On the Computation of Degrees-of-Freedom: A Didactic Perspective

Article · January 2005

DOI: 10.1115/DETC2005-84109

2 authors:

Massimo Cavacece
Università degli studi di Cassino e del Lazio Meridionale

Leonardo Vita
INAIL Istituto Nazionale per l'Assicurazione contro gli Infortuni sul Lavoro

Proceedings of IDETC'05
2005 ASME International Design Engineering Technical Conferences and Computers and Information in Engineering Conference
Long Beach, California, USA, September 24-28, 2005

DETC2005-84109

ON THE COMPUTATION OF DEGREES-OF-FREEDOM:A DIDACTIC PERSPECTIVE

E. Pennestrì *, M. Cavacece, L. Vita
Department of Mechanical Engineering
University of Rome Tor Vergata
Via del Politecnico 1, 00133 Rome (Italy)
Email: pennestri@mec.uniroma2.it
Tel. +39-0672597138 Fax. +39-062021351

ABSTRACT

Some didactic issues associated with the computation of degrees-of-freedom (d.o.f.) are herein discussed. In particular, the paper reports different definitions and methodologies of computation of this important parameter. It is also proposed an analytical approach to the d.o.f. computation of planar figures subjected to unilateral constraints. Mechanisms with variable kinematic structure are included in the present analysis. Some ambiguities in the d.o.f. definition are pointed out.

1 Introduction

The concept of degree of freedom (d.o.f.) is of practical use, but it must be acknowledged that has its own limitation. In fact, for its computation important simplifying hypotheses for the modeling of the mechanical system under analysis must be introduced. These hypotheses may cause a significant difference between the model and actual mechanism behavior. For instance, there is large class of mechanisms (denoted as *overconstrained mechanisms*) whose mobility is due to precise proportions of their parts, input links, geometric configuration. Moreover, under certain (singular) configurations, due to a sudden d.o.f. variation, numerical results of computer programs may be unreliable when precautions are not taken. Finally, joint tolerances, elasticity often play a determinant role in the mechanisms mobility.

Kinematics is the science of constrained motion. Thus, it is of practical interest to determine how many independent inputs must be prescribed in a mechanism in order to obtain a constrained motion of all the links.

This type of analysis can be preliminarily carried out by means of simple formulas requiring only the knowledge of the number of links l, the number j and nature of kinematic pairs. In particular these formulas are usually obtained subtracting the number of constraints imposed by the kinematic pairs from the degrees-of-freedom of the free moving links. However, they may fail to provide the correct answer.

Thus, the training of a mechanical engineer should make him/her aware:

- of the hypotheses introduced when investigating the mobility of a mechanism or when computing its d.o.f.;
- of the theoretical limits of some topological formulas;
- of the causes and numerical effects due to the d.o.f. variation;
- of some guidelines useful to identify idle or redundant d.o.f.;
- on how to recognize and identify the critical configuration of a mechanism;
- on how to compute the d.o.f. of an overconstrained mechanism;

In some simple cases the engineer should be trained to compute the link proportions that ensure the mechanism mobility of overconstrained linkages.

Despite the importance of the topic, standard textbooks of theory of machines (e.g. [1, 3]) usually dedicate very little space on methodologies for completing the listed tasks.

*Address all correspondence to this author.

On the contrary, the scientific literature records important monographies (e.g. [4, 5]) entirely dedicated to the topic (e.g. [4, 5]). However, the higly specialized nature of these contributions does not always help to an elementary and didactic effective introduction to the subject.

The main purpose of this paper is to report the choices and experiences of the authors when teaching this topic at a second year mechanical engineering course and call the attention on some contradictory definitions of d.o.f. The authors hope that the paper contents may give hints on more effective didactic approaches to the d.o.f. computation.

Somewhat novel analytical approaches of mobility analysis, for planar figures subjected to unilateral constraints and variable kinematic structure mechanisms, are also discussed.

Considered the tutorial purpose of this paper, are omitted those treatments requiring a knowledge outside the common theoretical background of an undegraduate engineering student.

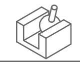
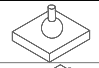
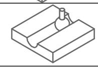
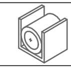
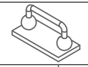
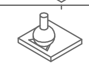
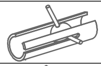
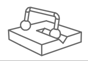
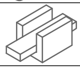
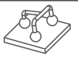
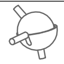
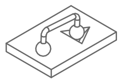
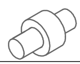
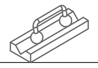
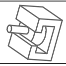
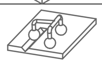
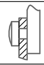
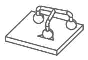
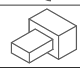
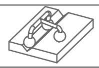
The paper is divided in the following parts:

- Review of some definitions of d.o.f.
- Review of frequently used topological formulas and of their extensions.
- State the mathematical bases for the matrix d.o.f. analysis.
- Proposal of an analytical method for computing the d.o.f. of mechanisms with variable kinematic structure.
- Comparison of the effectiveness of different definitions when the d.o.f. of a mechanism is assessed.
- Mobility analysis of planar figures with unilateral constraints.
- Case-studies.
- Conclusions

The following nomenclature is herein adopted:

- F :degrees-of-freedom of the mechanism (d.o.f.);
- f_i : degrees-of-freedom of the i^{th} kinematic pair;
- l : number of links (frame included);
- L_{ind} :number of independent circuits;
- j : number of kinematic pairs;
- j_i : number of kinematic pairs with i degrees-of-freedom;
- m_i : i^{th} independent, scalar, displacement variable of mechanisms (associated with the relative displacements at a joint);
- M total number of independent, scalar, displacement variables;
- p_i : number of kinematic pairs which introduce i degrees-of-constraint;
- λ: (mobility number) degree-of-freedom of space within which the mechanism operates *e.g.* (=3 for planar and spherical space), (=6 spatial space);
- λ_i: number of independent, scalar, loop-closure equations associated with the i^{th} independent loop.

Table 1. Classification of kinematic pairs according to their d.o.f. (Adapted from [22])

Degrees of freedom	Free rotations	Free translations	Name	Kinematic pair: Form closure	Kinematic pair: Force closure
5	3	2	*Sphere-plane*		
4	3	1	*Sphere-groove*		
	2	2	*Cylinder-plane*		
3	3	0	*Spheric*		
	2	1	*Sphere-slotted cylinder*		
	1	2	*Planar*		
2	2	0	*Slotted spheric*		
	2	0	*Toric*		
	1	1	*Cylindric*		
	1	1	*Slotted cylinder*		
1	1	0	*Revolute*		
	0	1	*Prismatic*		

2 Kinematic structure and kinematic pairs classification

Through the *kinematic structure* analysis are gathered all the essential informations about which link is connected to the remaining links and to the nature of kinematic joints. This is a the first step in mechanical systems analysis. For this task the correspondence between graphs and mechanisms seems very appropriate.

The kinematic pairs can be classified according to their degrees-of-freedom (see Table 1).

3 Some definitions of the term degree-of-freedom

The mobility analysis requires a correct and complete definition of what is meant with the term *degrees-of-freedom* of a mechanical system.

The following list of definitions has been compiled from textbooks:

Definition 1 :
"If $(\delta q_1, \delta q_2, \ldots, \delta q_n)$ are arbitrary infinitesimal increments of the coordinates in a dynamical system these will define a possible displacement if the system is holonomic, while for non-holonomic systems a certain number, say m of equations must be satisfied between them in order that they may correspond to a possible displacement. The number $(n-m)$ is called the *number of degrees of freedom* of the system." ([2], p.34)
Definition 2 :
"The number of degrees of freedom of a system is the number of independent variables that must be specified to define completely the condition of the system. In the case of kinematic chains, it is the number F of independent pair variables needed to completely define the relative positions of all links." ([6], p.133)
Definition 3 :
"By degrees of freedom we mean the number of independent inputs required to determine the position of all links of the mechanism with respect to ground." ([3], p. 16)
Definition 4 :
"Grübler was the first to study the relationship between the mobility of a plane four-bar linkage and the degrees of freedom of the individual members and joints. The degrees of freedom in a plane kinematic chain can be found by adding the number of degrees of freedom for the links in the mechanism taken separately and then subtracting the degrees of freedom lost as the links are assembled." ([9], p.103)
Definition 5 :
"In general, there will be N explicit equations of constraint associated with a given system; they may be expressed in the form

$$F_i(\psi_1, \psi_2, \ldots, \psi_M, t) = 0 \quad (i = 1, 2, \ldots, N) \tag{1}$$

Lagrangian coordinates may be chosen in a wide variety of ways. However, for holonomic systems the minimum number of such variables needed to define the position of every particle in the system is, *by definition*, the *degree of freedom* F. $\cdots$ When we use more Lagrangian variables than the minimum required for a complete set, we say we are using redundant coordinates. For each redundant coordinate introduced, there exists one explicit equation of constraint. $\cdots$ If M Lagrangian variables are chosen for a system with a maximum of C independent equations of constraints among them, it is possible, in principle, to compute C of the coordinates from a knowledge of the other $(M-C)$ coordinates. Accordingly, the system has

$$F = M - C \tag{2}$$

degrees of freedom. If there is any doubt about the number of independent[1] explicit constraint equations associated with a given choice of M, one can always calculate F from equation (1) (for the associated discrete particle model)." ([7], p.265-266)

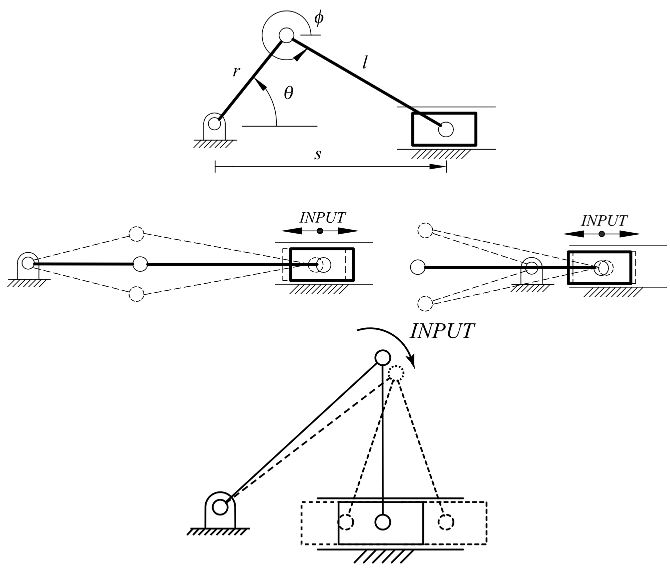


Figure 1. Some critical configurations of the slider-crank mechanism

Definition 6 :
...the number of parameters needed to specify the configuration of a mechanism, in terms of the number of links and joints and the freedom of movement allowed at each joint. This number is the *degree of freedom* or *mobility* of the mechanism. Changing the values of these parameters changes the configuration of the mechanism. Thus, if we view the set of all configurations available to the mechanism as a manifold in a higher dimensional space, then the mobility of the mechanism is the dimension of this manifold. ([10], p.67)

We can divide the mobility criteria of a mechanism into two categories:

- topological;
- analytical.

Mobility criteria based on mechanism topology allow to compute the mobility depending solely on the number of links, joints and joints type.

The analytical criteria require a more sophisticated approach, often based on calculus and geometry.

4 A review of some topological formulas for computing the degrees of freedom

Due to space limitations, only the formulas that are believed to be most useful for didactic purposes are herein mentioned.

Grübler formula (1883)

The Grübler formula is likely one of the oldest relation for

[1] A more formal determination of the number of independent equations of constraint is given by the rank of the Jacobian matrix formed from Eqs. (1).

computing the degrees of freedom of a linkage, although Grübler himself recognized in his writings that both Chebyshev (1869) and Sylvester (1874) were aware of equivalent relations.

$$F = 3\,(l-1) - 2j_1 - j_2\ . \tag{3}$$

The formula is valid for planar mechanisms only.

Somov-Malyshev formula (1923)

$$F = 6\,(l-1) - 5p_5 - 4p_4 - 3p_3 - 2p_2 - p_1\ . \tag{4}$$

In this form, the formula is valid for spatial mechanisms. Interesting variations of (4) to accommodate redundant constraints mechanisms are discussed by Ruzinov [18].

Kutzbach formula (1933)

$$F = \lambda\,(l-j-1) + \sum_{i=1}^{j} f_i\ . \tag{5}$$

The formula can be applied both to planar and spatial mechanisms by choosing properly the value of λ.

Sometime in a mechanism there are degrees-of-freedom which do not have any effect on the mathematical relationship between input and output links. These are called *idle* or *passive* degrees of freedom. For example, a binary link connected to adjacent spherical joints (S-S) can rotate freely about the axis through the centers of the spheres. However, such movement has not influence on the motion of the remaining links.

Let C represent cylindrical pair, E, plane pair, R, revolute pair and S spherical pair. The Table 2 summarizes some cases of binary links whose motion can be associated with passive d.o.f.

Kutzbach's formula can be modified as follows [19]

$$F = \lambda\,(l-j-1) - f_p + \sum_{i=1}^{j} f_i\ , \tag{6}$$

where f_p are the passive d.o.f. in the mechanism. Passive d.o.f. may appear also in spatial mechanisms such as the RSSR mechanism.

Buchsbaum-Freudenstein (1970)

This formula is valid for gear drive mechanisms [12] only. Substituting in (5) [8]

$$L_{ind} = j - l + 1\ , \tag{7}$$

$$j_G = L_{ind}\ , \tag{8}$$

Table 2. Binary links and passive d.o.f.

Adjacent kinematic pairs	Passive d.o.f.
SS	Rotation about the axes through the centers of spheres.
SC	Rotation about the axis of the cylindric pair through the center of S.
SE	Rotation about the axis $\perp$ to E through the center of S.
EE	Translation along an axis parallel to the axis common to both E planes. There are 3 passive d.o.f. when the planes are parallel (1 rotation and 2 translations.)

one obtains

$$F = j_R - j_G\ , \tag{9}$$

where j_R and j_G are the number of geared and revolute pairs, respectively.

Freudenstein-Alizade (1975)

The value of F is computed as a difference between the number of independent scalar displacement variables and independent scalar closure equations:

$$F = \sum_{i=1}^{M} m_i - \sum_{i=1}^{L} \lambda_i \tag{10}$$

In order to simplify the application of (10) three particular cases can be considered:

1. Displacement variables are in 1:1 correspondence with kinematic pairs d.o.f.
   The following equality hold:

$$\sum_{i=1}^{M} m_i = \sum_{i=1}^{j} f_i\ . \tag{11}$$

2. The number of independent scalar loops is the same for each mechanism circuit.
   The following equality hold:

$$\lambda_i = \lambda \quad (i = 1, 2, \ldots, L_{ind}) \tag{12}$$

3. Previous conditions hold simultaneously.
   The equation (10) can be rewritten as follows

$$F = \sum_{i=1}^{j} f_i - \lambda L_{ind} \; . \qquad (13)$$

   Considered (7), from (13), one obtains (5).

5 Matrix method of d.o.f. analysis

Let us consider a finite number n of generalized coordinates q_k $(k = 1, \cdots, n)$, which define the positions of all the links of a mechanism, and let p the number of independent equations that can be established between the infinitesimal variations $(\delta q_1, \delta q_2, \cdots, \delta q_n)$.

Then, according to Whittaker [2], the mobility of a mechanical system is obtained from

$$F = n - p \; . \qquad (14)$$

This expression requires the computation of p.

If the set of equations

$$\begin{cases} \psi_1 (q_1,\ q_2, \cdots, q_n) = 0 \\ \psi_2 (q_1,\ q_2, \cdots, q_n) = 0 \\ \cdots \\ \psi_m (q_1,\ q_2, \cdots, q_n) = 0 \end{cases} \qquad (15)$$

can be established between the coordinates q_k, then the following theorem hold [14][2]:

Theorem. *Given m compatible functions* $\psi_j\, (j = 1,\ 2, \cdots, m)$ *of any number n of variables* $q_k\, (k = 1,\ 2,\ \cdots, n)$*, if the rank of the Jacobian matrix is r, then there are* $m - r$ *relations (and not more) between the* ψ_j *which do not involve the* q_k*.*

As a corollary, if the functions are independent (*i.e.* $r = m$) there exists not any relation between them.

The theorem just stated supply us a criterion for testing the existence of functions of the type

$$F (\psi_1,\ \psi_2, \cdots, \psi_m) = 0 \; . \qquad (16)$$

involving the ψ's only and not the q's.

[2]The tight connection between the d.o.f. definition of E.T. Whittaker and the following theorems is also by witnessed by the fact that the Whittaker himself recommended the complete translation of the Italian text of T. Levi-Civita *Lezioni di calcolo differenziale assoluto* [14].

Once the absence of relations of the type (16) has been ascertained, one can proceed to the computation of p.

For this purpose let us partition[3] the vector $\{q\}$ in

1. *dependent* coordinates

$$\{y\} = \left\{ y_1\ y_2\ \cdots\ y_m \right\}^T \; ,$$

2. *independent* coordinates

$$\{x\} = \left\{ x_1\ x_2\ \cdots\ x_F \right\}^T \; ,$$

The theorem of existence of implicit functions states that [15,16]

Theorem. *Let* $\psi_1,\ \psi_2, \cdots, \psi_m$ *denote real single-valued compatible functions of a finite number of variables* $(q_1,\ q_2, \cdots, q_n)$*.*

If the following conditions hold simultaneously

1. $\left\{q^{(0)}\right\} = \left\{ x_1^{(0)}\ x_2^{(0)}\ \cdots\ x_F^{(0)}\ y_1^{(0)}\ y_2^{(0)}\ \cdots\ y_m^{(0)} \right\}^T$ *is a solution of the system of equations*

$$\begin{cases} \psi_1 \left(q_1^{(0)},\ q_2^{(0)}, \cdots, q_n^{(0)}\right) = 0 \; , \\ \psi_2 \left(q_1^{(0)},\ q_2^{(0)}, \cdots, q_n^{(0)}\right) = 0 \; , \\ \cdots \\ \psi_m \left(q_1^{(0)},\ q_2^{(0)}, \cdots, q_n^{(0)}\right) = 0 \; ; \end{cases} \qquad (17)$$

2. *the* $\psi_1,\ \psi_2, \cdots, \psi_m$ *and all their first partial derivatives are continuous over a neighborhood* $\left\{q^{(0)}\right\}$*;*
3. *the determinant of the Jacobian*

$$J\left(\frac{\psi_1, \cdots, \psi_m}{y_1, \cdots, y_m}\right) = \begin{bmatrix} \frac{\partial \psi_1}{\partial y_1} & \frac{\partial \psi_1}{\partial y_2} & \cdots & \frac{\partial \psi_1}{\partial y_m} \\ \frac{\partial \psi_2}{\partial y_1} & \frac{\partial \psi_2}{\partial y_2} & \cdots & \frac{\partial f_m}{\partial y_m} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \psi_m}{\partial y_1} & \frac{\partial \psi_m}{\partial y_2} & \cdots & \frac{\partial \psi_m}{\partial y_m} \end{bmatrix} \qquad (18)$$

   is different than zero;

then the (15)*, within a neighborhood of* $\left\{q^{(0)}\right\}$ *define* $(y_1,\ y_2, \cdots, y_m)$ *as single-valued functions of* $(x_1,\ x_2, \cdots, x_F)$*.*

If the conditions mentioned by the previously stated theorem are all satisfied, then

$$p = m \; , \qquad (19a)$$

[3]At the beginning the value of F is conjectured. The subsequent analysis is aimed to verify the conjecture.

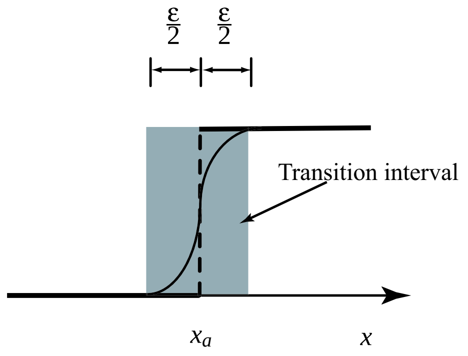


Figure 2. Approximation to Heaviside step function when $x = x_a$

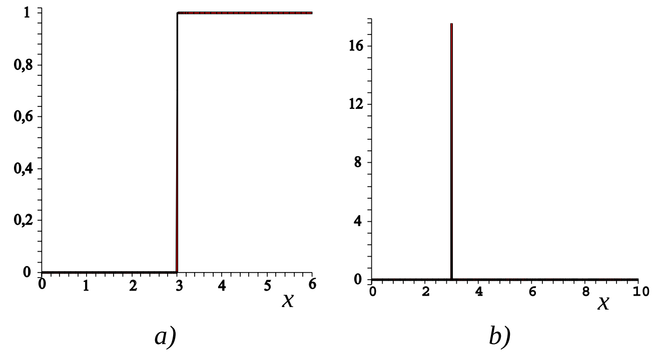


Figure 3. Plots of $L(x)$ and $L'(x)$ when $x_a = 3, n = 3, \varepsilon = 10^{-2}$

else,

$$p = r \ , \tag{19b}$$

when (18) has rank r. Although its reliability is higher than formulas based on topology, at the textbook level, this approach has been rarely reported [7, 17].

The matrix approach can evidence how geometry, link positions, and input link affect the degree of freedom of a mechanism.

Since the criterion discussed in this section requires the evaluation of constraint equations derivatives, the conclusions on the d.o.f. value are limited to a given configuration and limited to infinitesimal displacements.

The configurations of a mechanism without a full rank of the Jacobian are named *critical*. If rank deficiency is maintained for a finite range of movement, then the critical form is said *permanent*, otherwise *instantaneous*.

6 Degrees of freedom analysis of variable kinematic structure and intermittent motion mechanisms

In this section it is hinted the use of logical functions for d.o.f. computation of mechanisms with variable kinematic structure or intermittent motion. We assume that the matrix method is adopted.

Logical functions are an useful mathematical tool for modeling the kinematics and the dynamics of intermittent and variable kinematic structure mechanisms. For our purposes, the occurrence of discontinuities of kinematic structure can be treated by introducing *ad hoc* logic conditions that regulate the type and the number of kinematic constraints that must be taken into account. Let $L(x)$ be a continuous function

$$L(x) = \frac{1}{2}\frac{|x|^{2n+1} + x^{2n+1}}{|x|^{2n+1} + \frac{1}{2}\left[|x-\varepsilon|^{2n+1} - (x-\varepsilon)^{2n+1}\right]} = \begin{cases} 0 & x \le 0 \\ \frac{1}{2} & y = \frac{\varepsilon}{2} \\ 1 & x \ge \varepsilon \end{cases} \tag{20}$$

where $\varepsilon > 0$ is the amplitude of the *transition interval* (see Figure 2) from one state to another and n is chosen so as to assure continuity of any derivative, of order d, which will be true if

$$2n + 1 > d \ . \tag{21}$$

Equation (20) approximates the ideal Heaviside step function $H(x)$

$$H(x) = \begin{cases} 0 & \text{se } x < 0 \\ 1 & \text{se } x > 0 \ . \end{cases} \tag{22}$$

The first and second derivatives of (20) give an approximation of the δ Dirac's and doublet functions, respectively. The step function at abscissa $x = x_a$ is obtained substituting in (20) $(x - x_a)$ at x. Several investigations confirmed the reliability and accuracy of dynamic analysis results through the use of logical functions.

7 Case studies

The double slider-crank

By means of the theorem on the existence of implicit functions, one can find the dimensions of the linkage which ensure a permanent critical form of the mechanism. Alternatively, one can find both dimensions and configuration for an instantaneous critical form.

The loop constraint equations for this mechanism are

$$\{\Psi\} = \begin{Bmatrix} r_1\cos\theta_1 + r_{21}\cos\theta_2 - s_3 \\ r_1\sin\theta_1 + r_{21}\sin\theta_2 \\ r_1\cos\theta_1 - r_{22}\cos\theta_2 \\ r_1\sin\theta_1 - r_{22}\sin\theta_2 - s_4 \end{Bmatrix} \tag{23}$$

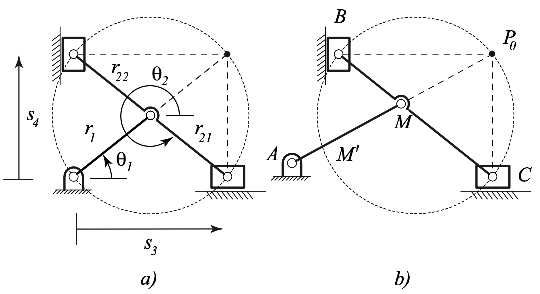


Figure 4. Double slider-crank in *a)* permanent and *b)* instantaneous critical form

The determinant of the Jacobian matrix is

$$|J| = \left| \frac{\Psi_1, \Psi_2, \Psi_3, \Psi_4}{\theta_1, \theta_2, s_3, s_4} \right| = r_{22} \sin\theta_2 \cos\theta_1 + r_{21} \sin\theta_1 \cos\theta_2 \tag{24}$$

This will be always zero when $r_{22} = r_{21}$ and $\theta_1 + \theta_2 = 2\pi$. It must be observed that (24) is fulfilled when the normals to the trajectory paths of points M, B and C simultaneously converge in only one point P (center of instantaneous rotation). The coupler will have an instantaneous mobility. Applying Euler-Savary equation, one conclude that the mobility can be up to second order infinitesimal displacements when $AM = P_0M^2/M'M$.

Mobility of a figure subjected to unilateral constraints

An interesting problem is the assessment of the degree of constraint of a planar figure. The problem received a graphical solution by Reuleaux [13, 17]. The matrix method for computing the d.o.f. is herein applied. Without loss of generality, let us assume that the planar figure:

- is an ellipse with

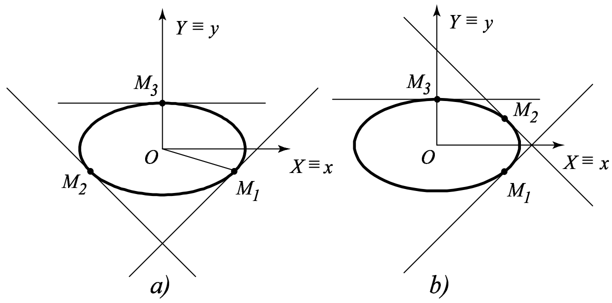
$$x_M = a\cos\tau \quad y_M = b\ sin\tau\ , \tag{25}$$

  as parametric equation;
- has initially three points of tangency with straight lines.

Our purpose is to investigate the slopes m_1, m_2 and m_3 of the three straight lines which fully constrain the ellipse.

The coordinates x_{M_i}, y_{M_i} of tangency points are located solving the equations

$$\tan\tau_i = -\frac{b}{am_i} \tag{26}$$



w.r.t. τ_i $(i = 1, 2, 3)$ and making use of (25).

A moving Cartesian system is attached to the ellipse. The origin (X_0, Y_0) is in the center of ellipse and the moving axes are directed as shown in Figure 7 and initially aligned with fixed axes (*i.e.* $\theta = 0$).

The absolute coordinates of the tangency points are

$$\begin{array}{l} X_{M_i} = X_0 + x_{M_i}\cos\theta - y_{M_i}\sin\theta \\ Y_{M_i} = Y_0 + y_{M_i}\sin\theta + y_{M_i}\cos\theta \end{array} \quad (i = 1, 2, 3) \tag{27}$$

When the points on the figure are initially in contact with the straight lines, any infinitesimal displacement of the figure must satisfy the following kinematic conditions:

$$\begin{Bmatrix} \delta X_{M_i} \\ \delta Y_{M_i} \end{Bmatrix}^T \begin{Bmatrix} n_{x_i} \\ n_{y_i} \end{Bmatrix} \geq 0\ , \quad (i = 1, 2, 3) \tag{28}$$

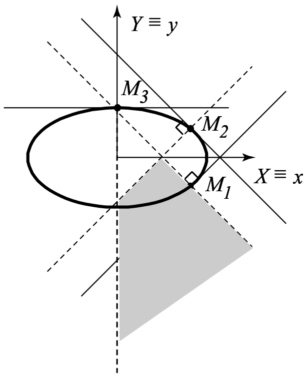
where $\{n\} = \left\{ n_{x_i}\ n_{y_i} \right\}$ is the versor of the normal to the i^{th} straight line oriented toward the inside of the figure. The equality sign holds when the i^{th} contact point is required to maintain contact with the straight line.

In matrix notation, eq. (28) can be rewritten as follows

$$\begin{bmatrix} n_{x_1}\ n_{y_1}\ (n_{y_1}a\cos\tau_1 - n_{x_1}b\sin\tau_1) \\ n_{x_2}\ n_{y_2}\ (n_{y_2}a\cos\tau_2 - n_{x_2}b\sin\tau_2) \\ n_{x_3}\ n_{y_3}\ (n_{y_3}a\cos\tau_3 - n_{x_3}b\sin\tau_3) \end{bmatrix} \begin{Bmatrix} \delta X_0 \\ \delta Y_0 \\ \delta\theta \end{Bmatrix} \geq \{0\} \tag{29}$$

When the above linear inequalities system does not have any feasible solution, then the figure is fully constrained.

A noteworthy case is when equalities apply and the coefficient matrix of (29) does not have a full rank. From the application of theorem of existence of implicit functions we deduce that the figure may have an infinitesimal displacement. This case is depicted in Figure 7 a) where the slopes of the straight lines are $m_1 = 1$, $m_2 = -1$ and $m_3 = 0$. The normals to the velocities of M_1, M_2 and M_3 simultaneously converge in only one point (center of instantaneous rotation).



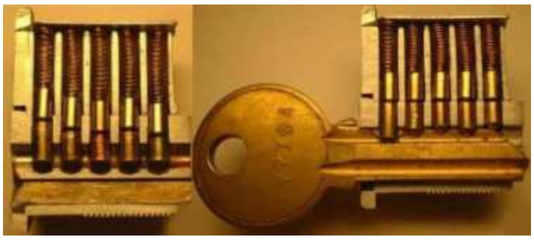


Figure 5. The Yale pin tumbler lock mechanism [21].

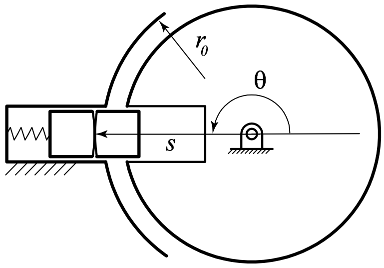
In the case depicted in Figure 7 b), (29) has the following solution

$$\delta X_0 \leq 0\ , \quad (30a)$$
$$\delta Y_0 \leq 0\ , \quad (30b)$$
$$\frac{\sqrt{2}}{2}\delta X_0 - \frac{\sqrt{2}}{2}\delta Y_0 \leq .769219158 1\delta\theta \quad (30c)$$
$$\frac{\sqrt{2}}{2}\delta X_0 + \frac{\sqrt{2}}{2}\delta Y_0 \leq -.769219158 1\delta\theta \quad (30d)$$

A geometric interpretation of the result obtained is shown in Figure 7. The feasible area of a center of instantaneous rotation of the figure is within the grey area and only c.c.w. rotations are admitted (*i.e.*$\delta\theta > 0$). In particular, the last two equations of eqs. (30) are the equations of the dotted straight lines through M_1 and M_2. This result is consistent with the graphical constructions described by Reuleaux.

The Yale type lock mechanism
The cylinder lock, shown in Figure 5, has five pins which are cut through. When the proper key is inserted in the lock these cuts all line up allowing the cylinder to be rotated and the lock to be opened.

A simplified planar model of this mechanism is shown in Figure 7 The kinematics of this mechanism can be described by



means of the following constraint equations

$$\{\Psi\} = \begin{Bmatrix} [1 - L_1(s - r_0)](\theta - \pi) \\ [1 - L_1(s - r_0)](s - p(t)) \\ L_1(s - r_0)(\theta - \alpha(t)) \\ L_1(s - r_0)(s - s_0) \end{Bmatrix} \quad (31)$$

where logic functions are used. The first two constraints are valid when the cylinder is not allowed to rotate.

8 Discussion

The d.o.f. definitions and computation criteria reviewed in this paper can be divided in two broad categories:

1. those based only on kinematic structure analysis;
2. those based on analytical criteria.

Due to their simplicity, the first type is always discussed and included under various algebraic forms in textbooks. The second type is less frequently reported.

Although the approaches for computing the d.o.f. are based on different simplifying hypotheses, it should be acknowledged that there are significant differences in the *definitions* of d.o.f. herein reviewed. These affect the d.o.f. estimate.

For example, in the slider-crank shown in Figure 1, only one variable is required to specify the relative positions of all links or to determine their *positions* w.r.t. the ground link. Thus, according to Definitions 2, 3 and 4, this linkage has $F = 1$ d.o.f.

However, since the slider-crank is in critical configurations (see Figure 1), from Definitions 1 or 5, one would conclude that the mechanism has instantaneously $F = 2$ two d.o.f. In fact, for a given infinitesimal displacement of the input link, the *infinitesimal displacements* of the remaining links are not uniquely defined.

This ambiguity follows directly from the different definitions of d.o.f. and not from simplifying hypotheses.

REFERENCES

[1] Shigley, J.E., Uicker, J.J., 1980, *Theory of Machines and Mechanisms*, McGraw-Hill Book Company.

[2] Whittaker, E.T., 1937, *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies*, Cambridge University Press, Fourth Edition.
[3] Sandor, G.N., Erdman, A.G., 1984, *Advanced Mechanism Design*, vol.I and II Prentice-Hall Inc..
[4] Reshetov, L., 1982, *Self-aligning Mechanisms*, Mir Publishers.
[5] Phillips, J., 1984, *Freedom in Machinery*, Cambridge University Press, vol.I and II.
[6] Hartenberg, R.S., Denavit, J., 1964, *Kinematic Synthesis of Linkages*, McGraw-Hill.
[7] Paul, B., 1979, *Kinematics and Dynamics of Planar Machinery*, Prentice-Hall.
[8] Paul, B., 1960, A Unified Criterion for the Degree of Constraint of Plane Kinematic Chains, *ASME Journal of Applied Mechanics*, vol.27, pp.196-200.
[9] Suh, C.H., Radcliffe, C.W., 1978, *Kinematics and Mechanism Design*, John Wiley and Sons.
[10] McCarthy, J.M., 1990, *Introduction to Theoretical Kinematics*, The MIT Press.
[11] Freudenstein, F., Alizade, R., 1975, On the Degree of Freedom of Mechanisms with Variable General Constraint, *Proceedings Fourth World Congress on the Theory of Machines and Mechanisms*, New Castle Upun Tyne, vol.I, pp.51-56.
[12] Freudenstein, F., Buchsbaum, F., 1970, Synthesis of Kinematic Structure of Geared Kinematic Chains and other Mechanisms, *Journal of Mechanisms*, vol.5, pp.357-392.
[13] Reuleaux, F., 1876, *The Kinematics of Machinery*, Dover Publications, Inc., New York, p.98. (Reprint by Dover Publications, 1977)
[14] Levi-Civita, T., 1926, *The Absolute Differential Calculus*, Blackie and Son Ltd, (Reprint by Dover Publications, 1977)
[15] Freudenstein, F., 1962, On the Variety of Motions Generated by Mechanisms, *ASME Journal of Engineering for Industry*, B4 Series B, pp.156-160.
[16] Litvin, F.L., 1980, Application of Implicit Function System Existence for Analysis and Synthesis of Linkages, *Mechanism and Machine Theory*, vol.15, pp.115-125.
[17] Di Benedetto, A., Pennestrì, E., 1993, *Introduction to Mechanism Kinematics*, Casa Editrice Ambrosiana, vol. I. (in Italian)
[18] Ruzinov, L.D., 1968, *Design of Mechanisms by Geometric Transformations*, London Iliffe Books Ltd.
[19] Tsai, L.W., 2001, *Mechanism Design: Enumeration of Kinematic Structure According to Function*, CRC Press.
[20] Söylemez, E., Logical Functional Design and Optimization of Mechanisms and Mechanical Systems, Doctoral Dissertation, Columbia University, 1974.
[21] Web site http://www.yalelock.com
[22] Soni, A.H., *Mechanisms Synthesis and Analysis*, Malabar, FL: Krieger Publishing, 1986. Reprint of 1974 edition.