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q-Functions and Distributions, Operational and Umbral Methods

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Abstract: The use of non-standard calculus means have been proven to be extremely powerful for studying old and new properties of special functions and polynomials. These methods have helped to frame either elementary and special functions within the same logical context. Methods of Umbral and operational calculus have been embedded in a powerful and efficient analytical tool, which will be applied to the study of the properties of distributions such as Tsallis, Weibull and Student's. We state that they can be viewed as standard Gaussian distributions and we take advantage of the relevant properties to infer those of the aforementioned distributions.

1. Introduction

In a previous paper, the umbral formalism has been exploited to express the functions associated with q-calculus in terms of elementary functions [1]. We have, in particular, seen how q-Bessel type functions [2] can be formally expressed in terms of ordinary exponential or Gaussian functions [3]. The idea put forward in these studies has been that of developing a slight variant of the Rota–Roman Umbral Calculus (*UC*) [4,5], has been that of merging algebraic, operational and *UC* tools to get the Indicial Umbral Calculus (*IUC*) [6], which allows significant advantages in computation.

To better frame the discussion we consider the cylindrical q-Bessel functions of n-th order in [7].

Definition 1. *We introduce the cylindrical q-Bessel functions of n-th order $J_n(x)_q$ through the series expansion:*

$$J_n(x)_q = \sum_{k=0}^{\infty} \frac{(-1)^k}{[k]_q! \, [k+n]_q!} \left(\frac{x}{2}\right)^{2k+n}, \qquad \forall n \in \mathbb{N}, \forall x \in \mathbb{R}, \forall q \in \mathbb{R} : q \neq 0, \tag{1}$$

where

$$[n]_q = \frac{q^n - 1}{q - 1}, \qquad 0 < q < 1 , \tag{2}$$

and

$$[n]_q! = \begin{cases} 1 & n = 0 \\ \prod_{r=1}^{n} [r]_q & n \geq 1 \end{cases}. \tag{3}$$

The corresponding q-Gamma function [7–10] extending the notion of q-factorial to non-integers is given by:

$$\Gamma_q(x) = (1-q)^{1-x} \prod_{s=0}^{\infty} \frac{1-q^{s+1}}{1-q^{s+x}}, \qquad \forall q \in \mathbb{R} : \mid q \mid < 1 , \tag{4}$$

and for non-negative integers n it is verified that:

$$\Gamma_q(n) = [n-1]_q! \tag{5}$$

as for the ordinary factorial.

In the following proposition we will use the Umbral Claculus, for an extensive discussion of the method, see [3].

Proposition 1. *Considering the case with $n = 0$, we write the umbral image of the q-Bessel function of 0-order $J_0(x)_q$ as:*

$$J_0(x)_q = e^{-\hat{c}_q \left(\frac{x}{2}\right)^2} \phi_0, \qquad \forall x, q \in \mathbb{R} : \mid q \mid < 1 , \tag{6}$$

where ϕ_0 is the so-called umbral vacuum [3] and $\hat{c}_q$ is an umbral operator defined in such a way that:

$$\hat{c}_q^{\,k}\, \phi_0 = \frac{\Gamma(k+1)}{\left(\Gamma_q(k+1)\right)^2}, \qquad \forall k, q \in \mathbb{R} : \mid q \mid < 1. \tag{7}$$

Proof. $\forall x, k, q \in \mathbb{R} : \mid q \mid < 1$, by using Equations (1), (5) and (7), we get:

$$J_0(x)_q = \sum_{k=0}^{\infty} \frac{(-1)^k}{[k]_q!^2}\left(\frac{x}{2}\right)^{2k} = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(k+1)}{k!\left(\Gamma_q(k+1)\right)^2}\left(\frac{x}{2}\right)^{2k} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \hat{c}_q^k \left(\frac{x}{2}\right)^{2k} \phi_0 = e^{-\hat{c}_q\left(\frac{x}{2}\right)^2}\phi_0 .$$

$\square$

Example 1. *A paradigmatic example is the derivation of q-Bessel integrals so we find, for example (we checked Equation (8) numerically and we found that it is verified for $0.8 < q < 1$. For lower values of q, numerical instabilities in the integration procedure prevent a safe conclusion), the result below.*

$$\int_{-\infty}^{\infty} J_0(x)_q\, dx = \int_{-\infty}^{\infty} e^{-\hat{c}_q\left(\frac{x}{2}\right)^2} dx\, \phi_0 = 2\sqrt{\pi}\hat{c}_q^{-\frac{1}{2}}\phi_0 = 2\sqrt{\pi}\frac{\Gamma\left(\frac{1}{2}\right)}{\left(\Gamma_q\left(\frac{1}{2}\right)\right)^2} = \frac{2\pi}{\left(\Gamma_q\left(\frac{1}{2}\right)\right)^2}. \tag{8}$$

These few remarks summarize the essential features of the formalism, which will be exploited in the forthcoming parts of the paper.

2. *q*-Distributions

The *q*-Gaussian Tsallis and the *q*-Weibull distributions [11,12] can be profitably expressed in terms of umbral forms. Within this framework, they are recognized as the images of Gaussian and exponentials, respectively, by the use of the methods put forward in the introductory section. Leaving aside, for the moment, their statistical environment, we use this section to establish the relevant mathematical properties within this "unconventional" context. We will see how other type of distributions, involving, for example, the Weibull distributions [13] or generalized forms of Dirac or Bose–Einstein distribution, can be mathematically treated by the use of techniques borrowed from the operational calculus [6].

Before entering the specific topics of this paper we should remember that functions like those belonging to the Bessel family can be "downgraded" to Gaussian like forms, while Lorentz distributions can be viewed as Gaussians or vice-versa. This is a by-product of the *Indicial Umbral Calculus* (*IUC*), originally developed in [3,6,8–10,13].

IUC is a different flavor of the calculus by Roman and Rota [4,5], embedding operational and algebraic methods which push its computational capabilities a little bit further. We summarize here the mathematical tools we are going to use in the following; in particular we start from the following example, involving an umbral restyling of the Lorentz function.

Example 2 (*q*-exponential). *Let $e_q(x)$ be the q-exponential*

$$e_q(x) = [1 + (1-q)x]^{\frac{1}{1-q}}, \qquad -\infty < q < 1. \tag{9}$$

In the following, we introduce non conventional distributions which can be written in terms of the q-exponential through the function $e_Q(-x^2)$ with $Q = 1 - q$.

The use of the Newton binomial formula yields the following series expansion for the function defined in Equation (9).

$$e_Q(-x^2) = \Gamma\left(1 + Q^{-1}\right) \sum_{s=0}^{\infty} \frac{(-1)^s x^{2s}}{s!} \frac{Q^s}{\Gamma(1 + Q^{-1} - s)}, \qquad \forall x \in \mathbb{R} :\mid x \mid< \frac{1}{1-q}. \tag{10}$$

It should be noted that the above series converges for $\mid \sqrt{Q} \mid \leq 1$, which coincides with the interval of definition of the function in Equation (9). In Ref. [3], it has been suggested the following q-exponential umbral images

$$e_Q(-x^2) = e^{-\hat{d}x^2} \psi_0 , \tag{11}$$

where the umbral operator $\hat{d}$ and the associated vacuum ψ_0 [4,5], are defined in such a way that

$$\hat{d}^\alpha \psi_0 = Q^\alpha \frac{\Gamma\left(1 + Q^{-1}\right)}{\Gamma(1 + Q^{-1} - \alpha)}, \qquad \forall \alpha \in \mathbb{R}. \tag{12}$$

indeed, by expanding the exponential in Equation (11), we find:

$$e^{-\hat{d}x^2} \psi_0 = \sum_{s=0}^{\infty} \frac{(-1)^s x^{2s}}{s!} \hat{d}^s \psi_0 = \sum_{s=0}^{\infty} \frac{(-1)^s x^{2s} Q^s}{s!} \frac{\Gamma\left(1 + Q^{-1}\right)}{\Gamma(1 + Q^{-1} - s)} = e_Q(-x^2). \tag{13}$$

The same result can be obtained by using the Newton Binomial expansion of the q-exponential defined in Equation (9).

Example 3. *According to the rules of the umbral calculus (12), namely treating $\hat{d}$ as an ordinary algebraic quantity, we can work out the associated integral as (the change in the bounds of the integral is due to the fact that the umbral image of $e_Q(-x^2)$ is a Gaussian and $\hat{d}$ is treated as an ordinary constant. Within this representation, x is not limited within the bounds specified by the extreme of the integral. The previous assumptions are better expressed by the scheme reported below $e_Q(-x^2) \to e_0(\hat{d}x^2)$ and with $\lim\limits_{q\to 1} Q^{-\frac{1}{2}} = \infty$.)*

$$\int_{-\frac{1}{\sqrt{1-q}}}^{\frac{1}{\sqrt{1-q}}} e_Q(-x^2)\, dx = \int_{-\infty}^{\infty} e^{-\hat{d}x^2}\, dx\, \psi_0 = \sqrt{\pi}\, \hat{d}^{-\frac{1}{2}} \psi_0 = \sqrt{\frac{\pi}{Q}} \frac{\Gamma\left(1 + Q^{-1}\right)}{\Gamma\left(\frac{3}{2} + Q^{-1}\right)}. \tag{14}$$

We accordingly introduce the so-called Tsallis distribution:

$$T_Q(x,\sigma) = \frac{1}{\sqrt{2\,\pi_Q}\,\sigma}\, e_Q\left(-\frac{x^2}{2\sigma^2}\right), \qquad \pi_Q = \frac{\pi}{Q}\left(\frac{\Gamma\left(1 + Q^{-1}\right)}{\Gamma\left(\frac{3}{2} + Q^{-1}\right)}\right)^2. \tag{15}$$

By Tsalis distribution we indicate a q-exponential distribution of the type introduced in [11,12] to study generalizations of the Boltzmann–Gibbs entropy.

The umbral formalism offers a noticeable flexibility and simplicity in its mathematical handling. The associated moments are therefore easily obtained but, before getting their explicit form, we consider the determination of the successive derivatives of $e_Q(-x^2)$, which are computed as follows.

Proposition 2. *Let*

$$H_m(x,y) = m! \sum_{r=0}^{\lfloor \frac{m}{2} \rfloor} \frac{x^{m-2r} r^r}{(m-2r)! r!}, \qquad \forall x,y \in \mathbb{R}, \forall m \in \mathbb{N}, \tag{16}$$

be the two variable Hermite polynomials [14,15] with the property [3,6]

$$\partial_x^n e^{-ax^2} = H_n(-2ax, -a) e^{-ax^2}. \tag{17}$$

The successive derivatives of $e_Q(-x^2)$ can be expressed in terms of two variables Hermite polynomials

$$\partial_x^m e_Q(-x^2) = H_m(-2\hat{d}x, -\hat{d}) e^{-\hat{d}x^2} \psi_0, \tag{18}$$

and eventually

$$\partial_x^m e_Q(-x^2) = (-1)^m m! \sum_{r=0}^{\lfloor \frac{m}{2} \rfloor} \frac{(2x)^{m-2r}(-1)^r Q^{m-r}}{(m-2r)! r!} \; {}_{m-r}e_Q(-x^2). \tag{19}$$

Proof. By using Equations (11) and (17), we can recast the successive derivatives of $e_Q(-x^2)$ as follows:

$$\partial_x^m e_Q(-x^2) = \partial_x^m e^{-\hat{d}x^2} \psi_0 = H_m(-2\hat{d}x, -\hat{d}) e^{-\hat{d}x^2} \psi_0, \qquad \forall x \in \mathbb{R}.$$

We expand the last term of previous equation and apply the umbral calculus:

$$H_m(-2\hat{d}x, -\hat{d}) e^{-\hat{d}x^2} \psi_0 = (-1)^m m! \sum_{r=0}^{\lfloor \frac{m}{2} \rfloor} \frac{(-1)^r (2x)^{m-2r}}{(m-2r)! r!} \hat{d}^{m-r} \psi_0 \, e^{-\hat{d}x^2}$$

$$= (-1)^m m! \sum_{r=0}^{\lfloor \frac{m}{2} \rfloor} \frac{(-1)^r (2x)^{m-2r}}{(m-2r)! r!} \sum_{s=0}^{\infty} \frac{(-1)^s x^{2s}}{s!} \hat{d}^{s+m-r} \psi_0.$$

We define the q-exponential of order k as:

$$_k e_Q(-x^2) := \sum_{s=0}^{\infty} \frac{(-1)^s x^{2s}}{s!} \hat{d}^{s+k} \psi_0, \tag{20}$$

which, by using Equation (12), becomes

$$\partial_x^m e_Q(-x^2) = H_m(-2\hat{d}x, -\hat{d}) e^{-\hat{d}x^2} \psi_0 = (-1)^m m! \sum_{r=0}^{\lfloor \frac{m}{2} \rfloor} \frac{(-1)^r (2x)^{m-2r}}{(m-2r)! r!} \sum_{s=0}^{\infty} \frac{(-1)^s x^{2s}}{s!} \hat{d}^{s+m-r} \psi_0 =$$

$$= (-1)^m m! \sum_{r=0}^{\lfloor \frac{m}{2} \rfloor} \frac{(-1)^r (2x)^{m-2r}}{(m-2r)! r!} \sum_{s=0}^{\infty} \frac{(-1)^s x^{2s}}{s!} Q^{s+m-r} \frac{\Gamma(1+Q^{-1})}{\Gamma(1+Q^{-1}-s-m+r)} = \tag{21}$$

$$= (-1)^m m! \sum_{r=0}^{\lfloor \frac{m}{2} \rfloor} \frac{(-1)^r (2x)^{m-2r}}{(m-2r)! r!} \; {}_{m-r}e_Q(-x^2)$$

(we note that the function $_k e_Q(-x^2)$ can be expressed through other series too): $_k e_Q(-x^2) = \Gamma(1+Q^{-1}) \sum_{s=0}^{\infty} \frac{(-1)^s x^{2s}}{s!} \frac{Q^s}{\Gamma(1+Q^{-1}-(s+k))}$. $\square$

Let us now look at the problem of evaluating the "momenta" of the q-Tsallis distribution. For this aim, we propose the following example.

Example 4. *We define the quantities*

$$I_n(\delta) = \int_{-\infty}^{\infty} (x + \delta)^n e^{-\hat{d}x^2} \, dx \, \psi_0, \qquad \forall n \in \{\mathbb{N} \setminus 0\} \tag{22}$$

where δ is a constant, introduced for future convenience. The integrals $I_n(\delta)$ are easily obtained by using, for example, the method of the generating function, thus finding:

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} I_n(\delta) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \int_{-\infty}^{\infty} (x + \delta)^n e^{-\hat{d}x^2} \, dx \, \psi_0 = \int_{-\infty}^{\infty} e^{(x+\delta)t} e^{-\hat{d}x^2} \, dx \, \psi_0 = e^{\delta t} \sqrt{\frac{\pi}{\hat{d}}} \, e^{\frac{t^2}{4\hat{d}}} \, \psi_0. \tag{23}$$

According to the Hermite generating function [6,15]:

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} H_k(x, y) = e^{xt + yt^2}, \tag{24}$$

we obtain

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} I_n(\delta) = \sqrt{\pi} \, \hat{d}^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n\left(\delta, \frac{\hat{d}^{-1}}{4}\right) \psi_0. \tag{25}$$

By equating the t-like power terms, we find

$$\begin{aligned}
I_n(\delta) &= \sqrt{\pi} \, n! \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \frac{\delta^{n-2r}}{(4)^r (n-2r)! r!} \hat{d}^{-\left(r+\frac{1}{2}\right)} \psi_0 \\
&= \sqrt{\frac{\pi}{Q}} n! \Gamma\left(1 + Q^{-1}\right) \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \frac{\delta^{n-2r}}{(4Q)^r (n-2r)! r! \Gamma\left(\frac{3}{2} + Q^{-1} + r\right)}.
\end{aligned} \tag{26}$$

We have so far provided an idea of how q and *IUC* can be embedded to obtain results of practical interest. In the forthcoming section, we will further extend the discussion by including more general examples.

3. Final Comments

The q-Weibull distribution writes in terms of the q- exponential as:

$$f(x; q, \lambda, \kappa) = \begin{cases} (2 - q)\dfrac{\kappa}{\lambda}\left(\dfrac{x}{\lambda}\right)^{\kappa-1} e_q\left(-\left(\dfrac{x}{\lambda}\right)^{\kappa}\right) & x \geq 0, \\ 0 & x < 0, \end{cases} \tag{27}$$

where $q < 1, \kappa > 0$ is shape and $\lambda > 0$ is the scale parameter, respectively. The use of the umbral formalism allows to cast (27) in the form:

$$f(x; q, \lambda, \kappa) = (1 + Q)\frac{\kappa}{\lambda}\left(\frac{x}{\lambda}\right)^{\kappa-1} e^{-\left(\frac{1}{Q}-1\right)\hat{d}\left(\frac{x}{\lambda}\right)^{\kappa}} \psi_0. \tag{28}$$

The average values and other specific quantities are easily obtained using the Umbral formalism. We note indeed that:

$$\langle x^m \rangle = (1+Q)\frac{\kappa}{\lambda}\frac{1}{\lambda^{\kappa-1}}\int_0^\infty x^{\kappa+m-1}e^{-\left(\frac{1}{Q}-1\right)\hat{d}\left(\frac{x}{\lambda}\right)^\kappa}dx\ \psi_0 =$$

$$= (1+Q)\kappa\lambda^m\int_0^\infty \sigma^{\kappa+m-1}e^{-\left(\frac{1}{Q}-1\right)\hat{d}\sigma^\kappa}d\sigma\ \psi_0 =$$

$$= (1+Q)\kappa\lambda^m\frac{\left(\left(\frac{1}{Q}-1\right)\hat{d}\right)^{-\left(1+\frac{m}{\kappa}\right)}\left(\Gamma\left(1+\frac{m}{\kappa}\right)\right)}{\kappa}\psi_0 =$$

$$= (1+Q)\lambda^m(1-Q)^{-\left(1+\frac{m}{\kappa}\right)}\Gamma\left(1+\frac{m}{\kappa}\right)\frac{\Gamma\left(1+Q^{-1}\right)}{\Gamma\left(2+Q^{-1}+\frac{m}{\kappa}\right)} =$$

$$= (2-q)\lambda^m q^{-\left(1+\frac{m}{\kappa}\right)}\left(\Gamma\left(1+\frac{m}{\kappa}\right)\right)\frac{\Gamma\left(1+(1-q)^{-1}\right)}{\Gamma\left(2+(1-q)^{-1}+\frac{m}{\kappa}\right)} =$$

$$= (2-q)\lambda^m q^{-\left(1+\frac{m}{\kappa}\right)}B\left(1+\frac{m}{\kappa},1+(1-q)^{-1}\right). \tag{29}$$

We have mentioned the *q*-exponential in Equation (9), with the *q* parameter bounded in $-\infty < q < 1$. This interval is not unique and indeed it is also defined within the limits $1 < q < 3$. In this case, a non umbral treatment can be efficient as well.

By making use of the Laplace transform identity,

$$x^{-\mu} = \frac{1}{\Gamma(\mu)}\int_0^\infty e^{-sx}s^{\mu-1}ds, \tag{30}$$

we can write the *q*-Gaussian as:

$$e_q(-x^2) = (1+\mid -A\mid x^2)^{-\frac{1}{A}} = \frac{1}{\Gamma(A^{-1})}\int_0^\infty e^{-s(1+Ax^2)}s^{A^{-1}-1}ds, \qquad A = q - 1. \tag{31}$$

The relevant calculation can be done again by exploiting the properties of the Gaussian integrals, thus finding, for example:

$$\int_{-\infty}^\infty e_q(-x^2)dx = \frac{1}{\Gamma(A^{-1})}\int_0^\infty e^{-s}s^{A^{-1}-1}\left(\int_{-\infty}^\infty e^{-sAx^2}dx\right)ds = \frac{\sqrt{\pi}}{\sqrt{A}\ \Gamma(A^{-1})}\int_0^\infty e^{-s}s^{A^{-1}-\frac{3}{2}} =$$

$$= \frac{\sqrt{\pi}}{\sqrt{A}\ \Gamma(A^{-1})}\Gamma\left(A^{-1}-\frac{1}{2}\right), \qquad 1 < q < 3. \tag{32}$$

Without entering further comments on the Laplace transform treatment of *q*-exponentials, we like to mention the *t*-Students distribution, which, albeit amenable for an umbral calculus restyling, can be profitably studying the last outlined technique.

Example 5. *The Student's t-distribution is used in statistics to obtain the mean value of a normal distribution when the available sample is small and the standard deviation is not available. The associated probability density function writes [16]*

$$S(t;\nu) = \frac{1}{\sqrt{\nu}\ B\left(\frac{1}{2},\frac{\nu}{2}\right)}\left(1+\frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}}, \tag{33}$$

where $B(a,b)$ is the Euler B-function [14]. The use of the Laplace transform method [6] allows us to cast Equation (33) in the form:

$$S(t;\nu) = \frac{1}{\sqrt{\nu}\ B\left(\frac{1}{2},\frac{\nu}{2}\right)\Gamma\left(\frac{\nu+1}{2}\right)}\int_0^\infty e^{-s\left(1+\frac{t^2}{\nu}\right)}s^{\frac{\nu-1}{2}}ds, \tag{34}$$

which can be exploited to derive the relevant properties in straightforward terms.

We can prove that the Student's distribution in the form given in Equation (34) *is correctly normalized by keeping the relevant infinite integral which yields:*

$$
\int_{-\infty}^{\infty} S(t; \nu) dt = \frac{1}{\sqrt{\nu}\, B\left(\frac{1}{2}, \frac{\nu}{2}\right) \Gamma\left(\frac{\nu+1}{2}\right)} \int_0^{\infty} \left(\int_{-\infty}^{\infty} e^{-s\frac{t^2}{\nu}} dt \right) e^{-s} s^{\frac{\nu-1}{2}} ds =
$$
$$
= \frac{\sqrt{\pi}}{B\left(\frac{1}{2}, \frac{\nu}{2}\right) \Gamma\left(\frac{\nu+1}{2}\right)} \int_0^{\infty} e^{-s} s^{\frac{\nu}{2}-1} ds = \frac{\sqrt{\pi}}{B\left(\frac{1}{2}, \frac{\nu}{2}\right) \Gamma\left(\frac{\nu+1}{2}\right)} \Gamma\left(\frac{\nu}{2}\right) = 1.
$$

(35)

This is just an illustration of how the technique works.

The calculus of higher order distribution moments proceeds along the same lines and we find, for example,

$$
m_2 = \int_{-\infty}^{\infty} t^2 S(t; \nu) dt = \frac{1}{2} \frac{\nu\, \Gamma\left(\frac{\nu}{2}-1\right)}{\Gamma\left(\frac{\nu}{2}\right)} = \frac{\nu}{\nu-2}\,, \qquad \forall \nu \in \mathbb{R} : \nu \neq 2.
$$

(36)

In this paper, we have gone through the use of umbral and operational methods applied to q-functions. We have indicated a general procedure to obtain detailed information from the distributions, in terms of momenta (including those of higher order). We have also worked out a general method for dealing with the properties of q-Bessel functions by indicating a fairly straightforward procedure to calculate the relevant integrals. In a forthcoming paper, we will extend this method of analysis to a wider class of q-functions.

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