □

For an ideal $I \subseteq R$, denote by $\mathcal{C}^e_R \cdot I$ the following ideal of R:

$$\mathcal{C}^e_R \cdot I = (\varphi(F^e_* f) \mid \varphi \in \mathcal{C}^e_R, f \in I).$$

Cartier operators and the ideals of the form $\mathcal{C}^e_R \cdot I$ prove useful to give a characterization of the Frobenius roots.

Proposition 4.18 ([BMS08, Proposition 2.5]). Let R be an F-finite ring of characteristic $p > 0$. Fix an integer $e \geq 0$ and suppose that $F^e_* R$ is a free R-module with basis $\{F^e_* x_1, \ldots, F^e_* x_n\}$.

Let $I = (f_1, \dots, f_r)$ be an ideal of R and for each generator write

$$F^e_* f_i = \sum_{j=1}^{n} f_{ij} F^e_* x_j.$$

Then the e-th Frobenius root of I is generated by the f_{ij}, that is to say,

$$I^{[1/p^e]} = \mathcal{C}^e_R \cdot I = (f_{ij} \mid 1 \leq i \leq r, 1 \leq j \leq n).$$

In particular, $\mathcal{C}^e_R \cdot I$ does not depend on the basis.

Proof. For each basis element $F^e_* x_i$, let $(F^e_* x_i)^* \colon F^e_* R \to R$ be the map given by

$$(F^e_* x_i)^*(F^e_* x_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases}$$

and extend it linearly to $F^e_* R$. It follows that $(F^e_* x_i)^*$ is the dual of $F^e_* x_i$, and that it is a Cartier operator of level e. Furthermore, each Cartier operator $\varphi \in \mathcal{C}^e_R$ is an R-linear combination of the $(F^e_* x_i)^*$, thus the second equality follows and we need only prove that

$$I^{[1/p^e]} = (f_{ij} \mid 1 \leq i \leq r, 1 \leq j \leq n).$$

On the one hand, by assumption one has $f_i = \sum_{j=1}^n f_{ij}^{p^e} x_j$ for each generator f_i of I, therefore $I \subseteq (f_{ij} \mid 1 \leq i \leq r, 1 \leq j \leq n)^{[p^e]}$. It follows from the minimality of the Frobenius root that $I^{[1/p^e]} \subseteq (f_{ij} \mid 1 \leq i \leq r, 1 \leq j \leq n)$. For the converse inclusion, choose generators for the ideal $I^{[1/p^e]}$, that is $I^{[1/p^e]} = (g_1, \dots, g_m)$. Then since $I \subseteq (I^{[1/p^e]})^{[p^e]}$, one can write $f_i = \sum_{k=1}^m \tilde{f}_{ik} g_k^{p^e}$ for some $\tilde{f}_{ik} \in R$. In turn, write $F^e_* \tilde{f}_{ik} = \sum_{j=1}^n h_{ijk} F^e_* x_j$ for some $h_{ijk} \in R$. Then $F^e_* f_i$ reads

$$F^e_* f_i = \sum_{k=1}^{m} g_k F^e_* \tilde{f}_{ik} = \sum_{j=1}^{n} \left(\sum_{k=1}^{m} h_{ijk} g_k \right) F^e_* x_j,$$

and since the $F^e_* x_j$ are a basis, it follows that $f_{ij} = \sum_{k=1}^m h_{ijk} g_k \in I^{[1/p^e]}$, thus proving that $(f_{ij} \mid 1 \leq i \leq r, 1 \leq j \leq n) \subseteq I^{[1/p^e]}$. □

Let R be a regular F-finite ring containing a perfect field $\mathbb{F}$ of characteristic $p > 0$. Recall that given an integer $e \geq 0$, $\mathcal{D}^{(e)}_R = \operatorname{Hom}_{R^{p^e}}(R, R)$ denotes the set of $\mathbb{F}$-linear differential operators on R of level e (see Section 3.2). Given an ideal $I \subseteq R$, we denote by $\mathcal{D}^{(e)}_R \cdot I$ the following ideal of R:

$$\mathcal{D}^{(e)}_R \cdot I = (\xi(f) \mid \xi \in \mathcal{D}^{(e)}_R, f \in I).$$

As Theorem 3.32 suggests and the proposition below confirms, there is a close interplay between differential operators, p^e-linear maps and Cartier operators.

Proposition 4.19 ([QG21a, Lemma 2.3])**.** Let R be a regular F-finite ring, let $I, J \subseteq R$ be ideals and fix an integer $e \geq 0$. Then:

(1) One has that $\mathcal{D}_R^{(e)} \cdot I = (\mathcal{C}_R^e \cdot I)^{[p^e]}$.
(2) One has that $\mathcal{C}_R^e \cdot I \subseteq J$ if and only if $I \subseteq J^{[p^e]}$.
(3) One has that $\mathcal{D}_R^{(e)} \cdot I = \mathcal{D}_R^{(e)} \cdot J$ if and only if $\mathcal{C}_R^e \cdot I = \mathcal{C}_R^e \cdot J$.

Proof. (1) For the proof of this statement it will be convenient the characterization of Cartier operators as p^{-e}-linear maps $\varphi\colon R \to R$. To show $(\mathcal{C}_R^e \cdot I)^{[p^e]} \subseteq \mathcal{D}_R^{(e)} \cdot I$, we note that the ideal $(\mathcal{C}_R^e \cdot I)^{[p^e]}$ is generated by elements of the form $\varphi(f)^{p^e} = (F^e\varphi)(f)$, where $f \in R$ and $\varphi \in \mathcal{C}_R^e$. Now the e-th iteration of the Frobenius is a p^e-linear map and, by Theorem 3.32, the composition $F^e\varphi$ is a differential operator of level e, whence the inclusion follows.

As for $\mathcal{D}_R^{(e)} \cdot I \subseteq (\mathcal{C}_R^e \cdot I)^{[p^e]}$, a differential operator $\xi \in \mathcal{D}_R^{(e)} = \operatorname{Hom}_{R^{p^e}}(R, R)$ can be factored as $\xi = \psi\varphi$, where ψ is a p^e-linear map and $\varphi \in \mathcal{C}_R^e$. Furthermore, ψ is of the form $\psi(f) = rf^{p^e}$ for some $r \in R$ by Proposition 3.28; in particular each $r \in R$ induces a p^e-linear map. As a result, $\mathcal{D}_R^{(e)} \cdot I$ is generated by elements of the form $(\psi\varphi)(f) = r\varphi(f)^{p^e}$, which proves the containment.

(2) Suppose that $\mathcal{C}_R^e \cdot I \subseteq J$; then by definition of Frobenius root, $I \subseteq (\mathcal{C}_R^e \cdot I)^{[p^e]} \subseteq J^{[p^e]}$. Conversely, if $I \subseteq J^{[p^e]}$, by the minimality of Frobenius roots one has $\mathcal{C}_R^e \cdot I \subseteq J$.

(3) Suppose that $\mathcal{D}_R^{(e)} \cdot I = \mathcal{D}_R^{(e)} \cdot J$, which by part 4.19.(1) is equivalent to $(\mathcal{C}_R^e \cdot I)^{[p^e]} = (\mathcal{C}_R^e \cdot J)^{[p^e]}$. Applying Frobenius roots to both sides and using Proposition 4.21 yields $\mathcal{C}_R^e \cdot I = \mathcal{C}_R^e \cdot J$. The reverse implication folllows from part 4.19.(1). □

The proposition above allows us to give a characterization of Frobenius roots in terms of Cartier operators in a more general setting than when $F_*^e R$ is a free R-module:

Proposition 4.20. Let R be a regular F-finite ring of characteristic $p > 0$ and let $I \subseteq R$ be an ideal. Denote by $\mathcal{D}_R$ the ring of $\mathbb{F}$-linear differential operators on R, where $\mathbb{F}$ is a perfect field of characteristic $p > 0$ contained in R. For each integer $e \geq 0$,

$$\mathcal{C}_R^e \cdot I = I^{[1/p^e]}.$$

Proof. As the identity map of R is a differential operator of level zero and $\mathcal{D}_R^{(0)} \subseteq \mathcal{D}_R^{(e)}$, one has that $I \subseteq \mathcal{D}_R^{(e)} \cdot I = (\mathcal{C}_R^e \cdot I)^{[p^e]}$, where the second equality is due to Proposition 4.19. The minimality of Frobenius roots implies that $I^{[1/p^e]} \subseteq \mathcal{C}_R^e \cdot I$. In order to show that the containment is in fact an equality, it suffices to verify it locally, i.e. check that it holds for every localization of R at a prime ideal $\mathfrak{p} \subseteq R$. At the local ring $R_\mathfrak{p}$, the ideal I extends to the ideal $IR_\mathfrak{p}$, hence without loss of generality we may assume that R is a regular local ring. Now $F_*^e R$ is flat by Kunz's theorem, and is defined over a a local ring, thus it is locally free, which puts us in the situation of Proposition 4.18, from where the equality follows. □

In view of the proposition above, the notations $I^{[1/p^e]}$ and $\mathcal{C}_R^e \cdot I$ for Frobenius roots coincide. Hereinafter, we shall almost exclusively use second notation. To conclude this section we prove several handy properties of Frobenius roots.

Proposition 4.21. Let R be a regular F-finite ring of characteristic $p > 0$ and let $I, J \subseteq R$ be ideals. For each integer $e \geq 0$,

$$I \cdot (\mathcal{C}_R^e \cdot J) = \mathcal{C}_R^e \cdot (I^{[p^e]} \cdot J).$$

In particular, $\mathcal{C}_R^e \cdot I^{[p^{d+e}]} = I \cdot (\mathcal{C}_R^e \cdot I^{[p^d]})$.

Proof. The ideal $I \cdot (\mathcal{C}_R^e \cdot J($ is generated by elements of the form $f\varphi(F_*^e g)$, with $f \in I$, $g \in J$ and $\varphi \in \mathcal{C}_R^e$. As φ is R-linear, one has $f\varphi(F_*^e g) = \varphi(F_*^e f^{p^e} g)$, which is an element in the ideal $\mathcal{C}_R^e \cdot (I^{[p^e]} \cdot J)$. The other inclusion follows from the same observation. The second statement is a consequence of the first and Proposition 4.11. □

Proposition 4.22 ([QG21b, Proposition II.48])**.** Let R be a regular F-finite ring of characteristic $p > 0$ and $I \subseteq R$ an ideal. Let $d, e \geq 0$ be integers.

(1) One has that $\mathcal{C}_R^e \cdot \mathcal{C}_R^d \cdot I = \mathcal{C}_R^{d+e} \cdot I$.
(2) One has that $\mathcal{C}_R^e \cdot I = \mathcal{C}_R^{d+e} \cdot I^{[p^d]}$.

Proof. (1) The inclusion ($\subseteq$) follows from Proposition 4.17 because $\mathcal{C}_R^e \circ \mathcal{C}_R^d \subseteq \mathcal{C}_R^{d+e}$. For the converse, let $F^d \colon R \to F_*^d R$ be the d-th iteration of the Frobenius, i.e. the homomorphism $f \mapsto F_*^d f^{p^d} = f F_*^d 1$. Fix an splitting $\sigma \colon F_*^d R \to R$ of F^d, that is to say, a map satisfying $\sigma \circ F^d = \mathrm{Id}_R$, which exists by [QG21b, Proposition II.6]. Note that the right-hand side is generated by elements of the form $\varphi(F_*^{d+e} f)$, where $\varphi \in \mathcal{C}_R^{d+e}$ and $f \in R$, therefore $\varphi(F_*^{d+e} f) = (\sigma \circ F^d \circ \varphi)(F_*^{d+e} f)$. The containment follows from the observation that $F^d \circ \varphi \in \mathcal{C}_R^e$ is a p^{-e}-linear map and, by definition of the splitting, one has that $\sigma \in \mathcal{C}_R^d$.

(2) The equality follows from Proposition 4.21 and (1), since

$$\mathcal{C}_R^e \cdot I = \mathcal{C}_R^e \cdot (I \cdot \mathcal{C}_R^e \cdot R) = \mathcal{C}_R^e \cdot \mathcal{C}_R^d I^{[p^d]} = \mathcal{C}_R^{d+e} \cdot I^{[p^d]}.$$ □

Proposition 4.23. Let R be a regular F-finite ring of characteristic $p > 0$ and let $I \subseteq R$ be an ideal.

(1) If $f \colon R \to S$ is a homomorphism of regular F-finite rings, $\mathcal{C}_S^e \cdot (f(I)S) \subseteq f(\mathcal{C}_R^e \cdot I)S$.
(2) If $f \colon R \xrightarrow{\sim} S$ is an isomorphism of regular F-finite rings, $\mathcal{C}_S^e \cdot (f(I)S) = f(\mathcal{C}_R^e \cdot I)S$.

Proof. (1) By definition of Frobenius root, $I \subseteq (\mathcal{C}_R^e \cdot I)^{[p^e]}$, hence

$$f(I)S \subseteq f((\mathcal{C}_R^e \cdot I)^{[p^e]})S = f(\mathcal{C}_R^e \cdot I)^{[p^e]} S = (f(\mathcal{C}_R^e \cdot I)S)^{[p^e]}$$

and, by the minimality, one concludes that $\mathcal{C}_S^e \cdot (f(I)S) \subseteq f(\mathcal{C}_R^e \cdot I)S$.

(2) The containment ($\subseteq$) is due to part (1). For the converse consider the ideal $J = f(I)S$; then applying part (1) gives $\mathcal{C}_R^e \cdot (f^{-1}(J)R) \subseteq f^{-1}(\mathcal{C}_S^e \cdot J)R$, that is to say,

$$\mathcal{C}_R^e \cdot I \subseteq f^{-1}(\mathcal{C}_S^e \cdot f(I)S)R \subseteq f^{-1}(f(\mathcal{C}_R^e \cdot I)S) = \mathcal{C}_R^e \cdot I,$$

whence $\mathcal{C}_R^e \cdot I = f^{-1}(\mathcal{C}_S^e \cdot f(I)S)R$, and the assertion follows. □

Proposition 4.24 ([BMS08, Lemma 2.7])**.** Let R be a regular F-finite ring and $I \subseteq R$ an ideal. If $W \subseteq R$ is a multiplicative subset, then for each integer $e \geq 0$,

$$W^{-1}(\mathcal{C}_R^e \cdot I) = \mathcal{C}_{W^{-1}R}^e \cdot (W^{-1}I).$$

Proof. The ideal $\mathcal{C}_R^e \cdot I$ may be viewed as the image of the map $\Phi\colon \mathcal{C}_R^e \otimes_R R \to R$ sending $\varphi \otimes f \mapsto \varphi(F_*^e f)$. This gives a short exact sequence of R-modules

$$\mathcal{C}_R^e \otimes_R I \xrightarrow{\Phi} \mathcal{C}_R^e \cdot I \longrightarrow 0$$

and, since localization is an exact functor, by localizing at the multiplicative set W one obtains an exact sequence of $W^{-1}R$-modules:

$$\mathcal{C}_{W^{-1}R}^e \otimes_{W^{-1}R} W^{-1}I \xrightarrow{W^{-1}\Phi} W^{-1}(\mathcal{C}_R^e \cdot I) \longrightarrow 0.$$

The isomorphism of the left piece follows from $W^{-1}(\mathcal{C}_R^e \otimes_R I) \cong \mathcal{C}_{W^{-1}R}^e \otimes_{W^{-1}R} W^{-1}I$ and

$$W^{-1}\mathcal{C}_R^e = W^{-1}\operatorname{Hom}_R(F_*^e R, R) \cong \operatorname{Hom}_{W^{-1}R}(F_*^e W^{-1}R, W^{-1}R) = \mathcal{C}_{W^{-1}R}^e.$$

Now the localized map $W^{-1}\Phi$ sends $\psi \otimes g \mapsto \psi(F_*^e g)$, hence its image is $\mathcal{C}_{W^{-1}R}^e \cdot (W^{-1}I)$. As the sequence is exact, there is an equality of $W^{-1}R$-modules $\mathcal{C}_{W^{-1}R}^e \cdot (W^{-1}I) = W^{-1}(\mathcal{C}_R^e \cdot I)$, and thus an equality of ideals in the ring $W^{-1}R$, which proves the proposition. ☐

Proposition 4.25. Let $R = \mathbb{F}[x_1, \dots, x_m]$ and $S = \mathbb{F}[x_1, \dots, x_m, \dots, x_n]$ be polynomial rings over a perfect field $\mathbb{F}$ of characteristic $p > 0$, and let $I \subseteq R$ be an ideal. For each integer $e \geq 0$,

$$(\mathcal{C}_R^e \cdot I)S = \mathcal{C}_S^e \cdot (IS).$$

Proof. By Proposition 2.33, $F_*^e R$ is a free R-module and $F_*^e S$ is a free S-module. Let $\mathcal{B}(F_*^e R) = \{F_*^e x_1^{a_1} \cdots x_m^{a_m} \mid 0 \leq a_1, \dots, a_m \leq p^e\}$ and $\mathcal{B}(F_*^e S) = \{F_*^e x_1^{b_1} \cdots x_n^{b_n} \mid 0 \leq b_1, \dots, b_n \leq p^e\}$ be the standard bases, which satisfy $\mathcal{B}(F_*^e R) \subseteq \mathcal{B}(F_*^e S)$. For the sake of simplicity in notation, denote by $F_*^e x^a$ an element in the basis $\mathcal{B}(F_*^e R)$.

Let $f_1, \dots, f_r \in R$ be generators for the ideal I; note that the extension IS is generated by the same elements. Since $R \subseteq S$ and $\mathcal{B}(F_*^e R) \subseteq \mathcal{B}(F_*^e S)$, the expression of the $F_*^e f_i$ in both basis is the same, namely

$$F_*^e f_i = \sum_{F_*^e x^a} f_{ia} F_*^e x^a.$$

On the one hand, $\mathcal{C}_R^e \cdot I = (f_{ia} \mid 1 \leq i \leq r, F_*^e x^a \in \mathcal{B}(F_*^e R))$ by Proposition 4.18, and the extension $(\mathcal{C}_R^e \cdot I)S$ is generated by the same elements. On the other hand, $\mathcal{C}_S^e \cdot I$ is generated by the f_{ia} as well, from where the equality $(\mathcal{C}_R^e \cdot I)S = \mathcal{C}_S^e \cdot (IS)$ follows. ☐

4.3. The ν-invariants

In this and subsequent sections, we present some well-known algebraic invariants involved in the positive characteristic setting of Bernstein-Sato theory. The first invariant that we study are the ν-invariants of an ideal $\mathfrak{a}$ of a regular ring of characteristic $p > 0$. These were introduced by Mustaţă , Takagi and Watanabe in [MTW05], with the purpose of studying the non-vanishing of the direct summands of a local cohomology module associated to the ideal $\mathfrak{a}$.

Given an ideal I in a ring R, the powers of I give a descending filtration $\cdots \subseteq I^3 \subseteq I^2 \subseteq I$. In a well-behaved situation, such as when $I \neq 0$ and R is reduced ring with no nilpotents, the powers of I are all different. Suppose, in addition, that R is a regular F-finite ring. Then one can compute the e-th Frobenius root of each power of I, giving in turn a descending chain

$$\cdots \subseteq \mathcal{C}_R^e \cdot I^3 \subseteq \mathcal{C}_R^e \cdot I^2 \subseteq \mathcal{C}_R^e \cdot I.$$

In this situation, however, Frobenius roots of consecutive powers of I may be equal, or different, i.e. there is a "jump". This behavior is illustrated by the following example:

Example 4.26. Let $R = \mathbb{F}[x_1, \ldots, x_n]$ be a polynomial ring over a perfect field $\mathbb{F}$ of characteristic $p > 0$, and let $\mathfrak{m} = (x_1, \ldots, x_n)$ be the homogeneous maximal ideal. Fix an integer $e \geq 0$ and let $N = n(p^e - 1)$; we claim that $\mathcal{C}_R^e \cdot \mathfrak{m}^N = R$. Indeed, one has that $(x_1 \cdots x_n)^{p^e-1} \in \mathfrak{m}^N$ and $F_*^e(x_1 \cdots x_n)^{p^e-1}$ is an element in the standard basis of $F_*^e R$ as given in Proposition 2.33 and Definition 2.34. It follows from Proposition 4.18 that $1 \in \mathcal{C}_R^e \cdot \mathfrak{m}^N$, which proves the claim. In particular,

$$\mathcal{C}_R^e \cdot \mathfrak{m}^N = \mathcal{C}_R^e \cdot \mathfrak{m}^{N-1} = \cdots = \mathcal{C}_R^e \cdot \mathfrak{m}^2 = \mathcal{C}_R^e \cdot \mathfrak{m} = R.$$

In contrast, $\mathcal{C}_R^e \cdot \mathfrak{m}^{N+1} = \mathfrak{m}$. By Proposition 4.19 the containment $\mathcal{C}_R^e \cdot \mathfrak{m}^{N+1} \subseteq \mathfrak{m}$ is equivalent to $\mathfrak{m}^{N+1} \subseteq \mathfrak{m}^{[p^e]}$, and the latter holds by Proposition 4.10. Regarding the converse, for each indeterminate x_i that generates $\mathfrak{m}$ one has $x_i(x_1 \cdots x_n)^{p^e-1} \in \mathfrak{m}^{N+1}$, and $F_*^e x_i(x_1 \cdots x_n)^{p^e-1} = x_i F_*^e(x_1 \cdots \hat{x}_i \cdots x_n)^{p^e-1}$, where $\hat{x}_i$ denotes that x_i is not in the product. Again by Proposition 4.18, it follows that $x_i \in \mathcal{C}_R^e \cdot \mathfrak{m}^{N+1}$, which shows that $\mathcal{C}_R^e \cdot \mathfrak{m}^{N+1} = \mathfrak{m}$.

Proposition 4.27. Let R be a Noetherian ring of characteristic $p > 0$, and let $\mathfrak{a}, J \subseteq R$ be ideals such that $\mathfrak{a} \subseteq \operatorname{rad} J$. For each integer $e \geq 0$, there exists $\ell \in \mathbb{Z}_{\geq 0}$ such that $\mathfrak{a}^\ell \subseteq J^{[p^e]}$.

Proof. By assumption the ideals $\mathfrak{a}$ and J are finitely generated, hence there exists an integer $n \geq 0$ such that $\mathfrak{a}^n \subseteq J$. Furthermore, the families of ideals $\{J^s\}_{s \geq 0}$ and $\{J^{[p^e]}\}_{e \geq 0}$ are cofinal by Proposition 4.10, thus one can choose $m \geq 0$ large enough so that $J^m \subseteq J^{[p^e]}$. By letting $\ell = mn$, it follows that $\mathfrak{a}^\ell \subseteq J^{[p^e]}$. □

Definition 4.28 ([MTW05])**.** Let R be a regular F-finite ring and fix an ideal $\mathfrak{a} \subseteq R$. Let $J \subseteq R$ be an ideal containing $\mathfrak{a}$ in its radical. Given an integer $e \geq 0$, define

$$\nu_{\mathfrak{a}}^J(p^e) := \max\left\{\ell \geq 0 \mid \mathfrak{a}^\ell \not\subseteq J^{[p^e]}\right\}.$$

The set of *ν-invariants of $\mathfrak{a}$ of level e*, denoted by $\nu_{\mathfrak{a}}^{\bullet}(p^e)$, is the set of integers $\nu_{\mathfrak{a}}^{J}(p^e)$ obtained as J ranges through the ideals of R containing $\mathfrak{a}$ in its radical, that is

$$\nu_{\mathfrak{a}}^{\bullet}(p^e) := \left\{ \nu_{\mathfrak{a}}^{J}(p^e) \;\middle|\; J \subseteq R \text{ such that } \mathfrak{a} \subseteq \operatorname{rad} J \right\}.$$

Example 4.29. Following Example 4.26, we have shown that

$$\nu_{\mathfrak{m}}^{\mathfrak{m}}(p^e) = n(p^e - 1).$$

In fact, the application of Theorem 5.47 to $\mathfrak{m}$ viewed as the ideal of maximal minors of a row matrix shows that the set of ν-invariants of $\mathfrak{m}$ of level e is

$$\nu_{\mathfrak{m}}^{\bullet}(p^e) = \{ sp^e + n(p^e - 1) \mid s \in \mathbb{Z}_{\geq 0} \}.$$

Proposition 4.30 ([QG21b, Proposition IV.12])**.** Let R be a regular F-finite ring and $\mathfrak{a} \subseteq R$ be an ideal. Then the set of ν-invariants of $\mathfrak{a}$ of level $e \geq 0$ is

$$\begin{aligned} \nu_{\mathfrak{a}}^{\bullet}(p^e) &= \left\{ \ell \in \mathbb{Z}_{\geq 0} \;\middle|\; \mathcal{C}_R^e \cdot \mathfrak{a}^{\ell+1} \neq \mathcal{C}_R^e \cdot \mathfrak{a}^{\ell} \right\} \\ &= \left\{ \ell \in \mathbb{Z}_{\geq 0} \;\middle|\; \mathcal{D}_R^{(e)} \cdot \mathfrak{a}^{\ell+1} \neq \mathcal{D}_R^{(e)} \cdot \mathfrak{a}^{\ell} \right\}. \end{aligned}$$

Proof. The second equality follows from Proposition 4.19, thus we need only show that $\nu_{\mathfrak{a}}^{\bullet}(p^e) = \left\{ \ell \in \mathbb{Z}_{\geq 0} \;\middle|\; \mathcal{C}_R^e \cdot \mathfrak{a}^{\ell+1} \neq \mathcal{C}_R^e \cdot \mathfrak{a}^{\ell} \right\}$. Let $\ell = \nu_{\mathfrak{a}}^{J}(p^e)$, that is to say, $\mathfrak{a}^{\ell} \not\subseteq J^{[p^e]}$ and $\mathfrak{a}^{\ell+1} \subseteq J^{[p^e]}$. By Proposition 4.19, this is equivalent to $\mathcal{C}_R^e \cdot \mathfrak{a}^{\ell} \not\subseteq J$ and $\mathcal{C}_R^e \cdot \mathfrak{a}^{\ell+1} \subseteq J$. Since $\mathcal{C}_R^e \cdot \mathfrak{a}^{\ell+1} \subseteq \mathcal{C}_R^e \cdot \mathfrak{a}^{\ell}$ and there exists an element $f \in \mathcal{C}_R^e \cdot \mathfrak{a}^{\ell} - J$, it follows that the containment is strict.

Conversely, let $\ell \in \mathbb{Z}_{\geq 0}$ be an integer such that $\mathcal{C}_R^e \cdot \mathfrak{a}^{\ell+1} \subsetneq \mathcal{C}_R^e \cdot \mathfrak{a}^{\ell}$, which is equivalent to $\mathfrak{a}^{\ell+1} \subsetneq (\mathcal{C}_R^e \cdot \mathfrak{a}^{\ell})^{[p^e]}$ by Proposition 4.19, and let $J := \mathcal{C}_R^e \cdot \mathfrak{a}^{\ell+1}$. As the identity map of R is a differential operator of level 0, it follows that $\mathfrak{a}^{\ell+1} \subseteq \mathcal{D}_R^{(e)} \cdot \mathfrak{a}^{\ell+1}$, which implies $\mathfrak{a}^{\ell+1} \subseteq J^{[p^e]}$. In particular, J contains $\mathfrak{a}$ in its radical. Moreover, $\mathfrak{a}^{\ell} \not\subseteq J^{[p^e]}$, for otherwise $\mathcal{C}_R^e \cdot \mathfrak{a}^{\ell} \subseteq J = \mathcal{C}_R^e \cdot \mathfrak{a}^{\ell+1}$, but this contradicts the original assumption, thereby $\ell = \nu_{\mathfrak{a}}^{J}(p^e)$. ☐

Proposition 4.31. Let R be a regular F-finite ring and let $I = (f_1, \ldots, f_n) \subseteq R$ be an ideal generated by n elements. For each integer $e \geq 0$,

$$\nu_I^I(p^e) \leq n(p^e - 1).$$

Proof. This follows from Proposition 4.10 since $I^{n(p^e-1)+1} \subseteq I^{[p^e]}$. ☐

To finish the section and to illustrate how the computations work, we compute the ν-invariants of a monomial ideal in a polynomial ring, that is, an ideal generated by monomials.

Example 4.32. Let $R = \mathbb{F}[x, y]$ be a polynomial ring over a perfect field $\mathbb{F}$ of characteristic $p > 0$, consider the ideal $\mathfrak{a} = (x^2, y^3)$ and fix an integer $e \in \mathbb{Z}_{\geq 0}$. For $n \in \mathbb{Z}_{\geq 0}$, one has that $\mathfrak{a}^n = (x^{2i}y^{3j} \mid i + j = n)$, thus the Frobenius root $\mathcal{C}_R^e \cdot \mathfrak{a}^n$ is generated by monomials of the form $x^a y^b$. Given integers $a, b \geq 0$, one can compute $\nu(a, b) = \max\{\ell \geq 0 \mid x^a y^b \in \mathcal{C}_R^e \cdot \mathfrak{a}^{\ell}\}$. Since

$x^a y^b \notin \mathcal{C}_R^e \cdot \mathfrak{a}^{\nu(a,b)+1}$, it follows that $\nu(a,b)$ is ν-invariant of $\mathfrak{a}$ of level e. Since the ideals $\mathcal{C}_R^e \cdot \mathfrak{a}^n$ are generated by the monomials $x^a y^b$, it follows that all the ν-invariants are of this form, that is, $\nu_{\mathfrak{a}}^{\bullet}(p^e) = \{\nu(a,b) \mid a, b \in \mathbb{Z}_{\geq 0}\}$. Fix some integers $a, b \in \mathbb{Z}_{\geq 0}$, then $x^a y^b \in \mathcal{C}_R^e \cdot \mathfrak{a}^n$ if and only the following conditions are met:

(a) There exist integers $0 \leq r, s < p^e$ such that $x^{ap^e+r} y^{bp^e+s} \in \mathfrak{a}^n$.
(b) There exist non-negative integers $i + j = n$ with $2i \leq ap^e + r$ and $3j \leq bp^e + s$.

As the sum $i+j=n$ must be maximized, one takes $r = s = p^e - 1$, thus the conditions become $2i \leq (a+1)p^e - 1$ and $3j \leq (b+1)p^e - 1$, therefore

$$i = \left\lfloor \frac{(a+1)p^e - 1}{2} \right\rfloor, \quad j = \left\lfloor \frac{(b+1)p^e - 1}{3} \right\rfloor,$$

and the optimum is $n = \nu(a,b) = i + j$. The values of i and j depend on a, b, p and e.

First take $p = 2$. Then $(a+1)p^e - 1 \equiv 1 \pmod 2$, hence $i = \frac{(a+1)p^e-2}{2}$. The value of j depends on the choice of $e \in \mathbb{Z}_{\geq 0}$:

- If $2 \mid e$, one has $p^e \equiv 1 \pmod 3$.
  - When $b \equiv 0 \pmod 3$, $(b+1)p^e - 1 \equiv 0 \pmod 3$, thus $j = \frac{(b+1)p^e-1}{3}$.
  - When $b \equiv 1 \pmod 3$, $(b+1)p^e - 1 \equiv 1 \pmod 3$, thus $j = \frac{(b+1)p^e-2}{3}$.
  - When $b \equiv 2 \pmod 3$, $(b+1)p^e - 1 \equiv 2 \pmod 3$, thus $j = \frac{(b+1)p^e-3}{3}$.
- If $2 \nmid e$, one has $p^e \equiv 2 \pmod 3$.
  - When $b \equiv 0 \pmod 3$, $(b+1)p^e - 1 \equiv 1 \pmod 3$, thus $j = \frac{(b+1)p^e-2}{3}$.
  - When $b \equiv 1 \pmod 3$, $(b+1)p^e - 1 \equiv 0 \pmod 3$, thus $j = \frac{(b+1)p^e-1}{3}$.
  - When $b \equiv 2 \pmod 3$, $(b+1)p^e - 1 \equiv 2 \pmod 3$, thus $j = \frac{(b+1)p^e-3}{3}$.

The sum $n = \nu(a,b) = i + j$ can be arranged in table form as follows:

	$b \equiv 0 \pmod 3$	$b \equiv 1 \pmod 3$	$b \equiv 2 \pmod 3$
$2 \mid e$	$\frac{(3a+2b+5)p^e-8}{6}$	$\frac{(3a+2b+5)p^e-10}{6}$	$\frac{(3a+2b+5)p^e-12}{6}$
$2 \nmid e$	$\frac{(3a+2b+5)p^e-10}{6}$	$\frac{(3a+2b+5)p^e-8}{6}$	$\frac{(3a+2b+5)p^e-12}{6}$

Table 4.1. The ν-invariants of $\mathfrak{a} = (x^2, y^3)$ when $p = 2$.

When $p = 3$, $(b+1)p^e - 1 \equiv 2 \pmod 3$, thus $j = \frac{(b+1)p^e-3}{3}$ for all $b \geq 0$. Since $p^e \equiv 1 \pmod 2$, one has:

- When $a \equiv 0 \pmod 2$, $(a+1)p^e - 1 \equiv 0 \pmod 2$, thus $i = \frac{(a+1)p^e-1}{2}$.

- When $a \equiv 1 \pmod 2$, $(a+1)p^e - 1 \equiv 1 \pmod 2$, thus $i = \frac{(a+1)p^e-2}{2}$.

The remaining cases to study are when $p \equiv 1 \pmod 6$ or $p \equiv 5 \pmod 6$. A similar analysis to the ones before show that the ν-invariants are given by:

	$b \equiv 0 \pmod 3$	$b \equiv 1 \pmod 3$	$b \equiv 2 \pmod 3$
$a \equiv 0 \pmod 2$	$\frac{(3a+2b+5)p^e-9}{6}$	$\frac{(3a+2b+5)p^e-9}{6}$	$\frac{(3a+2b+5)p^e-9}{6}$
$a \equiv 1 \pmod 2$	$\frac{(3a+2b+5)p^e-12}{6}$	$\frac{(3a+2b+5)p^e-12}{6}$	$\frac{(3a+2b+5)p^e-12}{6}$

Table 4.2. The ν-invariants of $\mathfrak{a} = (x^2, y^3)$ when $p = 3$.

	$b \equiv 0 \pmod 3$	$b \equiv 1 \pmod 3$	$b \equiv 2 \pmod 3$
$a \equiv 0 \pmod 2$	$\frac{(3a+2b+5)p^e-5}{6}$	$\frac{(3a+2b+5)p^e-7}{6}$	$\frac{(3a+2b+5)p^e-9}{6}$
$a \equiv 1 \pmod 2$	$\frac{(3a+2b+5)p^e-8}{6}$	$\frac{(3a+2b+5)p^e-10}{6}$	$\frac{(3a+2b+5)p^e-12}{6}$

Table 4.3. The ν-invariants of $\mathfrak{a} = (x^2, y^3)$ when $p \equiv 1 \pmod 6$.

		$b \equiv 0 \pmod 3$	$b \equiv 1 \pmod 3$	$b \equiv 2 \pmod 3$
$2 \mid e$	$a \equiv 0 \pmod 2$	$\frac{(3a+2b+5)p^e-5}{6}$	$\frac{(3a+2b+5)p^e-7}{6}$	$\frac{(3a+2b+5)p^e-9}{6}$
	$a \equiv 1 \pmod 2$	$\frac{(3a+2b+5)p^e-8}{6}$	$\frac{(3a+2b+5)p^e-10}{6}$	$\frac{(3a+2b+5)p^e-12}{6}$
$2 \nmid e$	$a \equiv 0 \pmod 2$	$\frac{(3a+2b+5)p^e-7}{6}$	$\frac{(3a+2b+5)p^e-5}{6}$	$\frac{(3a+2b+5)p^e-9}{6}$
	$a \equiv 1 \pmod 2$	$\frac{(3a+2b+5)p^e-10}{6}$	$\frac{(3a+2b+5)p^e-8}{6}$	$\frac{(3a+2b+5)p^e-12}{6}$

Table 4.4. The ν-invariants of $\mathfrak{a} = (x^2, y^3)$ when $p \equiv 5 \pmod 6$.

4.4. Bernstein-Sato roots

The next algebraic invariant that we introduce are Bernstein-Sato roots. To begin with, we recall some definitions concerning the p-adic integers. We refer the interested reader to [P] for a more detailed construction. Throughout this section, let $p \in \mathbb{Z}$ denote a prime number.

Definition 4.33. The *p-adic valuation* on $\mathbb{Z}$ is the map $v_p \colon \mathbb{Z} \to \mathbb{Z} \cup \{+\infty\}$ given by

$$v_p(n) = \max\left\{k \in \mathbb{Z}_{\geq 0} \;\middle|\; p^k \mid n\right\} \quad \text{for } n \neq 0,$$

and $v_p(0) = +\infty$. One may extend this to a map $v_p \colon \mathbb{Q} \to \mathbb{Z} \cup \{+\infty\}$ as follows:

$$v_p\left(\frac{a}{b}\right) := v_p(a) - v_p(b).$$

Definition 4.34. The *p-adic absolute value* is the function $|\cdot|_p \colon \mathbb{Q} \to \mathbb{R}_{\geq 0}$ given by

$$|x|_p := \begin{cases} p^{-v_p(x)} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

This induces the *p-adic metric*, which is the function $d_p \colon \mathbb{Q} \times \mathbb{Q} \to \mathbb{R}_{\geq 0}$ defined by

$$d_p(x, y) := |x - y|_p.$$

In the same way that one constructs the field $\mathbb{R}$ of real numbers as the completion of $\mathbb{Q}$ with respect to the Eucliden distance, one can complete $\mathbb{Q}$ with respect to a p-adic metric, which results in the field of p-adic numbers.

Definition 4.35. The *field* $\mathbb{Q}_p$ *of p-adic numbers* is the completion of $\mathbb{Q}$ with respect to the p-adic metric. The *ring of p-adic integers* is

$$\mathbb{Z}_p := \left\{x \in \mathbb{Q}_p \;\middle|\; |x|_p \leq 1\right\}.$$

In Section 4.1, we introduced the Bernstein-Sato polynomial or b-function of a polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$, which is a polynomial $b_f(s) \in \mathbb{C}[s]$. Moreover, we mentioned that the same polynomial $b_{\mathfrak{a}}(s)$ may be constructed for an ideal $\mathfrak{a} \subseteq \mathbb{C}[x_1, \ldots, x_n]$. These b-functions and, as every polynomial with complex coefficients, come equipped with a set of roots and for each root a multiplicity.

Bernstein-Sato roots are the positive characteristic analogue of the b-function roots. More precisely, let R be a regular F-finite ring of characteristic $p > 0$. To an ideal $\mathfrak{a} \subseteq R$, one associates the b-function, which is an ideal in the algebra $C(\mathbb{Z}_p, \mathbb{F}_p)$ of continuous functions $\mathbb{Z}_p \to \mathbb{F}_p$. The Bernstein-Sato roots of $\mathfrak{a}$ are precisely the roots of its b-function. In this context, however, there is no notion of multiplicity of a root. An alternative characterization of Bernstein-Sato roots in terms of ν-invariants was given by Quinlan-Gallego [QG21b]. It is this definition to which we adhere:

Definition 4.36 ([QG21b, Theorem IV.17])**.** Let R be a regular F-finite ring of characteristic $p > 0$ and let $\mathfrak{a} \subseteq R$ be an ideal $\mathfrak{a} \subseteq R$. A p-adic integer $\alpha \in \mathbb{Z}_p$ is a *Bernstein-Sato root* of $\mathfrak{a}$ if and only if α is the p-adic limit of a sequence of ν-invariants $(\nu_e)_{e=0}^{\infty} \subseteq \mathbb{Z}_{\geq 0}$ with $\nu_e \in \nu_{\mathfrak{a}}^{\bullet}(p^e)$.

Observation 4.37. As a remark to the definition above, we say that $\alpha \in \mathbb{Z}$ is the p-adic limit of the sequence $(\nu_e)_{e=\infty}^{\infty} \subseteq \mathbb{Z}_{\geq 0}$ if and only if for every integer $k \geq 0$, there exists $e_0 = e_0(k) \geq 0$ such that $p^k \mid \alpha - \nu_e$ for all $e \geq e_0$. In terms of the p-adic metric, this means that $d_p(\alpha, \nu_e) \leq p^{-k}$ for all $e \geq e_0$ and, since k can be taken to be arbitrarily large, $d_p(\alpha, \nu_e) \to 0$ as $e \to \infty$.

Furthermore, it is possible to relate Bernstein-Sato roots to the roots of the b-function in characteristic zero in the sense made precise below. Fix an ideal $\mathfrak{a} \subseteq \mathbb{Z}[x_1, \ldots, x_n]$, denote by $\mathfrak{a}_{\mathbb{C}}$ its extension to the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$ and let $b_{\mathfrak{a}}(s)$ be the corresponding b-function. For a prime number $p \in \mathbb{Z}$, let $\mathfrak{a}_p$ be the reduction modulo p to the ring $\mathbb{F}_p[x_1, \ldots, x_n]$. One then has the following theorem:

Theorem 4.38 ([QG21b, Theorem VI.3])**.** Suppose that $\alpha \in \mathbb{Z}_p$ is a Bernstein-Sato root of $\mathfrak{a}_p$ for infinitely many prime numbers $p \in \mathbb{Z}$. Then α is a root of the b-function $b_{\mathfrak{a}}(s)$.

Example 4.39. Let $R = \mathbb{F}_p[x_1, \ldots, x_n]$ be a polynomial ring over a perfect field of characteristic $p > 0$ and let $\mathfrak{m}$ be the homogeneous maximal ideal. In Example 4.26 we showed that $\nu_{\mathfrak{m}}^{\mathfrak{m}}(p^e) = n(p^e - 1)$ for each integer $e \geq 0$. By letting $\nu_e = n(p^e - 1)$, one obtains a sequence $(\nu_e)_{e=0}^{\infty} \subseteq \mathbb{Z}_{\geq 0}$ of ν-invariants with p-adic limit $\nu_e \to \alpha = -n$. It follows that $\alpha = -n$ is a Bernstein-Sato root of $\mathfrak{m}$ and, in view of [QG21b, Theorem VI.3], a root of the b-function in characteristic zero as well.

Example 4.40. Let $\mathbb{F}$ be a perfect field of characteristic $p > 0$, $R = \mathbb{F}[x, y]$ a polynomial ring and fix the ideal $\mathfrak{a} = (x^2, y^3)$. In Example 4.32 we computed the ν-invariants of $\mathfrak{a}$ of level $e \in \mathbb{Z}_{\geq 0}$, which are given by formulas of the type

$$\nu(a, b, e) = \frac{(3a + 2b + 5)p^e - k(a, b, e)}{6},$$

where $a, b, k(a, b) \in \mathbb{Z}_{\geq 0}$. Furthermore, we argued as $a, b \in \mathbb{Z}_{\geq 0}$ range over the non-negative integers, one obtains all the ν-invariants of $\mathfrak{a}$ of level $e \geq 0$.

Let $(\nu_e)_{e=0}^{\infty}$ be a sequence of ν-invariants of $\mathfrak{a}$, with $\nu_e \in \nu_{\mathfrak{a}}^{\bullet}(p^e)$. Each term in the sequence is of the form $\nu_e = \nu(a_e, b_e, e)$ and, in order for it to have a p-adic limit, it is clear that there must be an integer $e_0 \geq 0$ such that $k(a_{e_0}, b_{e_0}, e_0) = k(a_e, b_e, e)$ for all $e \geq e_0$. It follows that the Bernstein-Sato roots of $\mathfrak{a}$ are of the form $-k(a, b, e)/6$. Depending on the characteristic, one finds the following:

- When $p = 2$, $\mathrm{BSR}(\mathfrak{a}) = \{-2, -5/3, -4/3\}$
- When $p = 3$, $\mathrm{BSR}(\mathfrak{a}) = \{-2, -3/2\}$
- When $p \equiv 1 \pmod 6$, $\mathrm{BSR}(\mathfrak{a}) = \{-2, -5/3, -3/2, -4/3, -7/6, -5/6\}$.
- When $p \equiv 5 \pmod 6$, $\mathrm{BSR}(\mathfrak{a}) = \{-2, -5/3, -3/2, -4/3, -7/6, -5/6\}$.

In particular, for all primes $p \geq 5$, the sets of Bernstein-Sato roots coincide.

4.5. The F-pure threshold and test ideals

The log-canonical threshold of a polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$ at a point $x \in \mathbb{C}^n$ where $f(x) = 0$, is defined as the supremum of $\lambda \in \mathbb{R}_{>0}$ such that the function $1/|f|^{2\lambda}$ is integrable on a neighborhood of the point x (see Section 4.1).

Let R be a ring of characteristic $p > 0$. In this setting it is not possible to use analysis tools to measure the singularities of polynomials or ideals in R. Instead, one uses algebraic tools to define the F-pure threshold, which is the measure of singularity analogous to the log-canonical threshold. It was introduced by Takagi and Watanabe [TW04], and thereafter related to the Bernstein-Sato polynomial by Mustaţă , Takagi and Watanabe [MTW05]. Recall that if I, J are ideals of R satisfying $I \subseteq \text{rad } J$, for each integer $e \geq 0$ one defines

$$\nu_I^J(p^e) = \max\left\{\ell \geq 0 \;\middle|\; I^\ell \not\subseteq J^{[p^e]}\right\}.$$

Definition 4.41 ([MTW05])**.** Let R be a regular F-finite ring of characteristic $p > 0$. Let $I \subseteq R$ be an ideal and let $\mathfrak{m} \subseteq R$ be a maximal ideal such that $I \subseteq \mathfrak{m}$. The *F-pure threshold* of I with respect to $\mathfrak{m}$ is

$$\text{fpt}_{\mathfrak{m}}(I) := \lim_{e \to \infty} \frac{\nu_I^{\mathfrak{m}}(p^e)}{p^e}.$$

When $\mathfrak{m}$ is understood from the context, we shall denote it simply by $\text{fpt}(I)$.

The F-pure threshold of I may be defined at any ideal $J \subseteq R$ containing I in its radical. Nonetheless, for our purposes, it suffices to consider the case when $J = \mathfrak{m}$ is a maximal ideal.The proposition below shows that the definition is well-defined.

Proposition 4.42. Let R be a regular F-finite ring of characteristic $p > 0$ and fix an ideal $I \subseteq R$. Let $\mathfrak{m} \subseteq R$ be a maximal ideal such that $I \subseteq \mathfrak{m}$. For each integer $e \geq 0$,

$$\nu_I^{\mathfrak{m}}(p^{e+1}) \geq p\nu_I^{\mathfrak{m}}(p^e).$$

In particular, $(\nu_I^{\mathfrak{m}}(p^e)/p^e)_{e=0}^{\infty}$ is a bounded monotone sequence.

Proof. Let $\ell = \nu_I^{\mathfrak{m}}(p^e)$; then there exists $f \in I^\ell - \mathfrak{m}^{[p^e]}$, thus $\mathcal{C}_R^e \cdot f \not\subseteq \mathfrak{m}$ by Proposition 4.19. It follows from Proposition 4.22 that $\mathcal{C}_R^e \cdot f = \mathcal{C}_R^e \cdot \mathcal{C}_R \cdot f^p = \mathcal{C}_R^{e+1} \cdot f^p \subsetneq \mathfrak{m}$, hence $f^p \notin \mathfrak{m}^{[p^{e+1}]}$ and $f^p \in I^{p\ell} - \mathfrak{m}^{[p^{e+1}]}$. As a result, $\nu_I^{\mathfrak{m}}(p^{e+1}) \geq p\ell = p\nu_I^{\mathfrak{m}}(p^e)$, which proves that the sequence $(\nu_I^{\mathfrak{m}}(p^e)/p^e)_{e=0}^{\infty}$ is monotone. By letting N be the number of generators of I, one has that $\nu_I^{\mathfrak{m}}(p^e) \leq N(p^e - 1)$ as a consequence of Proposition 4.31, therefore the sequence is bounded above by N. □

Moreover, we have the following connection between the characteristic zero and the positive characteristic settings:

Theorem 4.43 ([MTW05, Theorem 3.4]). Let $A = \mathbb{Z}[a^{-1}]$ be the localization of $\mathbb{Z}$ at some non-zero integer $a \in \mathbb{Z}$ and fix a non-zero ideal $\mathfrak{a} \subseteq A[x_1, \dots, x_n]$ such that $\mathfrak{a} \subseteq (x_1, \dots, x_n)$. Let $\mathfrak{a}_{\mathbb{Q}} := \mathfrak{a} \cdot \mathbb{Q}[x_1, \dots, x_n]$ be the extension of $\mathfrak{a}$ and, for each prime $p \in \mathbb{Z}$, let $\mathfrak{a}_p = \mathfrak{a} \cdot \mathbb{F}_p[x_1, \dots, x_n]_{(x_1,\dots,x_n)}$ be the reduction of $\mathfrak{a}$ modulo p. Then

$$\operatorname{lct}(\mathfrak{a}) = \lim_{p\to\infty} \operatorname{fpt}(\mathfrak{a}).$$

A family of invariants closely related to the F-pure threshold are the test ideals, which are the counterparts of the multiplier ideals defined in Section 4.1. Test ideals were introduced in the context of tight closure theory by Hara and Yoshida [HY03]. Afterwards, Blickle, Mustaţă and Smith [BMS08] gave an alternative definition in terms of Frobenius roots.

Recall that, given a real number $x \in \mathbb{R}$, the ceiling of x, denoted by $\lceil x \rceil$, is the smallest integer greater or equal to x.

Definition 4.44 ([BMS08, Definition 2.9]). Let R be a regular F-finite ring of characteristic $p > 0$ and let $\mathfrak{a} \subseteq R$ be an ideal. For a real number $\lambda \geq 0$, the *test ideal of* $\mathfrak{a}$ *with exponent* λ is defined as

$$\tau(\mathfrak{a}^\lambda) := \bigcup_{e=0}^{\infty} \mathcal{C}_R^e \cdot \mathfrak{a}^{\lceil \lambda p^e \rceil}.$$

One may view the test ideals as a family of ideals of R indexed by the positive real numbers.

Proposition 4.45. Let R be a regular F-finite ring of characteristic $p > 0$ and let $\mathfrak{a} \subseteq R$ be an ideal. Then for each integer $e \geq 0$,

$$\mathcal{C}_R^e \cdot \mathfrak{a}^{\lceil \lambda p^e \rceil} \subseteq \mathcal{C}_R^{e+1} \cdot \mathfrak{a}^{\lceil \lambda p^{e+1} \rceil}.$$

In particular, $\tau(\mathfrak{a}^\lambda) = \mathcal{C}_R^e \cdot \mathfrak{a}^{\lceil \lambda p^e \rceil}$ for some integer $e \gg 0$.

Proof. Let $n = \lceil \lambda p^e \rceil$, that is to say, $n - 1 < \lambda p^e \leq n$. Then $(n-1)p < \lambda p^{e+1} \leq pn$, thus $\lceil \lambda p^{e+1} \rceil \leq \lceil \lambda p^e \rceil p$. It follows from [QG21b, Proposition II.48] that

$$\mathcal{C}_R^e \cdot \mathfrak{a}^{\lceil \lambda p^e \rceil} = \mathcal{C}_R^{e+1} \cdot (\mathfrak{a}^{\lceil \lambda p^e \rceil})^{[p]} \subseteq \mathcal{C}_R^{e+1} \cdot \mathfrak{a}^{\lceil \lambda p^e \rceil p} \subseteq \mathcal{C}_R^{e+1} \cdot \mathfrak{a}^{\lceil \lambda p^{e+1} \rceil}.$$

The second statement follows from the fact that R is a Noetherian ring. □

Proposition 4.46. Let R be a regular F-finite ring of characteristic $p > 0$ and $f \in R$. Then for each pair of integers $n \geq 0$ and $e \geq 0$,

$$\tau(f^{n/p^e}) = \mathcal{C}_R^e \cdot f^n.$$

Proof. The inclusion $(\supseteq)$ is clear. For the converse, fix an integer $E \geq e$. By Proposition 4.19, the containment $\mathcal{C}_R^E \cdot f^{np^{E-e}} \subseteq \mathcal{C}_R^e \cdot f^n$ is equivalent to $f^{np^{E-e}} \in (\mathcal{C}_R^e \cdot f^n)^{[p^E]}$. As a consequence of Proposition 4.11, the latter may be written as $(\mathcal{C}_R^e \cdot f^n)^{[p^E]} = (\mathcal{C}_R^e \cdot f^n)^{[p^e][p^{E-e}]} = (\mathcal{D}_R^{(e)} \cdot f^n)^{[p^{E-e}]}$. As the identity of R is a differential operator of level 0 and $\mathcal{D}_R^{(0)} \subseteq \mathcal{D}_R^{(e)}$, one has that $f^n \in \mathcal{D}_R^{(e)} \cdot f^n$, from where the inclusion follows. □

Proposition 4.47 ([BFS13, Proposition 3.26])**.** Let R be a regular F-finite ring of characteristic $p > 0$ and let $I \subseteq R$ be a non-zero ideal.

(1) For a real number $\lambda > 0$ sufficiently small, $\tau(I^\lambda) = R$.
(2) If $\lambda' \geq \lambda$, then $\tau(I^{\lambda'}) \subseteq \tau(I^\lambda)$.
(3) The F-pure threshold of I is $\mathrm{fpt}(I) = \sup\{\lambda \in \mathbb{R}_{\geq 0} \mid \tau(I^\lambda) = R\}$.

Proof. (1) We claim that there exists an integer $e_0 \geq 0$ such that $\nu_I^{\mathfrak{m}}(p^e) \geq 1$ for all $e \geq e_0$. For the sake of contradiction, suppose that $\nu_I^{\mathfrak{m}}(p^e) = 0$ for all $e \geq 0$, that is, $I \subseteq \mathfrak{m}^{[p^e]}$ for all $e \geq 0$. The families of ideals $\{\mathfrak{m}^s\}_{s\geq 0}$ and $\{\mathfrak{m}^{[p^e]}\}_{e\geq 0}$ are cofinal by Proposition 4.10, therefore

$$I \subseteq \bigcap_{e=0}^{\infty} \mathfrak{m}^{[p^e]} \subseteq \bigcap_{s=0}^{\infty} \mathfrak{m}^s = 0,$$

where the last equality follows from Krull's interesection theorem, but this contradicts the assumption that I be a non-zero ideal.

Now let $e \gg 0$ be large enough so that $\nu_I^{\mathfrak{m}}(p^e) \geq 1$, and choose $\lambda > 0$ small enough so that $\lambda p^e \leq \nu_I^{\mathfrak{m}}(p^e)$. It follows that $R = \mathcal{C}_R^e \cdot I^{\nu_I^{\mathfrak{m}}(p^e)} \subseteq \mathcal{C}_R^e \cdot I^{\lceil \lambda p^e \rceil}$, so $\tau(I^\lambda) = R$.

(2) For each integer $e \geq 0$, $\mathcal{C}_R^e \cdot I^{\lceil \lambda' p^e \rceil} \subseteq \mathcal{C}_R^e \cdot I^{\lceil \lambda p^e \rceil}$, therefore $\tau(I^{\lambda'}) \subseteq \tau(I^\lambda)$.

(3) Choose $\varepsilon > 0$ and define $\lambda = \mathrm{fpt}(I) - \varepsilon$. Since $(\nu_I^{\mathfrak{m}}(p^e)/p^e)_{e=0}^{\infty}$ is a monotone sequence with limit $\mathrm{fpt}(I)$, there exists an integer $e_0 = e_0(\varepsilon) \geq 0$ such that for all $e \geq e_0$,

$$\lambda < \frac{\nu_I^{\mathfrak{m}}(p^e)}{p^e} \leq \mathrm{fpt}(I).$$

In consequence, $\lceil \lambda p^e \rceil \leq \nu_I^{\mathfrak{m}}(p^e)$ and $R = \mathcal{C}_R^e \cdot I^{\nu_I^{\mathfrak{m}}(p^e)} \subseteq \mathcal{C}_R^e \cdot I^{\lceil \lambda p^e \rceil}$, thus $\tau(I^\lambda) = R$, which implies $\sup\left\{\lambda \in \mathbb{R}_{\geq 0} \mid \tau(I^\lambda) = R\right\} \leq \mathrm{fpt}(I)$. On the other hand, for $\mu > \mathrm{fpt}(I)$ one has that $\nu_I^{\mathfrak{m}}(p^e) < \lceil \mu p^e \rceil$ for all integers $e \geq 0$, thereby $\mathcal{C}_R^e \cdot I^{\lceil \mu p^e \rceil} \subsetneq \mathcal{C}_R^e \cdot I^{\nu_I^{\mathfrak{m}}(p^e)} = R$ and $\tau(I^\mu) \subsetneq R$, which gives the converse inequality. ☐

It follows from Proposition 4.47 that the F-pure threshold of an ideal is its smallest F-jumping number. As multiplier ideals, test ideals form a right semi-continuous family of ideals of R. This motivates the following definition:

Definition 4.48. Let R be a regular F-finite ring of characteristic $p > 0$ and let $I \subseteq R$ be an ideal. A real number $\lambda \in \mathbb{R}_{\geq 0}$ is an *F-jumping number* of I if $\tau(I^\lambda) \neq \tau(I^{\lambda-\varepsilon})$ for all $\varepsilon > 0$. The set of F-jumping numbers of I is denoted by $\mathrm{FJN}(I)$.

When I is an ideal in a regular F-finite algebra essentially of finite type over a field of characteristic $p > 0$, it is known that the set of F-jumping numbers is rational and discrete [BMS08, Theorem 3.1]. As we shall see in Chapter 5, the same is true when one studies ideals of maximal minors in polynomial rings over field of prime characteristic.

As the proposition below states, it suffices to compute the test ideals $\tau(I^\lambda)$ and the F-jumping numbers for λ in a bounded interval of the real line:

Proposition 4.49. Let R be an F-finite ring of characteristic $p > 0$ and let $I = (f_1, \dots, f_r)$ be an ideal generated by r elements. For each real number $\lambda \geq r$,

$$\tau(I^{\lambda+1}) = I\tau(I^{\lambda}).$$

Proof. See [QG21b, Proposition II.54]. □

For further properties of the F-pure threshold, test ideals and F-jumping numbers, we invite the interested reader to consult [AMJNB21; BFS13]. We conclude the chapter by completing the calculations started in Examples 4.32 and 4.40.

Example 4.50. Let $R = \mathbb{F}[x, y]$ be a polynomial ring over a perfect field $\mathbb{F}$ of characteristic $p > 0$ and let $\mathfrak{a} = (x^2, y^3)$. In Example 4.32, given $e \in \mathbb{Z}_{\geq 0}$ we computed the integers $\nu(a, b, e) = \max\{\ell \in \mathbb{Z}_{\geq 0} \mid x^a y^b \in \mathcal{C}_R^e \cdot \mathfrak{a}^{\ell}\}$, given by formulas of the form

$$\nu(a, b, e) = \frac{(3a + 2b + 5)p^e - k(a, b, e)}{6}$$

where $k(a, b, e) \in \mathbb{Z}_{\geq 0}$ with $\frac{k(a,b,e)}{p^e} \to 0$ as $e \to \infty$. In addition, we showed that the set of ν-invariants is given by the $\nu(a, b, e)$, that is, $\nu_{\mathfrak{a}}^{\bullet}(p^e) = \{\nu(a, b, e) \mid a, b \in \mathbb{Z}_{\geq 0}\}$.

Fix some integers $a, b \in \mathbb{Z}_{\geq 0}$ and choose a real number $0 \leq \lambda < \frac{3a+2b+5}{6}$. Since the sequence $(\nu(a, b, e)/p^e)_{e=0}^{\infty} \subseteq \mathbb{Z}_{\geq 0}$ is monotone for $e \gg 0$ and has limit $\frac{3a+2b+5}{6}$, there exists $e_0 \geq 0$ depending on λ such that $\lambda p^e < \nu(a, b, e)$ for all $e \geq e_0$. Letting $e \geq 0$ be large enough, one has that $x^a y^b \in \mathcal{C}_R^e \cdot \mathfrak{a}^{\lceil \lambda p^e \rceil} = \tau(\mathfrak{a}^{\lambda})$. It follows that

$$\operatorname{FJN}(\mathfrak{a}) = \left\{ \frac{3a + 2b + 5}{6} \;\middle|\; a, b \in \mathbb{Z}_{\geq 0} \right\}.$$

In particular, $\operatorname{fpt}(\mathfrak{a}) = \frac{5}{6}$ and the set of F-jumping numbers of $\mathfrak{a}$ is independent of the characteristic, whereas the ν-invariants are not. Based on the characterization of the ν-invariants of $\mathfrak{a}$, the computation of the test ideals $\tau(\mathfrak{a}^{\lambda})$ is immediate. These are depicted in Fig. 4.1:

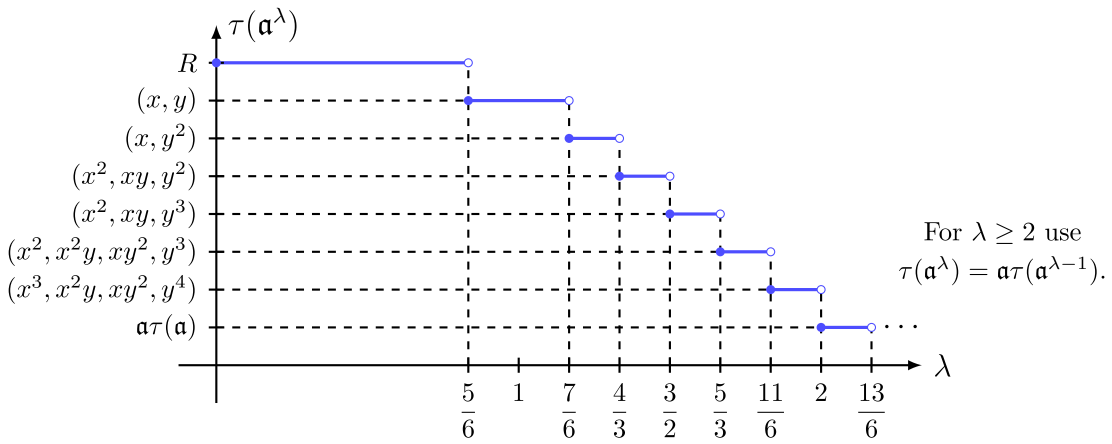

Figure 4.1

Chapter 5

Bernstein-Sato Theory for Determinantal Ideals

5.1. Determinantal rings and determinantal ideals

Throughout this chapter let B denote a Noetherian commutative ring. Choose integers $m, n \geq 1$ and consider the polynomial ring $R = B[x_{11}, \ldots, x_{1n}, \ldots, x_{m1}, \ldots, x_{mn}]$ in mn indeterminates. These can be arranged in matrix form as

$$X = (x_{ij}) = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{m1} & \cdots & x_{mn} \end{pmatrix}.$$

We say that $X = (x_{ij})$ is a generic matrix of indeterminates over the ring B but, since we deal exclusively with these kind of matrices, we shall refer to them simply as matrices of indeterminates. For brevity, we denote the polynomial ring by $R = B[X]$.

Definition 5.1. Let $X = (x_{ij})$ be an $n \times n$ matrix of indeterminates over a ring B. The *determinant* of X is the polynomial

$$\det X = \sum_{\sigma \in \mathbb{S}_n} \operatorname{sgn}(\sigma)\, x_{1\sigma(1)} \cdots x_{n\sigma(n)},$$

in the ring $B[X]$, where $\mathbb{S}_n$ denotes the symmetric group of degree n and $\operatorname{sgn}(\sigma)$ is the sign of the permutation $\sigma \in \mathbb{S}_n$.

Definition 5.2. Let $X = (x_{ij})$ be an $m \times n$ matrix of indeterminates over a ring B, and let $R = B[X]$ be a polynomial ring. Fix an integer $1 \leq t \leq \min(m, n)$.

(1) A $t \times t$ *submatrix* of X is a matrix of the form

$$\begin{pmatrix} x_{a_1 b_1} & \cdots & x_{a_1 b_t} \\ \vdots & \ddots & \vdots \\ x_{a_t b_1} & \cdots & x_{a_t b_t} \end{pmatrix},$$

where $1 \leq a_1, \ldots, a_t \leq m$ and $1 \leq b_1, \ldots, b_t \leq n$ are integers.

(2) A *t-minor* of X, or *minor of size t* of X, is the determinant of a $t \times t$ submatrix of X.

(3) The *ideal of t-minors* of X, denoted by $I_t(X)$, is the ideal of R generated by the t-minors of X. When X is understood from the context, we shall simply write I_t.

By convention, a 0-minor is an empty product, thus it is equal to $1 \in B$.

Observation 5.3. Let B be a ring and let $X = (x_{ij})$ and $X' = (x'_{ij})$ be matrices of indeterminates over B, of sizes $m \times n$ and $n \times m$, respectively. Denote by $R = B[X]$ and $R' = B[X']$ the corresponding polynomial rings. We note that the map that leaves B fixed and sends $x_{ij} \mapsto x'_{ji}$ gives a ring isomorphism $R \xrightarrow{\sim} R'$. In addition, this isomorphism maps the ideal $I_t(X) \subseteq R$ of t-minors of X to the ideal $I_t(X') \subseteq R'$ of t-minors of X'.

In view of the above remark, hereinafter we shall assume that an $m \times n$ matrix of indetem-inates X satisfies $1 \leq m \leq n$, that is to say, "it has more columns than rows".

Definition 5.4. Let $X = (x_{ij})$ be an $m \times n$ matrix of indeterminates over a ring B.

(1) A *submaximal* minor of X is a minor of X of size $t = m - 1$.
(2) A *maximal* minor of X is a minor of X of size $t = m$.

Notation 5.5 ([BV88])**.** Let $X = (x_{ij})$ be a matrix of indeterminates of size $m \times n$ over a ring B, with $m \leq n$, let $R = B[X]$ be a polynomial ring and fix an integer $1 \leq t \leq m$. Given integers $1 \leq a_1, \ldots, a_t \leq m$ and $1 \leq b_1, \ldots, b_t \leq t$, we define

$$[a_1, \ldots, a_t \mid b_1, \ldots, b_t] := \det \begin{pmatrix} x_{a_1 b_1} & \cdots & x_{a_1 b_t} \\ \vdots & \ddots & \vdots \\ x_{a_t b_1} & \cdots & x_{a_t b_t} \end{pmatrix}.$$

Note that the rows are determined by the choice of a_i's, and the columns are given by the choice of b_j's. When $t = m$, all the rows are chosen, thus we simply write $[b_1, \ldots, b_m]$.

Since the determinant of a matrix is invariant (up to a sign) under row and column swaps, we may assume that the indices are given in ascending order, that is, $1 \leq a_1 < \cdots < a_t \leq m$ and $1 \leq b_1 < \cdots < b_t \leq n$.

Example 5.6. Let $X = (x_{ij})$ be the following 3×3 matrix of indeterminates

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}.$$

over a field $\mathbb{K}$, and let $R = \mathbb{K}[X]$ be the corresponding polynomial ring.

(a) The 1-minors of X are the monomials $x_{11}, x_{12}, \ldots, x_{33}$, hence the ideal I_1 of 1-minors of X is the homogeneous maximal ideal of R.
(b) The 2-minors of X, i.e. the submaximal minors, have the form $[a_1, a_2 \mid b_1, b_2]$. For instance,

choosing rows 1 and 2, and columns 2 and 3, yields the 2-minor

$$[1,2 \mid 2,3] = \det \begin{pmatrix} x_{12} & x_{13} \\ x_{22} & x_{23} \end{pmatrix} = x_{12}x_{23} - x_{13}x_{22}.$$

It follows that the ideal I_2 of 2-minors X has 9 generators, which are listed below:

$$\begin{aligned} &[1,2 \mid 1,2] = x_{11}x_{22} - x_{12}x_{21}, && [1,2 \mid 1,3] = x_{11}x_{23} - x_{13}x_{21}, && [1,2 \mid 2,3] = x_{12}x_{23} - x_{13}x_{22}, \\ &[1,3 \mid 1,2] = x_{11}x_{32} - x_{12}x_{31}, && [1,3 \mid 1,3] = x_{11}x_{33} - x_{13}x_{31}, && [1,3 \mid 2,3] = x_{12}x_{33} - x_{13}x_{32}, \\ &[2,3 \mid 1,2] = x_{21}x_{32} - x_{22}x_{31}, && [2,3 \mid 1,3] = x_{21}x_{33} - x_{23}x_{31}, && [2,3 \mid 2,3] = x_{22}x_{33} - x_{23}x_{32}. \end{aligned}$$

- The only 3-minor of X is the determinant

$$\det X = x_{11}x_{22}x_{33} + x_{13}x_{21}x_{32} + x_{12}x_{23}x_{31} - x_{13}x_{22}x_{31} - x_{11}x_{23}x_{32} - x_{12}x_{21}x_{33},$$

thus $I_3 = (\det X)$ is a principal ideal of R.

Observation 5.7. We note that a t-minor of an $m \times n$ matrix of indeterminates $X = (x_{ij})$ corresponds bijectively to a choice of t different columns and t different rows of X. As a result, the ideal of t-minors of X has

$$\binom{m}{t}\binom{n}{t}$$

generators.

Definition 5.8. Let $X = (x_{ij})$ be an $m \times n$ matrix of indeterminates over a ring B, $m \leq n$. A *determinantal ring* is a commutative ring R that is isomorphic to a quotient ring of the form $B[X]/I_t(X)$. These rings are commonly denoted by $R_t(X) = B[X]/I_t(X)$.

Let V and W be n-dimensional and m-dimensional vector spaces over a field $\mathbb{K}$, respectively. After choosing bases for V and W, a $\mathbb{K}$-linear map $\varphi\colon V \to W$ can be represented by an $m \times n$ matrix

$$M = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

with entries in $\mathbb{K}$. One is usually interested in computing properties of φ, such as the rank, or eigenvalues and eigenspaces when $V \cong W$. To this purpose, a common procedure in linear algebra is to perform row operations on M in order to obtain an easier to handle matrix that is equivalent to M. A similar method can be applied to matrices of indeterminates.

Consider the following $m \times n$ matrix of indeterminates

$$X = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{m1} & \cdots & x_{mn} \end{pmatrix},$$

defined over a ring B, and let $R = B[X]$ be the corresponding polynomial ring. By localizing R at the indeterminate x_{mn}, $R[x_{mn}^{-1}] = B[X][x_{mn}^{-1}]$, one can "eliminate" the indeterminate x_{1n} from X via row operations as follows:

$$\begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{m1} & \cdots & x_{mn} \end{pmatrix} \longmapsto \begin{pmatrix} x_{11} - x_{m1}x_{1n}x_{mn}^{-1} & \cdots & x_{1,n-1} - x_{m,n-1}x_{1n}x_{mn}^{-1} & 0 \\ x_{21} & \cdots & x_{2,n-1} & x_{2n} \\ \vdots & \ddots & \vdots & \vdots \\ x_{m1} & \cdots & x_{m,n-1} & x_{mn} \end{pmatrix}$$

Repeating the same procedure with the remaining rows results in the following matrix:

$$\begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{m1} & \cdots & x_{mn} \end{pmatrix} \longmapsto \begin{pmatrix} x_{11} - x_{m1}x_{1n}x_{mn}^{-1} & \cdots & x_{1,n-1} - x_{m,n-1}x_{1n}x_{mn}^{-1} & 0 \\ x_{21} - x_{m1}x_{2n}x_{mn}^{-1} & \cdots & x_{2,n-1} - x_{m,n-1}x_{2n}x_{mn}^{-1} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ x_{m-1,1} - x_{m1}x_{m-1,n}x_{mn}^{-1} & \cdots & x_{m-1,n-1} - x_{m,n-1}x_{m-1,n}x_{mn}^{-1} & 0 \\ x_{m1} & \cdots & x_{m,n-1} & x_{mn} \end{pmatrix}$$

The upper left $(m-1) \times (n-1)$ matrix is simpler to study and carries a wealth of information of X in the sense made precise by the following results:

Observation 5.9 ([BV88])**.** Let B be a commutative ring, let $U = (u_{ij})$ be an $m \times n$ matrix of elements in B, $m \leq n$, and suppose that u_{mn} is a unit in B. Consider the $(m-1) \times (n-1)$ matrix $\widetilde{U} = (\tilde{u}_{ij})$ with entries given by

$$\tilde{u}_{ij} = u_{ij} - u_{mj}u_{in}u_{mn}^{-1} \quad \text{for } 1 \leq i \leq m-1,\ 1 \leq j \leq n-1.$$

Denote by $[\cdot \mid \cdot]_U$ and by $[\cdot \mid \cdot]_{\widetilde{U}}$ the minors of U and $\widetilde{U}$, respectively.

(1) Fix an integer $1 \leq t \leq m$. Given integers $1 \leq a_1, \ldots, a_t \leq m-1$ and $1 \leq b_1, \ldots, b_t \leq n-1$, one has

$$[a_1, \ldots, a_t, m \mid b_1, \ldots, b_t, n]_U = u_{mn}[a_1, \ldots, a_t \mid b_1, \ldots, b_t]_{\widetilde{U}}.$$

(2) For each integer $1 \leq t \leq m$, $I_{t-1}(\widetilde{U}) = I_t(U)$.

Proposition 5.10 ([BV88, Proposition 2.4])**.** Let $X = (x_{ij})$ and $Y = (y_{ij})$ the matrices of indeterminates over a ring B of sizes $m \times n$ and $(m-1) \times (n-1)$, respectively, $m \leq n$. Then the substitutions

$$\begin{aligned} x_{ij} &\longmapsto y_{ij} + x_{mj}x_{in}x_{mn}^{-1} && \text{for } 1 \leq i \leq m-1 \text{ and } 1 \leq j \leq n-1, \\ x_{in} &\longmapsto x_{in} && \text{for } 1 \leq i \leq m-1, \\ x_{mj} &\longmapsto x_{mj} && \text{for } 1 \leq j \leq n, \end{aligned}$$

induce an isomorphism of rings

$$\varphi \colon B[X][x_{mn}^{-1}] \longrightarrow B[Y][x_{m1}, \ldots, x_{mn}, x_{1n}, \ldots, x_{m-1,n}][x_{mn}^{-1}],$$

which maps the extension of $I_t(X) \subseteq B[X]$ to the extension of $I_{t-1}(Y) \subseteq B[Y]$ for each integer $1 \leq t \leq m$.

Proof. Consider the following substitutions:

$$\begin{aligned} y_{ij} &\longmapsto x_{ij} - x_{mj}x_{in}x_{mn}^{-1} && \text{for } 1 \le i \le m-1 \text{ and } 1 \le j \le n-1, \\ x_{in} &\longmapsto x_{in} && \text{for } 1 \le i \le m-1, \\ x_{mj} &\longmapsto x_{mj} && \text{for } 1 \le j \le n. \end{aligned}$$

This induces a ring homomorphism $\psi\colon B[Y][x_{m1},\dots,x_{mn},x_{1n},\dots,x_{m-1,n}][x_{mn}^{-1}] \to B[X][x_{mn}^{-1}]$, which is clearly the inverse of φ. From Observation 5.9 one has that $I_t(X)\,B[X][x_{mn}^{-1}] = I_{t-1}(\tilde{X})$, where $\tilde{X}$ is the $(m-1)\times(n-1)$ matrix with entries $\tilde{x}_{ij} = x_{ij} - x_{mj}x_{in}x_{mn}^{-1} =: y_{ij}$ for $1 \le i \le m-1$ and $1 \le j \le n-1$, thereby $I_{t-1}(\tilde{X})$ coincides with the extension of $I_{t-1}(Y)$. □

The proposition above gives a powerful tool to study ideals of minors, as it allows one to perform induction on the tuple (m,n,t). As we shall see in Sections 5.4 and 5.6, many properties of interest to us are shared between the ideals of minors corresponding to the tuples of integers (m,n,t), $(m+1,n+1,t+1)$, $(m+2,n+2,t+2)$, ... In addition, one has the following handy results:

Proposition 5.11. Let $X = (x_{ij})$ be an $m \times n$ matrix of indeterminates over a commutative ring B, $m \le n$. Then determinantal ideals of minors of X give the following chain of ideals in the polynomial ring $B[X]$:

$$I_m \subseteq I_{m-1} \subseteq \cdots \subseteq I_2 \subseteq I_1.$$

Proof. Choose an integer $1 \le t \le m-1$ and let $\delta = [a_1,\dots,a_{t+1} \mid b_1,\dots,b_{t+1}]$ be a $(t+1)$-minor. Expanding it along the first row yields

$$\delta = \sum_{k=1}^{t+1} (-1)^{k+1} x_{a_1 b_k} [a_2,\dots,a_{t+1} \mid b_1,\dots,\widehat{b_k},\dots,b_{t+1}],$$

where $\widehat{b_k}$ means that the integer b_k has been removed. Each summand is a multiple of a t-minor, thus $\delta \in I_t$ and $I_{t+1} \subseteq I_t$, which proves the proposition. □

Theorem 5.12 ([HE71, Theorem 1]). Let X be an $m \times n$ matrix of indeterminates over a commutative ring B. If B is a Noetherian domain, then I_t is a prime ideal for each $1 \le t \le m$.

5.2. Monomial orders

Monomial orders prove to be a useful tool in commutative algebra. For instance, these can be used to “highlight” a monomial μ of a polynomial f, and ensure that μ^n is a monomial of f^n for every integer $n \ge 1$. Furthermore, monomial orders lie in the very heart of the theory of Gröbner bases, which allow the use of computational methods in algebra.

In this section we present the basics of monomial orders in polynomial rings over a field $\mathbb{K}$. The reference is [E04, Chapter 15].

Notation 5.13. It will be convenient to use multi-index notation throughout this section. If $a = (a_1, \dots, a_n) \in \mathbb{Z}^n_{\geq 0}$ is an n-tuple of non-negative integers, x^a will denote the monomial

$$x^a := x_1^{a_1} \cdots x_n^{a_n}.$$

Under this notation, the product of two monomials x^a and x^b is $x^a x^b = x^{a+b}$, where $a+b$ is the component-wise sum of the multi-indices.

Definition 5.14. Let $R = \mathbb{K}[x_1, \dots, x_n]$ be a polynomial ring. A *term* in R is an element of the form αx^a, where $\alpha \in \mathbb{K}$ and $a = (a_1, \dots, a_n) \in \mathbb{Z}^n_{\geq 0}$ is a multi-index.

It is necessary to establish a total order on the indeterminates of the polynomial ring $\mathbb{K}[x_1, \dots, x_n]$ before defining monomial orders. The following order is considered

$$x_1 > x_2 > \cdots > x_n,$$

and it shall be the implicit ordering unless otherwise stated. Observe that if the ordering is different, the variables can be renamed so as to achieve the order above.

Definition 5.15. A *monomial order* on a polynomial ring $R = \mathbb{K}[x_1, \dots, x_n]$ is a total order $>$ on the set of monomials of S such that if x^a, x^b, $x^c \in R$ are monomials with $x^c \neq 1$, then

$$x^a > x^b \Longrightarrow x^{a+c} > x^{b+c} > x^b.$$

Definition 5.16. Let $S = \mathbb{K}[x_1, \dots, x_n]$ be a polynomial ring over a field $\mathbb{K}$ and let $>$ be a monomial order on S. Given a polynomial $f \in S$, the *initial term of* f, denoted by $\text{in}_>(f)$, is the greatest term of f with respect to the monomial order $>$. When the monomial order $>$ on S is clear from the context, we will denote the initial term by $\text{in}(f)$.

It is convenient to apply this order to terms as well: if x^a, $x^b \in S$ are monomials with $x^a > x^b$ and $0 \neq \alpha, \beta \in \mathbb{K}$, we write $\alpha x^a > \beta x^b$. Some examples of monomial orders are in order to fix ideas. Let $\mathbb{K}$ denote a field:

Definition 5.17. The *lexicographic order* $>_{\text{lex}}$ is a monomial order on $\mathbb{K}[x_1, \dots, x_n]$ defined as follows:

$$x^a >_{\text{lex}} x^b \text{ if and only if } a_i > b_i \text{ for the first index with } a_i \neq b_i.$$

Definition 5.18. The *homogeneous lexicographic order* $>_{\text{hlex}}$ is the monomial order on the polynomial ring $\mathbb{K}[x_1, \dots, x_n]$ defined by:

$$x^a >_{\text{hlex}} x^b \text{ if and only if } \begin{cases} \deg(x^a) > \deg(x^b) \text{ or,} \\ \deg(x^a) = \deg(x^b) \text{ and } a_i > b_i \text{ for the first index with } a_i \neq b_i. \end{cases}$$

Definition 5.19. The *graded reverse lexicographic order* $>_{\text{rlex}}$ is the monomial order on the polynomial ring $\mathbb{K}[x_1, \dots, x_n]$ defined by:

$$x^a >_{\text{rlex}} x^b \text{ if and only if } \begin{cases} \deg(x^a) > \deg(x^b) \text{ or,} \\ \deg(x^a) = \deg(x^b) \text{ and } a_i < b_i \text{ for the last index with } a_i \neq b_i. \end{cases}$$

Example 5.20. Let $R = \mathbb{K}[x, y, x]$ a polynomial ring with the usual ordering $x > y > z$.

(a) One has $x^2 >_{\text{lex}} yz^2$, but $yz^2 >_{\text{hlex}} x^2$ and $yz^2 >_{\text{rlex}} x^2$ since $\deg(yz^2) > \deg(x^2)$.

(b) Consider the monomials x^3, $xyz \in S$, that correspond to the multi-indices $(3, 0, 0)$ and $(1, 1, 1)$, respectively. Then $x^3 >_{\text{lex}} xyz$, $x^3 >_{\text{hlex}} xyz$ and also $x^3 >_{\text{rlex}} xyz$.

(c) Consider the degree 4 monomials y^3z, $xyz^2 \in S$. In the lexicographic and homogeneous lexicographic orders $xyz^2 >_{\text{lex}} y^3z$ and $xyz^2 >_{\text{hlex}} y^3z$, whereas under the graded reverse lexicographic order one has $y^3z >_{\text{rlex}} xyz^2$.

Example 5.21. Let $\mathbb{K}[x, y, u, v]$ be a polynomial ring with $x > y > u > v$, and define

$$f = \det \begin{pmatrix} x & y \\ u & v \end{pmatrix} = xv - yu.$$

With respect to the lexicographic and the homogeneous lexicographic orders, $\text{in}_{>_{\text{lex}}}(f) = \text{in}_{>_{\text{hlex}}}(f) = xv$. On the contrary, under the graded reverse lexicographic order, one has that $\text{in}_{>_{\text{rlex}}}(f) = -yu$.

Proposition 5.22. Let $R = \mathbb{K}[x_1, \ldots, x_n]$ be a polynomial ring and let $>$ be a monomial order on S. Then for any polynomials $f, g \in S$,

$$\text{in}(fg) = \text{in}(f)\,\text{in}(g).$$

Proof. The conclusion is clear when either of the polynomials is zero, thus suppose that both are non-zero. Let $\text{in}(f) = \alpha x^a$ and $\text{in}(g) = \gamma x^c$ be the initial terms of f and g, respectively, with $\alpha, \gamma \neq 0$, and write $f = \alpha x^a + \sum_{i=1}^r \beta_i x^{b_i}$, $g = \gamma x^c + \sum_{j=1}^s \delta_j x^{d_j}$. The product fg reads

$$fg = \alpha\gamma x^{a+c} + \sum_{i=1}^r \beta_i \gamma x^{b_i+c} + \sum_{j=1}^s \alpha\delta_j x^{a+d_j} + \sum_{i=1}^r \sum_{j=1}^s \beta_i \delta_j x^{b_i+d_j}.$$

Note that it might be necessary to collect terms in this expression, however this form suffices for the purpose of computing the initial term. By definition of monomial order,

$$\begin{aligned} \alpha x^a > \beta_i x^{b_i} &\implies \alpha\gamma x^{a+c} > \beta_i \gamma x^{b_i+c} \quad \text{for } 1 \leq i \leq r, \\ \gamma x^c > \delta_j x^{d_j} &\implies \alpha\gamma x^{a+c} > \alpha\delta_j x^{a+d_j} \quad \text{for } 1 \leq j \leq s. \end{aligned}$$

Likewise, $\alpha\gamma x^{a+c} > \beta_i \delta_j x^{b_i+d_j}$ for all $1 \leq i \leq r$ and $1 \leq j \leq s$, which proves the proposition. □

5.3. The F-pure threshold of a determinantal ideal

In [MSV14], Miller, Singh and Varbaro computed the F-pure threshold of a determinantal ideal in a polynomial ring over a field of positive characteristic.

Theorem 5.23 ([MSV14, Theorem 1.2]). Let $X = (x_{ij})$ be an $m \times n$ matrix of indeterminates over a field $\mathbb{F}$ of positive characteristic. Let $R = \mathbb{F}[X]$ be a polynomial ring and $1 \leq t \leq m$ an integer. The F-pure threshold of the ideal $I_t \subseteq R$ of t-minors of X is

$$\operatorname{fpt}(I_t) = \min \left\{ \frac{(m-k)(n-k)}{t-k} \;\middle|\; k = 0, \dots, t-1 \right\}.$$

In this section we reproduce this calculation. We begin by introducing some preliminary concepts and results, giving little to no proof.

Definition 5.24. Let R be a commutative ring and let $I \subseteq R$ be an ideal. The *integral closure* of I, denoted by $\overline{I}$, is the set of elements $x \in R$ that satisfy an equation of the form

$$x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n = 0,$$

for some integer $n \geq 1$, where $a_i \in I^i$ for each $0 \leq i \leq n$.

Observation 5.25. Some remarks regarding the integral closure of an ideal are in order:

(a) The integral closure $\overline{I}$ of an ideal $I \subseteq R$, is again an ideal of R.

(b) Let $R = \mathbb{K}[x_1, \dots, x_n]$ be a polynomial ring over a field $\mathbb{K}$ and let $I \subseteq R$ be an ideal. Let $f \in R$ be a homogeneous polynomial and suppose that $f \in \overline{I}$. Then there exists a homogeneous polynomial $g \in R$ such that $gf^\ell \in I^\ell$ for all integers $\ell \geq 1$.

The interested reader may consult the proofs of these facts and further observations in [HS06].

Theorem 5.26 (Briançon-Skoda theorem, [AH01, Theorem 1.2]). Let R be a regular local ring and let I be an ideal generated by r elements. Then for all $n \geq r$,

$$\overline{I^n} \subseteq I^{n-r+1}.$$

We bring in a notation introduced in [BV88, Chapter 10]. Given an integer t, define the function γ_t on a minor δ of a matrix of indeterminates X by

$$\gamma_t(\delta) := \begin{cases} 0 & \text{if } \deg \delta < t, \\ \deg \delta - t + 1 & \text{if } \deg \delta \geq t, \end{cases}$$

and extend it to a product $\delta_1 \cdots \delta_h$ of minors of X by

$$\gamma_t(\delta_1 \cdots \delta_h) := \sum_{i=1}^{h} \gamma_t(\delta_i).$$

Under mild hypothesis, the theorems below give explicit formulas for the symbolic powers and integral closures of determinantal ideals.

Theorem 5.27 ([BV88, Theorem 10.4]). Let B be an integral domain. Then for all integers $1 \leq t \leq m$ and $k \geq 0$, the k-th symbolic power of I_t is generated by the products of minors μ such that $\gamma_t(\mu) \geq k$. Equivalently,

$$I_t^{(k)} = \sum I_{t+\kappa_1-1} \cdots I_{t+\kappa_s-1},$$

where the sum is taken over all $\kappa_1, \dots, \kappa_s \geq 1$, $s \leq k$, such that $\kappa_1 + \cdots + \kappa_s \geq k$. Furthermore, $\mu \in I_t^{(k)}$ if and only if $\gamma_t(\mu) \geq k$.

Theorem 5.28 ([B91, Theorem 1.3])**.** Let X be an $m \times n$ matrix of indeterminates over an integral domain B, $m \leq n$. Then the primary decomposition of the integral closure of I_t^s in $B[X]$ is given by

$$\overline{I_t^s} = \bigcap_{j=1}^{t} I_j^{((t-j+1)s)}.$$

Theorem 5.29 (Bruns, [MSV14, Theorem 2.1])**.** Let $s \geq 1$ be an integer and let $\delta_1, \dots, \delta_h$ be minors of X. If $h \leq s$ and $\sum_{i=1}^{h} \deg \delta_i = ts$, then

$$\delta_1 \cdots \delta_h \in \overline{I_t^s}.$$

Proof. In view of [B91, Theorem 1.3] (see Theorem 5.28), one needs to show that $\delta_1 \cdots \delta_h \in I_j^{((t-j+1)s)}$ for each $j = 1, \dots, t$. By [BV88, Theorem 10.4] (see Theorem 5.27), this is equivalent to show that $\gamma_j(\delta_1 \cdots \delta_h) \geq (t - j + 1)s$, which follows from

$$\gamma_j(\delta_1 \cdots \delta_h) = \sum_{i=1}^{h} \gamma_j(\delta_i) = \sum_{\substack{i=1 \\ \deg \delta_i \geq j}}^{h} (\deg \delta_i - j + 1) \geq \sum_{i=1}^{h} (\deg \delta_i - j + 1) = (t - j + 1)s. \qquad \square$$

Proposition 5.30. Let $X = (x_{ij})$ be an $m \times n$ matrix of indeterminates over a field $\mathbb{K}$, $m \leq n$. Let I_t be the ideal of t-minors of X in the polynomial ring $R = \mathbb{K}[X]$. Then R_{I_t} is a regular local ring of dimension $(m - t + 1)(n - t + 1)$.

Proof. The ring R is regular because it is a polynomial ring over a field, thus by definition of regularity, the localized ring R_{I_t} is regular local of dimension equal to the height of I_t. The grade of I_t is $\operatorname{grade} I_t = (m - t + 1)(n - t + 1)$ by [BV88, Theorem 2.5] and, since R is a Cohen-Macaulay ring, height and grade of an ideal coincide, from where the result follows. $\square$

Observation 5.31. Fix a field $\mathbb{F}$ of characteristic $p > 0$, let X be an $m \times n$ matrix of indeterminates over $\mathbb{F}$ and $R = \mathbb{F}[X]$ a polynomial ring. Let I_t be the ideal of t-minors of X. Then for each integer $e \geq 0$, the only associated prime of $I_t^{[p^e]}$ is I_t. This follows from the flatness of the Frobenius endomorphism; for instance, see [ILL+07, Corollary 21.11].

Lemma 5.32. Let X be an $m \times n$ matrix of indeterminates over a field $\mathbb{F}$ of characteristic p and let I_k the ideal of k-minors of X in the polynomial ring $R = \mathbb{F}[X]$. Fix an integer $e \geq 0$. If $\mathfrak{a} \subseteq R$ is an ideal such that $\mathfrak{a}R_{I_k} \subseteq I_k^{[p^e]} R_{I_{k+1}}$, then $\mathfrak{a} \subseteq I_k^{[p^e]}$.

Proof. The ideal $\mathfrak{a}R_{I_k}$ is generated by elements of the form $a/1$, where $a \in \mathfrak{a}$. By assumption, there exists $f \in I_k^{[p^e]}$ and $u \in R - I_k$ such that $a/1 = f/u$, hence $(au - f)v = 0$ for some $v \in R - I_k$, but since R is a domain, one has that $au = f \in I_k^{[p^e]}$. Viewing $R/I_k^{[p^e]}$ as an R-module, one has that $u \cdot \overline{a} = \overline{f} = 0$. The set of zero-divisors on $R/I_k^{[p^e]}$ is equal to the union of the associated primes of $I_k^{[p^e]}$. By the flatness of the Frobenius endomorphism, the only associated prime of $I_k^{[p^e]}$ is I_k. As $u \in R - I_k$, it follows that $\overline{a} = 0$, that is, $a \in I^{[p^e]}$, which proves the assertion. □

Now we are in position to prove [MSV14, Theorem 1.2]. The proof consists in giving an upper bound and a lower bound for the ν-invariant $\nu_{I_t}^{\mathfrak{m}}(p^e)$ of I_t at the homogeneous maximal ideal $\mathfrak{m}$ of the polynomial ring $R = \mathbb{F}[X]$. These bounds have constant terms which do not depend on p^e, thus they vanish when taking limits to compute the F-pure threshold.

Proposition 5.33 ([MSV14])**.** Let $X = (x_{ij})$ be matrix of indeterminates of size $m \times n$, $m \leq n$, over a field $\mathbb{F}$ of characteristic p and set $R = \mathbb{F}[X]$. Denote by $\mathfrak{m} = (x_{11}, \dots, x_{mn}) \subseteq R$ the homogeneous maximal ideal and let $I_t \subseteq R$ be the ideal of t-minors of X. Then there exists an integer $N \geq 0$ such that for all $e \geq 0$,

$$\nu_{I_t}^{\mathfrak{m}}(p^e) \leq \frac{(m-k)(n-k)}{t-k}(p^e - 1) + N$$

for all integers $0 \leq k \leq t-1$.

Proof. Choose an integer $0 \leq k \leq t-1$ and let δ_k and δ_t be minors of X of sizes k and t, respectively. Then $\delta_k^{t-k-1}\delta_t$ is a product of $t-k$ minors and has degree $k(t-k-1)+t = (k+1)(t-k)$, hence $\delta_k^{t-k-1}\delta_t \in \overline{I_{k+1}^{t-k}}$ by [MSV14, Theorem 2.1] (see Theorem 5.29). Because δ_t is an arbitrary minor of size t, one has that $\delta_k^{t-k-1} I_t \subseteq \overline{I_{k+1}^{t-k}}$. By the Briançon-Skoda theorem (see Theorem 5.26) there exists an integer $N \geq 0$ such that for all $\ell \geq 1$,

$$\left(\delta_k^{t-k-1} I_t\right)^{N+\ell} \subseteq I_{k+1}^{(t-k)\ell}.$$

The ideals of minors satisfy $I_t \subseteq I_{k+1} \subseteq I_k$ because of Proposition 5.11, therefore in the localization $R_{I_{k+1}}$ the extension of I_t is a proper ideal, whereas δ_k becomes a unit, whence

$$I_t^{N+\ell} R_{I_{k+1}} \subseteq I_{k+1}^{(t-k)\ell} R_{I_{k+1}}.$$

Now the maximal ideal $I_{k+1}R_{I_{k+1}}$ of $R_{I_{k+1}}$ is generated by $(m-k)(n-k)$ elements by Proposition 5.30 and, by Proposition 4.31,

$$I_t^{N+\ell} R_{I_{k+1}} \subseteq I_{k+1}^{(t-k)\ell} R_{I_{k+1}} \subseteq I_{k+1}^{[p^e]} R_{I_{k+1}}$$

for all integers satisfying $(t-k)\ell \geq (m-k)(n-k)(p^e-1)+1$. As the only associated prime of $I_{k+1}^{[p^e]}$ is I_{k+1}, by applying Lemma 5.32 we recover the inclusion of ideals in R:

$$I_t^{N+\ell} \subseteq I_{k+1}^{[p^e]} \subseteq \mathfrak{m}^{[p^e]}.$$

It follows that the ν-invariant $\nu_{I_t}^{\mathfrak{m}}(p^e)$ is bounded above by

$$\nu_{I_t}^{\mathfrak{m}}(p^e) \leq \frac{(m-k)(n-k)}{t-k}(p^e - 1) + N. \qquad \square$$

Consider the homogeneous lexicographic order (see Section 5.2) on a polynomial ring $R = \mathbb{K}[X]$ over a field $\mathbb{K}$, with the following ordering on the indeterminates:

$$\begin{array}{ccccccccc} x_{11} & > & x_{12} & > & \cdots & > & x_{1n} & > \\ x_{21} & > & x_{22} & > & \cdots & > & x_{2n} & > \\ \vdots & & \vdots & & \vdots & & \vdots & \\ x_{m1} & > & x_{m2} & > & \cdots & > & x_{mn}. & \end{array}$$

Under this ordering, the initial term of a minor is the product of the indeterminates in the main diagonal, as shown in the following lemma.

Lemma 5.34. Let $X = (x_{ij})$ be an $m \times n$ matrix of indeterminates over a field $\mathbb{K}$, $m \leq n$. Fix $1 \leq t \leq m$ and choose integers $1 \leq a_1 < \cdots < a_t \leq m$ and $1 \leq b_1 < \cdots < b_t \leq n$. Let $>$ be the homogeneous lexicographic order on the polynomial ring $\mathbb{K}[X]$. Then

$$\operatorname{in}([a_1, \ldots, a_t \mid b_1, \ldots, b_t]) = x_{a_1 b_1} \cdots x_{a_t b_t}.$$

Proof. The t-minor reads $[a_1, \ldots, a_t \mid b_1, \ldots, b_t] = \sum_{\sigma \in \mathbb{S}_t} \operatorname{sgn}(\sigma) \prod_{i=1}^{t} x_{a_i b_{\sigma(i)}}$, where $\mathbb{S}_t$ is the symmetric group of degree t. Let $\sigma \in \mathbb{S}_t$ be a permutation different from the identity. Then there exists an integer $1 \leq i \leq t$ such that $\sigma(i) \neq i$, and is minimal with this property, that is, $\sigma(j) = j$ for all $1 \leq j < i$. In consequence, one has $\sigma(i) > i$, thus $x_{a_i b_i} > x_{a_i b_{\sigma(i)}}$ and $x_{a_1 b_1} \cdots x_{a_{i-1} b_{i-1}} x_{a_i b_i} > x_{a_1 b_{\sigma(1)}} \cdots x_{a_{i-1} b_{\sigma(i-1)}} x_{a_i b_{\sigma(i)}}$. As a result, $x_{a_1 b_1} \cdots x_{a_t b_t} > x_{a_1 b_{\sigma(1)}} \cdots x_{a_t b_{\sigma(t)}}$ for all permutations $\sigma \in \mathbb{S}_t - \{\mathrm{Id}\}$, which proves the lemma. □

Notice that if the integers $1 \leq a_1, \ldots, a_t \leq m$ and $1 \leq b_1, \ldots, b_t \leq n$ specifying the rows and the columns of the t-minor are not given in ascending order, the integers can be swapped until they are in ascending order. As the determinant of a matrix is invariant up to a sign under row and column swaps, the initial term is invariant up to a sign as well.

Example 5.35. Let $X = (x_{ij})$ be a 5×7 matrix of indeterminates over a field $\mathbb{K}$ and let δ be the 4-minor $\delta = [1, 3, 4, 5 \mid 2, 5, 6, 7]$. With respect to the homogeneous lexicographic order on $R = \mathbb{K}[X]$ one has

$$\operatorname{in}(\delta) = \operatorname{in} \begin{pmatrix} x_{12} & x_{15} & x_{16} & x_{17} \\ x_{32} & x_{35} & x_{36} & x_{37} \\ x_{42} & x_{45} & x_{46} & x_{47} \\ x_{52} & x_{55} & x_{56} & x_{57} \end{pmatrix} = x_{12} x_{35} x_{46} x_{57}.$$

Instead, suppose that the rows or columns are not given in ascending order, for instance let $\delta' = [4, 5, 1, 3 \mid 5, 2, 6, 7]$. One needs 5 transpositions in order to make the indices appear in ascending order, hence $\delta' = (-1)^5 \delta = -\delta$, and the initial term reads $\operatorname{in}(\delta') = -x_{12} x_{35} x_{46} x_{57}$.

Observation 5.36. We may graphically depict a minor as follows. Represent the $m \times n$ matrix of indeterminates $X = (x_{ij})$ as an $m \times n$ array of dots, in such a way that the (i, j)

dot corresponds to x_{ij}. Then a t-minor $\delta = [a_1, \dots, a_t \mid b_1, \dots, b_t]$ is drawn in the array of dots by means of a line connecting the indeterminates in the initial term of δ. As $\mathrm{in}(\delta)$ involves t variables and no two of them are in the same row or column, the line completely determines the minor. In the case of Example 5.35, the 4-minor $[1,3,4,5 \mid 2,5,6,7]$ is depicted as:

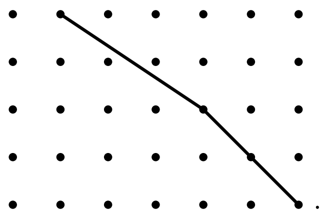

Figure 5.1

In order to obtain a lower bound on $\nu_{I_t}^{\mathfrak{m}}(p^e)$, one exhibits an element in some power of I_t not in $\mathfrak{m}^{[p^e]}$. We shall introduce further notation to this end. For each integer $0 \le k \le m$, define

$$\Delta_k := \prod_{i=0}^{n-m} [i+1, \dots, i+m] \prod_{j=2}^{m-k} [j, \dots, m \mid 1, \dots, m-j+1] \cdot [1, \dots, m-j+1 \mid n-m+j, \dots, n],$$

which is a product of $n+m-2k-1$ minors. One readily checks that $\deg \Delta_k = mn - k^2 - k$. Furthermore, for $1 \le k \le m$ define

$$\Delta'_k := \Delta_k \cdot [m-k+1, \dots, m \mid 1, \dots, k].$$

It is a product of $n+m-2k$ minors and has degree $\deg \Delta'_k = mn - k^2$.

Example 5.37. We display the products of minors to help with notation. The initial terms of Δ and Δ'_k are the product of the indeterminates in the diagonals drawn. In the case $k=0$, the initial term of Δ_0 is the product of all the indeterminates.

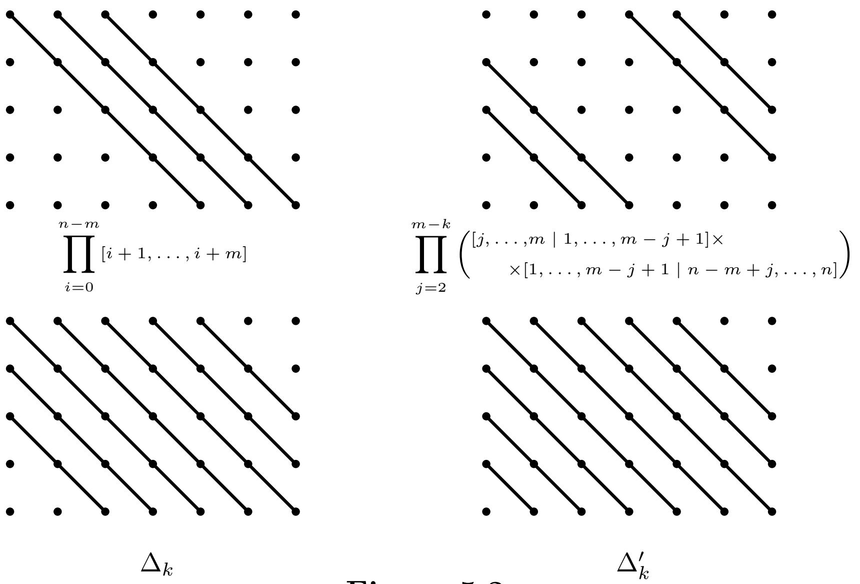

Figure 5.2

Lemma 5.38 ([MSV14, Lemma 2.2])**.** Fix integers $1 \leq t \leq m \leq n$. Let k be the least integer in the interval $[0, t-1]$ such that

$$\frac{(m-k)(n-k)}{t-k} \leq \frac{(m-(k+1))(n-(k+1))}{t-(k+1)},$$

where we view an integer divided by zero as infinity. Define $u = t(m+n-2k) - mn + k^2$. Then $t - k - u \geq 0$.

Proof. The integer $k \in [0, t-1]$ satisfying the hypothesis clearly exists. By manipulating the inequality, one gets

$$t - k - t(m+n-2k) + mn - k^2 \geq 0,$$

from where the assertion follows. □

Proposition 5.39 ([MSV14])**.** Let $X = (x_{ij})$ be a matrix of indeterminates of size $m \times n$, $m \leq n$, defined over a field $\mathbb{F}$ of characteristic $p > 0$ and set $R = \mathbb{F}[X]$. Denote by $\mathfrak{m} = (x_{11}, \ldots, x_{mn}) \subseteq R$ the homogeneous maximal ideal and let $I_t \subseteq R$ be the ideal of t-minors of X. Let k be the least integer in the interval $[0, t-1]$ such that

$$\frac{(m-k)(n-k)}{t-k} \leq \frac{(m-(k-1))(n-(k-1))}{t-(k-1)}.$$

Then there exists a homogeneous polynomial $f \in R$ such that for all integers $e \gg 0$,

$$\nu_{I_t}^{\mathfrak{m}}(p^e) \geq (m-k)(n-k)\left(\frac{p^e - 1 - \deg f}{t-k} - 1\right).$$

Proof. The integer k exists by [MSV14, Lemma 2.2] (see Lemma 5.38) and, by letting $u = t(m+n-2k) - mn + k^2$, one has that $t - k - u \geq 0$. Define the product of minors Δ by

$$\Delta \coloneqq \begin{cases} \Delta_0^t & \text{if } k = 0, \\ \Delta_k^u \left(\Delta_k'\right)^{t-k-u} & \text{if } k \geq 1 \text{ and } u \geq 0, \\ \Delta_{k-1}^{-u} \left(\Delta_k'\right)^{t-k+u} & \text{if } k \geq 1 \text{ and } u < 0. \end{cases}$$

When $k = 0$, Δ is a product of $t(m+n-1)$ minors, has degree tmn and, from $t - u \geq 0$, one has that $t(m+n-1) \leq mn$. For $k \geq 1$, Δ is a product of $(m-k)(n-k)$ minors and has degree $t(m-k)(n-k)$. In all cases, it follows from [MSV14, Theorem 2.1] (see Theorem 5.29) that

$$\Delta \in \overline{I_t^{(m-k)(n-k)}},$$

therefore there exists a homogeneous polynomial $f \in R$ such that for all integers $\ell \geq 1$,

$$f\Delta^\ell \in I_t^{(m-k)(n-k)\ell}.$$

By Proposition 5.22, the initial term of Δ with respect to the homogeneous lexicographic order reads

$$\operatorname{in}(\Delta) = \begin{cases} \operatorname{in}(\Delta_0)^t & \text{if } k = 0, \\ \operatorname{in}(\Delta_k)^u \operatorname{in}(\Delta'_k)^{t-k-u} & \text{if } k \geq 1 \text{ and } u \geq 0, \\ \operatorname{in}(\Delta_{k-1})^{-u} \operatorname{in}(\Delta'_k)^{t-k+u} & \text{if } k \geq 1 \text{ and } u < 0, \end{cases}$$

and it is clear that every variable involved in the initial term $\operatorname{in}(\Delta)$ has exponent at most $t-k$. As a result, as long as $\ell \geq 1$ satisfies

$$\deg f + \ell(t-k) \leq p^e - 1,$$

one has that $\operatorname{in}(f\Delta^\ell) = \operatorname{in}(f)\operatorname{in}(\Delta)^\ell \notin \mathfrak{m}^{[p^e]}$, hence

$$f\Delta^\ell \in I_t^{(m-k)(n-k)\ell} - \mathfrak{m}^{[p^e]},$$

and $\nu_{I_t}^{\mathfrak{m}}(p^e) \geq (m-k)(n-k)\ell$. Taking $\ell \geq 1$ to be maximal among the integers satisfying the inequality above, it follows that

$$\nu_{I_t}^{\mathfrak{m}}(p^e) \geq (m-k)(n-k)\left(\frac{p^e - 1 - \deg f}{t-k} - 1\right). \qquad \square$$

Proof of Theorem 5.23. By Propositions 5.33 and 5.39, the ν-invariant $\nu_{I_t}^{\mathfrak{m}}(p^e)$ is bounded by

$$(m-k)(n-k)\left(\frac{p^e - 1 - \deg f}{t-k} - 1\right) \leq \nu_{I_t}^{\mathfrak{m}}(p^e) \leq N + \frac{(m-k)(n-k)}{t-k}(p^e - 1)$$

for all integers $e \gg 0$. Dividing by p^e and taking the limit when $e \to \infty$, by definition of the F-pure threshold and from the choice of the integer k in the interval $[0, t-1]$ from [MSV14, Lemma 2.2] (see Lemma 5.38),

$$\operatorname{fpt}(I_t) = \min\left\{ \frac{(m-k)(n-k)}{t-k} \;\middle|\; k = 0, \ldots, t-1 \right\}. \qquad \square$$

5.4. The ν-invariants of ideals of maximal minors

As introduced in Chapter 1, Lőrincz, Raicu, Walther and Weyman computed the Bersntein-Sato polynomial of the ideal of maximal minors in [LRWW17].

Theorem 5.40 ([LRWW17, Theorem 4.1])**.** Let $X = (x_{ij})$ be an $m \times n$ matrix of indeterminates of size $m \times n$, $m \leq n$, defined over $\mathbb{C}$. Let I_m be the ideal of maximal minors of X in the polynomial ring $\mathbb{C}[X]$. The b-function of I_m is given by

$$b_{I_m}(s) = \prod_{i=n-m+1}^{n} (s+i).$$

In this section we prove the positive characteristic analogue, namely, the computation of the ν-invariants and the Frobenius roots of powers of the ideals of maximal minors of matrices of indeterminates over perfect fields of positive characteristic. The proof is given by a sequence of lemmas.

Lemma 5.41. Let $X = (x_{ij})$ and $Y = (y_{ij})$ be matrices of indeterminates of sizes $m \times n$ and $(m-1) \times (n-1)$, respectively, over a perfect field $\mathbb{F}$ of characteristic $p > 0$, with $m \leq n$. Define the rings $R = \mathbb{F}[X]$, $S = \mathbb{F}[Y]$ and $Q = R[x_{mn}^{-1}]$. Fix an integer $1 \leq t \leq m$, let $I = I_t(X) \subseteq R$ be the ideal of t-minors of X and $J = I_{t-1}(Y) \subseteq S$ the ideal of $(t-1)$-minors if Y. Then S injects in Q and for all integers $e \geq 0$ and $\ell \geq 0$,

$$(\mathcal{C}_R^e \cdot I^\ell)Q = (\mathcal{C}_S^e \cdot J^\ell)Q.$$

Proof. Consider the polynomial ring $T = \mathbb{K}[Y][x_{m1}, \ldots, x_{mn}, x_{1n}, \ldots, x_{m-1,n}][x_{mn}^{-1}]$ and let

$$\psi = \varphi^{-1} \colon T = \mathbb{K}[Y][x_{m1}, \ldots, x_{mn}, x_{1n}, \ldots, x_{m-1,n}][x_{mn}^{-1}] \xrightarrow{\cong} Q = \mathbb{K}[X][x_{mn}^{-1}]$$

be the isomorphism from [BV88, Proposition 2.4] (see Proposition 5.10). The ring S injects in the polynomial ring $\mathbb{K}[Y][x_{m1}, \ldots, x_{mn}, x_{1n}, \ldots, x_{m-1,n}]$ and, as it is a domain, the localization map is an injection, thus the composition $S \hookrightarrow T \xrightarrow{\cong} Q$ is an injection. This gives a diagram

$$\begin{array}{ccc} R & \hookrightarrow & Q \\ & & \cong \big\uparrow \psi \\ S & \hookrightarrow & T. \end{array}$$

For each integer $\ell \geq 0$ the isomorphism ψ maps the extension of I^ℓ to the extension of J^ℓ, that is, $I^\ell Q = J^\ell Q$. It follows that for each integer $e \geq 0$,

$$\begin{aligned} (\mathcal{C}_R^e \cdot I^\ell)Q &= \mathcal{C}_Q^e \cdot (I^\ell Q) && \text{(Proposition 4.24)} \\ &= \mathcal{C}_Q^e \cdot (J^\ell Q) && \text{(Proposition 5.10)} \\ &= (\mathcal{C}_T^e \cdot (J^\ell T))Q && \text{(Proposition 4.24)} \\ &= ((\mathcal{C}_S^e \cdot J^\ell)T)Q && \text{(Proposition 4.25)} \\ &= (\mathcal{C}_S^e \cdot J^\ell)Q, \end{aligned}$$

which proves the lemma. □

Lemma 5.42. Let $X = (x_{ij})$ be an $m \times n$ matrix of indeterminates over a perfect field $\mathbb{F}$ of positive characteristic. Let I_m be the ideal of maximal minors of X in the polynomial ring $R = \mathbb{F}[X]$. Then for all integers $\ell \geq 1$, x_{mn} is a non-zero-divisor modulo I_m^ℓ, that is to say, $f x_{mn} \in I_k^\ell$ implies that $f \in I_k^\ell$.

Proof. By [BC03, Proposition 2.2], x_{mn} is a non-zero-divisor modulo $I_m^{(\ell)}$ and, by [BC03, Corollary 2.3], symbolic and usual powers of I_m coincide, therefore x_{mn} is regular modulo I_m^ℓ. □

Lemma 5.43. Let R be a ring and let $I, J \subseteq R$ be ideals. Let $f \in R$ be an element such that $IR_f \subseteq JR_f$. If f is a non-zero-divisor on R/J, then $I \subseteq J$.

Proof. The ideal IR_f is generated by the elements a/f^n, where $a \in I$ and $n \geq 0$ is an integer. By assumption, for each $a \in I$ there exists $b \in J$ and an integer $n \geq 0$ such that $a/1 = b/f^n$, hence $(af^n - b)f^m = 0$ for some $m \gg 0$, thus $af^{n+m} = bf^m \in J$. As a result, $a \in (J \colon f^{n+m}) = J$, where the second equality follows from f being a non-zero-divisor on R/J. $\square$

Lemma 5.44. Let $R = \mathbb{F}[x_1, \ldots, x_n]$ be a polynomial ring over a field $\mathbb{F}$ of characteristic $p > 0$ and let $\mathfrak{m} = (x_1, \ldots, x_n) \subseteq R$ be the homogeneous maximal ideal. Fix an integer $e \geq 1$. Then for all integers $s \geq 0$,

$$\mathcal{C}_R^e \cdot \mathfrak{m}^{(s-1)p^e + n(p^e-1)+1} \subseteq \mathfrak{m}^s.$$

Proof. Given an integer $\ell \geq 1$, $\mathfrak{m}^\ell$ is the ideal of R generated by the monomials of degree ℓ. Let $x^a \in \mathfrak{m}^{(s-1)p^e+n(p^e-1)+1}$ be a monomial of degree $\deg x^a = |a| = (s-1)p^e + n(p^e-1)+1$. In the standard basis of $F_*^e R$ as a free R-module (see Definition 2.34), x^a is expressed as $F_*^e x^a = F_*^e x^{bp^e + r} = x^b F_*^e x^r$, where $F_*^e x^r$ belongs to the standard basis, thus $\deg x^r = |r| \leq n(p^e - 1)$. Since $|a| = |b|p^e + |r|$, one has

$$(s-1)p^e + n(p^e-1) + 1 \leq |b|p^e + n(p^e-1),$$

therefore $|b| \geq s - 1 + \frac{1}{p^e}$. As $|b|$ is an integer, one has that $|b| \geq s$, hence $x^b \in \mathfrak{m}^s$. From [BMS08, Proposition 2.5] (see Proposition 4.18), a subset of the monomials of degree $\geq s$ generate the Frobenius root $\mathcal{C}_R^e \cdot \mathfrak{m}^{(s-1)p^e+n(p^e-1)+1}$, thus it is a subset of $\mathfrak{m}^s$. $\square$

Let $X = (x_{ij})$ be an $m \times n$ matrix of indeterminates. In what follows, we will consider the following ordering on the variables:

$$\begin{array}{ccccccccc} x_{11} & > & x_{12} & > & \cdots & > & x_{1n} & > \\ x_{21} & > & x_{22} & > & \cdots & > & x_{2n} & > \\ \vdots & & \vdots & & \ddots & & \vdots & \\ x_{m1} & > & x_{m2} & > & \cdots & > & x_{mn}. & \end{array}$$

Proposition 5.45. Let $X = (x_{ij})$ be an $m \times n$ matrix of indeterminates over a field $\mathbb{F}$ of characteristic $p > 0$, and let $R = \mathbb{F}[X]$ be a polynomial ring. For each $i = 0, 1, \ldots, n-m$ let

$$\delta_i = [1+i, \ldots, m+i] = \det \begin{pmatrix} x_{1,1+i} & \cdots & x_{1,m+1} \\ \vdots & \ddots & \vdots \\ x_{m,1+i} & \cdots & x_{m,m+1} \end{pmatrix}.$$

Furthermore, define $\Delta = (\delta_0\, \delta_1 \cdots \delta_{n-m})^{p^e-1}$, and let $>$ be the homogeneous lexicoprahic order on R. Then:

(1) For each $i = 0, 1, \dots, n-m$, the initial term of δ_i is $\operatorname{in}(\delta_i) = x_{1,1+i} \cdots x_{m,m+i}$.
(2) The initial term of Δ reads

$$\operatorname{in}(\Delta) = (\operatorname{in}(\delta_0)\operatorname{in}(\delta_1)\cdots\operatorname{in}(\delta_{n-m}))^{p^e-1} = \prod_{i=1}^{n-m} (x_{1,1+i}\cdots x_{m,1+i})^{p^e-1},$$

thus it is an element of the standard basis of F^e_*R.
(3) In the standard basis of F^e_*R, $F^e_*\Delta$ reads

$$F^e_*\Delta = F^e_*\operatorname{in}(\Delta) + \sum_{F^e_*\mu \neq F^e_*\operatorname{in}(\Delta)} g_\mu F^e_*\mu, \quad \text{where } g_\mu \in R.$$

Example 5.46. In order to help intuition, we depict the principal diagonal of the minors $\delta_0, \delta_1, \delta_2$ and δ_3 for a 3×7 matrix.

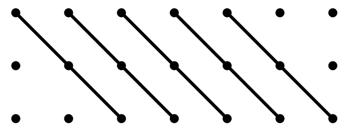

Figure 5.3

Proof of Proposition 5.45. (1) The assertion follows from Lemma 5.34.

(2) By Proposition 5.22, the initial term of the product is equal to the product of the initial terms. It follows that the following element is in the standard basis:

$$\operatorname{in}(\Delta) = \prod_{i=0}^{n-m} \operatorname{in}(\delta_i)^{p^e-1} = \prod_{i=0}^{n-m} (x_{1,1+i}\cdots x_{m,m+i})^{p^e-1}.$$

(3) Let $\alpha x^a = \alpha x_{11}^{a_{11}} x_{12}^{a_{12}} \cdots x_{mn}^{a_{mn}}$ be a monomial of Δ, with $\alpha \in \mathbb{F} - \{0\}$, and suppose that $F^e_*\alpha x^a = gF^e_*\operatorname{in}(\Delta)$, where $g \in R$. Since Δ is a homogeneous polynomial of degree $m(n-m+1)(p^e-1)$, $\alpha x^a = g^{p^e}\operatorname{in}(\Delta)$ is a monomial of the same degree. As a result, $(g^{p^e}+1)\operatorname{in}(\Delta)$ is a term of Δ, which equals $\operatorname{in}(\Delta)$, but this means $g = 0$ and $\alpha = 0$, a contradiction. $\square$

Theorem 5.47 (The ν-invariants of ideals of maximal minors)**.** Let $X = (x_{ij})$ be a matrix of indeterminates of size $m \times n$, $m \leq n$, over a perfect field $\mathbb{F}$ of characteristic $p > 0$. Fix an integer $e \geq 0$ and let I_m be the ideal of maximal minors of X in the polynomial ring $R = \mathbb{F}[X]$.
(1) For all integers $0 \leq k \leq (n-m+1)(p^e-1)$, $\mathcal{C}^e_R \cdot I^k_m = R$.
(2) For all integers $s \geq 1$,

$$\begin{aligned}\mathcal{C}^e_R \cdot I_m^{sp^e+(n-m+1)(p^e-1)} &= \mathcal{C}^e_R \cdot I_m^{sp^e+(n-m+1)(p^e-1)-1} = \mathcal{C}^e_R \cdot I_m^{sp^e+(n-m+1)(p^e-1)-2} = \cdots \\ &= \mathcal{C}^e_R \cdot I_m^{(s-1)p^e+(n-m+1)(p^e-1)+2} = \mathcal{C}^e_R \cdot I_m^{(s-1)p^e+(n-m+1)(p^e-1)+1} = I^s_m.\end{aligned}$$

(3) The set of ν-invariants of I_m of level e is

$$\nu^\bullet_{I_m}(p^e) = \{sp^e + (n-m+1)(p^e-1) \mid s \in \mathbb{Z}_{\geq 0}\}$$

(4) If $s \geq 0$ is an integer and $J = I^{s+1}_m$, then $\nu^J_{I_m}(p^e) = sp^e + (n-m+1)(p^e-1)$.

Proof. (1) Let $\Delta = (\delta_0 \cdots \delta_{n-m})^{p^e-1}$ be defined as in Proposition 5.45. It is a product of $n-m+1$ minors of X raised to p^e-1, hence $\Delta \in I_m^{(n-m+1)(p^e-1)}$ and $\mathcal{C}_R^e \cdot \Delta \subseteq \mathcal{C}_R^e \cdot I_m^{(n-m+1)(p^e-1)}$. From Proposition 4.18 and the expression of $F_*^e\Delta$ in the standard basis of F_*^eR given in Proposition 5.45, one has that $1 \in \mathcal{C}_R^e \cdot \Delta$, from where the assertion follows.

(2) The Frobenius roots give an ascending chain

$$\mathcal{C}_R^e \cdot I_m^{sp^e+(n-m+1)(p^e-1)} \subseteq \mathcal{C}_R^e \cdot I_m^{sp^e+(n-m+1)(p^e-1)-1} \subseteq \cdots \subseteq \mathcal{C}_R^e \cdot I_m^{(s-1)p^e+(n-m+1)(p^e-1)+1},$$

hence it suffices to show that $I_m^s \subseteq \mathcal{C}_R^e \cdot I_m^{sp^e+(n-m+1)(p^e-1)}$ and $\mathcal{C}_R^e \cdot I_m^{(s-1)p^e+(n-m+1)(p^e-1)+1} \subseteq I_m^s$. As for the first inclusion, by Proposition 4.21,

$$I_m^s = I_m^s\, \mathcal{C}_R^e \cdot I_m^{(n-m+1)(p^e-1)} = \mathcal{C}_R^e \cdot \left(I_m^{s[p^e]}\, I_m^{(n-m+1)(p^e-1)} \right) \subseteq \mathcal{C}_R^e \cdot I_m^{sp^e+(n-m+1)(p^e-1)}.$$

In order to prove the second inclusion, we proceed by induction on m. For a row matrix, i.e. $m = 1$ the result follows from Lemma 5.44, thus suppose that $m \geq 2$. Consider the rings $S = \mathbb{K}[Y]$ and $Q = R[x_{mn}^{-1}]$, where $Y = (y_{ij})$ is an $(m-1)\times(n-1)$ matrix of indeterminates over $\mathbb{F}$. By letting $J = I_{m-1}(Y) \subseteq S$ be the ideal of maximal minors of Y, it follows from Lemma 5.41 that $(\mathcal{C}_R^e \cdot I_m^\ell)Q = (\mathcal{C}_S^e \cdot J^\ell)Q$ for all integers $\ell \geq 0$. Observe that $\operatorname{fpt}(I_m) = \operatorname{fpt}(J) = n-m+1$, therefore

$$\begin{aligned}
(\mathcal{C}_R^e \cdot I_m^{(s-1)p^e+(n-m+1)(p^e-1)+1})Q &= (\mathcal{C}_R^e \cdot J^{(s-1)p^e+(n-m+1)(p^e-1)+1})Q && \text{(Lemma 5.41)}\\
&= J^sQ && \text{(Induction)}\\
&= I_m^sQ. && \text{(Proposition 5.10)}
\end{aligned}$$

By Lemma 5.42, x_{mn} is a non-zero divisor modulo I_m^s, therefore applying Lemma 5.43 one recovers the inclusion in the ring R, that is $\mathcal{C}_R^e \cdot I_m^{(s-1)p^e+(n-m+1)(p^e-1)+1} \subseteq I_m^s$.

(3) It follows from (1) and (2) that for all integers $s \geq 0$,

$$I^{s+1} = \mathcal{C}_R^e \cdot I^{sp^e+(n-m+1)(p^e-1)+1} \subsetneq \mathcal{C}_R^e \cdot I^{sp^e+(n-m+1)(p^e-1)} = I^s,$$

thereby the ν-invariants of I of level e are given by $sp^e + (n-m+1)(p^e-1)$ for $s \in \mathbb{Z}_{\geq 0}$.

(4) Immediate from (3). □

Corollary 5.48. Under the assumptions of Theorem 5.47, the F-pure threshold of I_m with respect to the homogeneous maximal ideal $\mathfrak{m}$ of R is $\operatorname{fpt}(I_m) = n-m+1$.

Proof. Since $\nu_{I_m}^{\mathfrak{m}}(p^e) = (n-m+1)(p^e-1)$ for each positive integer $e \geq 0$, one has

$$\operatorname{fpt}(I_m) = \lim_{e\to\infty} \frac{\nu_{I_m}^{\mathfrak{m}}(p^e)}{p^e} = n-m+1.$$

We note, however, that this fact can be deduced directly from [MSV14, Theorem 1.2] (see Theorem 5.23). □

Corollary 5.49. Under the assumptions of Theorem 5.47, the only Bernstein-Sato root of the ideal I_m is $\alpha = -\mathrm{fpt}(I_m) = -(n-m+1)$.

Proof. Let $(t_d)_{d=0}^{\infty} \subseteq \mathbb{Z}_{\geq 0}$ be a sequence of non-negative integers and define

$$\nu_d := t_d p^d + (n-m+1)(p^d-1) \in \nu_{I_m}^{\bullet}(p^d).$$

Then $(\nu_d)_{d=0}^{\infty}$ is a sequence of ν-invariants with p-adic limit $\nu_d \to -\mathrm{fpt}(I_m) = -(n-m+1)$ as $d \to \infty$. By [QG21b, Theorem IV.17] (see Definition 4.36), $\alpha = -\mathrm{fpt}(I_m)$ is a Bernstein-Sato root of I_m. Any sequence of ν-invariants is of this form, thus the only Bernstein-Sato root of I_m is $\alpha = -\mathrm{fpt}(I_m)$. □

The corollary above allows us to answer a question raised in [QG21a]:

Question 5.50 ([QG21a, Question 6.16])**.** Suppose that the F-pure threshold α of an ideal $\mathfrak{a} \subseteq R$ lies in $\mathbb{Z}_{(p)}$. Is the largest Bernstein-Sato root of $\mathfrak{a}$ equal to $-\alpha$?

We readily see that, for ideals of maximal minors in any prime characteristic, the answer to the question is yes.

The computation of Frobenius roots and ν-invariants of ideals of maximal minors allows one to determine the test ideals. For the remainder of this section, given a real number $\lambda \geq 0$, let $\{\lambda\}$ denote its fractional part.

Theorem 5.51 ([H14, Theorem 3.2])**.** Let $X = (x_{ij})$ be an $m \times n$ matrix of indeterminates over a perfect field $\mathbb{F}$ of characteristic $p > 0$, with $m \leq n$. Let I_m be the ideal of maximal minors of X in the polynomial ring $R = \mathbb{F}[X]$.

(1) For each real number $\lambda \geq \mathrm{fpt}(I_m)$, $\tau(I_m^{\lambda}) = I_m^{\lfloor \lambda \rfloor - \mathrm{fpt}(I_m)+1}$.
(2) The set of F-jumping numbers of I_m is $\mathrm{FJN}(I_m) = \{\lambda \in \mathbb{Z}_{n-m+1}\}$.

Proof. (1) Since R is a Noetherian ring, the chain of ideals

$$\mathcal{C}_R^0 \cdot I_m^{\lceil \lambda p^0 \rceil} \subseteq \mathcal{C}_R^1 \cdot I_m^{\lceil \lambda p \rceil} \subseteq \mathcal{C}_R^2 \cdot I_m^{\lceil \lambda p^2 \rceil} \subseteq \cdots \subseteq \mathcal{C}_R^e I_m^{\lceil \lambda p^e \rceil} \subseteq \cdots$$

eventually stabilizes. In view of Theorem 5.47, the chain stabilizes to a power of I_m. For $\lambda < \mathrm{fpt}(I_m) = n-m+1$ one has $\tau(I^{\lambda}) = R$, hence suppose that $\lambda \geq \mathrm{fpt}(I_m)$. Then the chain stabilizes to I_m^{s+1} for some integer $s \geq 0$. As a result, for all $e \gg 0$ one has

$$sp^e + \mathrm{fpt}(I_m)(p^e-1) < \lambda p^e \leq (s+1)p^e + \mathrm{fpt}(I_m)(p^e-1).$$

Rearranging the inequalities gives

$$\lambda - \mathrm{fpt}(I_m)\frac{p^e-1}{p^e} - 1 \leq s < \lambda - \mathrm{fpt}(I_m)\frac{p^e-1}{p^e}.$$

These inequalities determine a half-open interval of length 1 in the real line, thus it contains exactly one integer for $e \gg 0$. We claim that such integer is $s = \lfloor \lambda \rfloor - \mathrm{fpt}(I_m)$. On the one hand, by letting $\lambda = \lfloor \lambda \rfloor - \{\lambda\}$, the first inequality reads

$$\frac{\mathrm{fpt}(I_m)}{p^e} \leq 1 - \{\lambda\}\,,$$

which holds for all $e \geq 0$ large enough. On the other hand the inequality

$$\lfloor \lambda \rfloor - \mathrm{fpt}(I_m) < \lambda - \mathrm{fpt}(I_m)\frac{p^e - 1}{p^e}$$

holds for all $e \geq 0$. This proves that $\tau(I_m^\lambda) = I_m^{\lfloor \lambda \rfloor - \mathrm{fpt}(I_m) + 1}$.

(2) It follows from the formula for test ideals. □

The test ideals of I_m and its F-jumping numbers can be depicted as follows:

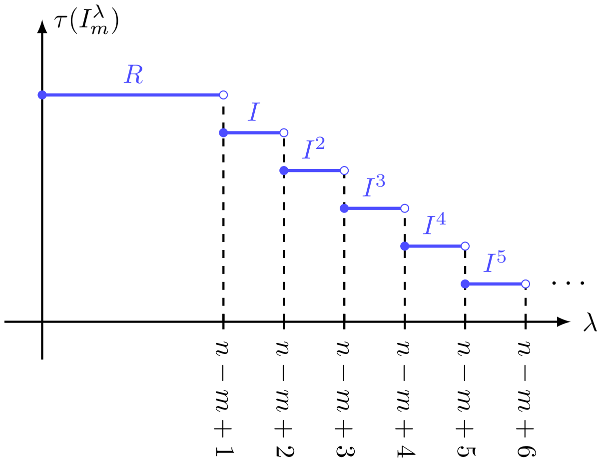

Figure 5.4

Recall that a commutative Noetherian ring R of characteristic $p > 0$ is F-finite if F_*R is a finitely generated R-module (see Section 2.3). Suppose that R is a regular F-finite ring and let $\mathfrak{a} \subseteq R$ be an ideal. Then for each real number $\lambda \geq 0$ the ascending chain of ideals

$$\mathcal{C}_R^0 \cdot \mathfrak{a}^{\lceil \lambda \rceil} \subseteq \mathcal{C}_R^1 \cdot \mathfrak{a}^{\lceil \lambda p \rceil} \subseteq \mathcal{C}_R^2 \cdot \mathfrak{a}^{\lceil \lambda p^2 \rceil} \subseteq \cdots \subseteq \mathcal{C}_R^e \cdot \mathfrak{a}^{\lceil \lambda p^e \rceil} \subseteq \cdots$$

stabilizes after finitely many steps. It is not known, however, at which step it stabilizes. This motivates the following definition:

Definition 5.52. Let R be a regular F-finite ring, $\mathfrak{a} \subseteq R$ an ideal and let $\lambda \geq 0$ be a real number. We define the *stabilization index* of $\mathfrak{a}$ with exponent λ, denoted by $S(\mathfrak{a}^\lambda)$, to be the least non-negative integer such that, for all integers $d \geq 0$,

$$\mathcal{C}_R^{S(\mathfrak{a}^\lambda)} \cdot \mathfrak{a}^{\lceil \lambda p^{S(\mathfrak{a}^\lambda)} \rceil} = \mathcal{C}_R^{S(\mathfrak{a}^\lambda)+e} \cdot \mathfrak{a}^{\lceil \lambda p^{S(\mathfrak{a}^\lambda)+e} \rceil}.$$

In particular, $\tau(\mathfrak{a}^\lambda) = \mathcal{C}_R^{S(\mathfrak{a}^\lambda)} \cdot \mathfrak{a}^{\lceil \lambda p^{S(\mathfrak{a}^\lambda)} \rceil}$.

Due to the particularly nice behavior of the ν-invariants of the ideals of maximal minors, it is possible to give an explicit formula for the stabilization index.

Proposition 5.53. Let $X = (x_{ij})$ be a matrix of indeterminates of size $m \times n$, $m \leq n$, defined over a perfect field $\mathbb{F}$ of characteristic $p > 0$. Let I_m be the ideal of maximal minors of X in the polynomial ring $\mathbb{F}[X]$. The stabilization index of I_m with exponent $\lambda \geq 0$ is

$$S(I_m^\lambda) = \begin{cases} \left\lceil \log_p \dfrac{\operatorname{fpt}(I_m)}{\operatorname{fpt}(I_m) - \lambda} \right\rceil & \text{if } 0 \leq \lambda < \operatorname{fpt}(I_m), \\ \left\lceil \log_p \dfrac{\operatorname{fpt}(I_m)}{1 - \{\lambda\}} \right\rceil & \text{if } \lambda \geq \operatorname{fpt}(I_m), \end{cases}$$

Proof. Since R is a Noetherian ring, the ascending chain of ideals

$$\mathcal{C}_R^0 \cdot I_m^{\lceil \lambda p^0 \rceil} \subseteq \mathcal{C}_R^1 \cdot I_m^{\lceil \lambda p \rceil} \subseteq \mathcal{C}_R^2 \cdot I_m^{\lceil \lambda p^2 \rceil} \subseteq \cdots \subseteq \mathcal{C}_R^e I_m^{\lceil \lambda p^e \rceil} \subseteq \cdots$$

eventually stabilizes. In view of Theorem 5.47, the chain stabilizes to a power I_m^s for some integer $s \geq 0$.

To begin with consider the case $0 \leq \lambda < \operatorname{fpt}(I_m)$. Since the test ideal $\tau(I_m^\lambda)$ is trivial, that is $\tau(I^\lambda) = R$, there is an integer $e_0 \geq 0$ such that $\mathcal{C}_R^e \cdot I_m^{\lceil \lambda p^e \rceil} = R$ for all $e \geq e_0$, and e_0 is minimal with respect to this property, i.e. $S(I_m^\lambda) = e_0$. As a result, by Theorem 5.47, one has that $\lambda p^e \leq \lceil \lambda p^e \rceil \leq \operatorname{fpt}(I_m)(p^e - 1)$, whence

$$S(I_m^\lambda) = \left\lceil \log_p \frac{\operatorname{fpt}(I_m)}{\operatorname{fpt}(I_m) - \lambda} \right\rceil.$$

Now suppose $\lambda \geq \operatorname{fpt}(I_m)$, then $\tau(I_m^\lambda) = I_m^{s+1}$ for some integer $s \geq 0$. As in the proof of Theorem 5.51, one concludes that s must satisfy

$$\lambda - \operatorname{fpt}(I_m)\frac{p^e - 1}{p^e} - 1 \leq s < \lambda - \operatorname{fpt}(I_m)\frac{p^e - 1}{p^e}$$

for all $e \gg 0$, and that the only integer doing so is $\lambda = \lfloor \lambda \rfloor - \operatorname{fpt}(I_m)$. The second inequality is true for all $e \geq 0$. Rearranging terms shows that the first inequality holds is equivalent to

$$\frac{\operatorname{fpt}(I_m)}{p^e} \leq 1 - \{\lambda\}.$$

It follows that the smallest integer $e \geq 0$ for which the inequality above is true is

$$S(I_m^\lambda) = \left\lceil \log_p \frac{\operatorname{fpt}(I_m)}{1 - \{\lambda\}} \right\rceil. \qquad \square$$

Observation 5.54. Regarding the stabilization index of I_m:

(1) The larger is the characteristic of the field $\mathbb{F}$, the faster the chain of ideals stabilizes.
(2) When $0 \leq \lambda < \operatorname{fpt}(I_m)$, the stabilization index diverges as $\lambda \to \operatorname{fpt}(I_m)^-$. The same occurs when $\lambda \geq \operatorname{fpt}(I_m)$ and $\{\lambda\} \to 1^-$.

5.5. Determinantal-type polynomials

Chronologically, first we were able to compute the ν-invariants and the Frobenius roots of powers of the determinant of an $n \times n$ matrix of indeterminates X over $\mathbb{F}_p$. By letting $f = \det X$ and setting $m = n$ in Theorem 5.47, one has:

(1) For all integers $s \geq 0$,

$$\mathcal{C}_R^e \cdot f^{sp^e+p^e-1} = \mathcal{C}_R^e \cdot f^{sp^e+p^e-2} = \cdots = \mathcal{C}_R^e \cdot f^{sp^e-p^e} = (f)^s.$$

(2) The set of ν-invariants of f of level e is $\nu_f^\bullet(p^e) = \{(s+1)p^e - 1 \mid s \in \mathbb{Z}_{\geq 0}\}$.

We observe that the same is true for a more general class of polynomials, which we refer to as determinantal-type polynomials. For this section we recover multi-index notation: if $a = (a_1, \ldots, a_n) \in \mathbb{Z}_{\geq 0}^n$ is an n-tuple of non-negative integers, we let x^a be the monomial

$$x^a = x_1^{a_1} \cdots x_n^{a_n}.$$

Definition 5.55. Let $R = B[x_1, \ldots, x_n]$ be a polynomial ring over a ring B.

(1) A *square-free monomial* is non-trivial monomial x^a of R, i.e. not a unit of R, such that $0 \leq a_1, \ldots, a_n \leq 1$.

(2) A *determinantal-type polynomial* is a non-zero polynomial whose monomials are square-free.

It follows from the definition that a determinantal-type polynomial is in the homogeneous maximal ideal $\mathfrak{m} = (x_1, \ldots, x_n)$ of R.

Example 5.56.

(a) If $X = (x_{ij})$ is an $n \times n$ matrix of indeterminates over a field $\mathbb{K}$, the determinant

$$\det X = \sum_{\sigma \in \mathbb{S}_n} \operatorname{sgn}(\sigma) x_{1,\sigma(1)} \cdots x_{n,\sigma(n)}$$

is a determinantal-type polynomial in the ring $\mathbb{K}[X]$.

(b) Let $X = (x_{ij})$ be a $2n \times 2n$ skew-symmetric matrix of indeterminates with zero diagonal, that is to say, $x_{ij} = -x_{ji}$ for all $1 \leq i, j \leq 2n$. The Pfaffian of X is the polynomial in $\mathbb{K}[X]$ given by

$$\operatorname{Pf} X = \frac{1}{2^n n!} \sum_{\sigma \in \mathbb{S}_{2n}} \operatorname{sgn}(\sigma) \prod_{i=1}^n x_{\sigma(2i-1),\sigma(2i)}.$$

One can show that $(\operatorname{Pf} X)^2 = \det X$. Since no variable appears twice in the same monomial, it is a determinantal-type polynomial.

(c) The elementary symmetric polynomials in $\mathbb{K}[x_1, \ldots, x_n]$, that is the polynomials given by

$$e_s = \sum_{1 \leq i_1 < \cdots < i_s \leq n} x_{i_1} \cdots x_{i_s}$$

for $1 \leq s \leq n$, are determinantal-type polynomials.

Recall that if $R = \mathbb{F}[x_1, \ldots, x_n]$ is a polynomial ring over a perfect field $\mathbb{F}$ of characteristic p, then for each $e \in \mathbb{Z}_{\geq 0}$, the R-module $F^e_* R$ is free with standard basis

$$\left\{ F^e_* x_1^{i_1} \cdots x_n^{i_n} \mid 0 \leq i_1, \ldots, i_n < p^e \right\}.$$

For a proof of this fact, see Proposition 2.33.

Lemma 5.57. Let $R = \mathbb{F}[x_1, \ldots, x_n]$ be a polynomial ring over a perfect field $\mathbb{F}$ of characteristic $p > 0$ and let $f \in R$ be a determinantal-type polynomial. Then for all integers $0 \leq r < p^e$, $F^e_* f^r$ is an $\mathbb{F}$-linear combination of elements in the standard basis.

Proof. Let $f = \sum_{i=1}^m \alpha_i x^{a_i} \in R$, where $\alpha_i \in \mathbb{K}$ and $a_i = (a_{i1}, \ldots, a_{in}) \in \{0,1\}^n$ for each integer $i = 1, \ldots, n$. By the multinomial theorem one has

$$f^r = \sum_{k_1 + \cdots + k_m = r} \binom{r}{k_1, \ldots, k_m} \prod_{i=1}^m \alpha_i^{k_i} x^{k_i a_i}.$$

The monomials in the expression above are of the form

$$\prod_{i=1}^m x^{k_i a_i} = x^{\sum_{i=1}^m k_i a_i} = x_1^{\sum_{i=1}^m k_i a_{i1}} \cdots x_n^{\sum_{i=1}^m k_i a_{in}},$$

and these are elements of the standard basis since for each $j = 1, \ldots, n$ one has

$$\sum_{i=1}^m k_i a_{ij} \leq \sum_{i=1}^m k_i = r \leq p^e - 1.$$

Thereby, up to collecting terms, in the standard basis $F^e_* f^r$ reads

$$F^e_* f^r = \sum_{k_1 + \cdots + k_m = r} \left(\binom{r}{k_1, \ldots, k_m} \prod_{i=1}^m \alpha_i^{k_i} \right)^{1/p^e} F^e_* x^{\sum_{i=1}^m k_i a_i},$$

which shows that the coefficients are in $\mathbb{K}$. As $F^e_* R$ is free and $f^r \neq 0$, at least one of the coefficients is non-zero. □

Theorem 5.58 (The ν-invariants of determinantal-type polynomials)**.** Let $R = \mathbb{F}[x_1, \ldots, x_n]$ be a polynomial ring over a perfect field $\mathbb{F}$ of characteristic p and let $f \in R$ be a determinantal-type polynomial. Fix an integer $e \geq 0$. Then:

(1) For all integers $s \geq 0$ and $0 \leq r < p^e$,

$$\begin{aligned} \mathcal{C}^e_R \cdot f^{sp^e + p^e - 1} &= \mathcal{C}^e_R \cdot f^{sp^e + p^e - 2} = \mathcal{C}^e_R \cdot f^{sp^e + p^e - 3} = \cdots \\ &= \mathcal{C}^e_R \cdot f^{sp^e + 1} = \mathcal{C}^e_R \cdot f^{sp^e} = (f)^s. \end{aligned}$$

(2) The ν-invariants of f of level e are $\nu^\bullet_f(p^e) = \{(s+1)p^e - 1 \mid s \in \mathbb{Z}_{\geq 0}\}$.

(3) If $s \geq 0$ is an integer and $J = (f)^{s+1}$, then $\nu^J_f(p^e) = (s+1)p^e - 1$.

Proof. (1) First consider the case when $s = 0$. We know that $F^e_* f^{p^e-1}$ is a linear combination of basis elements with coefficients in $\mathbb{K}$ by Lemma 5.57, thus by Proposition 4.18 one has $\mathcal{C}^e_R \cdot f^{p^e-1} = R$. Let $s \geq 0$ and $0 \leq r < p^e$ be arbitrary integers. Since the Frobenius roots give the ascending chain

$$\mathcal{C}^e_R \cdot f^{sp^e+p^e-1} \subseteq \mathcal{C}^e_R \cdot f^{sp^e+p^e-2} \subseteq \cdots \subseteq \mathcal{C}^e_R \cdot f^{sp^e+1} \subseteq \mathcal{C}^e_R \cdot f^{sp^e},$$

it is enough to verify that $(f)^s \subseteq \mathcal{C}^e_R \cdot f^{sp^e+p^e-1}$ and that $\mathcal{C}^e_R \cdot f^{sp^e} \subseteq (f)^s$. On the one hand, by Proposition 4.21,

$$(f)^s = (f)^s\, \mathcal{C}^e_R \cdot f^{p^e-1} = \mathcal{C}^e_R \cdot \left(f^{s[p^e]} f^{p^e-1}\right) = \mathcal{C}^e_R \cdot f^{sp^e+p^e-1}.$$

On the other hand, by Proposition 4.19, $\mathcal{C}^e_R \cdot f^{sp^e} \subseteq (f)^s$ is equivalent to $(f)^{sp^e} \subseteq (f)^{s[p^e]} = (f)^{sp^e}$.

(2) From (1), we have that for each integer $s \geq 0$,

$$(f)^{s+1} = \mathcal{C}^e_R \cdot f^{(s+1)p^e} \subsetneq \mathcal{C}^e_R \cdot f^{(s+1)p^e-1} = (f)^s,$$

hence the ν-invariants of f of level $e \geq 0$ are of the form $(s+1)p^e - 1$ for $s \in \mathbb{N}$.

(3) It follows at once from (2). ☐

Corollary 5.59. Let $R = \mathbb{F}[x_1, \ldots, x_n]$ be a polynomial ring over a perfect field $\mathbb{F}$ of characteristic $p > 0$ and let $f \in R$ be a determinantal-type polynomial. The F-pure threshold of f at the homogeneous maximal ideal $\mathfrak{m} = (x_1, \ldots, x_n)$ is $\operatorname{fpt}(f) = 1$.

Proof. One has that $\nu^{\mathfrak{m}}_f(p^e) = p^e - 1$ for each integer $e \geq 0$, thus by definition of the F-pure threshold,

$$\operatorname{fpt}(f) = \lim_{e\to\infty} \frac{\nu^{\mathfrak{m}}_f(p^e)}{p^e} = 1.$$ ☐

Corollary 5.60. Let $R = \mathbb{F}[x_1, \ldots, x_n]$ be a polynomial ring over a perfect field $\mathbb{F}$ of characteristic $p > 0$ and let $f \in R$ be a determinantal-type polynomial. The only Bernstein-Sato root of f is $\alpha = -\operatorname{fpt}(f) = -1$.

Proof. Let $(t_d)_{d=0}^{\infty} \subseteq \mathbb{Z}_{\geq 0}$ be a sequence of non-negative integers and set

$$\nu_d := (t_d + 1)p^d - 1 \in \nu^{\bullet}_f(p^d).$$

Then $(\nu_d)_{d=0}^{\infty}$ is a sequence of ν-invariants with p-adic limit $\nu_d \to \alpha = -\operatorname{fpt}(f) = -1$, thus $\alpha = -\operatorname{fpt}(f)$ is a Bernstein-Sato root of f by [QG21b, Theorem IV.17]. Any sequence of ν-invariants of f is of this form, therefore $\alpha - \operatorname{fpt}(f)$ is the only Bernstein-Sato root of f. ☐

Observe that Theorem 5.58 and its corollaries applied to the determinant of a matrix of indeterminates is a particular case of Theorem 5.47 and Corollary 5.49, respectively. Furthermore, we have the following connection to the characteristic zero case:

Corollary 5.61. Let $f \in \mathbb{Z}[x_1, \dots, x_n]$ be a determinantal-type polynomial and denote by f its extension to the polynomial ring $\mathbb{C}[x_1, \dots, x_n]$.

(1) The log-canonical threshold of f at the origin is $\operatorname{lct}(f) = 1$.
(2) A root of the b-function $b_f(s) \in \mathbb{C}[s]$ of f is -1.

Proof. For each prime $p \in \mathbb{Z}$, denote by $f_p \in \mathbb{F}_p[x_1, \dots, x_n]$ the reduction of f modulo p.

(1) By [MTW05, Theorem 3.4] (see Theorem 4.43), one has that $\operatorname{lct}(f) = \lim_{p\to\infty} \operatorname{fpt}(f_p) = 1$.

(2) Since $\alpha = -1$ is the only Bernstein-Sato root of f_p for each prime $p \in \mathbb{Z}$, it follows from [QG21b, Theorem VI.3] (see Theorem 4.38) that $\alpha = -1$ is a root of the b-function of f. □

Proposition 5.62. Let $R = \mathbb{F}[x_1, \dots, x_n]$ be a polynomial ring over a perfect field $\mathbb{F}$ of characteristic p and let $f \in R$ be a determinantal-type polynomial. Let $\lambda \geq 0$ be a real number and fix an integer $e \geq 0$. Then

$$\mathcal{C}_R^e \cdot f^{\lceil \lambda p^e \rceil} = \begin{cases} (f)^{\lfloor \lambda \rfloor} & \text{if } \{\lambda\} \leq (p^e - 1)/p^e, \\ (f)^{\lfloor \lambda \rfloor + 1} & \text{if } \{\lambda\} > (p^e - 1)/p^e, \end{cases}$$

where $\{\lambda\}$ denotes the fractional part of λ.

Proof. Write $\lambda = \lfloor \lambda \rfloor + \{\lambda\}$. Since $0 \leq \{\lambda\} < 1$, there exists an integer $0 \leq r < p^e$ such that $r \leq \{\lambda\}\, p^e < r + 1$. We distinguish two cases according to whether the inequalities are strict or not. First assume that $r = \{\lambda\}\, p^e$, that is, $\{\lambda\} \leq (p^e - 1)/p^e$. Then $\lfloor \lambda \rfloor p^e \leq \lambda p^e \leq \lfloor \lambda \rfloor p^e + p^e - 1$, and by Theorem 5.58,

$$(f)^{\lfloor \lambda \rfloor} = \mathcal{C}_R^e \cdot f^{\lfloor \lambda \rfloor p^e + p^e - 1} \subseteq \mathcal{C}_R^e \cdot f^{\lceil \lambda p^e \rceil} \subseteq \mathcal{C}_R^e \cdot f^{\lfloor \lambda \rfloor p^e} = (f)^{\lfloor \lambda \rfloor}.$$

Next suppose that both inequalities are strict. On the one hand, if $0 \leq r \leq p^e - 2$ one has $\{\lambda\} < (p^e - 1)/p^e$. In consequence, $\lfloor \lambda \rfloor p^e \leq \lambda p^e < \lfloor \lambda \rfloor p^e + p^e - 1$ and $\mathcal{C}_R^e \cdot f^{\lceil \lambda p^e \rceil} = (f)^{\lfloor \lambda \rfloor}$ as before. On the other hand, if $r = p^e - 1$, that is $(p^e - 1)/p^e < \{\lambda\} < 1$, it follows that $\lfloor \lambda \rfloor p^e + p^e - 1 < \lambda p^e < \lfloor \lambda \rfloor p^e + p^e$ and $\lceil \lambda p^e \rceil = (\lfloor \lambda \rfloor + 1)p^e$. Again by Theorem 5.58, $\mathcal{C}_R^e \cdot f^{\lceil \lambda p^e \rceil} = (f)^{\lfloor \lambda \rfloor + 1}$. □

Proposition 5.63. Let $R = \mathbb{F}[x_1, \dots, x_n]$ be a polynomial ring over a perfect field $\mathbb{F}$ of characteristic $p > 0$ and let $f \in R$ be a determinantal-type polynomial. Let $\lambda \geq 0$ be a real number. Then the stabilization index of f with exponent λ is given by

$$S(f^\lambda) = \left\lceil \log_p \frac{1}{1 - \{\lambda\}} \right\rceil.$$

Proof. First suppose that $\lambda \geq 0$ is an integer. Then by Proposition 5.62 one has

$$\mathcal{C}_R^e \cdot f^{\lceil \lambda p^e \rceil} = \mathcal{C}_R^e \cdot f^{\lambda p^e} = (f)^\lambda$$

for all integers $e \geq 0$, thus the stabilization index is $S(f^\lambda) = 0$. Next suppose that $\lambda \geq 0$ is not an integer. Since the sequence $((p^e - 1)/p^e)_{e=0}^\infty$ has limit 1 as $e \to \infty$, there exists some $E \geq 1$ such that

$$\frac{p^{E-1} - 1}{p^{E-1}} < \{\lambda\} \leq \frac{p^E - 1}{p^E}.$$

By Proposition 5.62 one has $\mathcal{C}_R^e \cdot f^{\lceil \lambda p^e \rceil} = (f)^{\lfloor \lambda \rfloor + 1}$ for all integers $0 \leq e \leq E - 1$, whereas $\mathcal{C}_R^e \cdot f^{\lceil \lambda p^e \rceil} = (f)^{\lfloor \lambda \rfloor}$ for all integers $e \geq E$. It follows that the stabilization index is E, which is given by

$$S(f^\lambda) = E = \left\lceil \log_p \frac{1}{1 - \{\lambda\}} \right\rceil. \qquad \square$$

As Corollary 5.59 shows, the F-pure threshold of the determinant of a matrix of indeterminates is 1. This shows that, Proposition 5.63 is a particular case of Proposition 5.53 applied to the determinant.

Theorem 5.64. Let $R = \mathbb{F}[x_1, \ldots, x_n]$ be a polynomial ring over a perfect field $\mathbb{F}$ of characteristic p and let $f \in R$ be a determinantal-type polynomial.

(1) For each real number $\lambda \geq 0$, $\tau(f^\lambda) = (f)^{\lfloor \lambda \rfloor}$.
(2) The set of F-jumping numbers of f is $\mathrm{FJN}(f) = \{\lambda \in \mathbb{Z}_{\geq 1}\}$.

Proof. (1) The sequence $((p^e - 1)/p^e)_{e=0}^\infty$ has limit 1 as $e \to \infty$, thus there exists $e_0 \geq 0$ such that $\{\lambda\} \leq (p^e - 1)/p^e$ for all $e \geq e_0$. Then $\mathcal{C}_R^e \cdot f^{\lceil \lambda p^e \rceil} = (f)^{\lfloor \lambda \rfloor}$ for all $e \geq e_0$ by Proposition 5.62, thus it follows that $\tau(f^\lambda) = (f)^{\lfloor \lambda \rfloor}$.

(2) Fix an integer $n \geq 0$. Then $\tau(f^\lambda) = (f)^{\lfloor \lambda \rfloor} = (f)^n$ for all real numbers $n \leq \lambda < n + 1$, whereas $\tau(f^{n+1}) = (f)^{n+1}$. Consequently $n + 1$ is an F-jumping number of f, from where the assertion follows. $\square$

5.6. Behavior of the F-pure threshold under induction

As previously mentioned, some properties of interest to us are preserved under induction on each component of the triple (m, n, t), where $m \times n$ is the size of the matrix and t is the size of the minors. Fix a field $\mathbb{F}$ of positive characteristic, and let J_i be the ideal of $(t + i)$-minors of a $(m + i) \times (n + i)$ matrix of indeterminates over $\mathbb{F}$. In this section we will show that if t is large enough with respect to m and n, then

$$\mathrm{fpt}(J_0) = \mathrm{fpt}(J_1) = \mathrm{fpt}(J_2) = \cdots,$$

thus the F-pure threshold "stabilizes". In contrast, if t is not large enough with respect to m and n, then there is an integer $\ell \geq 1$ such that

$$\mathrm{fpt}(J_0) > \mathrm{fpt}(J_1) > \cdots > \mathrm{fpt}(J_\ell) = \mathrm{fpt}(J_{\ell+1}) = \mathrm{fpt}(J_{\ell+2}) = \cdots.$$

Definition 5.65. Let $t \le m \le n$ be integers. Define the function $\theta\colon \mathbb{Z}^3 \times \mathbb{R} \to \mathbb{R}$ by

$$\theta(m,n,t;x) := \frac{(m-x)(n-x)}{t-x}.$$

Note that the F-pure threshold of the ideal $I_t(X)$ of t-minors of a matrix of indeterminates X of size $m \times n$, $m \le n$, defined over a field of prime characteristic is

$$\mathrm{fpt}(I_t(X)) = \min_{k=0,\dots,t-1} \theta(m,n,t;k).$$

(see [MSV14, Theorem 1.2]). For each choice of $m \le n$ with $t < m$, θ is a function of x of class $\mathcal{C}^1$ defined in $\mathbb{R} - \{t\}$. When $t = m$, i.e. when one considers maximal minors, the function reads $\theta(m,n,t;x) = n - x$, thus it is $\mathcal{C}^1$ everywhere.

Proposition 5.66. Let $t < m \le n$ be fixed integers and view $\theta(m,n,t;x)$ as a function of x.

(1) The point $x^* = t - \sqrt{(t-m)(t-n)}$ is a local minimum of $\theta(m,n,t;x)$.
(2) For $x < x^*$, $\theta(m,n,t;x)$ is strictly decreasing.
(3) For $x^* < x < t$, $\theta(m,n,t;x)$ is strictly increasing.

Proof. The partial derivative of θ with respect to x reads

$$\partial_x \theta = -\frac{(x-t+\sqrt{(t-m)(t-n)})(x-t-\sqrt{(t-m)(t-n)})}{(x-t)^2},$$

and it is vanishes when $x = x_\pm = t \pm \sqrt{(t-m)(t-n)}$. Since $t < m \le n$, the square root is a real number and we have $x_- < t < x_+$. Let $x^* = x_-$; we claim that θ has a local minimum at x^*. Indeed, on the one hand, $x^2 - 2tx + (m+n)t - mn < 0$ for all $x_- < x < x^+$, thus $\partial_x\theta > 0$ for $x_- < x < t$. On the other hand, $x^2 - 2tx + (m+n)t - mn > 0$ when $x < x_-$, hence $\partial_x\theta > 0$ for all $x < x_-$. $\square$

For the sake of clarity, we include the graph of $\theta(m,n,t;x)$ as a function of x:

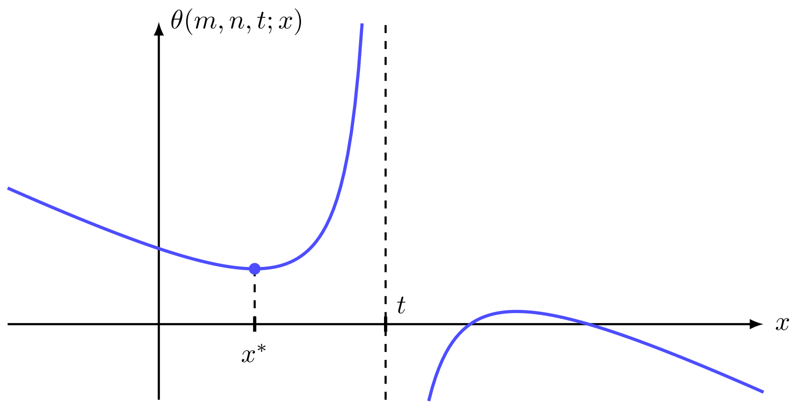

Figure 5.5

Lemma 5.67. Let $1 \leq t \leq m \leq n$ be integers. Then:

(1) For each integer $i \in \mathbb{Z}$, $\theta(m+i, n+i, t+i; x+i) = \theta(m, n, t; x)$.
(2) For each integer $i \in \mathbb{Z}$, $\theta(m+i, n+i, t+i; x) = \theta(m, n, t; x-i)$.

Proof. A couple of straightforward computations show that

$$\begin{aligned}\theta(m+i, n+i, t+i; x+i) &= \frac{(m+i-(x+i))(n+i-(x+i))}{t+i-(x+i)} \\ &= \theta(m, n, t; x),\end{aligned}$$

and

$$\begin{aligned}\theta(m+i, n+i, t+i; x) &= \frac{(m+i-x)(n+i-x)}{(t+i-x)} \\ &= \frac{(m-(x-i))(n-(x-i))}{t-(x-i)} \\ &= \theta(m, n, t; x-i).\end{aligned}$$

□

Notation 5.68. Let $X = (x_{ij})$ be an $m \times n$ matrix of indeterminates over a commutative ring B, with $m \leq n$, and let $i \geq -m+1$ be an integer. We define $X[i]$ as the matrix of indeterminates

$$X[i] := \begin{pmatrix} x_{11} & \cdots & x_{1,n+i} \\ \vdots & \ddots & \vdots \\ x_{m+i,1} & \cdots & x_{m+i,n+i} \end{pmatrix},$$

and we let $B[X[i]]$ be the corresponding polynomial ring. More explicitly:

(a) For $i = 0$, X is left unchanged, thus $X = X[0]$.
(b) When $i > 0$, $X[i]$ is the $(m+i) \times (n+i)$ matrix of indeterminates obtained by adjoining i rows below the m-th row of X, and i columns to the right of the n-th column.
(c) When $i < 0$, $X[i]$ is the $(m+i) \times (n+i)$ matrix of indeterminates obtained by removing the last i rows and columns of X. If $i \leq -m$ the matrix $X[i]$ is empty, thus $i \geq -m+1$.

To begin with, we look at the behavior under induction of the F-pure threshold of the ideal of maximal minors:

Proposition 5.69. Let $X = (x_{ij})$ be a matrix of indeterminates of size $m \times n$, $m \leq n$, defined over a field $\mathbb{F}$ of positive characteristic. Then for all integers $i \geq 0$,

$$\operatorname{fpt}(I_m(X)) = \operatorname{fpt}(I_{m+i}(X[i])) = n - m + 1.$$

Proof. Note that $\theta(m+i, n+i, m+i; x) = n+i-x$, thus the minimum on the set $[0, m+i-1] \cap \mathbb{Z}$ is achieved at $k = m+i-1$. which gives $\operatorname{fpt}(I_{m+i}(X[i])) = n+i-(m+i-1) = n-m+1$. □

Observe that the fact shown in the preceding proposition was already used in the proof of Theorem 5.47.(2).

In the succeeding propositions we shall study the behavior under induction of the F-pure threshold of the ideals of non-maximal minors. In order to avoid repeating the hypothesis, in the following propositions $X = (x_{ij})$ will be matrix of indeterminates of size $m \times n$, $m \leq n$, defined over a field $\mathbb{F}$ of positive characteristic. Moreover, for each integer $i \geq -t+1$, $I_{t+i}(X[i]) \subseteq \mathbb{F}[X[i]]$ will denote the ideal of $(t+i)$-minors of $X[i]$.

Proposition 5.70. Fix integers $1 < t < m \leq n$.

(1) If $x^* = t - \sqrt{(t-m)(t-n)} \leq 0$, then $\operatorname{fpt}(I_t(X)) = mn/t$.
(2) If $(m+n)t \leq mn$, then $\operatorname{fpt}(I_t(X)) = mn/t$.

Proof. (1) As shown in Proposition 5.66, $x^* = t - \sqrt{(t-m)(t-n)}$ is where $\theta(m,n,t;x)$ attains its local minimum. Since $x^* \leq 0$ and $\theta(m,n,t;x)$ is strictly increasing for $x^* < x < t$, we have that

$$\theta(m,n,t;0) = \frac{mn}{t} < \theta(m,n,t;k)$$

for all integers $1 \leq k < t$ from where the assertion follows.

(2) We note that $x^* \leq 0$ is equivalent to $(m+n)t \leq mn$. □

From the previous proposition and by definition of the F-pure threshold, one has:

$$\operatorname{fpt}(I_t(X)) = \begin{cases} \theta(m,n,t,0) & \text{if } x^* \leq 0, \\ \min\left\{\theta(m,n,t,\lfloor x^* \rfloor), \theta(m,n,t,\lceil x^* \rceil)\right\} & \text{otherwise.} \end{cases}$$

Recall that given a function $f\colon D \subseteq \mathbb{R} \to \mathbb{R}$, $\arg\min f$ denotes the subset of points of D where f attains its absolute minimum, that is:

$$\arg\min f := \{x \in D \mid f(x) \leq f(y) \text{ for all } y \in D\}.$$

Similarly, if E is a subset of D, we let

$$\arg\min\{f \mid x \in E\} := \arg\min f_{|_E} = \{x \in E \mid f(x) \leq f(y) \text{ for all } y \in E\}.$$

Proposition 5.71. Let $1 \leq t < m \leq n$ be fixed integers and suppose that

$$\kappa = \max\left(\arg\min\left\{\theta(m,n,t;k) \,\middle|\, k = 0,\dots,t-1\right\}\right) \geq 1.$$

Then for all integers $i \geq -\kappa+1$,

$$\operatorname{fpt}(I_{t+i}(X[i])) = \theta(m,n,t;\kappa).$$

Proof. We note that $\operatorname{fpt}(I_{t+i}(X[i])) = \min\{\theta(m+i,n+i,t+i;k) \mid k = 0,\dots,t+i-1\}$. Let x^* be the local minimum of $\theta(m,n,t;x)$ as a function of x given by Proposition 5.66. Since κ is the integer where $\theta(m,n,t;x)$ attains its minimum over the set $[0,t) \cap \mathbb{Z}$, it follows that $x^* \in (\kappa-1,\kappa+1) \subseteq \mathbb{R}_{>0}$, thus $\theta(m,n,t,\kappa) \leq \theta(m,n,t,\kappa-1)$ and $\theta(m,n,t,\kappa) \leq \theta(m,n,t,\kappa+1)$.

On the one hand, as $\kappa - 1 < x^*$ and, by Proposition 5.66, $\theta(m, n, t; x)$ is strictly decreasing as a function of x in $(-\infty, x^*) \subseteq \mathbb{R}$, hence $\theta(m, n, t; \kappa) \leq \theta(m, n, t; \kappa - \ell)$ for all integers $\ell \geq 1$. As a result, by Lemma 5.67,

$$\begin{aligned}\theta(m+i, n+i, t+i; \kappa+i) &= \theta(m, n, t; \kappa)\\ &\leq \theta(m, n, t; \kappa - \ell)\\ &= \theta(m+i, n+i, t+i; \kappa+i-\ell),\end{aligned}$$

for all integers $\ell \geq 1$. On the other hand $x^* < \kappa + 1$, and $\theta(m, n, t; x)$ is strictly increasing as a function of x in $(x^*, t) \subseteq \mathbb{R}$, thereby $\theta(m, n, t; \kappa) \leq \theta(m, n, t; \kappa+\ell)$ for all integers $1 \leq \ell \leq t-\kappa$, where we set $\theta(m, n, t; t) = \infty$. Again by Lemma 5.67,

$$\begin{aligned}\theta(m+i, n+i, t+i; \kappa+i) &= \theta(m, n, t; \kappa)\\ &\leq \theta(m, n, t; \kappa + \ell)\\ &= \theta(m+i, n+i, t+i; \kappa+i+\ell),\end{aligned}$$

for all integers $1 \leq \ell \leq t-\kappa$. This shows that $\theta(m+i, n+i, t+i; \kappa+i) \leq \theta(m+i, n+i, t+i; k)$ for all integers $k = 0, \ldots, t+i-1$, which proves the proposition. □

Proposition 5.72. Let $1 \leq t < m \leq n$ be integers. Suppose that

$$\arg\min \left\{ \theta(m, n, t; k) \,\middle|\, k = 0, \ldots, t-1 \right\} = 0,$$

and that $\theta(m, n, t; 0) \leq \theta(m, n, t; -1)$. Then for all integers $i \geq 1$,

$$\operatorname{fpt}(I_{t+i}(X[i])) = \theta(m, n, t; 0).$$

Proof. By assumption, $\operatorname{fpt}(I_t(X)) = \theta(m, n, t; 0)$. By Lemma 5.67, one has that

$$\begin{aligned}\theta(m+1, n+1, t+1; 1) = \theta(m, n, t; 0) \leq \theta(m, n, t; -1) = \theta(m+1, n+1, t+1; 0),\\ \theta(m+1, n+1, t+1; 1) = \theta(m, n, t; 0) \leq \theta(m, n, t; k) = \theta(m+1, n+1, t+1; k+1),\end{aligned}$$

for all integers $1 \leq k < t$, hence $\operatorname{fpt}(I_{t+1}(X[1])) = \theta(m+1, n+1, t+1; 1) = \operatorname{fpt}(I_t(X))$. This puts us in the situation of Proposition 5.71 since

$$\kappa = \max\left(\arg\min \left\{ \theta(m+1, n+1, t+1; k) \,\middle|\, k = 0, \ldots, t \right\} \right) = 1.$$

As a result, $\operatorname{fpt}(I_{t+i}(X[i])) = \operatorname{fpt}(I_t(X)) = \theta(m, n, t; 0)$ for all integers $i \geq 1$. □

In order to aid intuition with Propositions 5.71 to 5.73, it is useful to depict the behavior of the F-pure threshold and the integer at which θ attains its minimum as follows:

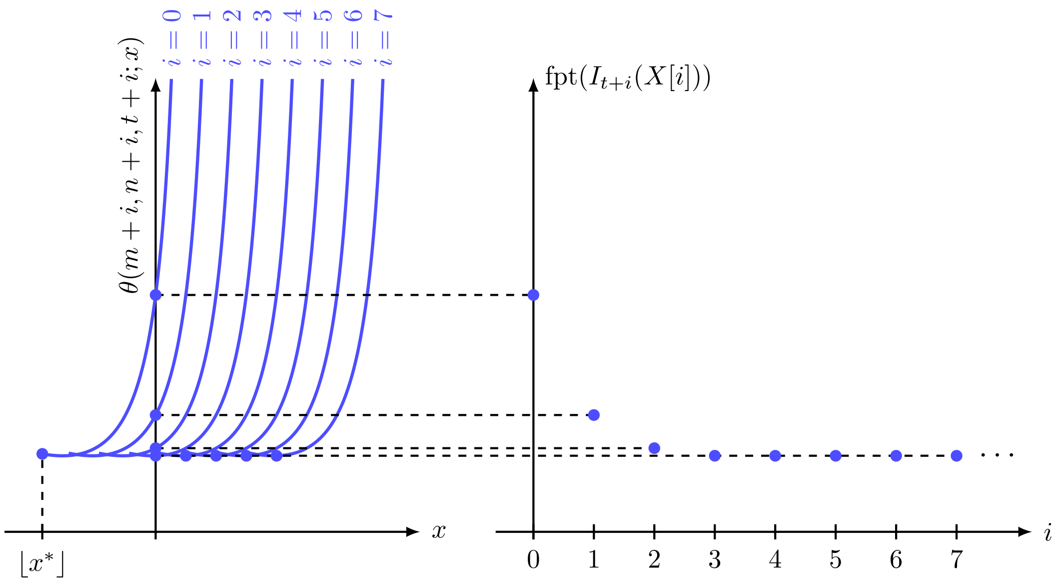

Figure 5.6

Proposition 5.73. Let $1 \leq t < m \leq n$ be integers and let $x^* = t - \sqrt{(t-m)(t-n)}$ be the local minimum of $\theta(m,n,t;x)$ as a function of x. Suppose that $x^* < 0$.

(1) If $\theta(m,n,t;\lfloor x^* \rfloor) < \theta(m,n,t;\lceil x^* \rceil)$, then

$$\mathrm{fpt}(I_t(X)) > \mathrm{fpt}(I_{t+1}(X[1])) > \cdots > \mathrm{fpt}(I_{t-\lfloor x^* \rfloor}(X[\lfloor x^* \rfloor])),$$

and, for all integers $i \geq 1$,

$$\mathrm{fpt}(I_{t-\lfloor x^* \rfloor}(X[-\lfloor x^* \rfloor])) = \mathrm{fpt}(I_{t-\lfloor x^* \rfloor+i}(X[-\lfloor x^* \rfloor + i])).$$

(2) If $\theta(m,n,t;\lceil x^* \rceil) \leq \theta(m,n,t;\lfloor x^* \rfloor)$, then

$$\mathrm{fpt}(I_t(X)) > \mathrm{fpt}(I_{t+1}(X[1])) > \cdots > \mathrm{fpt}(I_{t-\lceil x^* \rceil}(X[-\lceil x^* \rceil])),$$

and, for all integers $i \geq 1$,

$$\mathrm{fpt}(I_{t-\lceil x^* \rceil}(X[-\lceil x^* \rceil])) = \mathrm{fpt}(I_{t-\lceil x^* \rceil+i}(X[-\lceil x^* \rceil + i])).$$

Proof. By Lemma 5.67 we note that, for each integer $i \in \mathbb{Z}_{\geq 0}$, the graph of $\theta(m+i,n+i,t+i;x)$ as a function of x is the graph of $\theta(m,n,t;x)$ shifted by i units to the right. As a result, $\theta(m+i,n+i,t+i;x)$ attains its local minimum at

$$x^* + i = i + t - \sqrt{(t-m)(t-n)}.$$

(1) By assumption we have that $\lfloor x^* \rfloor < \lceil x^* \rceil$, for otherwise $\theta(m,n,t;\lfloor x^* \rfloor) = \theta(m,n,t;\lceil x^* \rceil)$. For each integer $0 \le i < \lfloor x^* \rfloor$, the local minimum of $\theta(m+i,n+i,t+i;x)$ is at $x^* + i \le 0$, hence by Proposition 5.70

$$\mathrm{fpt}(I_{t+i}(X[i])) = \theta(m+i,n+i,t+i;0) = \theta(m,n,t;-i),$$

where the second equality follows from Lemma 5.67. Since $\theta(m,n,t;x)$ is strictly increasing for $x^* < x < t$, it follows that

$$\theta(m,n,t;0) > \theta(m,n,t;-1) > \cdots > \theta(m,n,t;\lceil x^* \rceil) > \theta(m,n,t;\lfloor x^* \rfloor),$$

which proves the first part. From Proposition 5.66, $\theta(m,n,t;x)$ is strictly decreasing for $x < x^*$, thus $\theta(m,n,t;\lfloor x^* \rfloor - 1) \ge \theta(m,n,t;\lfloor x^* \rfloor) = \theta(m - \lfloor x^* \rfloor, n - \lfloor x^* \rfloor, t - \lfloor x^* \rfloor; 0)$, therefore

$$\kappa \coloneqq \arg\min \left\{ \theta(m - \lfloor x^* \rfloor, n - \lfloor x^* \rfloor, t - \lfloor x^* \rfloor; 0) \,\middle|\, k = 0, \dots, t - \lfloor x^* \rfloor - 1 \right\} = 0.$$

By Proposition 5.72, it follows that, for all integers $i \ge 1$,

$$\mathrm{fpt}(I_{t-\lfloor x^* \rfloor}(X[-\lfloor x^* \rfloor])) = \mathrm{fpt}(I_{t-\lfloor x^* \rfloor + i}(X[-\lfloor x^* \rfloor + i])).$$

(2) The argument is identical to the one used to prove (1). □

5.7. Initial ideals of determinantal ideals

In proving [MSV14, Theorem 1.2] and Theorem 5.47, the initial terms of products of minors have played an important role when computing the ν-invariants of ideals of minors. In this section we explore this idea in more detail. We refer the reader interested in Gröbner bases to [E04, Chapter 15].

Definition 5.74. Let $R = \mathbb{K}[x_1, \dots, x_n]$ be a polynomial ring over a field $\mathbb{K}$, let $I \subseteq R$ be an ideal and let τ be a monomial order on R. The *initial ideal* of I with respect to the monomial order τ is the ideal of R generated by the initial terms of the elements of I, that is, $\mathrm{in}_\tau(I) \coloneqq (\mathrm{in}_\tau(f) \mid f \in I)$. If the monomial order τ is clear from the context, we shall write $\mathrm{in}(f)$ and $\mathrm{in}(I)$.

If I is generated by polynomials $f_1, \dots, f_r$, it is not true in general that the initial ideal $\mathrm{in}(I)$ is generated by $\mathrm{in}(f_1), \dots, \mathrm{in}(f_n)$.

Definition 5.75. Let $R = \mathbb{K}[x_1, \dots, x_n]$ be a polynomial ring over a field $\mathbb{K}$, let $I \subseteq R$ be an ideal and let τ be a monomial order on R. A *Gröbner basis* of I with respect to the monomial order τ is a set of generators $f_1, \dots, f_r$ of I such that $\mathrm{in}(f_1), \dots, \mathrm{in}(f_r)$ generate the initial ideal $\mathrm{in}(I)$.

Example 5.76. Let $R = \mathbb{K}[x, y]$ be a polynomial ring over a field $\mathbb{K}$ and consider the ideal $I = (x + y, x - y)$, which is precisely the homogeneous maximal ideal of R. With respect to the lexicographic order on R, one has $\mathrm{in}(x+y) = \mathrm{in}(x-y) = x$ but $\mathrm{in}((x+y)-(x-y)) = 2y$, which is not an element of $(\mathrm{in}(x + y), \mathrm{in}(x - y)) = (x)$, thus $\{x + y, x - y\}$ is not a Gröbner basis of I. Instead, $\{x, y\}$ is a Gröbner basis for I.

Definition 5.77 ([BC03, Section 5])**.** Let $X = (x_{ij})$ be an $m \times n$ matrix of indeterminates over a field $\mathbb{K}$, and let $R = \mathbb{K}[X]$ be a polynomial ring. A *diagonal monomial order* τ on R, is a monomial order on R such that if $[a_1, \dots, a_t \mid b_1, \dots, b_t]$ is a t-minor of X with $1 \leq a_1 < \cdots < a_t \leq m$ and $1 \leq b_1 < \cdots < b_t \leq n$, then

$$\mathrm{in}_\tau([a_1, \dots, a_t \mid b_1, \dots, b_t]) = x_{a_1 b_1} \cdots x_{a_t b_t}.$$

Roughly speaking, if the indices determining the rows and columns of the minor are given in ascending order, a monomial order is diagonal if the initial term of the minor is the product of the elements in the diagonal. By Lemma 5.34, the homogeneous lexicographic order on $\mathbb{K}[X]$ is a diagonal monomial order. Likewise, the lexicographic order is also diagonal, since both orders coincide on homogeneous polynomials.

Proposition 5.78. Let $R = \mathbb{K}[x_1, \dots, x_n]$ be a polynomial ring over a field $\mathbb{K}$, let $I \subseteq R$ be an ideal and τ a monomial order on R. Then $\mathrm{in}(I)^\ell \subseteq \mathrm{in}(I^\ell)$ for all integers $\ell \geq 0$.

Proof. Choose polynomials $f_1, \dots, f_\ell \in I$ and let $\eta_i = \mathrm{in}(f_i)$ for each $i = 1, \dots, \ell$. Then the product $\eta_1 \cdots \eta_\ell = \mathrm{in}(f_1 \cdots f_\ell)$ is a generator of $\mathrm{in}(I)^\ell$, where the equality follows from Lemma 5.34. Since $f_1 \cdots f_\ell \in I^\ell$, it follows that $\eta_1 \cdots \eta_\ell \in \mathrm{in}(I^\ell)$, as desired. □

In general, the containment $\mathrm{in}(I^\ell) \subseteq \mathrm{in}(I)^\ell$ is not true for an arbitrary ideal $I \subseteq R$. Nonetheless, it does hold for ideals of maximal minors.

Theorem 5.79 ([C97, Theorem 2.1])**.** Let $X = (x_{ij})$ be an $m \times n$ matrix of indeterminates over a field $\mathbb{K}$ and let $R = \mathbb{K}[X]$ be a polynomial ring. The set $\{\delta_1 \cdots \delta_i \mid \delta_1, \dots, \delta_i \in I_m\}$ is a Gröbner basis of I_m^i for all $i \in \mathbb{Z}_{\geq 0}$ with respect to a diagonal monomial order τ. In particular $\mathrm{in}(I_m^i) = \mathrm{in}(I_m)^i$ for all $i \in \mathbb{Z}_{\geq 0}$.

Theorem 5.80. Let $X = (x_{ij})$ be an $m \times n$ matrix of indeterminates over a perfect field $\mathbb{F}$ of characteristic $p > 0$ and let $R = \mathbb{F}[X]$ be a polynomial ring. Let $\mathrm{in}(I_m)$ be the initial ideal of the ideal of maximal minors I_m with respect to a diagonal monomial order.

(1) For each integer $e \geq 0$, $\nu^{\mathfrak{m}}_{\mathrm{in}(I_m)}(p^e) = (n - m + 1)(p^e - 1)$.
(2) The F-pure threshold of $\mathrm{in}(I_m)$ is $\mathrm{fpt}(\mathrm{in}(I_m)) = n - m + 1$.

Proof. Let $\delta = [b_1, \dots, b_m]$ be a maximal minor of X, with $1 \leq b_1 < \cdots < b_m \leq n$. Then $1 \leq b_1 \leq n - m + 1$, for otherwise at least two of the b_i's are repeated. As a result, in each maximal minor of X one of the variables $x_{11}, x_{12}, \dots, x_{1,n-m+1}$ is involved.

(1) For each integer $j = 1, \dots, n-m+1$ define

$$\delta_j := [j, \dots, j+m-1] = \det \begin{pmatrix} x_{1j} & \cdots & x_{1,j+m-1} \\ \vdots & \ddots & \vdots \\ x_{mj} & \cdots & x_{m,j+m-1} \end{pmatrix},$$

which is a maximal minor with initial term

$$\eta_j := \operatorname{in}(\delta_j) = x_{1j}x_{2,j+1} \cdots x_{m,j+m-1}.$$

Then $\eta_1 \cdots \eta_{n-m+1}$ is a product of $n-m+1$ monomials, thus it lies in the ideal $\operatorname{in}(I_m)^{n-m+1}$, and each variable involved in it is raised to the power of 1. By letting $\mathfrak{m} = (x_{11}, \dots, x_{mn})$ be the homogeneous maximal ideal of R, for each integer $e \geq 0$ one has that

$$(\eta_1 \cdots \eta_{n-m+1})^{p^e-1)} \in \operatorname{in}(I_m)^{(n-m+1)(p^e-1)} - \mathfrak{m}^{[p^e]},$$

thus $\nu^{\mathfrak{m}}_{\operatorname{in}(I_m)}(p^e) \geq (n-m+1)(p^e-1)$. In order to show the converse inequality, let μ be a product of $(n-m+1)(p^e-1)+1$ initial terms of maximal minors of X. As noted above, the indeterminates $x_{11}, x_{12}, \dots, x_{1,n-m+1}$ are in μ. Furthermore, μ is divisible by a monomial of the form $x_{11}^{r_1}x_{12}^{r_2} \cdots x_{1,n-m+1}^{r_{n-m+1}}$, where $r_1, \dots, r_{n-m+1} \geq 1$ are integers satisfying $r_1 + \cdots + r_{n-m+1} = (n-m+1)(p^e-1)+1$. By the pigeonhole principle there is some $r_j \geq p^e$, whence $\mu \in \mathfrak{m}^{[p^e]}$, which proves that $\nu^{\mathfrak{m}}_{\operatorname{in}(I_m)}(p^e) \leq (n-m+1)(p^e-1)$ and thus the equality.

(2) By definition of the F-pure threshold of an ideal at $\mathfrak{m}$,

$$\operatorname{fpt}(\operatorname{in}(I_m)) = \lim_{e \to \infty} \frac{\nu^{\mathfrak{m}}_{\operatorname{in}(I_m)}(p^e)}{p^e} = n-m+1. \qquad \square$$

Observation 5.81. When dealing with the initial terms of t-minors with respect to diagonal monomial orders, one notices that some indeterminates are not involved in them. For instance, for a 7×9 matrix, the indeterminates excluded from the initial terms of the 4-minors are those within the triangles drawn below:

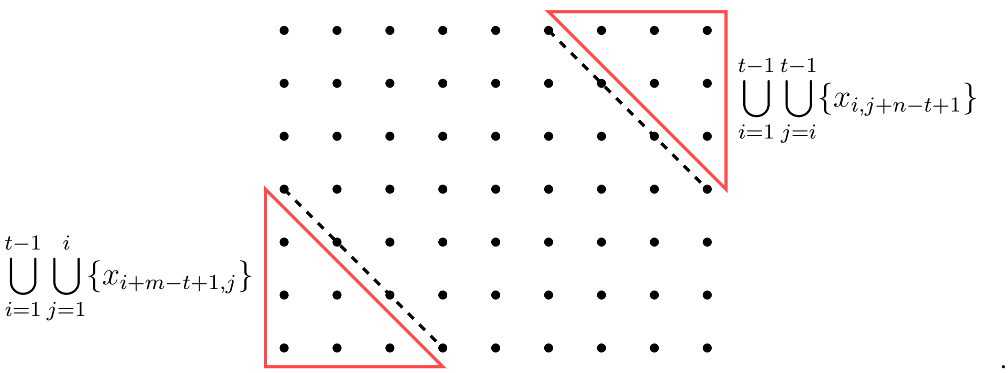

Figure 5.7

This remark is made precise in the following proposition.

Proposition 5.82. Let $X = (x_{ij})$ be an $m \times n$ matrix of indeterminates over a field $\mathbb{K}$ and let τ be a diagonal monomial order on the polynomial ring $R = \mathbb{K}[X]$. Fix an integer $1 \le t \le m$ and consider the subset of indeterminates

$$E = \underbrace{\left(\bigcup_{i=1}^{t-1} \bigcup_{j=1}^{i} \{x_{i+m-t+1,j}\} \right)}_{E_1} \cup \underbrace{\left(\bigcup_{i=1}^{t-1} \bigcup_{j=i}^{t-1} \{x_{i,j+n-t+1}\} \right)}_{E_2}.$$

Then:

(1) No indeterminate in E is involved in an initial term of a t-minor of X.

(2) Each indeterminate not in E is involved in an initial term of a t-minor of X.

It may be useful to consult Fig. 5.8 to assist with notation in the proof of Proposition 5.82. Observe that in Fig. 5.8 the matrix of dots is rotated so that the row number i is given by the x-axis, and the column number j is given by the y-axis.

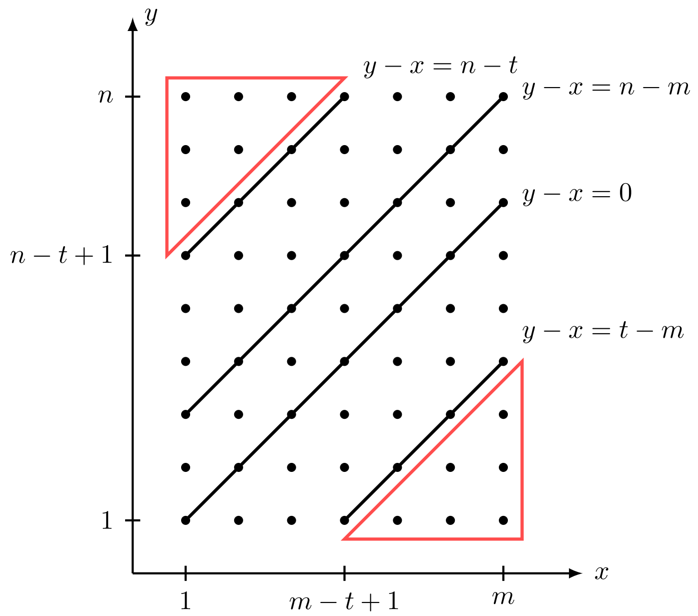

Figure 5.8

Proof of Proposition 5.82. (1) Let $\delta = [a_1, \ldots, a_t \mid b_1, \ldots, b_t]$; we may assume that the indices are given in ascending order. Let $x_{i+m-t+1,j}$ be a variable in E_1 involved in the minor δ. Suppose that it is involved in the initial term $\text{in}(\delta)$. Then there exists an integer $1 \le k \le t$ such that $a_k = i + m - t + 1$ and $b_k = j$. On the one hand,

$$a_k = i + m - t + 1 < a_{k+1} < \cdots < a_t \le m,$$

thus $t - k \le m - (i + m - t + 2) + 1 = t - i - 1$, that is, $i \le k - 1$. On the other hand,

$$1 \le b_1 < \cdots < b_{k-1} < b_k = j,$$

hence $k-1 \leq j-1$, i.e. $k \leq j$. One then finds that $1 \leq k \leq j \leq i \leq k-1$, a contradiction.

Likewise, let $x_{i,j+n-t+1}$ be a variable in E_2, involved in the minor δ, and suppose that it is in the initial term $\mathrm{in}(\delta)$. As before, there is an integer $1 \leq k \leq t$ with $a_k = i$ and $b_k = j+n-t+1$, hence

$$1 \leq a_1 < \cdots < a_{k-1} < a_k = i$$

and

$$b_k = j+n-t+1 < b_{k+1} < \cdots < b_t \leq n.$$

Both inequalities together give $k \leq i$ and $j \leq k-1$, hence $k \leq i \leq j \leq k-1$, a contradiction.

(2) Consider the lattice $\mathcal{L} = \{(x,y) \in \mathbb{Z}^2 \mid 1 \leq x \leq m,\ 1 \leq y \leq n\}$, define the subsets $\mathcal{E}_1 = \{(x,y) \in \mathcal{L} \mid y - x < t - m\}$ and $\mathcal{E}_2 = \{(x,y) \in \mathcal{L} \mid y - x > n - t\}$, and let $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$. By identifying the indeterminate x_{ij} and the lattice point (i,j), the matrix X is identified to $\mathcal{L}$, and the sets E_1 and E_2 are identified to $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively. Let x_{ij} be a variable not in E, that is, $(i,j) \in \mathcal{L} - \mathcal{E}$. Consider the line $V = \{(x,y) \in \mathbb{R}^2 \mid y - x = j - i\}$; we claim that $|V \cap \mathcal{L}| \geq t$. To prove it we consider several cases depending on $j - i$. In order to help intuition, we refer the reader to Fig. 5.8.

- If $t - m \leq j - i \leq 0$, then $V \cap \mathbb{Z} = \{(i-j+1, 1), (i-j+2, 2), \ldots, (m, m-i+j)\}$, thus it contains $m - i + j$ points of the lattice and, by assumption, $m - i + j \geq t$.
- If $0 < j - i \leq n - m$, one has $V \cap \mathbb{Z} = \{(1, j-i+1), (2, j-i+2), \ldots, (m, j-i+m)\}$, which contains $m \geq t$ points.
- If $n - m < j - i \leq n - t$, then $V \cap \mathbb{Z} = \{(1, j-i+1), (2, j-i+2), \ldots, (n-j+i, n)\}$. The intersection contains $n - j + i$ points of the lattice which is $\geq t$ by assumption.

Consequently, given a lattice point $(i,j) \in \mathcal{L} - \mathcal{E}$, one can choose $t-1$ additional points contained in V. The result is a collection of points $\{(a_1,b_1), \ldots, (a_t,b_t)\}$ satisfying $1 \leq a_1 < \cdots < a_t \leq m$ and $1 \leq b_1 < \cdots < b_t \leq n$. Furthermore, it is clear that there is some integer $1 \leq k \leq t$ with $a_k = i$ and $b_k = j$. The corresponding t-minor $[a_1, \ldots, a_t \mid b_1, \ldots, b_t]$ has x_{ij} in its initial term. □

Definition 5.83. Let $X = (x_{ij})$ be a matrix of indeterminates of size $m \times n$, $m \leq n$, and fix an integer $1 \leq t \leq m$. The set of *excluded indeterminates* of X is

$$E := \left(\bigcup_{i=1}^{t-1} \bigcup_{j=1}^{i} \{x_{i+m-t+1,j}\} \right) \cup \left(\bigcup_{i=1}^{t-1} \bigcup_{j=i}^{t-1} \{x_{i,j+n-t+1}\} \right).$$

Proposition 5.84. Let $X = (x_{ij})$ be a matrix of indeterminates of size $m \times n$, $m \leq n$, defined over a field $\mathbb{F}$ of characteristic $p > 0$. Fix an integer $1 \leq t < m$, let I_t be the ideal of t-minors in the polynomial ring $R = \mathbb{F}[X]$, and let $\mathrm{in}(I_t)$ be the initial ideal with respect to a diagonal monomial order on R. Then for each integer $e \geq 0$,

$$\nu^{\mathfrak{m}}_{\mathrm{in}(I_t)}(p^e) \geq \left(\left\lfloor \frac{m}{t} \right\rfloor n - t + 1 \right) (p^e - 1).$$

Proof. For each pair of integers $1 \le i \le \left\lfloor \frac{m}{t} \right\rfloor$ and $1 \le j \le n-t+1$, let

$$\delta_{ij} = [(i-1)t+1, \dots, it \mid j, \dots, t+j-1] \,.$$

Furthermore, for each pair of integers $1 \le i \le \left\lfloor \frac{m}{t} \right\rfloor - 1$ and $1 \le j \le t-1$, define

$$\varepsilon_{ij} = [(i-1)t+j+1, \dots, it+j \mid 1, \dots, t-j, n-j+1, \dots, n]$$

(see Example 5.85 for a sketch of the δ_{ij} and ε_{ij}). Denote by Δ the product of all the δ_{ij} and ε_{ij}. Then Δ is a product of

$$\left\lfloor \frac{m}{t} \right\rfloor (n-t+1) + \left(\left\lfloor \frac{m}{t} \right\rfloor - 1 \right)(t-1) = \left\lfloor \frac{m}{t} \right\rfloor n - t + 1$$

minors of size t of X. By construction, the initial term $\text{in}(\Delta)$, which is the product of the all the $\text{in}(\delta_{ij})$ and $\text{in}(\varepsilon_{ij})$, is a square-free monomial, whence

$$\text{in}(\Delta)^{p^e-1} \in \text{in}(I_t)^{\left(\lfloor \frac{m}{t} \rfloor n - t + 1\right)(p^e-1)} - \mathfrak{m}^{[p^e]}. \qquad \square$$

Example 5.85. In order to assist with notation in the proof of Proposition 5.84, we depict the δ_{ij} and the ε_{ij} for the 3-minors of an 8×8 matrix:

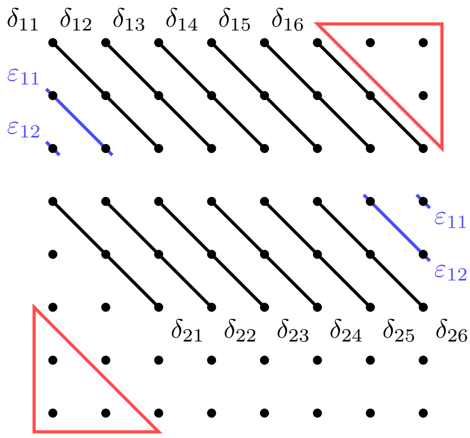

Figure 5.9

Suppose that t divides m. Then the lower bound on $\nu^{\mathfrak{m}}_{\text{in}(I_t)}(p^e)$ given by Proposition 5.84 is the best one possible, in the sense that all the indeterminates of $X = (x_{ij})$, except the excluded ones, are involved in the initial term of some δ_{ij} or some ε_{ij}. As the proposition below shows, the same is true when t divides n.

Proposition 5.86. Let $X = (x_{ij})$ be a matrix of indeterminates of size $m \times n$, $m \le n$, defined over a perfect field $\mathbb{F}$ of characteristic p. Fix an integer $1 \le t \le m$, let I_t be the ideal of t-minors of X in the polynomial ring $R = \mathbb{F}[X]$, and let $\text{in}(I_t)$ be the initial ideal with respect to a diagonal monomial order on R. Suppose that t divides n. Then:

(1) There is a product of $\frac{mn}{t} - t + 1$ minors such that each indeterminate of X not in the excluded indeterminates is in the initial term of some t-minor of X.
(2) Let $\mathfrak{m} \subseteq R$ be the homogeneous maximal ideal. For each integer $e \geq 1$,

$$\nu^{\mathfrak{m}}_{\mathrm{in}(I_t)}(p^e) \geq \left(\frac{mn}{t} - t + 1\right)(p^e - 1).$$

Proof. (1) For each pair of integers $1 \leq i \leq m - t + 1$ and $1 \leq j \leq \frac{n}{t}$, define

$$\delta_{ij} = [i, \ldots, i + t - 1 \mid (j-1)t + 1, \ldots, jt].$$

Furthermore, for each pair of integers $1 \leq i \leq t - 1$ and $1 \leq j \leq \frac{n}{t} - 1$, let

$$\varepsilon_{ij} = [1, \ldots, t - i, n - i + 1, \ldots, n \mid (j-1)t + 1 + i, \ldots, jt + i],$$

(see Example 5.87 for a sketch of the δ_{ij} and ε_{ij}). It is clear that every indeterminate not among the excluded ones is in the initial term of some δ_{ij} or ε_{ij}. Denote by Δ the product of all the δ_{ij} and ε_{ij}. Then Δ is a product of $\frac{n}{t}(m - t + 1) + \left(\frac{n}{t} - 1\right)(t - 1) = \frac{mn}{t} - t + 1$ minors.

(2) The initial term $\mathrm{in}(\Delta)$ is the product of all the initial terms $\mathrm{in}(\delta_{ij})$ and $\mathrm{in}(\varepsilon_{ij})$. Since each non-excluded indeterminate of X is involved in an initial term, and no indeterminate is in two different initial terms, it follows that $\mathrm{in}(\Delta)$ is a square-free monomial. As a result,

$$\mathrm{in}(\Delta) \in \mathrm{in}(I_t)^{\left(\frac{mn}{t} - t + 1\right)(p^e - 1)} - \mathfrak{m}^{[p^e]},$$

from where the assertion follows. □

Example 5.87. To assist with notation in the proof of Proposition 5.86, we show the δ_{ij} and ε_{ij} for the 3-minors of an 8×9 matrix of indeterminates:

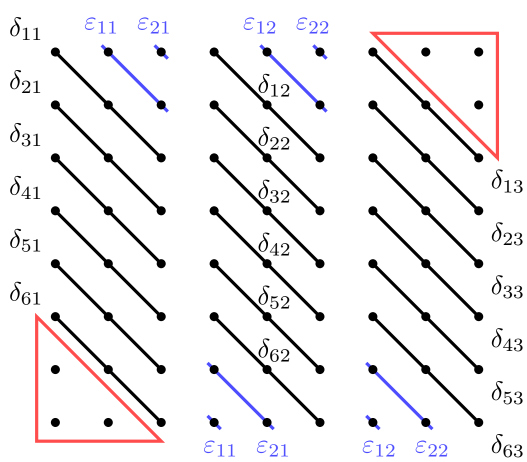

Figure 5.10

5.8. Ideals of t-minors

Let $X = (x_{ij})$ be an $m \times n$ matrix of indeterminates over a perfect field $\mathbb{F}$ of characteristic p, and let $I_m(X) \subseteq \mathbb{K}[X]$ be the ideal of maximal minors. Recall that the F-pure threshold of $I_m(X)$ is $\operatorname{fpt}(I_m(X)) = n - m + 1$, and that ν-invariants of level $e \geq 0$ are given by

$$\nu^{\bullet}_{I_m(X)}(p^e) = \{\nu(s) \coloneqq p^e + (n-m+1)(p^e-1) \mid s \in \mathbb{Z}_{\geq 0}\},$$

hence two consecutive ν-invariants satisfy $\nu(s+1) - \nu(s) = p^e$. In particular, we have that

$$\mathcal{C}^e_R \cdot I_m(X)^{(n-m+1)(p^e-1)} = R.$$

In order to prove this fact, we have showed that there is a product of $(n-m+1)(p^e-1)$ maximal minors that has a monomial in the standard basis of F^e_*R as a free R-module (see Definition 2.34 and Proposition 2.33). Furthermore, for each integer $s \in \mathbb{Z}_{\geq 0}$ we have that

$$\mathcal{C}^e_R \cdot I_m(X)^{\nu(s)+p^e} = \mathcal{C}^e_R \cdot I_m(X)^{\nu(s)+p^e-1} = \cdots = \mathcal{C}^e_R \cdot I_m(X)^{\nu(s)+1} = I_m(X)^{s+1}.$$

In order to prove this fact, it sufficed to show two inclusions: $I_m(X)^{s+1} \subseteq \mathcal{C}^e_R \cdot I_m(X)^{\nu(s)+p^e}$ and $\mathcal{C}^e_R \cdot I_m(X)^{\nu(s)+1} \subseteq I_m(X)^{s+1}$. As Proposition 5.88 shows, a variant of this inclusion is satisfied by every ideal in a regular F-finite ring.

Proposition 5.88. Let $R = \mathbb{F}[x_1, \ldots, x_n]$ be a polynomial ring over a field $\mathbb{F}$ of characteristic $p > 0$ and let $\mathfrak{m} = (x_1, \ldots, x_n)$ be the homogeneous maximal ideal. Let $\mathfrak{a} \subseteq \mathfrak{m}$ be an ideal, fix an integer $e \geq 0$ and let $\nu = \nu^{\mathfrak{m}}_{\mathfrak{a}}(p^e)$. Then for all integers $s \geq 0$

$$\mathfrak{a}^{s+1} \subseteq \mathcal{C}^e_R \cdot \mathfrak{a}^{\nu+(s+1)p^e}.$$

Proof. By assumption $\mathcal{C}^e_R \cdot \mathfrak{a}^{\nu} = R$, hence by Proposition 4.21,

$$\mathfrak{a}^{s+1} = \mathfrak{a}^{s+1} \cdot \mathcal{C}^e_R \cdot \mathfrak{a}^{\nu} = \mathcal{C}^e_R \cdot (\mathfrak{a}^{(s+1)[p^e]}\mathfrak{a}^{\nu}) = \mathcal{C}^e_R \cdot (\mathfrak{a}^{(s+1)[p^e]}\mathfrak{a}^{\nu}) \subseteq \mathcal{C}^e_R \cdot \mathfrak{a}^{\nu+(s+1)p^e}. \qquad \square$$

On the contrary, the inclusion $\mathcal{C}^e_R \cdot I_m(X)^{\nu(s)+1} \subseteq I_m(X)^{s+1}$ is more delicate. To deduce it, we proceeded by induction on m to reduce the problem to an $(m-1) \times (n-1)$ matrix Y; this is achieved by localizing at the variable x_{mn}. By the inductive hypothesis one has

$$\mathcal{C}^e_{\mathbb{K}[Y]} \cdot I_{m-1}(Y)^{sp^e+(n-m+1)(p^e-1)+1} \subseteq I_{m-1}(Y)^{s+1}.$$

In the localized ring Q, the extension of the powers of $I_m(X)$ correspond to the extension of the powers of $I_{m-1}(Y)$, that is $I_m(X)^{\ell}Q = I_{m-1}(Y)^{\ell}Q$ for all integers $\ell \geq 1$. A key fact in this step is that $\operatorname{fpt}(I_m(X)) = \operatorname{fpt}(I_{m-1}(Y)) = n - m + 1$, thus one can set $\ell = n - m + 1$. Afterwards, an inclusion of Frobenius roots in the localization is shown and, using that x_{mn} is a non-zero-divisor modulo $I_m(X)^s$, one recovers the aforementioned inclusion in the polynomial ring $\mathbb{K}[X]$.

In Section 5.6 we have shown that given a tuple (m, n, t), there is an integer $i_0 \in \mathbb{Z}$ (not necessarily non-negative) such that for all $i \geq i_0$, the F-pure threshold of the ideal $I_{t+i}(X[i])$ of $(t+i)$-minors of an $(m+i) \times (n+i)$ matrix of indeterminates $X[i]$ is stable in the sense that

$$\operatorname{fpt}(I_{t+i_0}(X[i_0])) = \operatorname{fpt}(I_{t+i_0+1}(X[i_0+1])) = \operatorname{fpt}(I_{t+i_0+2}(X[i_0+2])) = \cdots .$$

Suppose that $i_0 \leq 0$. Then a strategy to compute the ν-invariants of the ideal $I_t(X)$ is to reduce the problem to the case $(m+i_0, n+i_0, t+i_0)$, thus a seemingly important step in the computation of the ν-invariants of ideals of t-minors is completed. Nonetheless, it is not true in general that x_{mn} is a non-zero-divisor modulo $I_t(X)^\ell$ for all integers $\ell \geq 1$, thus this strategy cannot be applied.

In this section we give bounds on $\nu_{I_t}^{\mathfrak{m}}(p^e)$, since we believe that the remaining ν-invariants can be computed from this.

Proposition 5.89. Let $X = (x_{ij})$ be an $m \times n$ matrix of indeterminates over a perfect field $\mathbb{K}$ of characteristic $p > 0$. Let $R = \mathbb{K}[X]$ be the corresponding polynomial ring and $I_t(X) \subseteq R$ the ideal of t-minors of X. Suppose that $\operatorname{fpt}(I_t(X)) \notin \mathbb{Z}$, then:

(1) For infinitely many integers $e \geq 0$, $\nu_{I_t}^{\mathfrak{m}}(p^e) \geq \lfloor \operatorname{fpt}(I_t) \rfloor (p^e - 1)$
(2) For infinitely many integers $e \geq 0$, $\nu_{I_t}^{\mathfrak{m}}(p^e) \leq \lceil \operatorname{fpt}(I_t) \rceil (p^e - 1)$.

Proof. (1) Suppose that this is not the case, i.e. suppose that $\nu_{I_t}^{\mathfrak{m}}(p^e) < \lfloor \operatorname{fpt}(I_t) \rfloor (p^e - 1)$ for all integers $e \geq 0$. Then

$$\operatorname{fpt}(I_t) = \lim_{e \to \infty} \frac{\nu_{I_t}^{\mathfrak{m}}(p^e)}{p^e} = \lim_{e \to \infty} \lfloor \operatorname{fpt}(I_t) \rfloor \frac{p^e - 1}{p^e} = \lfloor \operatorname{fpt}(I_t) \rfloor,$$

but this is a contradiction since $\operatorname{fpt}(I_t) > \lfloor \operatorname{fpt}(I_t) \rfloor$.

(2) Analogous to part (1). □

Proposition 5.90. Let $X = (x_{ij})$ be a matrix of indeterminates of size $m \times n$, $m \leq n$, defined over a field $\mathbb{F}$ of characteristic $p > 0$. Fix an integer $1 \leq t \leq m$, let I_t be the ideal of t-minors of X in the polynomial ring $R = \mathbb{F}[X]$. Suppose that t divides m or divides n. Then for each integer $e \geq 0$,

$$\nu_{I_t}^{\mathfrak{m}}(p^e) \geq \left(\frac{mn}{t} - t + 1 \right) (p^e - 1).$$

Proof. By Propositions 5.84 and 5.86, there is a product Δ of $\frac{mn}{t} - t + 1$ minors of size t whose initial term is a square-free monomial under a diagonal monomial order on R. As a result, for each integer $e \geq 0$,

$$\Delta^{p^e - 1} \in I_t^{\left(\frac{mn}{t} - t + 1\right)(p^e - 1)} - \mathfrak{m}^{[p^e]},$$

thus $\nu_{I_t}^{\mathfrak{m}}(p^e) \geq \left(\frac{mn}{t} - t + 1 \right) (p^e - 1)$, as desired. □

Proposition 5.91. Let $X = (x_{ij})$ be a matrix of indeterminates of size $m \times n$, $m \leq n$, defined over a field $\mathbb{F}$ of characteristic $p > 0$, and let I_t be the ideal of t-minors of X in the polynomial ring $R = \mathbb{F}[X]$. For each integer $e \geq 0$,

$$\mathcal{C}_R^e \cdot I_t^{(m-t+1)(n-t+1)(p^e-1)+1} \subseteq I_t.$$

In particular, by letting $\mathfrak{m}$ be the homogeneous maximal ideal of R,

$$\nu_{I_t}^{\mathfrak{m}}(p^e) \leq (m-t+1)(n-t+1)(p^e-1).$$

Proof. The localized ring R_{I_t} is a regular local ring of dimension $(m-t+1)(n-t+1)$ (see Proposition 5.30), thus by Proposition 4.10,

$$I_t^{(m-t+1)(n-t+1)(p^e-1)+1} R_{I_t} \subseteq I_t^{[p^e]} R_{I_t}.$$

By Lemma 5.32 the only associated prime of $I_t^{[p^e]}$ is I_t, therefore one recovers the inclusion in the ring R, that is,

$$I_t^{(m-t+1)(n-t+1)(p^e-1)+1} \subseteq I_t^{[p^e]},$$

from where the assertion follows. □

So far we have only considered the homogeneous lexicographic order induced by the usual ordering on the variables, namely, $x_{11} > \cdots > x_{1n} > \cdots > x_{m1} > \cdots x_{mn}$. As the following example shows, weighted monomial orders prove to be a useful tool to compute ν-invariants.

Example 5.92. Let $X = (x_{ij})$ be the 3×3 matrix of indeterminates

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix},$$

defined over a field $\mathbb{F}$ of characteristic $p > 0$ and set $R = \mathbb{F}[X]$. Consider the 2-minors

$$\delta_1 = [1,2 \mid 1,2] = x_{11}x_{22} - x_{12}x_{21}, \quad \delta_2 = [1,2 \mid 1,3] = x_{12}x_{23} - x_{13}x_{22},$$
$$\delta_3 = [2,3 \mid 1,2] = x_{21}x_{32} - x_{22}x_{31}, \quad \delta_4 = [1,3 \mid 1,3] = x_{11}x_{33} - x_{13}x_{31},$$

and define $\Delta = \delta_1\delta_2\delta_3\delta_4$, which is an element in I_2^4. The product Δ computed over $\mathbb{Z}$ reads

$$\begin{aligned}\Delta = {} & x_{12}x_{13}^2x_{21}x_{22}^2x_{31}^2 - x_{11}x_{13}^2x_{22}^3x_{31}^2 - x_{12}^2x_{13}x_{21}x_{22}x_{23}x_{31}^2 \\ & + x_{11}x_{12}x_{13}x_{22}^2x_{23}x_{31}^2 - x_{12}x_{13}^2x_{21}^2x_{22}x_{31}x_{32} + x_{11}x_{13}^2x_{21}x_{22}^2x_{31}x_{32} \\ & + x_{12}^2x_{13}x_{21}^2x_{23}x_{31}x_{32} - \mathbf{x_{11}x_{12}x_{13}x_{21}x_{22}x_{23}x_{31}x_{32}} - x_{11}x_{12}x_{13}x_{21}x_{22}^2x_{31}x_{33} \\ & + x_{11}^2x_{13}x_{22}^3x_{31}x_{33} + x_{11}x_{12}^2x_{21}x_{22}x_{23}x_{31}x_{33} - x_{11}^2x_{12}x_{22}^2x_{23}x_{31}x_{33} \\ & + x_{11}x_{12}x_{13}x_{21}^2x_{22}x_{32}x_{33} - x_{11}^2x_{13}x_{21}x_{22}^2x_{32}x_{33} - x_{11}x_{12}^2x_{21}^2x_{23}x_{32}x_{33} \\ & + x_{11}^2x_{12}x_{21}x_{22}x_{23}x_{32}x_{33},\end{aligned}$$

which has a square-free monomial. Consider the matrix of weights

$$W = (w_{ij}) = \begin{pmatrix} 100 & 8 & 7 \\ 6 & 5 & 50 \\ 3 & 70 & -9000 \end{pmatrix}.$$

This gives the weight function $\mu\colon \mathbb{R}^9 \to \mathbb{R}$, $\mu(s_{11}, \dots, s_{33}) = s_{11}w_{11} + \cdots + s_{33}w_{33}$, which induces a weighted monomial order $>_\mu$ on R:

$$x^a >_\mu x^b \text{ if and only if } \mu(a) > \mu(b).$$

Under the monomial order $>_\mu$, the initial term of Δ is the one corresponding to the square-free monomial, that is, $\operatorname{in}_{>_\mu}(\Delta) = -x_{11}x_{12}x_{13}x_{21}x_{22}x_{23}x_{31}x_{32}$. Since the coefficient of the square-free monomial is -1, the initial term of the reduction mod p of Δ is $\operatorname{in}_{>_\mu}(\Delta) = -x_{11}\cdots x_{32}$. In consequence, by letting $\mathfrak{m} = (x_{11}, \dots, x_{33})$ be the homogeneous maximal ideal, for each integer $e \geq 0$ one has $\operatorname{in}_{>_\mu}(\Delta^{p^e-1}) = (x_{11}\cdots x_{32})^{p^e-1}$, which gives

$$\Delta^{p^e-1} \in I_2^{4(p^e-1)} - \mathfrak{m}^{[p^e]},$$

hence $\nu_{I_2}^{\mathfrak{m}}(p^e) \geq 4(p^e-1)$. By Proposition 5.91 one has that $\mathcal{C}_R^e \cdot I_2^{4(p^e-1)+1} \subseteq I_2$, thus $\nu_{I_2}^{\mathfrak{m}}(p^e) = 4(p^e-1)$. It follows from Proposition 5.88 that $I_2 \subseteq \mathcal{C}_R^e \cdot I_2^{4(p^e-1)+p^e}$, whence

$$\mathcal{C}_R^e \cdot I_2^{4(p^e-1)+p^e} = \mathcal{C}_R^e \cdot I_2^{4(p^e-1)+p^e-1} = \cdots = \mathcal{C}_R^e \cdot I_2^{4(p^e-1)+2} = \mathcal{C}_R^e \cdot I_2^{4(p^e-1)+1} = I_2.$$

From this computation one recovers the F-pure threshold of I_2, namely, $\operatorname{fpt}(I_2) = 4$. Furthermore, one deduces the following:

(a) For each real number $4 \leq \lambda < 5$, $\tau(I_2^\lambda) = I_2$. Indeed, since $\tau(I_2^\lambda) = \mathcal{C}_R^e \cdot I_2^{\lceil \lambda p^e \rceil}$ for some $e \gg 0$, in order for $\mathcal{C}_R^e \cdot I_2^{\lceil \lambda p^e \rceil}$ to be a known Frobenius root of I_2, one needs

$$4(p^e-1) < \lambda p^e \leq 4(p^e-1) + p^e$$

which is equivalent to

$$4\frac{p^e-1}{p^e} < \lambda \leq 4\frac{p^e-1}{p^e} + 1.$$

Since both inequalities must hold for all integers $e \geq 0$ large enough, by letting $e \to \infty$, it follows that $\lambda \in [4, 5) \subseteq \mathbb{R}$.

(b) Define $\nu_e := \nu_{I_2}^{\mathfrak{m}}(p^e) = 4(p^e-1)$. Then $(\nu_e)_{e=0}^\infty \subseteq \mathbb{Z}$ is a sequence of ν-invariants with p-adic limit $\nu_e \to -4$ as $e \to \infty$, therefore -4 is a Bernstein-Sato root of I_2 in any prime characteristic p. As it does not depend on p, it follows that -4 is a root of the b-function of I_2 in characteristic zero.

Example 5.93. Let $X = (x_{ij})$ be an $n \times n$ matrix of indeterminates over a perfect field $\mathbb{F}$ of characteristic $p > 0$, and let I_{n-1} be the ideal of $(n-1)$-minors of X in the polynomial ring $R = \mathbb{F}[X]$. Proceeding as in Example 5.92, one shows that $\mathcal{C}_R^e \cdot I_{n-1}^{4(p^e-1)+1} \subseteq I_{n-1}$. By Proposition 4.19, this is equivalent to $I_{n-1}^{4(p^e-1)+1} \subseteq I_{n-1}^{[p^e]}$, hence $\nu_{I_{n-1}}^{\mathfrak{m}}(p^e) \leq 4(p^e-1)$. Furthermore, using Section 5.6, one sees that $\operatorname{fpt}(I_{n-1}) = 4$ for all integers $n \geq 1$.

5.9. Open questions

Let X be a matrix of indeterminates of size $m \times n$, $m \leq n$, defined over a perfect field $\mathbb{F}$ of characteristic $p > 0$. Let I_t be the ideal of t-minors of X in the polynomial ring $R = \mathbb{F}[X]$ and fix $e \in \mathbb{Z}_{\geq 0}$. In view of Theorems 5.47 and 5.80 and Example 5.92, several natural questions arise:

Question 5.94. Is it true that for all integers $e \geq 0$ large enough, $\nu_{I_t}^{\mathfrak{m}}(p^e) = \lfloor \mathrm{fpt}(I_t)(p^e - 1) \rfloor$?

Question 5.95. For each $s \in \mathbb{Z}_{\geq 0}$, denote by $\nu(s)$ the s-th smallest ν-invariant of level e of I_t. In other words, $\nu(0) = \nu_{I_t}^{\mathfrak{m}}(p^e)$ and, for each $s \geq 1$, $\nu(s) = \min(\nu_{I_t}^{\bullet}(p^e) - \{\nu(0), \nu(1), \ldots, \nu(s-1)\})$. Is is true that $\nu_(s+1) - \nu(s) = p^e$?

Question 5.96.

(a) Let $\ell \geq 0$ be an integer; is the Frobenius root $\mathcal{C}_R^e \cdot I_t^{\ell}$ equal to some power I_t^k?

(b) Even stronger, following the notation of Question 5.95, let $\nu(s)$ and $\nu(s+1)$ be consecutive ν-invariants of level e of the ideal I_t. We wonder if the following equalities hold:

$$\mathcal{C}_R^e \cdot I_t^{\nu(s+1)} = \mathcal{C}_R^e \cdot I_t^{\nu(s+1)-1} = \cdots = \mathcal{C}_R^e \cdot I_t^{\nu(s)+2} = \mathcal{C}_R^e \cdot I_t^{\nu(s)+1} = I_t^s.$$

Question 5.97. Fix a diagonal monomial order τ on the polynomial ring $R = \mathbb{F}[X]$. It might be possible to relate properties of the ideal of t-minors to its initial ideal. More precisely:

(a) Is it true that $\mathrm{fpt}(I_t) = \mathrm{fpt}(\mathrm{in}_\tau(I_t))$?

(b) Is it true that $\nu_{I_t}^{\bullet}(p^e) = \nu_{\mathrm{in}_\tau(I_t)}^{\bullet}(p^e)$?

Question 5.98. Let $W = (w_{ij})$ be an $m \times n$ matrix of real numbers and set $\deg x_{ij} = w_{ij}$ for all $1 \leq i \leq m$, $1 \leq j \leq n$. Set $\nu := \nu_{I_t}^{\mathfrak{m}}(p^e)$ and let $\Delta = \delta_1 \cdots \delta_\nu$ be a product of ν minors of size t of X such that $\Delta \notin \mathfrak{m}^{[p^e]}$. Can one find a matrix of weights W such that $\mathrm{in}(\Delta)$ is an element in the standard basis of $F_*^e R$ as an R-module?

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Chapter A
Algorithms

A.1. Computation of the F-pure threshold of a determinantal ideal

The functions in the Python code below return the F-pure threshold of the ideal I_t of t-minors of an $m \times n$ matrix of indeterminates X defined over a field $\mathbb{F}$ of characteristic $p > 0$.

```
import math

# Returns (m - k)*(n - k)/(t - k)
def fpt_function(m, n, t, k):
    return (m - k)*(n - k)/(t - k)

# Returns the fpt of the ideal of t-minors
def fpt(m, n, t):
    # Check arguments
    if t < 1:
        print("Error: t < 1, function will return {-1,-1}.")
        return [-1, -1]
    if t > min(m,n):
        print("Error: t > min(m,n), function will return {-1,-1}.")
        return [-1, -1]
    if t == m:
        return [n - m + 1, m - 1]
    # Compute F-pure threshold (fpt)
    fpt = fpt_function(m, n, t, 0)
    arg_min = 0
    for k in range(1, t):
        fk = fpt_function(m, n, t, k)
        if fk < fpt:
            fpt = fk
            arg_min = k
    # Return statement
    return [fpt, arg_min]
```

A.2. Frobenius roots

Let $R = \mathbb{F}[x_1, \ldots, x_n]$ be a polynomial ring over a field $\mathbb{F}$ of characteristic $p > 0$ and fix an integer $e \geq 0$. The following functions coded in Macaulay2 compute the Frobenius root $\mathcal{C}_R^e \cdot I$ of an ideal $I \subseteq R$.

A.2.1. Monomialization

Let $x^a = x_1^{a_1} \cdots x_n^{a_n} \in R$ be a monomial. Each a_i may be written uniquely as $a_i = b_i p^e + r_i$, where $b_i, r_i \geq 0$ are integers and $0 \leq r_i < p^e$, therefore x^a can be expressed as

$$x_1^{a_1} \cdots x_n^{a_n} = (x_1^{b_1} \cdots x_n^{b_n})^{p^e} x_1^{r_1} \cdot x_n^{r_n}.$$

Equivalently, in the standard basis of $F_*^e R$ as a free R-module (see Definition 2.34), $x_1^{r_1} \cdots x_n^{r_n}$ is a basis element, therefore $F_*^e(x_1^{a_1} \cdots x_n^{a_n}) = (x_1^{b_1} \cdots x_n^{b_n}) F_*^e(x_1^{r_1} \cdots x_n^{r_n})$.

Given an integer $\ell \geq 0$, denote by $[\ell] \coloneqq \{0, 1, \ldots, \ell\}$. Then one can verify that the map $\varphi \colon [p^e - 1]^n \to [np^e - 1]$ sending $(r_1, \ldots, r_n) \mapsto \sum_{i=1}^n r_i p^{e(n-i)}$ is a bijection. By identifying the basis monomial $x^r = x_1^{r_1} \cdots x_n^{r_n}$ with its multi-index $r = (r_1, \ldots, r_n)$, one can keep track of the basis monomial by identifying it with the integer $\varphi(r)$.

The method monomialize carries out this operation: given a multi-index $a = (a_1, \ldots, a_n)$, it returns the multi-indices $b = (b_1, \ldots, b_n)$ and $r = (r_1, \ldots, r_n)$, as well as the corresponding basis element $\mathrm{bel}(r) \coloneqq \varphi(r)$.

```
-- Given a monomial, the function computes it in the std basis
monomialize = method();

monomialize(ZZ, List, List) := (pe, a, ve) -> (
    b := {};
    r := {};
    xb := 1;
    bel := 0;
    for i from 0 to (length a - 1) do (
        ai := a#i;
        bi := floor(ai / pe);
        ri := ai % pe;
        ti := bi * pe + ri;
        b = append(b, bi);
        r = append(r, ri);
        bel = bel + ri * ve#i;
        xb = xb * x_(i+1)^bi;
    );
    return (xb, r, bel);
);
```

A.2.2. Computation of Frobenius roots for principal ideals

Fix a polynomial $f = \sum_i \alpha_i x^{a_i} \in R$ and express it in the standard basis of $F^e_* R$ as a free R-module like $F^e_* f = \sum_{j=1}^m f_j F^e_* x^{r_j}$. It follows from [BMS08, Proposition 2.5] (see Proposition 4.18) that $\mathcal{C}^e_R \cdot f = (f_1, \ldots, f_m)$.

The method newFrobeniusRootPrincipal computes the coefficients $f_1, \ldots, f_m$. To this end, the function loops through the terms $\alpha_i x^{a_i}$ of f and calls the method monomialize to express them as $\alpha_i^{1/p^e} x^{b_i} F^e_* x^{r_i}$. Each monomial is identified by the basis elements x^{r_i}, which corresponds uniquely to the integer $\mathrm{bel}(r_i) = \varphi(r_i)$. Since different terms may have the same basis element $F^e_* x^{r_i}$, as the coefficients $\alpha_i^{1/p^e} x^{b_i}$ are stored in a list, they are compared with one another using the integer $\mathrm{bel}(r_i)$.

```
-- Frobenius root of a principal ideal
newFrobeniusRootPrincipal = method();
newFrobeniusRootPrincipal(RingElement, ZZ, List) := (f, pe, ve) -> (
    (M, C) := coefficients f;    -- Monomials of f and their Coefficients
    fRootGenList := {};          -- Generators of the Frobenius root of f
    -- Loop through all the monomials of f
    for j from 0 to (numColumns(M) - 1) do (
        -- Compute the monomial in the standard basis
        exponentsMonomial = (exponents M_(0,j))#0;
        (xb, r, bel) := monomialize(pe, exponentsMonomial, ve);
        xb = xb * C_(j,0);
        -- Introduce the monomial in the gens of the Frobenius root of f
        -- Check if there is a monomial with same basis element (same bel)
        k := 0;
        found := false;
        while(found == false and k < length fRootGenList) do (
            genk := fRootGenList#k;
            if (bel == genk#0) then (
                found = true;
                xb = xb + genk#1;
                fRootGenList = drop(fRootGenList, {k,k});
                fRootGenList = append(fRootGenList, {bel, xb});
                -- fRootGenList#k = {bel, genk#1 + xb};
            );
            k = k + 1;
        );
        -- If not, append element at the end of the list
        if (found == false) then (
            fRootGenList = append(fRootGenList, {bel, xb});
        );
    );
    return fRootGenList;
);
```

A.2.3. Computation of Frobenius roots

Let $I = (f_1, \ldots, f_r) \subseteq R$ be an ideal. As the action of Cartier operators is additive, one has that $\mathcal{C}_R^e \cdot I = \sum_{i=1}^r \mathcal{C}_R^e \cdot f_i$.

The method newFrobeniusRoot computes the Frobenius root $\mathcal{C}_R^e \cdot I$ as the sum of the $\mathcal{C}_R^e \cdot f_i$. To calculate the latter, it calls the method newFrobeniusRootPrincipal, thus obtaining a list of generators which later on are put together.

```
-- Frobenius root of an ideal
newFrobeniusRoot = method();
newFrobeniusRoot(ZZ, Ideal) := (e, I) -> (
    -- Vector of powers ( (p^e)^(N-1), (p^e)^(N-2), ..., (p^e), 1)
    ve := {1};
    for i from 0 to (N - 2) do (
        ve = prepend(p * ve#0, ve);
    );
    -- Lists of generators
    genIdeal := first entries gens I;    -- Generators of I
    genList := {};                       -- Generators of the Frob. root
    -- Loop through all generators
    for i from 0 to (length genIdeal - 1) do (
        -- i-th generator of I
        f := genIdeal#i;
        fRootGenList := newFrobeniusRootPrincipal(f, pe, ve);
        -- Merge lists
        for j from 0 to (length fRootGenList - 1) do (
            gen := fRootGenList#j;
            genList = append(genList, gen#1);
        );
    );
    J := ideal unique(genList);
    return J;
);
```

A.2.4. Example of computation of a Frobenius root

Let $X = (x_{ij})$ be a 3×4 matrix of indeterminates over $\mathbb{F}_3$ and set $R = \mathbb{F}_3[X]$. Let $I = I_3(X) \subseteq R$ be the ideal of maximal minors of X.

The example code below displays the calculation of the Frobenius root $\mathcal{C}_R^1 \cdot I^4$, using newFrobeniusRoot as well as the method frobeniusRoot from the TestIdeals package. We refer the reader interested in this Macaulay2 package to [FHK+19].

```
load "Monomialize.m2"
load "NewFrobeniusRoots.m2"
loadPackage "TestIdeals"

-- Variables
m = 3                                  -- Rows
n = 4                                  -- Columns
p = 3                                  -- Characteristic
e = 1                                  -- Exponent
pe = p^e;
R = GF(p)[x_1..x_(m*n)];               -- Polynomial ring over Fp
X = genericMatrix(R, m, n);            -- Matrix of indeterminates
I = minors(3, X);                      -- Ideal of minors
nu = (n - m + 1)*(p^e - 1);            -- nu-invariant

-- Power of the ideal of minors and generators
J = I^nu;
gensIdeal = first entries gens J; -- Generators of the ideal

-- Computation of Frobenius roots
froot1 = newFrobeniusRoot(e, J);  -- New Frobenius root
froot2 = frobeniusRoot(e, J);      -- TestIdeals package
```