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Voigt Transform and Umbral Image

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Abstract: In this paper, we show that the use of methods of an operational nature, such as umbral calculus, allows achieving a double target: on one side, the study of the Voigt function, which plays a pivotal role in spectroscopic studies and in other applications, according to a new point of view, and on the other, the introduction of a Voigt transform and its possible use. Furthermore, by the same method, we point out that the Hermite and Laguerre functions, extension of the corresponding polynomials to negative and/or real indices, can be expressed through a definition in a straightforward and unified fashion. It is illustrated how the techniques that we are going to suggest provide an easy derivation of the relevant properties along with generalizations to higher order functions.

Keywords: Voigt function; flattened optical beams; free electron laser; operators theory; umbral methods; special functions; Hermite polynomials

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1. Introduction

We begin the introductory section by providing an overview of the special functions and their umbral images.

One of the appealing features of the umbral (U) [1] treatment of special functions is that it provides a means to reduce the study of the relevant properties to that of the elementary families, by establishing an appropriate correspondence between the higher transcendental under study and a corresponding elementary function, usually indicated as the associated umbral image (UI) [2].

We note that the Gaussian is the UI of the cylindrical Bessel functions and that the Newton binomial realizes the UI of Hermite polynomials. Regarding this last point, we remind that, in a recent investigation (see Equation (2.2) in [3]), the following identity was stated:

$$H_n(x,y) = \left(x + {}_y\hat{h}\right)^n \theta_0. \tag{1}$$

Within such a context, the U-operator ${}_y\hat{h}$ acts on its corresponding vacuum θ_0 according to the prescription given below.

We introduce the Hermite function vacuum θ_0.

Definition 1. *Let:*

$$\theta(z) := \theta_z = y^{\frac{z}{2}} \left(\frac{\Gamma(z+1)}{\Gamma\left(\frac{1}{2}z+1\right)} \left|\cos\left(\frac{\pi}{2}z\right)\right| \right), \quad \forall z \in \mathbb{R}, \tag{2}$$

the Hermite function vacuum.

We can then establish the action of the umbral operator ${}_y\hat{h}$ according to the following proposition.

Proposition 1. *The umbral operator ${}_y\hat{h}^r$ acts on the vacuum θ_0 according to the rule:*

$$
\begin{aligned}
&{}_y\hat{h}^r\,\theta_0 := \theta_r, \quad \forall r \in \mathbb{R}, \\
&\theta_r = \frac{y^{\frac{r}{2}}\,r!}{\Gamma\left(\frac{r}{2}+1\right)}\left|\cos\left(r\frac{\pi}{2}\right)\right| = \begin{cases} 0, & r = 2s+1, \\ y^s\dfrac{(2s)!}{s!}\,, & r = 2s, \end{cases} \qquad \forall s \in \mathbb{Z}.
\end{aligned} \tag{3}
$$

Proof. $\forall r \in \mathbb{R}$, ${}_y\hat{h}^r\,\theta_0 := \theta_r$ where, according to Definition 1, it produces $\forall s \in \mathbb{Z}$:

$$
\begin{aligned}
\theta_r &= y^{\frac{r}{2}}\left(\frac{\Gamma(r+1)}{\Gamma\left(\frac{1}{2}r+1\right)}\left|\cos\left(\frac{\pi}{2}r\right)\right|\right) = y^{\frac{r}{2}}\frac{r!}{\frac{r}{2}+1}\begin{cases} 0, & r = 2s+1 \\ 1\,, & r = 2s \end{cases} \\
&= \begin{cases} 0, & r = 2s+1, \\ y^s\dfrac{(2s)!}{\Gamma(s+1)}\,, & r = 2s, \end{cases}
\end{aligned}
$$

□

The numbers $\dfrac{(2s)!}{s!} = 1, 2, 12, 120, 1680, \ldots$ are recognized as the quadruple factorial numbers, reported in *OEIS* sequence *A*001813.

Corollary 1. *The generating function of Hermite polynomials is accordingly easily obtained from Equation (3) as:*

$$
\sum_{n=0}^{\infty}\frac{t^n}{n!}H_n(x,y) = e^{xt}e^{\,{}_y\hat{h}\,t}\theta_0, \quad \forall t \in \mathbb{R}, \tag{4}
$$

which yields the ordinary expression $\sum_{n=0}^{\infty}\frac{t^n}{n!}H_n(x,y) = e^{xt+yt^2}$ by noting that:

$$
e^{\,{}_y\hat{h}\,t}\theta_0 = \sum_{r=0}^{\infty}\frac{t^r}{r!}\left({}_y\hat{h}\right)^r\theta_0 = e^{yt^2}, \quad \forall y, t \in \mathbb{R}. \tag{5}
$$

The U-operators can be raised to any (real or complex) index, so it is therefore natural to specify the real negative order Hermite functions (NOH) as:

$$
H_{-\nu}(x,y) = \left(x + {}_y\hat{h}\right)^{-\nu}\theta_0, \quad \forall \nu \in \mathbb{R}^+. \tag{6}
$$

They are no longer polynomials, but Hermite functions. The relevant NOH-function integral representation can be written by the use of the Laplace transform identity according to the following proposition.

Proposition 2 (NOH-integral representation)**.** *The relevant NOH-function integral representation can be recast as:*

$$
H_{-\nu}(x,y) = \frac{1}{\Gamma(\nu)}\int_0^{\infty}s^{\nu-1}e^{-sx}e^{-|y|s^2}ds, \quad \forall x, y \in \mathbb{R}, \forall \nu \in \mathbb{R}^+. \tag{7}
$$

Proof. $\forall x, y \in \mathbb{R}, \forall \nu \in \mathbb{R}^+$; by the use of the Laplace transform and Equation (5), we obtain

$$
\begin{aligned}
H_{-\nu}(x,y) &= \frac{1}{\left(x + {}_y\hat{h}\right)^{\nu}}\theta_0 = \int_0^{\infty}e^{-xs}\frac{s^{\nu-1}e^{-{}_y\hat{h}s}}{\Gamma(\nu)}ds\,\theta_0 = \frac{1}{\Gamma(\nu)}\int_0^{\infty}s^{\nu-1}e^{-xs}\left(e^{-s\,{}_y\hat{h}}\theta_0\right)ds \\
&= \frac{1}{\Gamma(\nu)}\int_0^{\infty}s^{\nu-1}e^{-sx}e^{-|y|s^2}ds.
\end{aligned}
$$

It should be noted that, to ensure the convergence of the integral, we must impose that ${}_{y}\hat{h} = {}_{-|y|}\hat{h}$, thus getting:

$$e^{-{}_{y}\hat{h}s} = e^{-ys^2}, \quad y > 0. \tag{8}$$

$Re(y) > 0$ if we consider complex arguments. □

The integral representation in (7) is the definition of the (real) negative order Hermite functions, currently quoted in the mathematical literature; see for example [4], where they are given in terms of a single variable. The link with the definition (7) is specified by the identities:

$$\begin{aligned} &H_{-\nu}(x) = H_{-\nu}(2x, -1), \\ &H_{-\nu}(x, y) = y^{-\frac{\nu}{2}} H_{-\nu}\left(\frac{x}{2\sqrt{y}}\right). \end{aligned} \tag{9}$$

A fairly straightforward application of the method is the explicit evaluation of the following integral.

Example 1. *Let*

$$I(\alpha, \beta) = \int_{-\infty}^{\infty} e^{-\alpha x^2 - \beta x^4} dx, \quad \forall \alpha \in \mathbb{R}, \forall \beta \in \mathbb{R}_0^+, \tag{10}$$

which can be written in terms of a Hermite function of order $-1/2$. We note indeed that:

$$I(\alpha, \beta) = \int_{-\infty}^{\infty} e^{-x^2(\alpha + {}_{\beta}\hat{h})} dx\, \theta_0, \tag{11}$$

which yields:

$$I(\alpha, \beta) = \sqrt{\frac{\pi}{\alpha + {}_{-|\beta|}\hat{h}}}\theta_0 = \sqrt{\pi}\, H_{-\frac{1}{2}}(\alpha, \beta) = \frac{1}{2}\sqrt{\frac{\alpha}{\beta}} e^{\frac{\alpha^2}{8\beta}} K_{\frac{1}{4}}\left(\frac{\alpha^2}{8\,\beta}\right), \tag{12}$$

where $K_\nu(x)$ is the Macdonald function (or modified Bessel function of the second kind) [4]:

$$K_\nu(x) = \frac{\pi}{2} \frac{I_{-\nu}(x) - I_\nu(x)}{\sin(\pi\nu)} \tag{13}$$

with:

$$I_n u(x) = \sum_{r=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2r+\nu}}{r!\Gamma(\nu + r + 1)} \tag{14}$$

the modified Bessel functions of first kind.

We underline that in [2,5,6], by using the $\hat{c}$-operator introduced in the following in Equation (18), the $K_\nu(x)$ and $I_\nu(x)$ umbral images were provided:

$$K_\nu(x) = \frac{\pi}{\sin(\nu\pi)} \left[\sinh\left(\nu \ln\left(\frac{\hat{c}x}{2}\right)\right)\right] e^{\hat{c}\left(\frac{x}{2}\right)^2} \varphi_0, \qquad I_\nu(x) = \left(\hat{c}\frac{x}{2}\right)^\nu e^{\hat{c}\left(\frac{x}{2}\right)^2} \varphi_0. \tag{15}$$

Example 2. *The use of the same procedure leads to the derivation of the infinite integral $\forall y \in \mathbb{R}, \forall m \in \mathbb{R}^+$:*

$$\begin{aligned} I_\nu(x, y \mid m) &= \int_0^\infty e^{-s^m(x + ys^m)} s^{\nu-1} ds = \int_0^\infty e^{-s^m(x + {}_{-|y|}\hat{h})} s^{\nu-1} ds\, \theta_0 \\ &= \frac{\Gamma\left(\frac{\nu}{m}\right)}{m} H_{-\frac{\nu}{m}}(x, y). \end{aligned} \tag{16}$$

The previous integrals, which can also be obtained by standard means from Equation (10) after an appropriate change of variable, suggest that the umbral procedure is reliable and that we can extend the method to more complicated examples, leading to previously unknown results.

Before entering further specific examples, we devote this introductory section to a description of the formalism we will use in the following and first note that the Laguerre functions can be introduced in an analogous fashion.

In the case of Laguerre polynomials as well, the corresponding UI is a Newton binomial introduced by using the umbral operator $\hat{c}$ and specified by the following identity (as was proven in [7], see Equation (3.1) and in the following).

Proposition 3 (Laguerre umbral image). *$\forall x, y \in \mathbb{R}, \forall n \in \mathbb{N}$, the Laguerre polynomials can be represented by an umbral image using the Newton binomial:*

$$L_n(x,y) = (y - \hat{c}x)^n \, \varphi_0 \tag{17}$$

where the operator $\hat{c}$ acts on the vacuum φ_0 so that:

$$\hat{c}^{\nu} \varphi_0 = \frac{1}{\Gamma(\nu+1)}. \tag{18}$$

Corollary 2. *Proposition 3 naturally yields the definition of the relevant negative (real) order functions as:*

$$L_{-\nu}(x,y) = (y - \hat{c}x)^{-\nu} \, \varphi_0 = \frac{1}{\Gamma(\nu)} \int_0^{\infty} s^{\nu-1} e^{-s(y-\hat{c}x)} ds \, \varphi_0, \quad \forall \nu \in \mathbb{R}^+. \tag{19}$$

Example 3. *By recalling that the zero-order cylindrical Bessel function in term of the $\hat{c}$ operator reads (see Equation (1.2.9) and Chapter 5 in [2]):*

$$J_0(x) = e^{-\hat{c}\left(\frac{x}{2}\right)^2} \varphi_0 = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{2r}}{r!^2}, \tag{20}$$

we obtain the corresponding integral transform:

$$L_{-\nu}(x,y) = \frac{1}{\Gamma(\nu)} \int_0^{\infty} s^{\nu-1} e^{-sy} J_0\left(2\sqrt{-xs}\right) ds. \tag{21}$$

Example 4. *A fairly immediate consequence is provided by the derivation of the integral:*

$$G(a,b) = \int_{-\infty}^{\infty} e^{-ax^2} J_0(bx) dx \tag{22}$$

which, on account of Equation (17), can be written as:

$$G(a,b) = \sqrt{\frac{\pi}{a + \hat{c}\left(\frac{b}{2}\right)^2}} = \sqrt{\pi} L_{-\frac{1}{2}}\left(-\frac{b^2}{4}, a\right). \tag{23}$$

Equation (23) is particularly interesting since, as will be shown in the concluding section, it can also be interpreted in terms of two variable generalized Bessel functions.

Before closing this introductory section, we discuss a further example, based on the definition of a family of polynomials belonging to the so-called hybrid family, because they share properties in between those of Hermite and Laguerre polynomials [8].

Example 5. *We consider indeed the following umbral definition:*

$$\Lambda_n(x,y) = \left(x + \sqrt{\hat{c}_{-|y|}} \hat{h}\right)^n \theta_0 \, \varphi_0 \tag{24}$$

in which the use of the product of two umbral operators ($\hat{c}, {}_{-|y|}\hat{h}$) does not create any ambiguity, since they act independently on Hermite (θ_0) and Laguerre (φ_0) vacua. It is therefore a straightforward matter to infer that, in terms of a truncated series, the polynomials defined in Equation (24) read:

$$\Lambda_n(x,y) = n! \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \frac{x^{n-2r} y^r}{(n-2r)! r!^2}. \tag{25}$$

Since the Λ_n polynomials stay between Hermite and Laguerre, they are expected to satisfy recurrences and differential equations reflecting such a peculiarity. It is indeed shown that they obey the partial differential equation (where $\partial_y y \partial_y$ is the so-called "Laguerre derivative"):

$$\begin{cases} \partial_y y \partial_y \, \Lambda_n(x,y) = \partial_x^2 \, \Lambda_n(x,y), \\ \Lambda_n(x,0) = x^n, \end{cases} \tag{26}$$

which can be viewed as a kind of generalized heat equation, yielding the following operational definition for the hybrid polynomials:

$$\Lambda_n(x,y) = J_0\left(2i\sqrt{y}\,\partial_x\right) x^n. \tag{27}$$

Technically speaking, the polynomials (25) are Appél-type polynomials, and the use of the previously foreseen procedure suggests that the negative order hybrid functions can be written as:

$$\Lambda_{-\nu}(x,y) = \left(x + \sqrt{\hat{c}}\; {}_{-|y|}\hat{h}\right)^{-\nu} \varphi_0 \, \theta_0 = \frac{1}{\Gamma(\nu)} \int_0^\infty s^{\nu-1} e^{-sx} J_0\left(2is\sqrt{y}\right) \, ds. \tag{28}$$

Infinite integrals containing, e.g., products of Bessel and exponential functions can therefore be expressed in terms of negative order Λ functions. We find for example that:

$$S(a,b) = \int_0^\infty e^{-ax} J_0(bx) dx = \Lambda_{-1}\left(a, -\frac{b^2}{4}\right). \tag{29}$$

After these introductory remarks, in which we fixed the main body of the formalism we are going to use in the following, we enter the principal topic of the paper, consisting of three further sections.

In Section 2, we will deal with the application of the method to the theory of Voigt functions and to their applications [9] and with their use to treat the free electron laser high gain equation [10].

Section 3 is devoted to the definition of higher order Hermite functions and to final comments providing further hints and lines for future extensions of the present investigation.

2. Voigt Functions, Hermite Functions, and Generalized Forms

Voigt functions find several applications in spectroscopy [9] and are defined by the convolution of the Gaussian and the Lorentzian distributions. Apart from the relevant applications in physics, they have raised a certain interest in mathematics for their relation with a number of special functions. The generalized forms of the Voigt transform has been proposed in a series of authoritative papers [11,12] in which an extended study of general properties was performed using a formalism mixing up different special functions of a multi-variable and multi-index type. This general point of view stimulated the following consideration.

In this section, we will explore the relevant link with the Hermite functions and study the associated consequences within the context of the formalism so far developed.

In the following, we denote the Voigt functions (VF) by $K(x,y,z)$, $L(x,y,z)$ and define them in terms of the integral representations (see Equation (7) in [13]). The current definition includes two variables only (x,z) and assumes $y = 1/4$.)

Definition 2. *$\forall x, z \in \mathbb{R}, \forall y \in \mathbb{R}^+$, we introduce the Voigt functions (VF) by $K(x,y,z)$, $L(x,y,z)$:*

$$K(x,y,z) = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-x\xi - y\xi^2} \cos(z\xi) \; d\xi,$$
$$L(x,y,z) = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-x\xi - y\xi^2} \sin(z\xi) \; d\xi \tag{30}$$

and the complex VF:

$$E(x,y,z) = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-(x-iz)\xi - y\xi^2} \, d\xi. \tag{31}$$

Observation 1. *If we note that VF in Equation (30) are the relevant real and imaginary parts, we can easily conclude that it is expressible in terms of of the complementary error function "$erfc(z)$" defined as:*

$$erfc(z) = 1 - erf(z) \tag{32}$$

where:

$$erf(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt \tag{33}$$

is the error (or Gauss error) function (Chapter 7 in [14]).

Proposition 4. *$\forall x, z \in \mathbb{R}$, $\forall y \in \mathbb{R}^+$, we get:*

$$E(x,y,z) = \frac{1}{2\sqrt{y}} e^{\frac{(x-iz)^2}{4y}} erfc\left(\frac{x-iz}{2\sqrt{y}}\right). \tag{34}$$

Proof. We can exploit Equations (6)–(7) to end up with the identity:

$$E(x,y,z) = \frac{1}{\sqrt{\pi}} H_{-1}(x-iz, y) = \frac{1}{\sqrt{\pi}} \left(x - iz + {}_y\hat{h}\right)^{-1} \theta_0 = \frac{1}{2\sqrt{y}} e^{\frac{(x-iz)^2}{4y}} erfc\left(\frac{x-iz}{2\sqrt{y}}\right).$$

□

Corollary 3. *We obtain the relevant derivatives by the use of the well-known properties of the Hermite functions and find that, $\forall m \in \mathbb{N}$:*

$$\partial_z^m E(x,y,z) = \frac{i^m \, m!}{\sqrt{\pi}} H_{-(m+1)}(x - iz, y). \tag{35}$$

The procedure we have envisaged allows unifying many of the previous analyses [15] aimed at getting different ways of expressing the VF in forms suitable for various specific applications.

To this end, we provide the following identities.

Definition 3 (Voigt-transform). *$\forall x, z \in \mathbb{R}$, $\forall y \in \mathbb{R}^+$, we introduce the Voigt (V-)transform of a function $f(z)$:*

$${}_V\hat{f}(x,y;z) = \int_0^\infty e^{-xt - yt^2} f(zt) \; dt. \tag{36}$$

Proposition 5. *The V-transform can be represented by using the Hermite polynomials:*

$${}_V\hat{f}(x,y;z) = \sum_{n=0}^\infty a_n H_{-(n+1)}(x,y) z^n, \qquad \forall x, z \in \mathbb{R}, \forall y \in \mathbb{R}^+ \tag{37}$$

Proof. Let us note that the V-transform can be viewed as a generalized form of transform involving the Gamma function (see Chapter 2.7 in [5]) by using of the identity:

$$t^{\,z\partial_z} f(z) = f(tz) \tag{38}$$

and then, we can get:

$$_V\hat{f}(x,y;z) = \int_0^\infty e^{-xt-yt^2} t^{z\partial_z}\, dt\; f(z) = \Gamma\left(z\partial_z + 1\right) H_{-(z\partial_z+1)}(x,y) f(z). \tag{39}$$

By expanding the function $f(z)$ in series and by using the property $f\left(z\partial_z\right) z^n = f(n) z^n$ [5], we can finally write:

$$\begin{aligned} {}_V\hat{f}(x,y;z) &= \sum_{n=0}^{\infty} a_n H_{-(n+1)}(x,y) z^n, \\ f(z) &= \sum_{n=0}^{\infty} a_n \frac{z^n}{n!}. \end{aligned} \tag{40}$$

□

Corollary 4. *According to the present formalism, the VF (30) is the V-transform (36) of the circular functions, thus reading:*

$$\begin{aligned} K(x,y,z) &= \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n H_{-(2n+1)}(x,y) z^{2n}, \\ L(x,y,z) &= \frac{z}{\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n H_{-(2n+2)}(x,y) z^{2n}. \end{aligned} \tag{41}$$

Example 6. *Let us furthermore note that the V-transform of a zero-order cylindrical Bessel function is:*

$$_V\hat{f}(x,y;z) = \sum_{n=0}^{\infty} \frac{(2n)!}{n!^2} (-1)^n H_{-(2n+1)}(x,y) \left(\frac{z}{2}\right)^{2n}. \tag{42}$$

Remark 1. *The generalization of the V-functions proposed in [16] can be viewed as V-transforms of different families of functions.*

In the previous section, we noted that Hermite functions of negative order can be defined by means of infinite integrals yielding the relevant integral representation; however, the use of the formalism we are proposing may be useful in a wider context as, e.g., for the evaluation of definite integrals, as shown below.

Example 7.

$$\begin{aligned} \int_0^x \xi^{\nu-1} e^{-a\xi - b\xi^2} d\xi &= \int_0^x \xi^{\nu-1} e^{-\left(a + {}_{-|b|}\hat{h}\right)\xi} d\xi\; \theta_0 = \frac{1}{\left(a + {}_{-|b|}\hat{h}\right)^{\nu}} \gamma\left(\nu, \left(a + {}_{-|b|}\hat{h}\right) x\right) \theta_0 \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n H_n(a,-b) x^{\nu+n}}{n!(\nu+n)}, \\ \gamma(\nu,x) &= \int_0^x \xi^{\nu-1} e^{-\xi} d\xi, \end{aligned} \tag{43}$$

with $\gamma(\nu,x)$ being the incomplete gamma function [4].

Further examples of the simplification of calculations, involving integrals of the type (43), allowed by the present formalism will be given in a forthcoming paper. Before closing this section, we mention a further problem of practical interest regarding the free electron laser physics.

Example 8 (Free electron laser)**.** *In this type of device, the growth of the electromagnetic field is ruled by an integro-differential equation, which can be cast in the form [10]:*

$$\begin{aligned} &\partial_\tau\, a = i\,\pi\, g_0 \int_0^\tau \tau' e^{-i\nu\tau' - \frac{1}{2}(\pi\mu_\varepsilon)^2\tau'^2} a(\tau-\tau')\, d\tau', \\ &a(0) = 1. \end{aligned} \tag{44}$$

Without taking into account the relevant physical meaning, we note that it is just an integro-differential equation of the Volterra type, which, according to our formalism, can be written as:

$$\begin{aligned} &\partial_\tau\, a = i\,\pi\, g_0 \int_0^\tau \tau' e^{-i\hat{\nu}\tau'} a(\tau-\tau')\, d\tau'\, \theta_0, \\ &\hat{\nu} = \nu + {}_{-\frac{1}{2}(\pi\mu_\varepsilon)^2}\hat{h}. \end{aligned} \tag{45}$$

Equation (45) is hardly amenable to an analytically treatment. The Mathematica code has allowed the derivation of the relevant solutions [10], which are however expressed in the form of a very complicated combination of special functions and is not particularly useful for the relevant physical understanding.

A way to get solutions of practical interest is that of providing a Volterra series expansion, which has also been implemented in Mathematica. We used the umbral methods of this paper to make a comparison with the results of [2,10]. Limiting ourselves to the first order solution in g_0, we find:

$$a \simeq a(0) + i\pi g_0 a_1 \tag{46}$$

where:

$$\begin{aligned} a_1 &= \frac{1}{(\pi\mu_\varepsilon)^5}\left\{(\pi\mu_\varepsilon)\, A_1 + A_2\left[erf\left(i\frac{\delta}{\sqrt{2}\,(\pi\mu_\varepsilon)}\right) - i\, erf\left(i\frac{\nu}{\sqrt{2}\,(\pi\mu_\varepsilon)}\right)\right]\right\}, \\ A_1 &= (\pi\mu_\varepsilon)^4 + i\,\delta - i\,\nu\, e^{-\frac{1}{2}(\pi\mu_\varepsilon)^2\tau^2 - i\tau\nu}, \\ A_2 &= \sqrt{\frac{\pi}{2}} e^{-\frac{\nu^2}{2(\pi\mu_\varepsilon)^2}}\left(\nu^2 - (\pi\mu_\varepsilon)^2(1 + i\,\tau\,\nu)\right), \\ \delta &= \nu - i(\pi\mu_\varepsilon)^2\tau, \qquad \nu > 0. \end{aligned} \tag{47}$$

Even though it was obtained after a significant amount of work, the expression in Equation (47) coincides with that derived with Mathematica. The computation of higher order terms in g_0 becomes heavily complicated to accomplish with the methods described here, notwithstanding the equality between the present treatment and the results from Mathematica is a further confirmation of the reliability of the procedure.

3. Final Comments and Applications

The discussion we developed in the previous sections yields a fairly interesting meaning employing symbolic methods for computational purposes. The technique we propose is, in some sense, self-generating, namely it can be mounted as a kind of Chinese box tool as illustrated below.

Example 9. *Let us therefore consider the following V-transform:*

$${}_V\hat{f}_\mu(x,y,z) = \int_0^\infty t^{\mu-1} e^{-xt-yt^2} J_0(zt)\, dt, \quad \forall x,y,z,\mu \in \mathbb{R} : y > 0, \tag{48}$$

which, according to the previous discussion, can formally be written as:

$$
\begin{aligned}
&{}_V\hat{f}_\mu(x,y,z) = \int_0^\infty t^{\mu-1+z\partial_z} e^{-xt-yt^2}\, dt\, J_0(z) = \sum_{n=0}^{\infty} \frac{\Gamma(2n+\mu)}{n!^2}(-1)^n H_{-(2n+\mu)}(x,y)\left(\frac{z}{2}\right)^{2n} = e^{-\hat{g}\left(\frac{z}{2}\right)^2}\phi_0, \\
&\hat{g}^n\phi_0 := \frac{\Gamma(2n+\mu)}{n!} H_{-(2n+\mu)}(x,y).
\end{aligned} \tag{49}
$$

Example 10. *The umbral form can, e.g., be exploited to derive the integral:*

$$\int_{-\infty}^{\infty} {}_V\hat{f}_\mu(x,y,z)\, dz = 2\sqrt{\pi}\, \hat{g}^{-\frac{1}{2}}\, \phi_0 = 2\Gamma(\mu-1)\, H_{-(\mu-1)}(x,y), \tag{50}$$

as also checked by a direct integration of Equation (48), namely:

$$\int_{-\infty}^{\infty} {}_V\hat{f}_\mu(x,y,z)\, dz = \int_0^\infty t^{\,\mu-1} e^{-xt-yt^2}\left(\int_{-\infty}^{\infty} J_0(zt)\, dz\right) dt = 2\int_0^\infty t^{\,\mu-2} e^{-xt-yt^2}\, dt. \tag{51}$$

The methods we envisaged are, according to the previous examples, eligible for a tool in a wide range of applications. To further stress this point of view, we consider the generalization of the functions by noting that the formalism offers noticeable degrees of freedom yielding a significant amount of new directions that can be developed and applied.

In the following, we discuss a further example of an application concerning the so-called Gaussian flattened beams (*GFB*).

Example 11 (Gaussian flattened beams)**.** *To study the GFB, we introduce the higher order Hermite polynomials, also called lacunary Hermite polynomials (or Kampé de Fériet or Gould Hopper polynomials), defined through the operational identity (see Equation (4.8.4) and the following in [5]):*

$$H_n^{(m)}(x,y) = e^{y\partial_x^m} x^n, \quad \forall m \in \mathbb{N}, \forall x,y \in \mathbb{R}, \tag{52}$$

are specified by the series:

$$H_n^{(m)}(x,y) = n! \sum_{r=0}^{\lfloor \frac{n}{m} \rfloor} \frac{x^{n-mr} y^r}{(n-mr)!\,r!}. \tag{53}$$

As was proven in [5] (Equation (9.5.1)), they can be reduced to the n-th power of a binomial by introducing the umbral operator ${}_y\hat{h}_m^r$ such that, $\forall r \in \mathbb{R}$,

$$
\begin{aligned}
&{}_y\hat{h}_m^r\, {}_m\theta_0 := {}_m\theta_r = \frac{y^{\frac{r}{m}}\, r!}{\Gamma\left(\frac{r}{m}+1\right)} A_{m,r}, \quad r = ms \\
&A_{m,r} = \begin{cases} 1, & s \in \mathbb{Z}, \\ 0, & s \notin \mathbb{Z}, \end{cases}
\end{aligned} \tag{54}
$$

which allows defining them as:

$$H_n^{(m)}(x,y) = \left(x + {}_y\hat{h}_m\right)^n {}_m\theta_0. \tag{55}$$

It is clearly evident that not too much effort is necessary to study the associated functions, which can be derived using the same procedure adopted for the second order case ($m = 2$) as, e.g., higher negative order:

$$H_{-\nu}^{(m)}(x,y) = (x + {}_{-|y|}\hat{h}_m)^{-\nu}\, {}_m\theta_0 = \frac{1}{\Gamma(\nu)} \int_0^\infty s^{\nu-1} e^{-sx} e^{\,{}_{-|y|}\hat{h}_m s}\, {}_m\theta_0\, ds \tag{56}$$

or:

$$H_{-\nu}^{(m)}(x,y) = \frac{1}{\Gamma(\nu)} \int_0^\infty s^{\nu-1} e^{-sx} e^{-ys^m}\, ds, \quad Re(y) > 0. \tag{57}$$

It is therefore evident that:

$$
\begin{aligned}
&I^{(m,1)}(x,y) = \int_0^\infty e^{-sx} e^{-s^m y} ds = H_{-1}^{(m-1)}(x,y), \\
&I^{(m,2)}(x,y) = \int_0^\infty e^{-s^2 x} e^{-s^m y} ds = \sqrt{\pi}\, H_{-1/2}^{(m-2)}(x,y), \\
&\ldots \\
&I^{(m,n)}(x,y) = \int_0^\infty e^{-s^n x} e^{-s^m y} ds = \frac{\Gamma\left(\frac{1}{n}\right)}{n} H_{-1/n}^{(m-n)}(x,y), \\
&m > n.
\end{aligned}
\tag{58}
$$

Super-Gaussians (SG) are used in optics to describe GFB. Limiting ourselves to the one-dimensional case, they are represented by an exponential function of the type:

$$
S_G(x,m) = e^{-x^m}, \tag{59}
$$

where, for simplicity, m is assumed to be an even integer. The relevant propagation has been treated by the use of effective methods' superposition employing Gaussian beams [17], which amount to the approximation of an SG beam with the superposition of a Gauss beam, whose transformations through a lens-like device are well known. The use of an alternative approach (even though less efficient than the flattened beam method) is suggested by means of a Fresnel transform [18], which is written:

$$
S_G(x,m;A,B,D) = \frac{1}{\sqrt{2\pi i B}} \int_{-\infty}^{+\infty} e^{\pm \frac{i}{2B}(Ay^2+2xy+Dx^2)} S_G(y,4)\, dy, \tag{60}
$$

where A, B, D are constants accounting for the optical elements constituting the transport line.

It is evident, according to the previous formalism, that the above integral can be cast in the form:

$$
S_G(x,m;A,B,D) = \frac{e^{\frac{iD}{2B}x^2}}{\sqrt{2\pi i B}} \left[H_{-1}^{(4,2)}\left(\frac{ix}{B}, \frac{iA}{2B}, 1\right) + H_{-1}^{(4,2)}\left(-\frac{ix}{B}, \frac{iA}{2B}, 1\right) \right], \tag{61}
$$

where the Hermite function on the right corresponds to the polynomial:

$$
H_n^{(4,2)}(x,y,z) = n! \sum_{r=0}^{\lfloor \frac{n}{4} \rfloor} \frac{H_{n-4r}(x,y)\, z^r}{(n-4r)!r!} \tag{62}
$$

also defined by the operational identity:

$$
H_n^{(4,2)}(x,y,z) = e^{z\, \partial_x^4} H_n(x,y) \tag{63}
$$

and extended to the associated functions defined as:

$$
H_{-\nu}^{(4,2)}(x,y,z) = \frac{1}{\Gamma(\nu)} \int_0^\infty s^{\nu-1} e^{xt+yt^2-zt^4} ds, \quad Re(y) > 0. \tag{64}
$$

The possibility we envisaged of using Hermite functions to study the propagation of SG beams needs various refinements to become an effective tool, notwithstanding that it provides a further proof of the possibility offered by these techniques.

Before closing this paper, we provide a further example concerning the Gauss–Weierstrass transform.

Example 12 (Gauss–Weierstrass transform)**.** *Within the present context, the Gauss–Weierstrass transform can be viewed as being generated from the operational identity:*

$$e^{\,y\,\partial_x^2} f(x) = \frac{1}{2\sqrt{\pi y}} \int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4y}} f(\xi)\, d\xi = \frac{1}{2\sqrt{\pi y}} \int_{-\infty}^{\infty} e^{-\frac{\xi^2}{4y}} e^{-\frac{x^2}{4y}+\frac{x\xi}{2y}} f(\xi)\, d\xi. \tag{65}$$

The use of the generating function of Hermite polynomials allows casting Equation (65) in the form:

$$\begin{aligned} &e^{\,y\,\partial_x^2} f(x) = \frac{1}{2\sqrt{\pi y}} \sum_{n=0}^{\infty} \frac{x^n}{n!} \left({}_H\hat{f}(y) \right), \\ &{}_H\hat{f}(y) := \int_{-\infty}^{\infty} e^{-\frac{\xi^2}{4y}} \left(\frac{\xi}{2y} + {}_{-\frac{1}{4|y|}}\hat{h} \right)^n f(\xi)\, d\xi\, \theta_0, \qquad\qquad y > 0, \end{aligned} \tag{66}$$

where ${}_H\hat{f}(y)$ is the Hermite transform of the function $f(x)$. The relevant properties as a function of y will be discussed elsewhere.

Example 13 (Laguerre transform)**.** *The same point of view can be followed to introduce the Laguerre transform, which can be derived from the identity:*

$$e^{-y\,\partial_x x \partial_x} f(x) = \int_0^{\infty} e^{-\sigma} e^{\frac{x}{y}} C_0\left(\frac{x}{y}\sigma \right) f(-y\sigma)\, d(\sigma). \tag{67}$$

The use of the generating function leads therefore to the identification of the Laguerre transform ${}_Lf(n)$ [19], as shown below:

$$\begin{aligned} &\int_0^{\infty} e^{-\sigma} e^{\frac{x}{y}} C_0\left(\frac{x}{y}\sigma \right) f(-y\sigma)\, d(\sigma) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{x}{y} \right)^n \, {}_Lf_n(y), \\ &{}_Lf_n(y) = \int_0^{\infty} e^{-\sigma} \left(1 - \hat{c}\sigma\right)^n f(-y\sigma)\, d\sigma\, \varphi_0. \end{aligned} \tag{68}$$

Either the Laguerre or Hermite transform is linked to the orthogonal properties of these families of polynomials, and the relevant definitions can be extended to higher order polynomials or to multivariate Hermite polynomials.

4. Conclusions

Hermite and Laguerre calculus, as we presented here, is a by-product of the umbral formalism. It has been shown to disclose noticeable potentialities in terms of flexibility and allows further extensions, which are worth being pursued. Within the context of this formalism, old and new concepts associated with various types of transforms acquire a unitary flavor and appear to be amenable for further extensions, which will be presented in forthcoming investigations.

Equation (25) may indeed be considered a new type of Gauss–Weierstrass transform by virtue of the umbral properties of the Bessel function, which is viewed also as an image of the Gaussian function.

The novelty of the paper comes from a general and unified view of addressing different problems, involving either special functions and transforms, within a unique operational point of view. The technique we considered benefits from algebraic, theoretic, and symbolic methods and yields a unitary thread connecting apparently uncorrelated topics.

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