


Article

# Fractional Derivatives, Memory Kernels and Solution of a Free Electron Laser Volterra Type Equation

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**Abstract:** The high gain free electron laser (FEL) equation is a Volterra type integro-differential equation amenable for analytical solutions in a limited number of cases. In this note, a novel technique, based on an expansion employing a family of two variable Hermite polynomials, is shown to provide straightforward analytical solutions for cases hardly solvable with conventional means. The possibility of extending the method by the use of expansion using different polynomials (two variable Legendre like) expansion is also discussed.

**Keywords:** free electron laser (FEL); Volterra equations; iterative solutions; Hermite polynomials; Legendre polynomials

**JEL Classification:** 78A60; 45D05; 37M99; 37M99; 33C45; 33C45

## 1. Introduction

The free electron laser (FEL) is a device capable of transforming the kinetic energy of a relativistic electron beam into electromagnetic radiation.

The constituting elements of the device are a beam of high energy electrons, injected into an alternating magnetic field where they emit laser like radiation whose growth is ruled by the following integro differential equation [1–3]

$$\begin{aligned} \partial_{\tau} a &= i \pi g_0 \int_0^{\tau} \tau' e^{-i \nu \tau'} - \frac{(\pi \mu_{\epsilon} \tau')^2}{2} a(\tau - \tau') d\tau', \\ a(0) &= 1 \end{aligned} \quad (1)$$

where  $a$  represents the laser field amplitude,  $g_0$  the small signal gain coefficient,  $\nu$  is linked to the laser frequency and the coefficient  $\mu_{\epsilon}$  is a parameter regulating the effects of the gain reduction due to the electrons' energy distribution.

The FEL equation reported in (1) belongs to the Volterra integro-differential family. It has played an important role in the design of modern FEL devices, it is the most elementary version of the FEL linear intensity growth equation. More elaborated versions include the e-beam transverse and longitudinal distributions, the magnetic field inhomogeneities and so on. Equation (1) captures the essential features of the FEL dynamics itself and more complicated expressions can be treated with the methods outlined below [1–3]. The relevant kernel can be viewed as a memory operator, it is indeed associated with the nature of the physics it describes, namely the amplification of a laser field.

The fractional derivative operators can be ascribed to the same category, they are indeed exploited to represent phenomena implying dissipation of energy or damage [4–6].

Following such a point of view the authors of references [7,8] have proposed a further definition of fractional derivative without a singular kernel, defined as

$$D_{ex}f(t) = \frac{m(\alpha)}{1-\alpha} \int_{\beta}^t f(\xi) e^{-\frac{t-\xi}{1-\alpha}} d\xi \tag{2}$$

where  $m(\alpha)$  is a normalization term and  $\alpha$  constant.

Without entering into further discussion regarding the use of this derivative in applications, we note that we can define the FEL fractional (temporal) derivative as

$$D_{FEL}f(t) = \int_0^{\tau} \xi f(\xi) e^{-i\nu(\tau-\xi) - \frac{(\pi\mu_\epsilon)^2}{2}(\tau-\xi)^2} d\xi \tag{3}$$

which can be considered conceptually analogous to the Caputo- Fabrizio fractional derivative [7].

The analytical/numerical method developed in this paper to deal with the solution of Equation (1) can be applied to problems involving derivatives of type (2), as discussed elsewhere.

We can write Equation (1) in a slightly different form, more suitable for our purposes

$$\begin{aligned} \partial_{\tau} a &= \hat{V}(\tau) a, \\ \hat{V}(\tau) &= i \pi g_0 \int_0^{\tau} \tau' e^{-i\nu\tau' - \frac{(\pi\mu_\epsilon)^2}{2}\tau'^2} d\tau'. \end{aligned} \tag{4}$$

The solution can accordingly be obtained by the iteration

$$\partial_{\tau} a_n(\tau) = \hat{V}^n a_0 \tag{5}$$

which can also be cast in the more familiar form [9,10]

$$\begin{aligned} \partial_{\tau} a_n &= i \pi g_0 \int_0^{\tau} \tau' e^{-i\nu\tau' - \frac{(\pi\mu_\epsilon)^2}{2}\tau'^2} a_{n-1}(\tau - \tau') d\tau', \\ a_0 &= 1. \end{aligned} \tag{6}$$

For later convenience we will call  $n$  the principal expansion index.

Even though the method is efficient and fast converging the integrals cannot be done analytically but requires a numerical treatment, which increases the computation times when a larger number of terms is involved.

## 2. Using the Hermite Expansion

A way out is the use of two variable Hermite polynomials defined as [11,12]

$$H_m(x, y) = m! \sum_{r=0}^{\lfloor \frac{m}{2} \rfloor} \frac{x^{m-2r} y^r}{(m-2r)! r!} \tag{7}$$

and specified by the generating function

$$\sum_{m=0}^{\infty} \frac{t^m}{m!} H_m(x, y) = e^{xt+yt^2} \tag{8}$$

$$\partial_{\tau} a_n = i \pi g_0 \sum_{m=0}^{\infty} \frac{1}{m!} \int_0^{\tau} \tau'^{(m+1)} H_m\left(-i\nu, -\frac{(\pi\mu_\epsilon)^2}{2}\right) a_{n-1}(\tau - \tau') d\tau', \tag{9}$$

The expansion index  $m$  is therefore nested into the principal index  $n$ . The integration procedure is now straightforward and we get

1. First Order Solution

$$a_1 = i \pi g_0 \sum_{m=0}^{\infty} \frac{H_m \left( -i v, -\frac{(\pi \mu_\epsilon)^2}{2} \right)}{m!} \int_0^\tau d\tau' \int_0^{\tau'} \tau''^{(m+1)} d\tau'', \tag{10}$$

which yields

$$a_1 = i \pi g_0 \sum_{m_1=0}^{\infty} \alpha_{m_1} \tau^{m_1+3}, \tag{11}$$

$$\alpha_{m_1} = \frac{H_{m_1}}{m_1!(m_1 + 2)(m_1 + 3)}.$$

The arguments of the Hermite have been omitted to avoid cumbersome expressions.

2. Second Order Solution

$$a_2 = (i \pi g_0)^2 \sum_{m_2=0}^{\infty} \frac{H_{m_2}}{m_2!} \int_0^\tau d\tau' \int_0^{\tau'} \tau''^{(m_2+1)} a_1(\tau' - \tau'') d\tau'', \tag{12}$$

which after some algebra yields

$$a_2 = (i \pi g_0)^2 \sum_{m_1, m_2=0}^{\infty} \alpha_{m_1, m_2} \tau^{m_1+m_2+6},$$

$$\alpha_{m_1, m_2} = \alpha_{m_1} \sum_{s=0}^{m_1+3} \binom{m_1+3}{s} \frac{(-1)^s H_{m_2}}{m_2!(m_2 + s + 2)(m_2 + m_1 + 6)} = \tag{13}$$

$$= \frac{H_{m_1} H_{m_2}}{m_2! m_1!(m_1 + 2)(m_1 + 3)(m_2 + m_1 + 6)} \sum_{s=0}^{m_1+3} \binom{m_1+3}{s} \frac{(-1)^s}{(m_2 + s + 2)}.$$

We can use the previous two contributions to check whether the results we obtain concerning e.g., the FEL gain agree with analogous conclusions already given in the literature.

It must be emphasized that further iterations are necessary, if one is interested to larger gain values. The technicalities of the expansion will be discussed in the forthcoming sections.

3. Higher Order Solution

The interesting aspect of the present nested procedure is that the  $n$ -th order can be computed in a modular way just looking at the symmetries of the expansion itself. The  $n$ -th order term indeed reads

$$a_n = (i \pi g_0)^n \sum_{m_1, \dots, m_n=0}^{\infty} \alpha_{m_1, \dots, m_n} \tau^{(\sum_{r=1}^n m_r + 3n)}, \tag{14}$$

$$\alpha_{m_1, \dots, m_n} = \alpha_{m_1, \dots, m_{n-1}} \sum_{s=0}^{(\sum_{r=1}^{n-1} m_r + 3(n-1))} \binom{\sum_{r=1}^{n-1} m_r + 3(n-1)}{s} \frac{H_{m_n}}{m_n!(m_n + s + 2)(\sum_{r=1}^n m_r + 3n)}.$$

A pivotal quantity in FEL physics is the gain defining the fractional intensity growth at the end of the interaction ( $\tau = 1$ ), namely

$$G(v, \mu_\epsilon) = \|a\|^2 - 1. \tag{15}$$

In the forthcoming section we will discuss the relevant numerical integration and compare the relevant results with those from a complete numerical integration.

### 3. Implementation and Comparison

The procedure has been implemented in Mathematica, but does not rely on its symbolic capabilities and thus can be implemented in any computer language, unless the analytical result is requested. In this case the calculation has to be done with a tool with symbolic capabilities.

The gain equation of order  $n$  is given from (15) combining solutions of (1) as

$$G(v, \mu_\epsilon) = \|a_0 + a_1 + a_2 + .. + a_n\|^2 - 1. \tag{16}$$

It requires the calculation of the norm of the quadratic sum of  $n + 1$  terms, of which one is constant (unity for  $a_0$ ) and  $n$ , according to the method outlined in previous section, are the Hermite polynomials expansion as of in Equations (11), (13) and (14). The expansion has to be truncated for computational reasons and it is easy to implement, by the literal point of view, but the performance is poor. The  $j$ -th expansion, in fact, requires the evaluation of  $M^j$  terms (if all expansions are truncated at the  $M$ -th term) for every  $j = 1 \dots n$ , that is  $\frac{M^{n+1} - 1}{M - 1}$  terms in total, which can be quite demanding. Algorithmic optimizations are thus needed.

It can be noted that some Hermite polynomials are the same in different occurrences, and could be evaluated once and then stored for the next occurrences.

The first optimization is to build a lookup table containing the Hermite polynomials or to define functions that remember their own values (which is a feature offered by Mathematica to implement transparently lookup tables).

The second optimization is to do index or exponent reordering with the aim to move factors in or out of the summation to avoid repetitions of the calculus, like in the r.h.s of Equation (13).

The resulting code is very compact and can be viewed in Figure A1 (Appendix A).

The comparison is reported in Figure 1 for a 2nd order gain function, where the present procedure (at different truncation levels) is confronted with those from a complete numerical integration and no differences are foreseeable. The truncation of the Hermite expansion obviously affects the speed and the approximation of the gain curve.

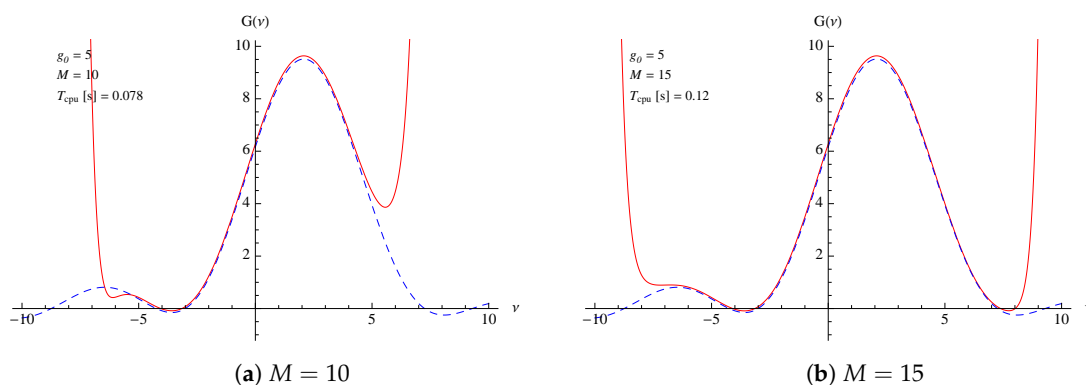
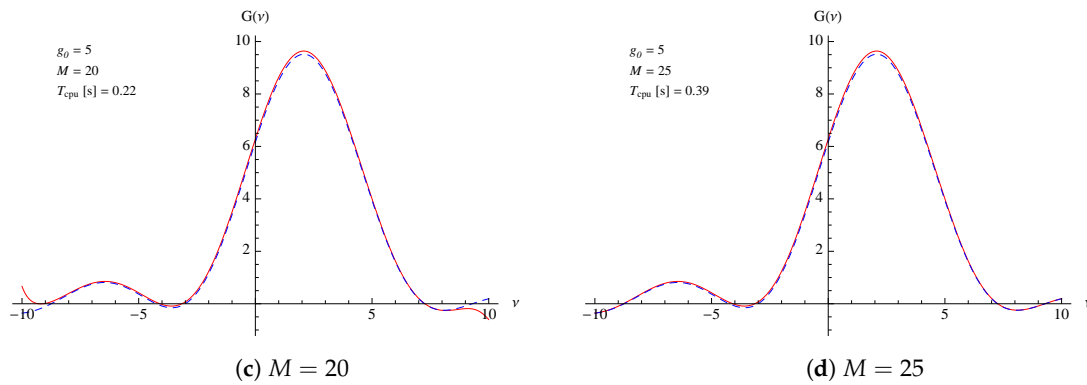
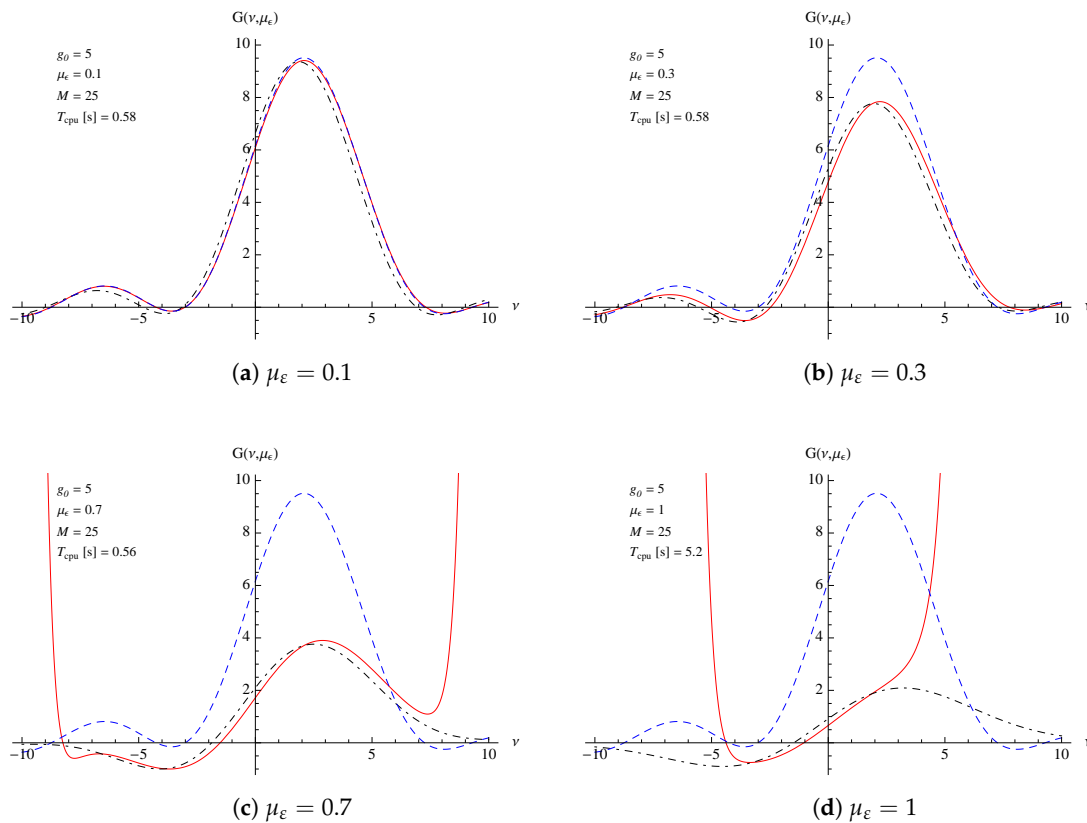


Figure 1. Cont.

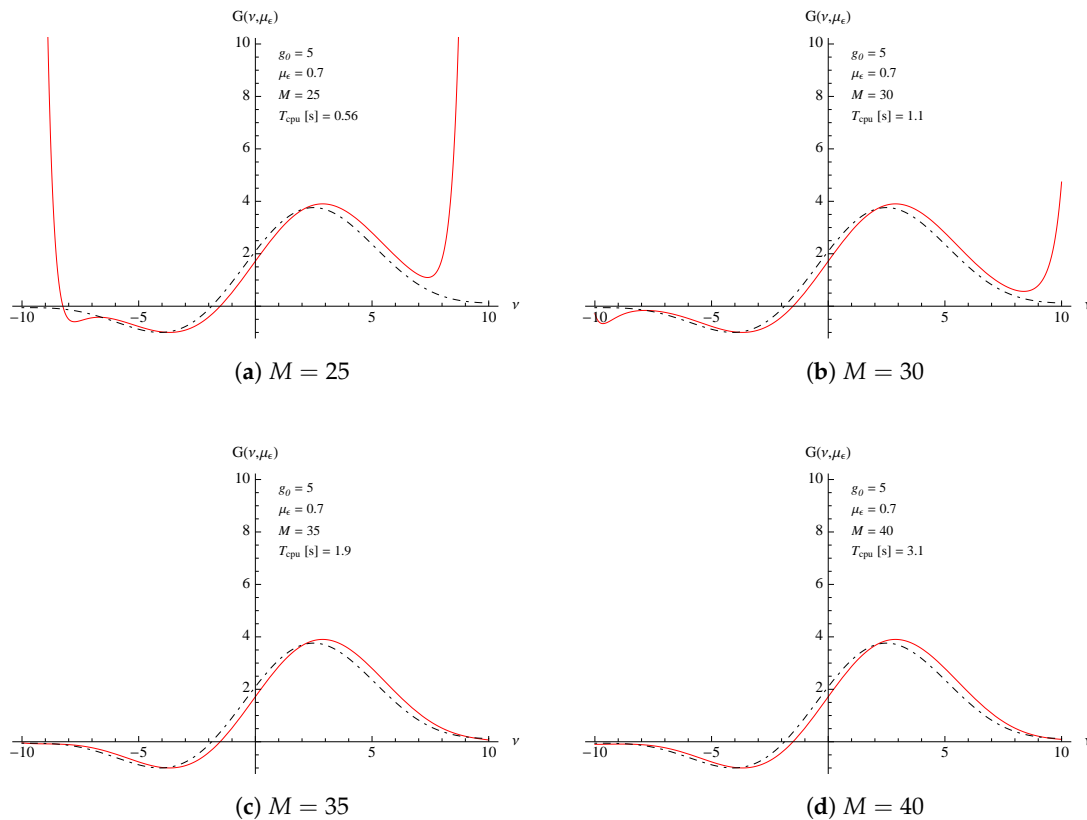


**Figure 1.** Comparison with the complete numerical integration (blue dashed line) at different truncation levels ( $M$ ) and with no broadening effects ( $\mu_\epsilon = 0$ ) and  $g_0 = 5$ . The CPU time used refers to an Intel I7 3 GHz processor.

The method allows also taking into account the broadening effect as shown in Figure 2. It has to be noted that for higher broadening effects an higher truncation level is needed to achieve a good approximation, as shown in Figure 3. In this case, the comparison is made with the expression discussed in Appendix B.



**Figure 2.** Comparison with analytical expression (black dash-dotted line with different broadening effect ( $\mu_\epsilon$ )) and with the complete numerical integration (blue dashed line, which has no broadening effect) and  $g_0 = 5$ . The CPU time used refers an Intel I7 3 GHz processor.



**Figure 3.** Comparison with analytical expression (black dash-dotted line) in the case of higher broadening effects ( $\mu_\epsilon = 0.7$ ) for different truncation levels ( $M$ ) and  $g_0 = 5$ . The CPU time used refers to an Intel I7 3 GHz processor.

The Hermite expansion approach let the integrals found in Equation (6) become polynomials and thus very easy to solve automatically. They are, of course, high in number but tractable also by a symbolic processor.

It is therefore possible to obtain, with the same degree of approximation, but with longer calculation times than the numeric case, also the full analytical expression for the curve.

The longer timings are then well payed back by the possibility to evaluate the curve on the fly with any combination of parameter values.

In Appendix A it can be found the symbolic expression of the curve for  $M = 10$  (which is inaccurate, but compact enough to be checked by visual inspection and corresponding to that plotted in Figure 1).

It is clear that in our procedure there are two levels of approximation, the first associated with the principal index expansion  $n$ , depending on the values of the small signal gain coefficient  $g_0$  which fixes how fast the laser intensity grows in time, the second is associated with the nested index  $m_n$ , depending on the values of the inhomogeneous parameter. It is worth noting that in actual FEL devices  $n = 4$  is largely sufficient and since  $\mu_\epsilon$  is not larger than 0.3 the index  $m_n$  is not required to exceed 30.

#### 4. Conclusions

The method we have foreseen is general enough to be applied to Volterra type equations with different kernels, provided that a corresponding expansion in terms of a suitable polynomial family is allowed.

To clarify the previous statement we consider the equation

$$\partial_{\tau} a(\tau) = \lambda \int_0^{\tau} \frac{\tau'}{1 + \alpha\tau' + \beta\tau'^2} a(\tau - \tau') d\tau' \quad (17)$$

In this case the problem can be solved by the use of a two variable Legendre like polynomials

$$P_n(x, y) = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(n-r)! x^{n-2r} y^r}{(n-2r)! r!} \quad (18)$$

with generating function

$$\sum_{n=0}^{\infty} t^n P_n(x, y) = \frac{1}{1 + x t + y t^2}. \quad (19)$$

The solution of the problem (17) looks, therefore, like the expression derived for Equation (1) with the two variable Legendre polynomials replacing the Hermite polynomials, namely

$$\partial_{\tau} a_n(\tau) = \lambda \sum_{m_n=0}^{\infty} P_{m_n}(-\alpha, -\beta) \int_0^{\tau} \tau'^{(m_n+1)} a_{n-1}(\tau - \tau') d\tau'. \quad (20)$$

In a forthcoming paper, dedicated to the physical aspects of the FEL, we will show how this family of polynomials allows the inclusion in the FEL high gain equation of the gain detrimental effects due to electron beam transverse and angular distributions.

**Author Contributions:** Each author has contributed equally to this work.

**Conflicts of Interest:** The authors declare no conflict of interest.

## Appendix A

Wolfram Mathematica<sup>TM</sup> code snippet, actually used to generate the curves in the figures.

```
(*Hermite polynomials*)
H[n_, x_, y_] :=
  (n!) * ((x^n / (n!)) + Sum[(x^(n-2s) * y^s) / (((n-2s)!) * (s!)), {s, 1, Floor[n/2]}]);

(*lookup table for Hermite polynomials to be used in the expansion*)
Htabular[order, v, mu] = Htabular[order_, v_, mu_] := H[order, -I * v, -1/2 * (Pi * mu)^2];

(*first order solution*)
a1[v_, g0_, mu_, tau_, M_] :=
  I * Pi * g0 * Sum[Htabular[n, v, mu] * tau^(n+3) / ((n+3) (n+2) (n!)), {n, 0, M}];

(*second order solution*)
a2[v_, g0_, mu_, tau_, M_] :=
  (I * Pi * g0)^2 * tau^6 * Sum[tau^n * (n+3)! * (Htabular[n, v, mu]) / (n! * (n+3) (n+2)) *
  Sum[tau^m * (Htabular[m, v, mu]) / (m! * (n+m+6)) *
  Sum[1 / (r! * (n+3-r)!) * (-1)^r / (r+m+2), {r, 0, n+3}], {m, 0, M}], {n, 0, M}];

(*gain function at a second order approximation*)
gain2[v_, g0_, mu_, M_] := Simplify[Norm[1 + a1[v, g0, mu, 1, M] + a2[v, g0, mu, 1, M]]^2 - 1];
```

Figure A1. Working Mathematica code.

If the code is run in symbolic mode, then an analytical expression can be found, as that on below

```

-1 + Norm[1 -  $\frac{1}{6227020800}$ 
  i g0  $\pi$  (-1.037836800 + 10.395 mu10  $\pi^{10}$  + 518.918400 i v + 155.675520 v2 - 34.594560 i v3 -
  61.77600 v4 + 92.640 i v5 + 120.120 v6 - 13.728 i v7 - 140.4 v8 + 130 i v9 + 11 v10 +
  945 mu8  $\pi^8$  (-156 + 130 i v + 55 v2) + 630 mu6  $\pi^6$  (2860 - 2288 i v - 936 v2 + 260 i v3 + 55 v4) +
  90 mu4  $\pi^4$  (-205.920 + 154.440 i v + 60.060 v2 - 16.016 i v3 - 327.6 v4 + 54.6 i v5 + 77 v6) +
  9 mu2  $\pi^2$  (17.297280 - 11.531520 i v - 4.118400 v2 + 1.029600 i v3 +
  200.200 v4 - 32.032 i v5 - 43.68 v6 + 520 i v7 + 55 v8)] -
 $\frac{1}{403291461126605635584000000}$  g02  $\pi^2$  (560.127029342507827200000 +
  108.056025 mu20  $\pi^{20}$  - 320.072588195718758400000 i v - 100.022683811162112000000 v2 +
  22.227263069147136000000 i v3 + 3.889771037100748800000 v4 -
  565.784878123745280000 i v5 - 70.723109765468160000 v6 + 7.77177030389760000
  i v7 + 763.298869132800000 v8 - 67.848788367360000 i v9 - 5.512714054848000 v10 +
  385.504479360000 i v11 + 23.999543568000 v12 - 1.352647296000 i v13 -
  69.621552000 v14 + 3.283737600 i v15 + 141.726000 v16 - 5.553600 i v17 - 193.700 v18 +
  5720 i v19 + 121 v20 + 98.232750 mu18  $\pi^{18}$  (-130 + 52 i v + 11 v2) + 1.488375 mu16
 $\pi^{16}$  (741.520 - 370.240 i v - 116.220 v2 + 21.736 i v3 + 2299 v4) + 4.422600 mu14  $\pi^{14}$ 
(-19.126800 + 9.715200 i v + 3.284400 v2 - 705.600 i v3 - 108.150 v4 + 12.012 i v5 + 847 v6) +
  28.350 mu12  $\pi^{12}$  (226.067441600 - 102.848345600 i v - 34.958123200 v2 + 7.556806400 i v3 +
  1.229009600 v4 - 158.005120 i v5 - 16.208920 v6 + 1.288144 i v7 + 68.123 v8) +
  3780 mu10  $\pi^{10}$  (-150.989254416000 + 49.510869408000 i v + 16.472127672000 v2 -
  3.310891584000 i v3 - 528.515988000 v4 + 69.242659200 i v5 + 7.574985600 v6 -
  698.006400 i v7 - 53.779050 v8 + 3.326180 i v9 + 140.723 v10) + 9450 mu8  $\pi^8$ 
(1.343758470912000 - 1.075006776729600 i v - 436.721503046400 v2 + 69.995519193600
  i v3 + 10.424998584000 v4 - 1.318168051200 i v5 - 142.978355520 v6 + 13.488583680
  i v7 + 1.112782320 v8 - 80.059200 i v9 - 4.945980 v10 + 250.536 i v11 + 8833 v12) +
  9000 mu6  $\pi^6$  (-28.064726097408000 + 21.588250844160000 i v + 8.481098545920000 v2 -
  2.261626278912000 i v3 - 459.392837904000 v4 + 54.711400423680 i v5 +
  5.658685032000 v6 - 523.200437760 i v7 - 43.336653360 v8 + 3.218483840 i v9 +
  214.021808 v10 - 12.655552 i v11 - 653.198 v12 + 28.028 i v13 + 847 v14) + 45 mu4  $\pi^4$ 
(96.318139966304256000 - 70.049556339130368000 i v - 26.268583627173888000 v2 +
  6.735534263377920000 i v3 + 1.323051373163520000 v4 - 211.6882197061632000 i v5 -
  28.6661130852096000 v6 + 2.7738181126656000 i v7 + 230.735769264000 v8 -
  17.213099040000 i v9 - 1.165531086720 v10 + 71.812815360 i v11 +
  4.015880960 v12 - 202.092800 i v13 - 8.970000 v14 + 334.048 i v15 + 8833 v16) +
  90 mu2  $\pi^2$  (-666.817892074414080000 + 444.545261382942720000 i v +
  155.590841484029952000 v2 - 37.718991874916352000 i v3 -
  7.072310976546816000 v4 + 1.088047842545664000 i v5 + 142.48245571456000 v6 -
  16.2837092081664000 i v7 - 1.6538142164544000 v8 + 134.699800435200 i v9 +
  9.5620228704000 v10 - 610.017408000 i v11 - 35.429634240 v12 + 1.881834240 i v13 +
  91.278720 v14 - 4.010240 i v15 - 156.390 v16 + 5148 i v17 + 121 v18)]2

```

Figure A2. Mathematica code for Hermite solution.

Appendix B

The comparison of the solution method with the case including the effect of the inhomogeneous broadening has been done using as benchmark the following first order analytical solution, sufficient for the range of parameters we have quoted in Figures 2 and 3.

$$a_1 = i \pi g_0 \sqrt{\frac{\pi}{2}} \frac{1}{(\pi \mu_\epsilon)^2} e^{-\frac{v^2}{2(\pi \mu_\epsilon)^2}} \cdot A_1 \tag{A1}$$

where

$$\begin{aligned}
 A_1 = & |v| \left\{ i \sqrt{\frac{2}{\pi}} (\pi \mu_\epsilon) \left( -v e^{\delta^2} + \delta e^{\frac{v^2}{2(\pi \mu_\epsilon)^2}} \right) + i \left[ (\pi \mu_\epsilon)^2 - v^2 \right] \operatorname{Erfi} \left( \frac{v}{\sqrt{2}(\pi \mu_\epsilon)} \right) + \right. \\
 & \left. + \left[ v \delta^* - i(\pi \mu_\epsilon)^2 \right] \operatorname{Erfi} \left( \frac{\delta}{\sqrt{2}(\pi \mu_\epsilon)} \right) \right\} - (\pi \mu_\epsilon)^2 v^2 + \operatorname{Erfi} \left( \frac{|v|}{\sqrt{2}(\pi \mu_\epsilon)} \right), \tag{A2} \\
 \delta = & v - i(\pi \mu_\epsilon)^2 \tau.
 \end{aligned}$$

Higher order terms and the framing within the FEL theory will be discussed elsewhere.



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