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Umbral Methods and Harmonic Numbers

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Abstract: The theory of harmonic-based functions is discussed here within the framework of umbral operational methods. We derive a number of results based on elementary notions relying on the properties of Gaussian integrals.

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1. Introduction

Methods employing the concepts and the formalism of umbral calculus have been exploited in [1] to guess the existence of generating functions involving harmonic numbers [2]. The conjectures put forward in [1] have been proven in [3,4] and further elaborated in [5], and these were extended to hyper-harmonic numbers in [6].

In this note, we use the same point of view as [1], by discussing the possibility of exploiting the formalism developed therein in a wider context.

The umbral methods we are going to describe have certain advantages with respect to the ordinary techniques. The key idea is that of exploiting the harmonic number index as a power exponent; such a "promotion" allows the possibility of reducing the associated computational technicalities to elementary algebraic manipulations. Series involving harmonic numbers can, e.g., be treated as formal series of known functions (exponential, Gaussian, rational, etc.), and the relevant properties can be exploited to carry out computations, which are significantly more cumbersome and involved when conventional methods are employed.

2. Harmonic Numbers and Generating Functions

The harmonic numbers are defined by means of the following partial sum [2]:

$$h_n := \sum_{r=1}^{n} \frac{1}{r}, \quad \forall n \in \mathbb{N}_0 . \tag{1}$$

The integral representation for this family of numbers can be derived using a standard procedure, tracing back to Euler, which is sketched below.

Proposition 1. *The use of elementary integral transform yields, for the finite sum in Equation (1), the identity:*

$$h_n = \sum_{r=1}^{n} \int_0^\infty e^{-s\,r} ds, \quad \forall n \in \mathbb{N}_0 , \tag{2}$$

thereby getting the n-th harmonic number through Euler's integral [7–9]:

$$h_n = \int_0^1 \frac{1-x^n}{1-x} dx. \tag{3}$$

Proof. $\forall n \in \mathbb{N}_0$, by applying the Laplace transform, the theorem of uniform convergence and the sum of a geometric series, we obtain:

$$\begin{aligned} h_n &= \sum_{r=1}^{n} \int_0^\infty e^{-sr} ds = \int_0^\infty \left[\left(\sum_{r=0}^{n} e^{-sr} \right) - 1 \right] ds \\ &= \int_0^\infty \frac{1-(e^{-s})^{n+1}}{1-e^{-s}} - 1\, ds = \int_{-\infty}^0 \frac{1-(e^s)^{n+1}}{1-e^s} - 1\, ds \\ &= \int_{-\infty}^0 \frac{e^{(n+1)s} - e^s}{e^s - 1} ds \end{aligned}$$

and by applying the change of variables $e^s \to x$, we eventually end up with:

$$h_n = \int_0^1 \frac{1-x^n}{1-x} dx.$$

□

According to [8], from this point onwards, the definition in Equation (3) can be so extended to any real value of n, and therefore, it can be exploited as an alternative definition holding for n a positive real.

Definition 1. *The function:*

$$\varphi_h(z) := \int_0^1 \frac{1-x^z}{1-x} dx, \quad \forall z \in \mathbb{R}^+, \tag{4}$$

is called the harmonic number umbral vacuum, or simply the vacuum.

Definition 2. *The operator:*

$$\hat{h} := e^{\partial_z} \tag{5}$$

realizes the vacuum shift operator, zbeing the domain's variable of the function on which the operator acts For a deeper introduction to umbral calculus, see [10,11]

Theorem 1. *The umbral operator, $\hat{h}^n$, $\forall n \in \mathbb{R}^+$, defines the harmonic numbers, h_n, as the action of the shift operator (5) on the vacuum (4):*

$$\hat{h}^n \varphi_h(z)\Big|_{z=0} := \hat{h}^n \varphi_{h_z}\Big|_{z=0} = h_n \tag{6}$$

or simply:

$$\begin{aligned} &\hat{h}^n \equiv h_n, \\ &h_0 = 0. \end{aligned} \tag{7}$$

Proof. $\forall n \in \mathbb{R}^+$, by applying the shift operator (5) on the vacuum (4), we obtain:

$$\begin{aligned} \hat{h}^n \varphi_{h_0} &= \hat{h}^n \varphi_{h_z}\Big|_{z=0} = e^{n\partial_z} \varphi_{h_z}\Big|_{z=0} = \varphi_{h_{z+n}}\big|_{z=0} = \int_0^1 \frac{1-x^{z+n}}{1-x} dx\Bigg|_{z=0} \\ &= \int_0^1 \frac{1-x^n}{1-x} dx = h_n. \end{aligned}$$

□

Properties 1. *$\forall n, m \in \mathbb{R}^+$, we get:*

$$\begin{aligned} &(i) \quad \hat{h}^n \hat{h}^m = \hat{h}^{n+m} \,, \\ &(ii) \left(\hat{h}^n\right)^m = \hat{h}^{nm} \,. \end{aligned} \tag{8}$$

The proof is a fairly direct consequence of the realization given in Equation (5).

Definition 3. *We call the Harmonic-Based Exponential Function (HBEF) the series:*

$$_h e(x) := e^{\hat{h}\,x}\varphi_{h_0} = 1 + \sum_{n=1}^{\infty} \frac{h_n}{n!}\, x^n. \tag{9}$$

This function, as already discussed in [1], has quite remarkable properties.

The relevant derivatives can accordingly be expressed as (see the concluding part of the paper for further comments):

$$\begin{aligned} &_h e(x, m) := \left(\frac{d}{dx}\right)^m {}_h e(x) = \hat{h}^m e^{\hat{h}\,x}\varphi_{h_0} = h_m + \sum_{n=1}^{\infty} \frac{h_{n+m}}{n!}\, x^n, \quad \forall x \in \mathbb{R}, \forall m \in \mathbb{N} \\ &_h e(x, k+m) = \left(\frac{d}{dx}\right)^m {}_h e(x, k), \quad \forall k \in \mathbb{N}, \end{aligned} \tag{10}$$

and according to Equation (9), we also find that:

$$\int_0^\infty {}_h e(-\alpha\, x)\, e^{-x} dx = \int_0^\infty e^{-(\alpha\,\hat{h}+1)\,x} dx = \frac{1}{\alpha\,\hat{h}+1}, \quad |\ \alpha\ | < 1. \tag{11}$$

Corollary 1. *By expanding the umbral function on the r.h.s. of Equation (11), we obtain:*

$$\frac{1}{\alpha\,\hat{h}+1} = 1 + \sum_{n=1}^{\infty} (-1)^n \alpha^n h_n, \quad |\ \alpha\ | < 1. \tag{12}$$

Proof. By using the Taylor expansion and Equation (7), for $|\ \alpha\ | < 1$, we have:

$$\frac{1}{\alpha\,\hat{h}+1} = \sum_{n=0}^{\infty} (-\alpha\hat{h})^n = 1 + \sum_{n=1}^{\infty} (-1)^n \alpha^n \hat{h}^n = 1 + \sum_{n=1}^{\infty} (-1)^n \alpha^n h_n,$$

□

This is an expected conclusion, achievable by direct integration, underscored here to stress the consistency of the procedure.

A further interesting example comes from the following "Gaussian" integral.

$$\int_{-\infty}^{\infty} {}_h e(-\alpha\, x)\, e^{-x^2} dx = \int_{-\infty}^{\infty} e^{-(\alpha\,\hat{h}\,x + x^2)}\, dx = \sqrt{\pi} e^{\frac{\alpha^2\,\hat{h}^2}{4}} \ \forall \alpha \in \mathbb{R}. \tag{13}$$

The last term in Equation (13) has been obtained by treating $\hat{h}$ as an ordinary algebraic quantity and then by applying the standard rules of the Gaussian integration.

We notice that, using Equation (9), we obtain:

$$_{h^2} e\left(\frac{\alpha^2}{4}\right) := e^{\frac{\hat{h}^2 \alpha^2}{4}} \varphi_{h_0} = 1 + \sum_{r=1}^{\infty} \frac{h_{2r}}{r!} \left(\frac{\alpha}{2}\right)^{2\,r}. \tag{14}$$

Let us now consider the following slightly more elaborate example, involving the integration of two "Gaussians", namely the ordinary case and its analogous HBEF.

Example 1.

$$\int_{-\infty}^{\infty} {}_{h}e(-\alpha\, x^2)\, e^{-x^2} dx = \int_{-\infty}^{\infty} e^{-(\hat{h}\,\alpha+1)x^2}\, dx\ \varphi_{h_0} = \sqrt{\frac{\pi}{1+\alpha\,\hat{h}}}\varphi_{h_0}, \quad |\ \alpha\ | < 1. \tag{15}$$

This last result, obtained after applying elementary rules, can be worded as follows: the integral in Equation (15) depends on the operator function on its r.h.s., for which we should provide a computational meaning. The use of the Newton binomial yields:

$$\begin{aligned}\sqrt{\frac{\pi}{1+\alpha\hat{h}}}\varphi_{h_0} &= \sqrt{\pi}\sum_{r=0}^{\infty}\binom{-\frac{1}{2}}{r}\left(\alpha\,\hat{h}\right)^r\varphi_{h_0} = \sqrt{\pi}\left(1+\sqrt{\pi}\sum_{r=1}^{\infty}\frac{\alpha^r h_r}{\Gamma\left(\frac{1}{2}-r\right)\,r!}\right)\\ &= \sqrt{\pi}\left(1+\sum_{r=1}^{\infty}\binom{2r}{r}\frac{(-\alpha)^r\,h_r}{2^{2r}}\right),\\ &|\alpha| < 1\,.\end{aligned} \tag{16}$$

The correctness of this conclusion has been confirmed by the numerical check, as well.

It is evident that the examples we have provided show that the use of concepts borrowed from umbral theory offers a fairly powerful tool to deal with the "harmonic-based" functions.

3. Harmonic-Based Functions and Differential Equations

In the following, we will further push the formalism to stress the associated flexibility.

We note indeed that the function:

$${}_{\sqrt{h}}e(x) := e^{\hat{h}^{\frac{1}{2}}\,x}\varphi_{h_0} = 1+\sum_{n=1}^{\infty}\frac{\left(\sqrt{\hat{h}}\,x\right)^n}{n!}\varphi_{h_0} = 1+\sum_{n=1}^{\infty}\frac{h_{n/2}}{n!}\,x^n, \forall x\in\mathbb{R}, \tag{17}$$

defines, $\forall\alpha\in\mathbb{R}$, an HBEF through the following Gauss transform:

$$\int_{-\infty}^{+\infty} {}_{\sqrt{h}}e(\alpha\,x)\,e^{-x^2}dx = \int_{-\infty}^{+\infty} e^{\hat{h}^{\frac{1}{2}}\,\alpha\,x-x^2}dx\ \varphi_{h_0} = \sqrt{\pi}e^{\hat{h}\,\left(\frac{\alpha}{2}\right)^2}\,\varphi_{h_0} = \sqrt{\pi}\ {}_{h}e\left(\left(\frac{\alpha}{2}\right)^2\right). \tag{18}$$

On the other side, Equation (17) can be expressed in terms of the HBEF, ${}_{h}e(x)$, using appropriate integral transform methods [12].

Definition 4. *If:*

$$g_{\frac{1}{2}}(\eta) = \frac{1}{2\sqrt{\pi\eta^3}}e^{-\frac{1}{4\eta}}, \quad \forall\eta\in\mathbb{R}^+, \tag{19}$$

is the Levy distribution of order $\frac{1}{2}$, *then [12]:*

$$e^{-p^{\frac{1}{2}}\,x} = \int_0^{\infty} e^{-p\,\eta\,x^2}\,g_{\frac{1}{2}}(\eta)\,d\eta\,, \quad \forall p\in\mathbb{R}^+, \tag{20}$$

is the associated Levy integral transform.

The use of Equations (17) and (19) allows us to write the following identity.

Corollary 2.

$$_{\sqrt{h}}e(-x) = \int_0^\infty {}_he(-\eta\, x^2)\, g_{\frac{1}{2}}(\eta)\, d\eta. \tag{21}$$

Proof.

$$_{\sqrt{h}}e(-x) = e^{-\hat{h}^{\frac{1}{2}}x}\varphi_{h_0} = \int_0^\infty e^{-\hat{h}\eta x^2} g_{\frac{1}{2}}(\eta)\, d\eta\, \varphi_{h_0} = \int_0^\infty {}_he(-\eta\, x^2)\, g_{\frac{1}{2}}(\eta)\, d\eta\, .$$

□

The possibility of defining $_{\sqrt[k]{h}}e(x)$ will be discussed elsewhere.

Theorem 2. *The function $_he(x)$ satisfies the first order non-homogeneous differential equation:*

$$\begin{cases} {}_he'(x) = \dfrac{d}{dx}{}_he(x) = {}_he(x) + \dfrac{e^x - x - 1}{x}, & \forall x \in \mathbb{R}_0, \\ {}_he(0) = 1. \end{cases} \tag{22}$$

Proof. Equation (10), for $m = 1$, yields:

$$_he'(x) =_h e(x, 1) = 1 + \sum_{n=1}^\infty \frac{h_{n+1}}{n!}\, x^n\, . \tag{23}$$

Since $h_{n+1} = h_n + \frac{1}{n+1}$, we find:

$$1 + \sum_{n=1}^\infty \frac{h_{n+1}}{n!}\, x^n = {}_he(x) + \frac{1}{x}\,(e^x - x - 1) \tag{24}$$

and hence, Equation (22) follows. □

Corollary 3. *The solution of Equation (22) yields for the HBEF the explicit expression in terms of ordinary special functions $\forall x \in \mathbb{R}^+$:*

$$\begin{aligned} &{}_he(x) = 1 + e^x\,(\ln(x) + E_1(x) + \gamma)\,, \\ &E_1(x) = \int_x^\infty \frac{e^{-t}}{t} dt, \\ &(\ln(x) + E_1(x) + \gamma) = -\sum_{n=1}^\infty \frac{(-x)^n}{n\, n!}, \end{aligned} \tag{25}$$

where γ is the Euler–Mascheroni–constant .

The previous expression is the generating function of harmonic numbers originally derived by Gosper (see [2,13]).

By iterating the previous procedure, we find the following general recurrence.

Corollary 4.

$$_he(x, m) = {}_he(x) + \sum_{r=0}^{m-1} \left(\frac{d}{dx}\right)^r \frac{e^x - 1 - x}{x}\, . \tag{26}$$

Definition 5. *The binomial expansion:*

$$h_n(x) := (x + \hat{h})^n \varphi_{h_0} = x^n + \sum_{s=1}^n \binom{n}{s}\, x^{n-s}\, h_s, \quad \forall x \in \mathbb{R}, \forall n \in \mathbb{N}_0, \tag{27}$$

specifies the harmonic polynomials.

They are easily shown to be linked to the HBEF by means of the generating function as follows.

Corollary 5.

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} h_n(x) = e^{x\,t}\,{}_h e(t), \quad \forall x, t \in \mathbb{R}. \tag{28}$$

Proof. It is readily checked that:

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} h_n(x) = \sum_{n=0}^{\infty} \frac{t^n}{n!} (x+\hat{h})^n \varphi_{h_0} = e^{t(x+\hat{h})} \varphi_{h_0} = e^{x\,t}\,{}_h e(t)\;.$$

□

According to Equation (28), $h_n(x)$ are recognized as Appél polynomials and satisfy the following recurrences.

Properties 2. *The properties below hold:*

$$(i)\;\; \frac{d}{dx} h_n(x) = n\, h_{n-1}(x), \quad \forall x \in \mathbb{R}, \tag{29}$$

$$\begin{aligned} &(ii)\;\; h_{n+1}(x) = (x+1)\, h_n(x) + f_n(x), \;, \\ &f_n(x) := \sum_{s=1}^{n} \frac{n!}{(n-s)!} \frac{x^{n-s}}{(s+1)!} = \int_0^1 (x+y)^n dy - x^n, \quad \forall x \in \mathbb{R}. \end{aligned} \tag{30}$$

Proof. The recurrence given in Equation (29) follows from the definition of the derivative itself since we treat h as an ordinary algebraic quantity. The proof of the identity (30) is slightly more elaborate; we note indeed that:

$$\begin{aligned} h_{n+1}(x) &= (x+\hat{h})(x+\hat{h})^n \varphi_{h_0} = (x+\hat{h}) \left(x^n + \sum_{s=1}^{n} \binom{n}{s} x^{n-s}\, \hat{h}^s \right) \varphi_{h_0} \\ &= x\, h_n(x) + 1 \cdot x^n + \sum_{s=1}^{n} \binom{n}{s} x^{n-s}\, \hat{h}^{s+1} \varphi_{h_0} \\ &= x\, h_n(x) + \left(x^n + \sum_{s=1}^{n} \binom{n}{s} x^{n-s}\, \hat{h}^s \right) \varphi_{h_0} + \sum_{s=1}^{n} \frac{n!\, x^{n-s}}{(n-s)!(s+1)!} \\ &= (x+1)\, h_n(x) + \sum_{s=1}^{n} \frac{n!\, x^{n-s}}{(n-s)!(s+1)!} \end{aligned}$$

and:

$$\begin{aligned} \sum_{s=1}^{n} \frac{n!}{(n-s)!} \frac{x^{n-s}}{(s+1)!} &= \sum_{s=1}^{n} \frac{n!}{s!(n-s)!} \frac{x^{n-s}}{s+1} y^{s+1} \Bigg|_{y=1} \\ &= \sum_{s=1}^{n} \binom{n}{s} x^{n-s} \int_0^1 y^s dy = \int_0^1 \left(\sum_{s=0}^{n} \binom{n}{s} x^{n-s} y^s - x^n \right) dy \\ &= \int_0^1 (x+y)^n dy - x^n\;. \end{aligned}$$

□

Corollary 6. *The identity:*

$$h_n(-1) = (-1)^n \left(1 - \frac{1}{n}\right), \quad \forall n \in \mathbb{N}, \tag{31}$$

follows from the Equation (30) *after setting* $x = -1$.

The further relationship:

$$h_n = 1 + \sum_{s=1}^{n} \binom{n}{s} h_s(-1), \quad \forall n \in \mathbb{N}_0, \tag{32}$$

is a consequence of the fact that $\hat{h}^n = ((\hat{h} - 1) + 1)^n$.

The harmonic Hermite polynomials (touched on in [1,3,14]) can also be written as follows.

Definition 6.

$$\begin{gathered}\sum_{n=0}^{\infty} \frac{t^n}{n!}\, {}_hH_n(x) = e^{x\,t}\, {}_he(t^2), \quad \forall x, t \in \mathbb{R}, \\ {}_hH_n(x) := H_n(x, \hat{h})\varphi_{h_0} = e^{\hat{h}\partial_x^2} x^n \varphi_{h_0} = x^n + n! \sum_{r=1}^{\lfloor \frac{n}{2} \rfloor} \frac{x^{n-2\,r} h_r}{(n-2\,r)!\,r!}.\end{gathered} \tag{33}$$

Properties 3. *The recurrences identity of the umbral Hermite polynomials:*

$$\begin{aligned}&(i)\ \frac{d}{dx} {}_hH_n(x) = n\, {}_hH_{n-1}(x)\,, \quad \forall x \in \mathbb{R}, \\ &(ii)\ {}_hH_{n+1}(x) = \left(x + 2\,\hat{h}\,\frac{d}{dx}\right) {}_hH_n(x)\varphi_{h_0} = \left(x + 2\frac{d}{dx}\right) {}_hH_n(x) + 2\,\alpha_n'(x)\,, \\ &\qquad \alpha_n(x) = n! \sum_{s=1}^{\lfloor \frac{n}{2} \rfloor} \frac{x^{n-2s}}{(s+1)!\,(n-2s)!}\,, \\ &\qquad \alpha_n'(x) = \frac{d}{dx}\alpha_n(x) = n\,\alpha_{n-1}(x)\,,\end{aligned} \tag{34}$$

are a by-product of the previous identities and a consequence of the monomiality principle discussed in [15].

Corollary 7. *The umbral Hermite satisfies the second order non-homogeneous ODE:*

$$\left(x\frac{d}{dx} + 2\left(\frac{d}{dx}\right)^2\right) {}_hH_n(x) = n\, {}_hH_n(x) - 2\,\alpha_n''(x). \tag{35}$$

4. Truncated Exponential Numbers and Final Comments

Before closing the paper, we want to stress the possibility of extending the present procedure to the truncated exponential numbers, namely:

$$e_n := \sum_{r=0}^{n} \frac{1}{r!}\,, \quad \forall n \in \mathbb{N}. \tag{36}$$

The relevant integral representation is written [16]:

$$e_\alpha := \frac{1}{\Gamma(\alpha+1)} \int_0^\infty e^{-s}(1+s)^\alpha ds\,, \tag{37}$$

which holds for $\alpha \in \mathbb{R}$, as well. For example, we find:

Example 2.

$$e_{-\frac{1}{2}} = \frac{e}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}, 1\right) \tag{38}$$

with $\Gamma\left(\frac{1}{2}, 1\right)$ *being the lower incomplete Gamma function.*

According to the previous discussion and to Equation (38), setting $\hat{e}^{\alpha} \leftrightarrow e_{\alpha}$, we also find that:

$$\begin{aligned} &\int_{-\infty}^{+\infty} e^{-\hat{e}\,x^2} dx = \sqrt{\pi} e_{-\frac{1}{2}}, \\ &e^{-\hat{e}\,x^2} = \sum_{r=0}^{\infty} (-1)^r \frac{e_r}{r!} x^{2\,r}\ . \end{aligned} \tag{39}$$

This last identity is a further proof that the implications offered by the topics treated in this paper are fairly interesting and deserve further and more detailed investigation, which will be more accurately treated elsewhere.

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