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Dissertation

Sharing and Cooperation in Markets and Organizations

Axiomatic Foundations for the Laws of Coalition Formation in Selected Scenarios

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Interactions involving multiple parties and necessitating their agreement are pervasive in both market and non-market settings. As the number of participants increases, these situations become progressively complex to describe and analyse. Despite the prolific nature of such scenarios, a comprehensive conceptual framework addressing such settings is often lacking.

The focus of this dissertation lies in a distinct type of multilateral interaction, where a commitment of a group, or a coalition, of participants is required for achieving a positive surplus. The analysis encompasses three scenarios, namely, government formation in parliamentary democracies, bilateral trading on a market with multiple buyers and sellers, and resource allocation in the US presidential campaign. This dissertation proposes an approach that provides axiomatic foundations for a theory of coalition formation in these settings and, for two of these scenarios, simultaneously provides an empirically accurate forecast methodology.



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List of Abbreviations

TU Transferable Utility

1 Introduction

Situations requiring an agreement of multiple participants represent a common type of market and non-market interactions. Examples of such scenarios include decision-making within international organizations on various issues, international trade agreements, government formation, market trading scenarios, and cartel agreements.

Complexity is an inherent characteristic of such multilateral interactions and increases with the number of parties involved, “while an explicit conceptual treatment of the subject is often lacking” (Zartman, 1994).

This dissertation focuses on a particular type of multilateral interaction, where the commitment of a group, or a coalition, of participants is required for a positive surplus to be realized. Compelling examples of such interactions include:

- Government formation in parliamentary democracies;
- Trading scenarios on a market with multiple buyers and sellers;
- Resource allocation in the US presidential campaign.

Two natural questions arise in these and similar scenarios. The first one is which composition of the coalition will emerge. In particular, which political parties will join the government coalition; which buyers will trade with which sellers; which states are prioritized in terms of resource allocation. The second question is how the members of the coalition will divide the surplus generated by the agreement (political power, a positive surplus generated by trade, and available resources, respectively).

For the selected scenarios of coalition formation, the contribution of the dissertation is to provide axiomatic foundations for the laws of coalition formation. Additionally, for the scenarios of government formation in parliamentary democracies and bilateral trading, we propose an empirically accurate forecast methodology predicting the coalition structure and surplus distribution.

1.1 Related Work and Research Gap

Ray and Vohra (2015) describe the nature of previous work on coalition formation as “fragmented”. The approaches aiming to develop formal foundations for the

theory of coalition formation can generally be categorized into two distinct strands of literature, namely, cooperative and non-cooperative game theory approaches.

The former focuses on predicting the outcomes arising from strategic interaction according to general inference rules abstracting from detailed knowledge of a specific scenario. This line of research traces back to the seminal contribution of Morgenstern and Von Neumann (1953), who formulated the notions of strategic stability in cooperative games and the minimal winning coalition. Further principal notions originating from this strand of literature include the notion of strategic influence, or the Shapley value, (Shapley, 1953), the core (Gillies, 1959), and the bargaining sets (Aumann & Maschler, 1961; Davis & Maschler, 1963; Mas-Colell, 1989).

Although the generality of cooperative approaches is a desirable feature in the context of modeling complex multi-sided interaction, these approaches show limited applicability to the cases of real-life coalition formation. In particular, the Shapley value employs an ex-ante approach to coalition formation – before any knowledge of which participants will join to form a coalition – and hence can neither be used to predict the actual composition of the emerging coalition, nor the power distribution among its members. The core is empty for many types of games (Lucas & Rabie, 1982) and cannot be applied to provide predictions in these cases. The bargaining sets, although always non-empty, vary in definition and are complex to calculate: “To date, there is no known practical method for computing the bargaining set in games with a large number of players” (Maschler et al., 2020, p. 790). These approaches also assume that a grand coalition, or a coalition of all participants, will form, which contradicts observed outcomes of real-life situations of coalition formation.

Non-cooperative approaches to coalition formation (Rubinstein, 1982; Baron & Ferejohn, 1989; Chatterjee et al., 1993; Okada, 1996; Seidmann & Winter, 1998) imply specifying all details of a bargaining protocol (sequence of moves, offers and replies over time) and finding an equilibrium in the game described. These approaches have limited applicability to the cases of complex multilateral interactions, as specifying all details of such interactions quickly becomes prohibitive.

Descriptive/ inductive theories are specific to each particular branch of science and their findings cannot be easily generalized to other areas. Moreover, these approaches do not provide theoretical foundations for the underlying mechanisms

of coalition formation and rely on multiple data inputs without clear prioritization criteria.

We would like to propose an approach that allows for the generality of cooperative game theory approaches and simultaneously provides accurate predictions when applied to real-life coalition formation scenarios. This has become possible with the recent methodological advances, the conditional Shapley value (Casajus & La Mura, 2020)

1.2 Methodology

This dissertation applies the axiomatic method, a method of constructing theory from basic concepts and principles, or axioms, as its methodological approach. The underlying axioms must be conceptually evaluated before a theory is tested.

1.2.1 The Shapley value

The Shapley value (Shapley, 1953) is a prominent notion originating from cooperative game theory, which over the years attracted sustained research interest and gave rise to a large and expanding body of literature (Shapley & Shubik, 1954; Banzhaf III, 1965; Straffin, 1977; Aumann & Shapley, 1974; Roth, 1977, and others). More recent work includes contributions by Hausken and Mohr (2001), Béal et al. (2018), and Basallote et al. (2020). Applied studies focus on taxation and redistribution (Aumann, 1994), cost allocation (Littlechild & Owen, 1973; Fragnelli et al., 2000), risk attribution (Tarashev et al., 2016), loss allocation in energy transmission networks (Bergantiños et al., 2019), supply chain management (Kemahlioglu-Ziya & Bartholdi III, 2011; Bartholdi III & Kemahlioglu-Ziya, 2005), portfolio performance attribution (Moehle et al., 2021), and other applications. Lundberg and Lee (2017), Ribeiro et al. (2016), Shrikumar et al. (2017), and Bach et al. (2015) discuss the applications of the Shapley value to machine learning.

Winter (2002, p. 1) describes the measure as a “solution remarkable not only for its attractive and intuitive definition but also for its unique characterization by a set of reasonable axioms”. The Shapley value can be understood as a player’s strategic influence in the coalition formation process based on the “chance they have of being critical to the success of a winning coalition”. The Shapley value can

also be interpreted as a measure of the opportunity cost that the participant faces when committing to a coalition agreement.

1.2.2 Formal definitions and terminology

We model coalition formation scenarios as strategic situations, and apply tools from cooperative game theory based on some “natural” conditions or axioms. Specifically, the term *game* shall be used to refer to a formal description of a strategic situation in the form of a transferable utility (TU) game, namely, a game of coalition formation in which unrestricted side payments are allowed.

For a given nonempty set of players N , a TU game is completely defined by its characteristic function v , a function that assigns a worth (collective payoff) to each non-empty subset (coalition) of players $S \subset N$. If enlarging a coalition never decreases its worth the game is said to be monotonic.

We consider scenarios where a commitment of a group of players, or a coalition, is required for the positive surplus to be realized. Shapley (1953) introduced four axioms that can be used to identify a broad class of such scenarios:

- **Efficiency:** Players share exactly the worth of the grand coalition;

Efficiency can be understood as a budget balance condition in scenarios involving payments or spending or as sovereignty in scenarios involving power distribution.

- **Symmetry:** 2 players whose contributions to any coalition are always identical receive the same payoff;

The symmetry axiom implies the equivalence of two players with an identical contribution in a coalition formation process.

- **Null player:** A player that, joining any coalition, always leaves its worth unchanged receives 0 payoff;

The null player axiom identifies redundant participants of a coalition formation process and assigns zero payoff to them.

- **Additivity:** The payoff of jointly playing 2 different games equals the sum of payoffs obtained by playing them separately.

We interpret the additivity axiom as context independence or absence of interference. The payoff a player receives in a given coalition formation process is (additively) independent of the payoff received by participating in any other parallel activity.

Shapley (1953) showed that the unique rule for payoff allocation that jointly satisfies the above conditions is given by:

$$Sh_i(v) := \sum_{S \subset N \setminus \{i\}} p(S)(v(S \cup \{i\}) - v(S)), \quad (1)$$

where $p(S)$ is the probability that player i gets to join coalition S under a random sequential coalition formation process.

The Shapley value embodies an ex-ante perspective on coalition formation, before any knowledge of the outcomes of coalition formation. The measure assumes that the grand coalition always forms, and, therefore, cannot be applied to the situations when non-trivial coalition structure is expected to emerge. For instance, in government formation scenarios it is rarely observed that all political parties that obtained seats in parliament participate in a government coalition.

This motivates the introduction of a conditional solution for transferable utility games (games with unrestricted side payments), embodying the interim perspective of an observer who has already received some information on the outcome of coalition formation process.

1.2.3 The conditional Shapley value

The conditional Shapley value (Casajus & La Mura, 2020) is a conditional extension of the Shapley value for monotonic games.

Apart from **Null Player**, **Symmetry** and **Additivity**, the conditional Shapley value satisfies

- **Conditional Efficiency:** If a coalition S forms the sum of payoffs received by its members must equal its worth $v(S)$;

The conditional efficiency implies budget balance or sovereignty for all coalitions including any subset of players.

- **Consistency:** As new information comes in, expected gains or losses occur in the same proportion for all coalition members

The Consistency of expectations property describes situations in which players i and j are both members of a given coalition S , which was already announced, and are both also members of a smaller coalition $T \subset S$. In this case, if T is announced, their expected payoff will change in the same direction and in equal proportion.

Let $\mathbb{M}(N)$ denote the set of all monotonic games and $\mathfrak{P}(N)$ denote the set of all coalition structures for N . A coalition structure for N is a partition \mathcal{P} of N .

Casajus and La Mura (2020) show that for all $v \in \mathbb{M}(N)$, $\mathcal{P} \in \mathfrak{P}(N)$, and $i \in N$ there exists a unique conditional solution for monotonic TU games that always satisfies conditional efficiency, symmetry, null player, additivity, and consistency. The conditional Shapley value, is given by:

$$cSh_i(v, \mathcal{P}) := \begin{cases} \frac{Sh_i(v) \cdot v(\mathcal{P}(i))}{\sum_{l \in \mathcal{P}(i)} Sh_l(v)}, & v(\mathcal{P}(i)) > 0 \\ 0, & v(\mathcal{P}(i)) = 0 \end{cases} \quad (2)$$

where $Sh_l(v)$ represents the ordinary (unconditional) Shapley value for player i in game v , and $cSh_i(v, \mathcal{P})$ denotes i 's expected payoff given that coalition S has already formed.

Unlike the original notion, the conditional Shapley value reflects an interim perspective, in which the value is expressed conditionally to all currently available information. In particular, we interpret the announcement that a certain coalition S forms as a commitment of players in S not to cooperate with any players outside of S . This does not mean, however, that all players within S will necessarily all cooperate with each other: it may still happen that a smaller coalition $T \subset S$ is further announced, whereas players in T make a commitment not to cooperate with any player outside. Hence, the setting allows for new information to be incorporated, in a way that refines previous information without contradicting it.

A coalition structure is a partition of the players into disjoint subsets, or components. The components represent the productive units or coalitions, which the players join to generate the worth v . A coalition structure is (cSh-) *stable* if there is no coalition that can deviate from it and make all its members strictly better off, when all players formulate their expectations according to the conditional Shapley value.

Casajus and La Mura show that, for any monotonic TU game, there exists at

least one stable coalition structure. Moreover, they show that all stable coalition structures can be constructed in the following way. One first assigns a rank $\Pi(S)$ to every (non-null) coalition S , defined by the ratio between its worth and the sum of expectations of its members:

$$\Pi(S) := \frac{v(S)}{\sum_{l \in S} Sh_l(v)}. \quad (3)$$

All stable coalition structures can then be iteratively generated by selecting at each step a coalition with highest rank, among those which only involve unassigned players, and assigning its members to it. The process continues until there are no more unassigned players.

The underlying concept behind the stability analysis of the conditional Shapley value can be explained in the following manner: When a coalition of players with initially low expectations is able to collectively generate a positive surplus together, it becomes highly challenging to prevent the realization of such a coalition.

The stability analysis of the conditional Shapley value offers a significant practical benefit. It allows to rank predictions according to their stability, where the more stable coalitions are more likely to emerge, which makes it possible to provide a definite prediction of the outcome of a coalition formation process.

1.3 Limitations and Overview of the Dissertation

The study has some limitations. The selected methodology, the conditional Shapley value, provides predictions based on the opportunity cost of players but may fail to capture certain behavioral aspects of negotiation. In particular, we observe that in the bilateral trading scenario, when the number of players on both sides of the market is close to equal, the players of the scarce side receive less on average than predicted by the conditional Shapley value.

A further limitation stems from the selected framework of TU games, which treats the medium of payoff as homogeneous currency, which may not always be strictly the case. In particular, in the government formation scenario, all regular ministries are treated as equivalent from the perspective of political parties.

An overview of the three articles is provided in Table 1:

Manuscript	Publication status
A Gamson-Shapley Laws: a Formal Approach to Parliamentary Coalition Formation	Presented at 10th Annual Conference of the European Political Science Association; IPSA 26th World Congress of Political Science; Nordic Political Science Association Virtual Congress 2021. Published in the journal “ <i>Humanities and Social Sciences Communications</i> ” (ISSN: 2662-9992), 2023, Volume 10, Number 1, Page 1. https://doi.org/10.1057/s41599-023-02207-7
B Bilateral trading: predicting matching and payoff distribution in markets for indivisible goods	Unpublished manuscript
C Resource Allocation and the Strategic Prioritization of Swing States in the US Presidential Campaign	Unpublished manuscript

Table 1: List of manuscripts

In the following, a summary of the articles regarding contribution and findings, research approach, and the originality are presented.

1.3.1 Article 1: Gamson-Shapley Laws: a Formal Approach to Parliamentary Coalition Formation

Contribution and findings: Our contribution is threefold. First, we provide axiomatic foundations for a version of Gamson’s Laws (“Gamson-Shapley” Laws), in which the critical resource is identified with strategic influence, as measured by the Shapley value (Shapley, 1953). Second, we test the forecast accuracy of the Gamson-Shapley approach versus the original Gamson Laws on a panel of thirty-three parliamentary democracies in the time frame from 2016 to 2020. By applying a paired-sample t -test we show that the Gamson-Shapley approach provides more accurate forecasts of both the composition of the winning coalition, and the power distribution within it, than the original Gamson’s Laws. Third, we propose an extension to the Gamson-Shapley approach to further improve the accuracy of forecast by addressing a key drawback of Gamson’s theory, namely, the inability to distinguish between strong coalitions with a narrow policy divergence from weak coalitions of compatible but heterogeneous members.

Research approach: We model coalition formation as a strategic process, and apply tools from cooperative game theory based on some “natural” conditions, or axioms.

Originality: Numerous attempts have been made to propose models of government formation and advance the understanding of the factors influencing this process. While these contributions yielded significant results, there still seems to be a lack of a general theory suitable for the forecast of coalition structure and power distribution within a winning coalition by a single methodological approach. In parliamentary systems, the government formation process reflects the opportunity costs arising from the contributions of participating political parties, vis-à-vis the rewards accruing from their role within the winning coalition. Therefore, the composition of a winning coalition and its power distribution are not only jointly determined, but also strategically interconnected. As a consequence, considering these two aspects separately may significantly reduce the accuracy of the resulting predictions. Moreover, clear and general foundations for a theory of government formation are required in order to establish the necessary and sufficient data input for an accurate model.

1.3.2 Article 2: Bilateral trading: predicting matching and payoff distribution in markets for indivisible goods

Contribution and findings: We develop a theory of bilateral trading for a class of scenarios obeying five natural principles (budget balance, symmetry, marginality, absence of interference, and consistency of expectations). Specifically, we use those principles to characterize stable buyer-seller matching and payoff distribution. The resulting theory is detail-free with respect to the type of market process under consideration: in particular, it applies to auctions, bilateral bargaining scenarios, spontaneous markets, or double auctions, as long as those market processes conform to our principles. Furthermore, the approach is applicable to situations of both complete and incomplete information. In scenarios with heterogeneous values, our approach captures a potential tension between stability and efficiency that in other approaches is only clearly identified in the presence of incomplete information. We demonstrate the empirical accuracy of our approach in predicting matching and payoff distribution, relative to existing alternatives, in a bilateral trading experiment with homogeneous buyers and sellers, on a data set of 1217

agreed deals in 12 negotiation rounds with a varying number of counterparts.

Research approach: We model bilateral trading with multiple buyers and sellers as a strategic process, and apply tools from cooperative game theory based on some “natural” conditions, or axioms.

Originality: While the role of transaction costs has been considered in some classes of models (e.g., in bargaining models such as Rubinstein (1982), Rubinstein and Wolinsky (1985), Gale (1987), and Mortensen and Wright (2002)), those approaches require a commitment to a specific trading protocol. By contrast, we study classes of bilateral trading scenarios as abstract market games (in the sense of Shapley and Shubik 1974), and introduce a set of natural principles that provide a detail-free characterization of the expected outcomes of the market process.

1.3.3 Article 3: Resource Allocation and the Strategic Prioritization of Swing States in the US Presidential Campaign

Contribution and findings: We model resource allocation during US presidential campaigns in a two-party system, with a specific emphasis on swing states where the election outcome remains uncertain. We provide axiomatic foundations for a model of resource allocation to swing states based on their strategic contribution of the number of the electoral votes towards attaining a majority.

Research approach: We model resource allocation scenario in a presidential campaign in a two-party system as a strategic situation, and apply tools from cooperative game theory based on some “natural” conditions or axioms.

Originality: Unlike previous contributions, the model assumes interim perspective on resource allocation and allows for consistent sequential adjustments. Moreover, the model provides different recommendations for the two parties and is parsimonious in terms of the data input and computation requirements. While we assumed a two-party system, with smaller modifications the model can incorporate the impact of minority candidates.

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2 Gamson-Shapley Laws: a Formal Approach to Parliamentary Coalition Formation

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Abstract

We consider a set of empirical assumptions formulated by Gamson (1961), namely, Gamson's Laws, which remain at the heart of government formation forecast in parliamentary systems. While the critical resource postulated in Gamson's approach is the proportion of votes received by each party, other versions of Gamson's Laws can be defined by a different choice of critical resource. We model coalition formation as a cooperative game, and provide axiomatic foundations for a version of Gamson's Laws in which the critical resource is identified with strategic influence, as measured by the Shapley value. We compare the empirical accuracy of the resulting Gamson-Shapley theory against the original Gamson's Laws in a panel of 33 parliamentary elections, and find that it leads to significantly more accurate predictions of both coalition structure and power distribution. Finally, we propose an extension of the Gamson-Shapley approach which also incorporates information about policy distance among coalition partners. In particular, we discuss the advantages of the extended approach in the context of the German elections in 1987 and 2017.

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2.1 Introduction

In parliamentary democracies a government is typically formed as a result of a closed-door negotiation process among representatives of political parties once the outcome of a parliamentary election becomes known. Studying the mechanisms of government formation bears particular importance, as it sheds light on the intrinsic mechanisms of democracy itself. Better understanding of the consequences of electoral preferences in the context of future government policies may lead to more aware voting behavior, and a more transparent coalition formation process. Further benefits may arise from reducing the time and complexity involved in reaching a coalition agreement, as well as providing guidance for the positioning of candidates and parties.

We would like to address two natural questions, that arise in the general scenario of a parliamentary election:

- Which composition of the government coalition will emerge?
- How will this coalition divide the ministerial appointments among its members?

Several early approaches supplied ample evidence of various factors influencing the outcome of coalition formation, including the importance of a smaller ideological range between the parties within coalitions (Axelrod, 1970; Swaan, 1973); key role of the median party (Warwick, 1994); and advantage in coalition formation of the political party with the highest representation in parliament (van Deemen, 1989). Other approaches emphasize the role of a formateur party (Austen-Smith & Banks, 1988; Baron, 1991); the advantage of an incumbent administration (Strom et al., 1994); the importance of pre-electoral commitments (Klingemann & Volkens, 1992; Tops & Dittrich, 1992); and the low likelihood of an “anti-system” party to join the government (Budge & Keman, 1990). Further approaches concentrate on the context and history of coalition negotiations, e.g. intra-party politics (Bäck, 2008; Pedersen, 2010; Meyer, 2012); pre-electoral coalitions (Golder, 2006; Debus, 2009), or the role of previous conflict (Tavits, 2008) as factors of coalition formation. These contributions enhance the understanding of the drivers of political coalition formation and provide a rich basis for the application of statistical methods. Yet, the predictive power of those approaches is limited by a variety of

confounding factors, which make it difficult to accurately predict the outcomes of coalition formation in the general case.

At the same time, several attempts to develop formal foundations for government formation forecast have been made. One of the early essential contributions comes from Morgenstern and Von Neumann (1953, p. 430), who formulated a notion of strategic stability in cooperative games, as well as the notion of a minimal winning coalition, which they define as “those winning coalitions in which no participant can be spared”. The latter was applied by Riker (1962, pp. 32–33) to government formation scenarios: “In social situations similar to n-person, zero-sum games with side-payments, participants create coalitions just as large as they believe will ensure winning and no larger”. However, Riker’s approach has not escaped criticism for producing ambiguous results and a lack of a definite prediction (Browne, 1971).

Another prominent contribution coming from cooperative game theory is the Shapley-Shubik power index (Shapley & Shubik, 1954). The authors introduced a measure of a player’s strategic influence in the coalition formation process based on the “chance they have of being critical to the success of a winning coalition”. Nevertheless, the approach has a limited application to real governments forecast, as the Shapley-Shubik index embodies an ex-ante perspective on coalition formation - before any knowledge of which parties will join to form a winning coalition, and hence can neither be used to predict the actual composition of the winning coalition, nor the power distribution among its members.

The work by Gamson (1961) is another prominent contribution towards developing a general theory of political coalition formation. Gamson sought inspiration from cooperative game theory; however, no applicable tools existed at the time. Hence, he formulated his theory as a set of empirical assumptions (Gamson’s Laws) about government formation.

Proportionality: The distribution of power within a winning coalition is expected to occur in direct proportion to the contributions of a critical resource by its members (Gamson, 1961, 376, para. 6).

Stability: The realized winning coalition must be “cheapest” in terms of overall resource contributions by its members (Gamson, 1961, 376, para. 9).

Stability can be understood as a consequence of the more fundamental Law of Proportionality. Gamson writes: “When a player must choose among alternative coalition strategies where the total payoff to a winning coalition is constant, he will maximize his payoff by maximizing his share. The theory states that he will do this by maximizing the ratio of his resources to the total resources of the coalition. Since his resources will be the same regardless of which coalition he joins, the lower the total resources, the greater will be his share. Thus, where the total payoff is held constant, he will favor the cheapest winning coalition”.

This formulation leaves open the matter of identifying the critical resource with respect to which, in Gamson’s approach, a winning coalition may be regarded as cheapest, providing an opportunity to state several versions of Gamson’s Laws. Gamson proposed to use the percentage of votes obtained by a political party in a general election as an easily accessible proxy for the underlying critical resource, and the proportion of ministry seats received as a proxy for power distribution.

Even though Gamson’s theory is broadly consistent with empirical data (Browne & Franklin, 1973; Warwick & Druckman, 2006) in terms of predicting power distribution within a winning coalition, the approach does not account for the range of political platforms within a coalition, and as a consequence only rarely identifies the correct composition of the new coalition. Furthermore, Gamson’s theory demonstrates systematic biases in power distribution forecast (Falcó-Gimeno & Indridason, 2013), especially in the case of smaller parties, as Gamson also recognized in his seminal work. Moreover, “theoretical underpinnings of this relationship, dubbed ‘Gamson’s Law’ due to its high level of predictability, are still wanting” (Bäck et al., 2011, p. 442).

Several other studies, also inspired by formal methods, concentrated on the role of party preferences in ministerial portfolio allocation. Based on a set of assumptions about the government formation process, Budge and Keman (1990) developed a system of hierarchical rules according to which governments are formed. To address the issue of ministry seats distribution, the authors classified political parties into five party families (conservative, liberal, religious, socialist, and single-issue) and derived preferences on ministerial portfolios for each family “based partly on the traditional group support, ideologies, and historical origins of the parties, and partly on previous analyses of their characteristic issue concerns” (p. 95). While significant supporting evidence was found, the approach is limited by the assumed

homogeneity of party preferences, and of the scope of ministerial portfolios across different countries (Bäck et al., 2011, p. 444). In addition to predicting a coalition structure, the portfolio allocation model of Laver and Shepsle (1996) can also predict the allocation of key ministerial positions. The model falls short, however, of predicting the distribution of the remaining ministry seats among member parties.

Subsequent approaches considered various empirically motivated drivers of coalition formation, in order to extract predictions from a statistical framework, in which a large set of potentially relevant variables is included. In particular, Martin and Stevenson (2001) tested a wide variety of hypotheses concerning the composition of the winning coalition, while Bäck et al. (2011) concentrated on portfolio allocation. Using regression analysis, Warwick (1996) predicts the probability of a single party becoming a formateur party or entering the government as a coalition partner.

Several other studies introduce bargaining models of coalition formation (Baron & Ferejohn, 1989; Morelli, 1999; Diermeier & Merlo, 2000). Yet, these approaches produce predictions contradicting empirical findings. Above all, they do not account for the approximate proportional distribution of cabinet portfolios, Gamson's Proportionality Law (Bassi, 2013). Bassi (2013) proposes a model of endogenous government formation that addresses these limitations but, as in the case of other non-cooperative approaches, has limited applicability to predicting real situations of coalition formation due to its reliance on a specific bargaining protocol.

Another stream of literature focuses on predicting power distribution in a government coalition. Those approaches (Ansolabehere et al., 2005; Warwick & Druckman, 2001; Mershon, 2001) avoid some of the limitations of Gamson's theory, and generally provide more accurate predictions.

While these contributions yielded significant results, there still seems to be a lack of a general theory suitable for the forecast of coalition structure and power distribution within a winning coalition by a single methodological approach. In parliamentary systems, the government formation process reflects the opportunity costs arising from the contributions of participating political parties, vis-à-vis the rewards accruing from their role within the winning coalition. Therefore, the composition of a winning coalition and its power distribution are not only jointly determined, but also strategically interconnected. As a consequence, considering these two aspects separately may significantly reduce the accuracy of the resulting

predictions. Moreover, clear and general foundations for a theory of government formation are required in order to establish the necessary and sufficient data input for an accurate model.

We concentrate on Gamson’s Laws, which, despite their flaws, remain an important benchmark in government formation forecast, as it provides a prediction of both coalition structure and power distribution within a winning coalition. Our contribution is threefold. First, we provide axiomatic foundations for a version of Gamson’s Laws (“Gamson-Shapley” Laws), in which the critical resource is identified with strategic influence, as measured by the Shapley value (Shapley, 1953). The resulting theory does not require committing to a specific bargaining protocol. By contrast, it applies to a general class of scenarios satisfying five natural conditions. Second, we test the forecast accuracy of the Gamson-Shapley approach versus the original Gamson Laws on a panel of thirty-three parliamentary democracies in the time frame from 2016 to 2020. By applying a paired-sample t test we show that the Gamson-Shapley approach provides more accurate forecasts of both the composition of the winning coalition, and the power distribution within it, than the original Gamson’s Laws. Third, we propose an extension to the Gamson-Shapley approach to further improve the accuracy of forecast by addressing a key drawback of Gamson’s theory, namely, the inability to distinguish between strong coalitions with a narrow policy divergence from weak coalitions of compatible but heterogeneous members.

The remainder of this paper is organized as follows. In Section 2.2 we describe the Gamson-Shapley approach to government formation forecast and detail its axiomatic foundations. In the third Section we present the results of a statistical comparison of the accuracy of the Gamson-Shapley theory versus the original version of Gamson’s Laws within the considered dataset. In the fourth Section we provide an extension of the Gamson-Shapley approach which also accounts for policy divergence within coalitions. The last Section contains a discussion on our results, and implications for government formation forecast.

2.2 An axiomatic foundation for Gamson's Laws

2.2.1 Game-theoretic preliminaries

We model coalition formation as a strategic process, and apply tools from cooperative game theory based on some “natural” conditions, or axioms. Specifically, the term *game* shall be used to refer to a formal description of a strategic situation in the form of a transferable utility (TU) game, namely, a game of coalition formation in which unrestricted transfers among players are allowed. For a given nonempty set of players N , a TU game is completely defined by its characteristic function v , a function that assigns a worth (collective payoff) to each non-empty subset (coalition) of players $S \subset N$. If enlarging a coalition never decreases its worth the game is said to be monotonic.

Shapley (1953) showed that there exists a unique rule to allocate payoffs to players in a TU game, which always satisfies four desirable properties:

- **Efficiency:** The players share exactly the worth of the grand coalition $v(N)$.
- **Symmetry:** Two players whose contributions to the worth of any coalition are always identical expect the same payoff.
- **Null Player:** A player that, joining any coalition, always leaves its worth unchanged expects zero payoff.
- **Additivity:** The payoff a player receives by jointly playing two different games equals the sum of payoffs obtained by playing them separately.

The unique payoff allocation rule that jointly satisfies the above conditions, known as the Shapley value, is given by

$$Sh_i(v) := \sum_{S \subset N \setminus \{i\}} p(S)(v(S \cup \{i\}) - v(S)), \quad (4)$$

where $Sh_i(v)$ is the payoff expected by player i in game v ; $p(S)$ is the probability that player i gets to join coalition S under a random sequential formation process.

Casajus and La Mura (2020) introduced the conditional Shapley value, a conditional extension of the original Shapley value for monotonic TU games. Unlike the original notion the conditional Shapley value reflects an interim perspective,

in which value is expressed conditionally to all currently available information. In particular, they interpret the announcement that a certain coalition S forms as a commitment of players in S not to cooperate with any players outside of S . This does not mean, however, that all players within S will necessarily all cooperate with each other: it may still happen that a smaller coalition $T \subset S$ is further announced, whereas players in T make a commitment not to cooperate with any player outside. Hence, the setting allows for new information to be incorporated, in a way that refines previous information without contradicting it.

Casajus and La Mura show that there exists a unique conditional solution for monotonic TU games that always satisfies, besides the Null Player, Symmetry and Additivity conditions, the following conditional extension of the Efficiency condition, and an additional Consistency condition which reflects the role of new information.

- **Conditional Efficiency:** If a coalition S is formed, the sum of payoffs expected by its members must equal its worth $v(S)$.
- **Consistency:** If a coalition S is already formed, and a smaller coalition $T \subset S$ is announced, the expected payoffs of players in T must be in the same proportion before and after the announcement.

The Consistency property describes situations in which players i and j are both members of a given coalition S , which was already announced, and are both also members of a smaller coalition $T \subset S$. In this case, if T is announced, their expected payoff will change in the same direction and in equal proportion.

The resulting solution, namely, the conditional Shapley value, is given by:

$$cSh_i(v, S) := \begin{cases} \frac{Sh_i(v) \cdot v(S)}{\sum_{l \in S} Sh_l(v)}, & v(S) > 0 \\ 0, & v(S) = 0 \end{cases} \quad (5)$$

where $Sh_i(v)$ represents the ordinary (unconditional) Shapley value for player i in game v , and $cSh_i(v, S)$ denotes i 's expected payoff given that coalition S has already formed.

A coalition structure is a partition of the players into disjoint subsets, or components. The components represent the productive units or coalitions, which the

players join to generate the worth v . A coalition structure is (cSh-) *stable* if there is no coalition that can deviate from it and make all its members strictly better off, when all players formulate their expectations according to the conditional Shapley value.

Casajus and La Mura show that, for any monotonic TU game, there exists at least one stable coalition structure. Moreover, they show that all stable coalition structures (up to participation of null players) can be constructed in the following way. One first assigns a rank $\Pi(S)$ to every non-null coalition S , defined by the ratio between its worth and the sum of expectations of its members:

$$\Pi(S) := \frac{v(S)}{\sum_{l \in S} Sh_l(v)}. \quad (6)$$

Intuitively, $\Pi(S)$ can be understood as a measure of the opportunity cost faced by potential participants when deciding whether to commit to forming coalition S .

All stable coalition structures can then be iteratively generated by selecting at each step a coalition with highest rank, among those which only involve unassigned players, and assigning its members to it. The process continues until there are no more unassigned players.

Observe that the conditional Shapley value for player i in coalition S can be written as the product of the unconditional Shapley value for i , and of the stability rank for S :

$$cSh_i(v, S) = Sh_i(v) \times \Pi(S). \quad (7)$$

If one interprets $cSh_i(v, S)$ as a measure of the *power* player i receives if coalition S forms in game v ; $Sh_i(v)$ as the player's strategic *influence* in the game; and $\Pi(S)$ as a measure of the *stability* of S , then the above relationship can be stated in more conceptual form as:

$$power(i, S) = influence(i) \times stability(S).$$

2.2.2 Gamson-Shapley Laws: a formal derivation

As a first step towards providing a formal derivation of a version of Gamson's Laws we represent the government coalition formation process as a game. In this context political parties are regarded as players, who can join other parties to form coalitions in view of producing a majority based on the seats obtained in the current election cycle.

The coalition formation process is subject to additional constraints arising from potential incompatibilities among party platforms, which may prevent some parties from participating together in a single coalition. We model such constraints in terms of a binary compatibility relation, and admit as feasible only coalitions which contain no incompatible members. In formulating such constraints we take into account explicit and public commitments of political parties during their respective campaigns and declared political stance. Our aim is to distinguish between credible commitments, which in our setting always reduce the number of potential coalition partners and, consequently, the bargaining power of parties that undertake them, and various other signals that political parties send during the election campaign (“cheap talk”).

A majority (or winning) coalition requires more than 50 per cent of seats. A minimal winning coalition is a majority coalition that cannot exclude any of its members while still keeping a majority of seats. As soon as a winning coalition is realized, it allocates all available ministerial posts in the cabinet among its members. We refer to scenarios in which any majority coalition receives a payoff of one as simple coalition formation scenarios. Any other coalition, if realized, receives a payoff of zero.

The assumption that every majority coalition of compatible partners would receive the same payoff, regardless of the closeness of their programs, is clearly a simplification. Yet, we adopt it in order to avoid ambiguities in the proper definition of closeness, which especially in the presence of multi-dimensional policy issues can be problematic, and to facilitate a comparison with the original version of Gamson’s Laws. Later on (in Section 2.5) we shall relax this assumption, and also consider the role of ideological dispersion in government formation. Similarly, we abstract from the role of party preferences in the allocation of specific positions in the cabinet, and assume that payoffs are represented by a single dimensional variable, namely, power (e.g., as represented by the share of ministries in a cabinet), that is equally desirable to all players and can be freely transferred among coalition members.

We now reformulate the conditions stated in Section 2.1 in the context of such simple coalition formation scenarios.

- **Sovereignty:** The power of any realized coalition is fully and exclusively dis-

tributed among its members.

- **Equivalence:** Two parties with equal level of support and identical compatibility constraints expect equal payoff from the outcome of the coalition formation process.
- **Redundancy:** A party that can enable no majority coalition expects zero payoff.
- **Context independence:** The payoff a party receives in the government formation process is (additively) independent of the payoff received by participating in any other parallel activity.
- **Consistency of expectations:** The expected payoffs in the presence of new information are affected in equal proportion for all members of a new coalition.

Let us now show that the validity of the two Gamson-Shapley Laws can be derived from these formal assumptions (Sovereignty, Equivalence, Redundancy, Context independence, and Consistency). If X_i denotes the critical resource in government formation contributed by player i to coalition S , and $\varphi_i(S)$ the payoff expected by i when coalition S forms, then Gamson's Laws can be formally stated as follows.

Proportionality: The distribution of power within a winning coalition is expected to occur in proportion to the overall resource contributions by its members:

$$\varphi_i(S) = \frac{X_i}{\sum_{l \in S} X_l}. \quad (8)$$

Stability: The realized winning coalition S^* must be cheapest in terms of overall resource contributions:

$$S^* \in \arg \min_{S \subset N} \sum_{l \in S} X_l. \quad (9)$$

We shall refer to the specific version of Gamson's Laws in which the critical resource is identified with strategic influence, as measured by the (unconditional) Shapley value (i.e., in which $X_i = Sh_i$), as *Gamson-Shapley Laws*.

We can now establish our main result: under the above conditions, a particular version of Gamson's Laws in which the critical resource is identified with the (unconditional) Shapley value hold in all simple coalition formation scenarios.

Theorem 1. *In simple coalition formation scenarios satisfying Sovereignty, Equivalence, Redundancy, Context independence, and Consistency of expectations, the Proportionality and Stability laws hold with $X_i = Sh_i$.*

Proof. Notice that the game associated with the simple political scenarios the conditions stated in Section 2.2.1 reduce to five conditions above (Sovereignty, Equivalence, Redundancy, Context independence, and Consistency). Therefore, the conditional expected payoffs for all players are identified by conditional Shapley value.

Note that in simple coalition formation scenarios, equation 2 reduces to the following form:

$$cSh_i(S) = \frac{Sh_i}{\sum_{l \in S} Sh_l}. \quad (10)$$

It follows that $\varphi_i(S) = cSh_i(S)$, therefore, the Gamson-Shapley Proportionality law follows for $X_i = Sh_i$.

Next, observe that a coalition structure is stable in the sense of Section 2.2.1 just in case the realized winning coalition S^* is an element of $\arg \max_{S \subset N} \Pi(S)$. Furthermore, observe that in simple games $v(S) = 1$ for every winning coalition, and hence the stability rank defined in Section 2.2.1 reduces to

$$\Pi(S) = \frac{1}{\sum_{l \in S} Sh_l}. \quad (11)$$

Considering that $\arg \max_{S \subset N} \Pi(S) = \arg \min_{S \subset N} \sum_{l \in S} Sh_l$, Stability immediately follows. \square

2.3 Gamson-Shapley vs. Gamson's Laws: a comparison of predictive accuracy

Predictions on government coalition structure and power distribution based on the Gamson-Shapley approach can be directly compared with those provided by the original version of Gamson's theory. For each of those two dimensions we specify a procedure to compare the predictive power of the two models (Gamson-Shapley vs. Gamson). Specifically, we conduct two paired-samples t-tests: one for the predicted

coalition structure, and one for the power distribution within the realized winning coalition.

Due to the fact that a party's ability to complete a majority does not depend sensitively on its size (consider a simple example where a coalition, which already has support of 40% of the seats, can be completed to a majority by a party with 11% or 13% of the seats), the Shapley value assigns the same level of strategic influence to political parties with similar level of support and identical set of potential coalition partners. For a comparison of the predicted coalition structure this means that, while Gamson's approach typically produces a total order of coalition structures, in many cases the conditional Shapley value only produces a preorder. To address this issue we utilize the following procedure:

- Coalitions are preordered according to the value of their stability coefficient.
- Coalitions with the same stability are assigned to the same equivalence class.
- We assign ordering for each equivalence class in such a way that the realized coalition receives the lowest rank.
- We assign an accuracy score, which is calculated according to the formula $r = \frac{\sum_{i=1}^k \Pi_i}{\sum_{j=1}^n \Pi_j}$, where k is the rank of the realized prediction, and n is the total number of possible majority coalitions.

Notice that the procedure we utilize (assigning lowest rank to the realized coalition) puts the Gamson-Shapley predictions at a relative disadvantage with respect to Gamson's theory. Alternatively, ranking predictions in each equivalence class according to a predefined random ordering would have been a more neutral approach.

The second test aims at comparing the accuracy of the predicted power distribution. This requires some assumption on the worth of a prime minister seat relative to a regular ministerial seat. Browne and Franklin (1973) found that a political party receiving the prime minister seat is often underpaid, with respect to its resource contribution, if the worth of the prime minister seat is set equal to the worth of other ministries. Warwick and Druckman (2001), as well as Ansolabehere et al. (2005), treat a prime minister seat as more valuable than other ministries, while other ministries are generally seen as equally valuable (Ansolabehere et al., 2005).

We assume that the value of a prime minister seat is given by

$$P = 1 + M \times \alpha, \tag{12}$$

where M is the number of regular ministerial positions in the cabinet, and α is a positive constant.

Note that we do not assume a constant worth of the prime minister seat (Ansolabehere et al. (2005) assigns a constant worth of 3 other ministries), as it would imply that the relative worth of prime minister seat is always lower in larger cabinets, which seems to be too restrictive.

Next, we consider the realized government coalition and apply the following procedure:

- First, we calculate the power distribution that each method would predict in case the given coalition is realized.
- Next, for each of the two approaches, we apply an accuracy score calculated as the sum of squared prediction errors compared to the actual power distribution.

In the case of minority governments, political parties granting external support are considered as coalition members with zero payoff (no ministerial seats assigned). Non-partisan ministries are assigned to a political party, if they were nominated by a specific party.

We considered the government formation process in thirty three parliamentary democracies in the time frame from 2016 to 2020. Following Warwick (1996), we did not include, as part of our sample, elections which led to a trivial (single-party) majority.

2.4 Results

For our panel we report the accuracy scores for the predicted coalition structure displayed in Table 2 (a lower score indicates a higher accuracy of the forecast). The paired-samples t-test show a significant difference in the accuracy scores for Gamson-Shapley ($M=0.353$, $SD=0.316$) and original Gamson's Laws ($M=0.425$, $SD=0.303$) predictions of the actual coalition structure at the 99% confidence level in the case of both one- and two-tailed t-test ($t(33) = 2.906$).

	Gamson-Shapley	Gamson's Laws
Croatia, 2020	0.013	0.016
Ireland, 2020	0.027	0.152
Slovakia, 2020	1.000	1.000
North Macedonia, 2020	0.409	0.346
Montenegro, 2020	0.040	0.120
Lithuania, 2020	0.137	0.125
Moldova, 2019	1.000	1.000
Estonia, 2019	0.441	0.427
Andorra, 2019	0.586	0.453
Finland, 2019	0.371	0.470
Spain, 2019	0.188	0.344
Belgium, 2019	0.447	0.554
Denmark, 2019	0.041	0.039
Austria, 2019	0.706	0.726
Slovenia, 2018	0.264	0.594
Sweden, 2018	0.237	0.411
Latvia, 2018	0.837	0.876
Luxembourg, 2018	0.032	0.057
Liechtenstein, 2017	0.818	0.820
Netherlands, 2017	0.055	0.057
Bulgaria, 2017	0.706	0.709
Norway, 2017	0.084	0.115
Germany, 2017	0.443	0.454
Austria, 2017	1.000	1.000
Czech, 2017	0.156	0.206
Iceland, 2017	0.327	0.403
Macedonia, 2016	0.162	0.239
Croatia, 2016	0.010	0.610
Slovakia, 2016	0.145	0.539
Montenegro, 2016	0.100	0.053
Iceland, 2016	0.095	0.090
Spain, 2016	0.354	0.596
Ireland, 2016	0.427	0.436
Total	11.657	14.037

Table 2: Accuracy score of the coalition structure prediction

For predictions on the power distribution we obtain the scores reported in Table 3 (a lower score indicates a higher accuracy of the forecast).

After running a paired-samples t-test we find a significant difference in the accuracy scores of Gamson-Shapley ($M=0.072$, $SD=0.210$) and original Gamson's Laws ($M=0.125$, $SD=0.366$) in predicting the power distribution within a winning coalition. The difference is significant at the 95% confidence level in the case of both one- and two-tailed t-test ($t(33) = 2.111$).

Our comparison is based on an estimated worth of the prime minister seat with $\alpha = 0.186$ for Gamson-Shapley and $\alpha = 0.095$ for Gamson's Laws, respectively, which we chose as best empirical fit for each of the two models. Significant difference is still found with a different choice of α .

	Gamson-Shapley	Gamson's Laws
Croatia, 2020	0.001	0.008
Ireland, 2020	0.023	0.004
Slovakia, 2020	0.007	0.002
North Macedonia, 2020	0.011	0.019
Montenegro, 2020	0.059	0.053
Lithuania, 2020	0.001	0.004
Moldova, 2019	0.135	0.240
Estonia, 2019	0.047	0.015
Andorra, 2019	0.031	0.021
Finland, 2019	0.018	0.012
Spain, 2019	0.010	0.046
Belgium, 2019	0.025	0.010
Denmark, 2019	0.053	0.298
Austria, 2019	0.005	0.000
Slovenia, 2018	0.017	0.047
Sweden, 2018	0.049	0.153
Latvia, 2018	0.036	0.033
Luxembourg, 2018	0.038	0.024
Liechtenstein, 2017	0.047	0.019
Netherlands, 2017	0.012	0.006
Bulgaria, 2017	0.006	0.006
Norway, 2017	0.003	0.003
Germany, 2017	0.001	0.003
Austria, 2017	0.080	0.071
Czech, 2017	0.047	0.041
Iceland, 2017	0.002	0.002
Macedonia, 2016	0.058	0.063
Croatia, 2016	0.019	0.292
Slovakia, 2016	0.009	0.004
Montenegro, 2016	0.036	0.052
Iceland, 2016	0.019	0.008
Spain, 2016	0.076	0.254
Ireland, 2016	0.253	0.308
Total	1.231	2.123

Table 3: Accuracy score of the power distribution prediction

2.5 An extension: variable coalition worth

As discussed above, the Shapley value assigns the same level of strategic influence to political parties with the same level of support and identical set of potential coalition partners. This may be viewed as an undesirable feature of a forecast methodology, as the method in some cases may assign the same level of stability to several government coalitions, thus failing to provide a definite prediction of the government formation outcome.

Moreover, in reality, the power of a government can depend on the range of the ideological position of the participating political parties. We, thus, would like to introduce the following modification of the stability coefficient to improve the accuracy of predictions based on the Gamson-Shapley approach:

$$\Pi(S)_{adj.} = \frac{1 - L(S)}{\sum_{i \in S} Sh_i}, \quad (13)$$

where $L(s) \in [0, 1]$ is a measure of ideological dispersion.

We use data from the Manifesto project (Burst et al., 2020) to calculate $L(s)$. This data source was selected for its consistent approach to coding and availability of data across countries and years, but is not an integral part of the extended Gamson-Shapley methodology. Other data sources on policy positions can be used as appropriate. We employ the following procedure:

- we use all five programmatic variables provided by the Manifesto Project (left-right position of a party, party position on planned economy, market economy, welfare, and international peace).
- we apply factor analysis to capture inter-dependencies between those five variables. We base further analysis on the extracted factors.
- we calculate the distance between positions of each pair of political parties as Euclidean distance D in the five-dimensional space.
- $L(S)$ is calculated according to the following formula:

$$L(S) = \frac{\sum_{i,j \in S} D_{ij}}{\sum_{i,j \in N} D_{ij}}, \quad (14)$$

where $\sum_{i,j \in S} D_{ij}$ is the sum of distances between the platforms of political parties in coalition S , and $\sum_{i,j \in N} D_{ij}$ is the sum of distances between the platforms of political parties in the grand coalition (a coalition which includes all parties).

$L(S)$ can be interpreted as a penalty for a wider range of political views within a coalition: the higher is the value of $L(S)$, the lower is the numerator of the adjusted stability coefficient.

2.6 Comparison with other approaches

We would like to compare predictions based on the proposed extension to Gamson-Shapley Laws with existing approaches to government formation forecast. For this purpose, we choose the Portfolio Allocation model by Laver and Shepsle (1996), which produces a similar output (a prediction of coalition structure). We also compare the accuracy of the standard Gamson-Shapley approach with its extended formulation.

2.6.1 A comparison with the standard Gamson-Shapley approach. Government formation in Germany in 2017

We analyze government formation process in Germany in 2017 and discuss the predictions based on the Gamson-Shapley approach and its extension to variable coalition worth.

The following seat distribution occurred as a result of the federal elections in Germany in 2017 (See Table 4). As no political party secured a single-party majority, a coalition government had to be formed.

Party		Number of seats
Christian Democratic Union/ Christian Social Union	CDU/CSU	246
Social Democratic Party	SPD	153
Alternative for Germany	AfD	94
Free Democratic Party	FDP	80
Left	Left	69
Greens	Greens	67
Total		709

Table 4: General election in Germany, 2017: seats distribution

For this forecast, we assume that no party besides CDU/CSU¹ would partner with the right-wing AfD as presented in Figure 1, where a line denotes bilateral compatibility between each pair of political parties.

¹As described in subsection 2.2, instead of relying on pre-electoral cheap talk, we estimate the distance between political parties in policy space using Manifesto project data (Burst et al., 2020) as a way to assess the strength of a potential coalition involving those parties.

While both CDU/CSU and AfD heavily criticized each other in the run-up to the election, they fell short of ruling out cooperation as a credible commitment. Furthermore, attempts for cooperation on the state level were made within the same election cycle (in 2020 Thomas Kemmerich (FDP) was elected as a Thuringian Minister-President with votes from the AfD, CDU, and FDP), which confirms the absence of a credible commitment.

Although in 2021 this coalition was ruled out on the federal level, for the analysis of the government formation in 2017 we only look at the explicit commitments at the time. Even though the prospect of CDU/CSU and AfD forming a coalition looked thin, these parties chose not to resolve the ambiguity, which suggests that they did so to strengthen their strategic positioning.

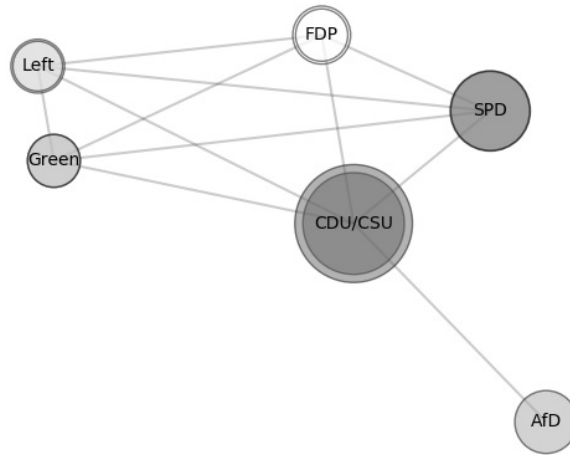


Figure 1: Constraints in coalition formation, Germany, 2017

A line denotes bilateral compatibility between each pair of political parties

Source: Own illustration

The original Gamson-Shapley methodology (not accounting for the distances between political positions of the parties) produces the forecast summarized in Table 5.

Coalition	Stability
SPD-FDP-Left-Green	1.818
CDU/CSU-SPD	1.538
CDU/CSU-FDP-Left	1.463
CDU/CSU-FDP-Green	1.463
CDU/CSU-Left-Green	1.463

Table 5: Minimal winning coalitions. Forecast based on the Gamson-Shapley laws

Here, a coalition of SPD, FDP, Left, and Green receives the highest stability rank, and is, therefore, the top prediction. This forecast does not take into account

that for the center-right FDP it could be difficult to find consensus on a variety of issues with other center-left and left parties. The correct prediction, a coalition of CDU/CSU and SPD, obtains the second-highest stability rank.

The extended Gamson-Shapley methodology produces the predictions of the government formation process in Germany as summarized in Table 6.

Coalition	Stability
CDU/CSU-SPD	1.396
SPD-FDP-Left-Green	1.356
CDU/CSU-FDP-Green	1.339
CDU/CSU-Left-Green	1.199
CDU/CSU-FDP-Left	1.185

Table 6: Minimal winning coalitions. Forecast based on the extension of the Gamson-Shapley laws

According to the extended Gamson-Shapley approach, the correct prediction, a coalition of CDU/CSU and SPD, obtains the highest stability rank. The stability rank of a coalition comprising SPD, FDP, Left, and Green is notably reduced reflecting the differences of the platforms of political parties.

Both approaches, extended and based on simple games, produce a rather accurate forecast of the power distribution in German government that formed after the 2017 elections (see Figure 2).

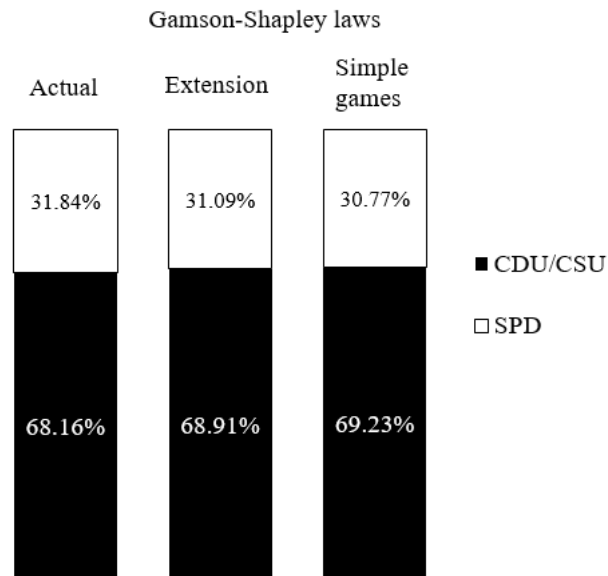


Figure 2: Actual and predicted power distribution in German government, 2017

Source: Own illustration

2.6.2 A comparison with the Portfolio Allocation model. Government formation in Germany in 1987

We consider a case study of government formation in Germany in 1987 analyzed by Laver and Shepsle (1996).

The number of seats obtained by each political party is shown in Table 7. None of the political parties has enough seats to secure a majority, so a coalition had to be formed.

Party		Number of seats
Christian Democratic Union/ Christian Social Union	CDU/CSU	223
Social Democratic Party	SPD	186
Free Democratic Party 2019	FDP	46
Greens	Greens	42
Total		497

Source: Laver and Shepsle (1996, 127, Table 6.1)

Table 7: General election in Germany, 1987: seats distribution

To provide a forecast of government formation, Laver and Shepsle (1996) apply their portfolio allocation model, which involves the following steps. First, they identified all winning government coalitions (both minimal and non-minimal), as summarized in Table 8.

CDU/CSU - SPD	CDU/CSU - SPD - FDP
CDU/CSU - FDP	CDU/CSU - FDP - Greens
CDU/CSU - Greens	CDU/CSU - SPD - Greens
SDP - FDP - Greens	CDU/CSU - SPD - FDP - Greens
laver1996making	

Source: Laver and Shepsle (1996, 127, Table 6.1)

Table 8: General election in Germany, 1987: winning coalitions

Second, Laver and Shepsle (1996), based on expert opinions, determine the key German cabinet portfolios to be Foreign Affairs and Finance, and hence identify foreign policy and economic policy as the two main policy dimensions. To place the political parties across these dimensions, the authors use the estimates of the German parties' positions obtained by Laver and Hunt (1992), as shown in Table 9.

Party	Economic policy (average)	Foreign policy
CDU/CSU	13.6	9.8
SPD	7.3	4.6
FDP	16.6	6.6
Greens	6.2	4.0

Source: Laver and Shepsle (1996, 129 – 131, Tables 6.2, 6.4)

Table 9: Mean position of German parties on selected policy dimensions

This allows the authors to construct a two-dimensional policy space and place the political parties across these dimensions, as depicted in Figure 3.

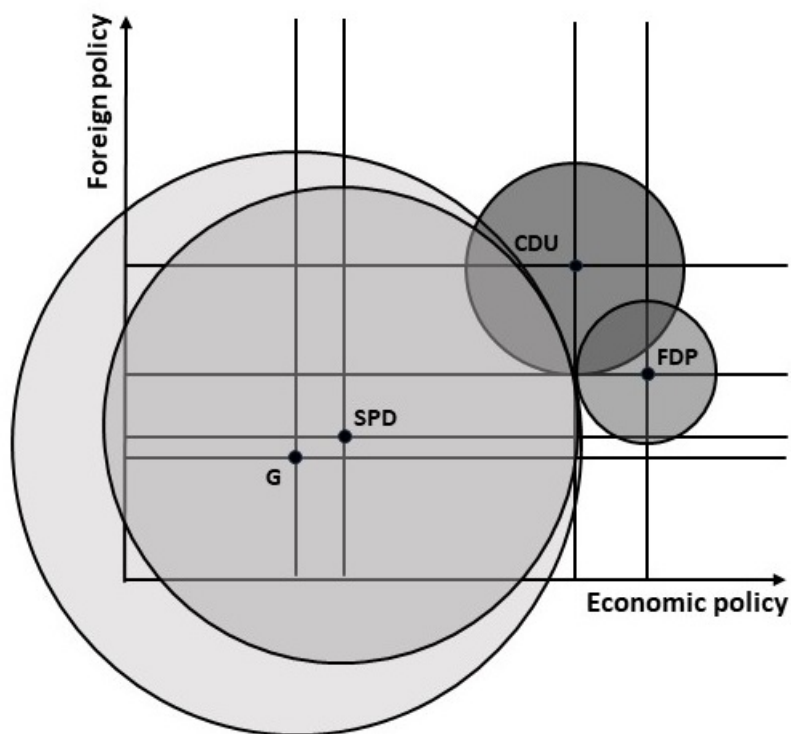


Figure 3: Two-dimensional German policy space: Indifference curves relative to the ideal point

Source: Own elaboration based on Laver and Shepsle (1996, 132, Figure 6.1)

The black dots on Figure 3 denote the ideal policy positions of the political

parties, while the possible allocations of the two ministerial portfolios are given by the line intersections. The indifference curves show the attitudes of each political party towards a potential government, where finance and foreign affairs portfolios are allocated to CDU/CSU and FDP respectively. Any dot inside the indifference curve is closer to a party's ideal position and, therefore, is more attractive. The intersection of the circles denotes the outcomes preferred by a group of political parties.

The authors call such CDU/CSU - FDP government dimension-by-dimension median portfolio allocation and by means of indifference analysis show that no other cabinet is preferred by winning coalitions of CDU/CSU - FDP, CDU/CSU - SPD, CDU/CSU - Greens, and Greens - SPD - FDP. The CDU/CSU - FDP cabinet is, thus, the coalition structure predicted by the model.

The portfolio allocation model by Laver and Shepsle (1996) provides a correct prediction of the coalition structure in the considered case of government formation, but is limited to the allocation of only two key ministerial portfolios and does not forecast the power distribution within a winning coalition.

We apply the extended Gamson-Shapley approach to the same case of government formation. We introduce no constraints in our analysis, as all of the parties are compatible with each other. Table 10 summarizes the stability ranks of the winning coalitions.

Coalition	Stability
CDU/CSU - FDP	1.288
CDU/CSU - SPD	1.215
SDP - FDP - Greens	1.184
CDU/CSU - Greens	1.160
CDU/CSU - SPD - Greens	0.676
CDU/CSU - SPD - FDP	0.648
CDU/CSU - FDP - Greens	0.549
CDU/CSU - SPD - FDP - Greens	0.000

Table 10: General election in Germany, 1987: winning coalitions' stability

As seen from Table 10, the extended Gamson-Shapley approach provides the correct prediction of the outcome of government formation process in Germany in 1987, as the CDU/CSU - FDP government receives the top stability rank. Furthermore, the methodology provides a prediction of the power distribution in the winning coalition that is within 4% of the actual result (see Figure 4).

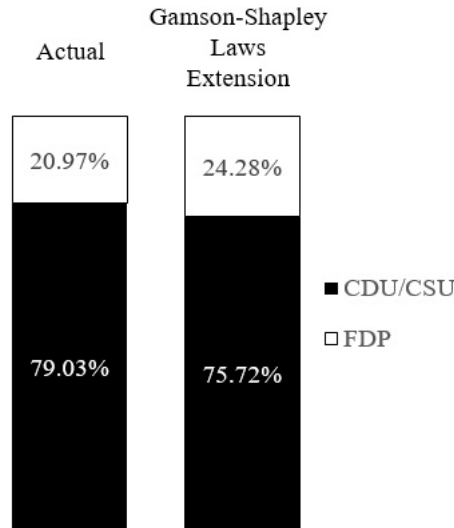


Figure 4: Actual and predicted power distribution in German government, 1987

Source: Own illustration

2.6.3 A comparison with Warwick (1996)

The Gamson-Shapley approach cannot be directly compared with statistical methods that infer the likelihood of a given coalition structure from empirical drivers. Yet, we propose the following procedure to compare the predictions provided by Gamson-Shapley approach and the model by Warwick (1996), which accounts of the likelihood of a political party to join a government:

- we rank all winning coalitions in terms of the stability coefficient starting from the top;
- from that distribution we obtain a probability of each winning coalition to come into power;

- based on this information, we obtain the marginal probability of each political party joining the government.

An explicit comparison of the predictive accuracy of the two approaches is beyond our scope, and hence we defer it to future work.

2.7 Discussion

We provided axiomatic foundations for a particular version of Gamson’s Laws, namely, Gamson-Shapley Laws, in which the critical resource in government formation is identified with strategic influence. The resulting theory does not require committing to a specific bargaining protocol. By contrast, it applies to a general class of scenarios satisfying five natural conditions. Compared with the original version of Gamson’s Laws we found that Gamson-Shapley Laws provide significantly more accurate forecasts both in terms of predicting the composition of the realized government coalition, as well as its internal power distribution.

Our analysis confirms that Gamson’s theory appears to provide less precise predictions than Gamson-Shapley especially in the presence of small party effects (e.g., Macedonia 2016) or minority scenarios (e.g., Sweden 2018 or Denmark 2015). Both approaches prove less accurate, when a broader coalition is formed than necessary for a majority (Slovakia 2020).

We then introduce a measure of closeness for the platforms of political parties in a coalition. The extension allows us to distinguish between ‘weak’ and ‘strong’ governments, and assign a higher power to governments with stronger affinity on policy issues. Taking the German elections of 1987 and 2017 as representative examples, we show that the extended Gamson-Shapley approach predicts government structure and power distribution with an accuracy that matches or exceeds that of other approaches.

A limitation of our approach, which cannot be easily overcome within the framework of TU games, is that we treat all regular ministries as equivalent from the perspective of political parties. Thus, we regard our approach as complementary, rather than alternative, to portfolio allocation models.

Finally, we note that the Gamson-Shapley approach can be easily integrated with methods based on regression analysis that aim to predict specific properties in the composition of the winning coalition, as in Martin and Stevenson (2001), or

the likelihood of a political party to join the government, as in Warwick (1996). Specifically, in the first class of models the Gamson-Shapley stability coefficient can be introduced as an additional explanatory variable in the regression, while in the latter strategic influence can be similarly integrated as an additional explanatory variable in the analysis.

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3 Bilateral trading: predicting matching and payoff distribution in markets for indivisible goods

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Abstract We study matching process and payoff distribution in bilateral trading scenarios with indivisible goods and multiple buyers and sellers, and develop a theory for the class of scenarios which obey five natural principles (budget balance, symmetry, marginality, absence of interference, and consistency of expectations). We use these principles to characterize stable buyer-seller matching and expected payoff distribution. Next, we demonstrate the empirical accuracy of our approach in predicting matching and payoff distribution in a simple bilateral trading experiment with homogeneous buyers and sellers relative to existing alternatives on a data set of 1217 agreed deals in 12 negotiation rounds with a varying number of counterparts. In scenarios with heterogeneous values, our approach captures a potential tension between stability and efficiency that in other approaches is only clearly identified in situations of incomplete information.

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3.1 Introduction

We consider bilateral trading scenarios with indivisible goods, in which market participants may have different valuations for the item and are interested in trading a single unit. Important questions to answer in those scenarios are who will trade with whom, how the trading pairs will divide the surplus, and under what conditions trades will be efficient.

The exact matching of buyers and sellers in a market remains incompletely specified by existing approaches, which only predict if a participant will trade or not. Furthermore, existing approaches fail to provide a definite prediction for the price or surplus distribution within each trading pair (Böhm-Bawerk, 1891), or only apply to specific settings (Rubinstein, 1982; Wilson, 1985; Satterthwaite & Williams, 1989). Previous work has also uncovered a tension between market outcomes and Pareto efficiency, in situations of incomplete information (Myerson & Satterthwaite, 1983), externalities (Meade, 1952), or when transaction costs (Coase, 1937; Arrow, 1970) are present. While market efficiency cannot be expected to hold in situations of incomplete information, the picture with respect to complete information scenarios is less clear. In those cases, market uncertainty may arise not from the lack of information about the type of other market participants but from the unknown outcome of the market process. Most approaches only assume or study market processes that are efficient under complete information about other participants' types.

We develop a theory of bilateral trading for a class of scenarios obeying five natural principles (budget balance, symmetry, marginality, absence of interference, and consistency of expectations). Specifically, we use those principles to characterize stable buyer-seller matching and payoff distribution. The resulting theory is detail-free with respect to the type of market process under consideration: in particular, it applies to auctions, bilateral bargaining scenarios, spontaneous markets, or double auctions, as long as those market processes conform to our principles. Furthermore, the approach is applicable to situations of both complete and incomplete information. In scenarios with heterogeneous values, our approach captures a potential tension between stability and efficiency that in other approaches is only clearly identified in the presence of incomplete information. We demonstrate the empirical accuracy of our approach in predicting matching and payoff distribution,

relative to existing alternatives, in a bilateral trading experiment with homogeneous buyers and sellers, on a data set of 1217 agreed deals in 12 negotiation rounds with a varying number of counterparts.

While the role of transaction costs has been considered in some classes of models (e.g., in bargaining models such as Rubinstein (1982), Rubinstein and Wolinsky (1985), Gale (1987), and Mortensen and Wright (2002)), those approaches require a commitment to a specific trading protocol. By contrast, we study classes of bilateral trading scenarios as abstract market games (in the sense of Shapley and Shubik 1974), and introduce a set of natural principles that provide a detail-free characterization of the expected outcomes of the market process.

The remainder of the paper is organized as follows: In Section 3.2, we discuss related contributions and highlight theoretical gaps. Next, we develop a theory for the class of bilateral trading scenarios obeying five natural principles (budget balance, symmetry, marginality, absence of interference, and consistency of expectations). In Section 3.3 we conduct an empirical test of the accuracy of our approach in predicting matching and payoff distribution relative to existing alternatives using data from a bilateral trading experiment with homogeneous buyers and sellers. In the last section we provide a discussion of our findings.

3.2 Related work

The formal analysis of bilateral trade scenarios dates back to the work of Edgeworth (1881), who provided a seminal contribution to the theory of payoff distribution in bilateral trading. However, his analysis was only limited to markets for divisible goods. Moreover, his approach did not account for the matching process of buyers and sellers on the market, or the exact surplus distribution within each trading pair.

The analysis of bilateral trading scenarios with indivisible goods traces back to the work of Böhm-Bawerk (1891), who studied a market for a homogeneous good with a number of buyers and sellers having different valuations (Böhm-Bawerk’s horse market). His approach provides bounds for the surplus distribution within each trading pair, based on the number of participants and their valuations, but does not offer a sharp prediction regarding the detailed composition of each trading pair, or the exact surplus distribution in each trade. Shapley and Shubik (1971) in their seminal contribution provide a formal analysis of Böhm-Bawerk’s horse mar-

ket along game-theoretic lines and coined the term market game. They conclude that the core (Gillies, 1959) has weaknesses as a solution concept when applied to bilateral trading scenarios, as it does not account for the ability of individuals or coalitions to obstruct outcomes, and hence does not fully reflect a buyer or seller's opportunity cost when committing to a trade.

Coase (1960) hypothesised that the impact of externalities can always be corrected through negotiation and contracting, predicting allocation efficiency. The hypothesis, which came to be known as Coase's Theorem, played a prominent role in the subsequent literature on bilateral trade. Farrell (1987, p. 113) summarized the broader implications of the Coase Theorem as follows: "if nothing obstructs efficient bargaining then people will negotiate until they reach Pareto-efficiency". This conclusion has not escaped various types of criticism, among them is the role of transaction costs (Coase, 1937; Arrow, 1970) and the problem of assigning externalities (Meade, 1952). Furthermore, in real life complete information about other participants' valuations, discount factors, and other elements of context is hardly ever achieved, so this assumption has limited applicability to predicting outcomes of trading situations.

The start of the modern analysis of bilateral trade scenarios can be attributed the two contributions by Nash (1950, 1953), where Nash bargaining solution and Nash demand game are presented. The former represents an axiomatic approach to modelling bilateral trading, and relies on the assumption of the efficient market outcomes. The latter can be seen as a simple and realistic model, yet limited to a single round of simultaneous demands.

Rubinstein (1982) provided a seminal non-cooperative model of bilateral bargaining, introducing a degree of commitment to every offer. The model predicts the payoff distribution within a trading pair with no alternative trading options. These results, however, only apply to the case of a single buyer and seller.

Myerson and Satterthwaite (1983) analyzed a class of scenarios and showed that in any fixed bilateral market mechanism asymmetric information leads to inefficient allocation. At the same time, existing theories say little about the possibility of inefficient allocation in the situation where market players have complete information about each others' valuations.

For the special case of bilateral markets, auctions, Vickrey (1961, 1962) established revenue equivalence, as well as efficient allocation, for the English, Dutch,

first-price and second-price auctions under the symmetric independent private values model.

3.3 Setup and results

3.3.1 A class of bilateral trade scenarios

We consider a market for indivisible goods with multiple buyers and sellers with different valuations, who are engaged in a market process. We do not specify in advance how the market process should be organized, and consider a variety of scenarios including auctions, bilateral bargaining (Rubinstein, 1982), double auctions (Satterthwaite & Williams, 1989), and other bilateral trading scenarios.

In the setting we consider, all participants are interested in trading at most a single unit of the good. Only trading pairs consisting of a buyer and a seller can create positive economic surplus. A buyer, whose value is below those of all sellers on the market, and a seller, whose value is higher than those of all buyers on the market, are called null.

We regard a buyer or seller's ex-ante expected payoff, before any commitments are made, as a measure of the opportunity cost that the participant faces when committing to a trade. Furthermore, we regard the total payoff expected by a buyer-seller pair before any commitments are made, relative to the total surplus available from the corresponding trade, as the opportunity cost associated to the formation of the corresponding trading pair.

The following principles identify a broad class of bilateral trading scenarios.

- **Budget balance:** In a trading pair, the price paid by a buyer equals the revenue received by the respective seller: taxes or subsidies are excluded.
- **Symmetry:** Market participants from the same side of the market (buyers or sellers) and with the same valuations expect equal payoff.
- **Marginality:** The expected payoff of a null participant is zero.
- **Absence of interference:** The payoff expected for a player from participating in one market does not depend on the outcomes from another market.
- **Consistency of expectations:** As new information comes in, expected payoffs of buyers and sellers in each potential trading pair vary in the same proportion.

The Consistency property describes situations in which two players are both members of a given coalition S , which was already announced, and are both also members of a smaller coalition $T \subset S$. In this case, if T is announced, their expected payoff will change in the same direction and in equal proportion.

We will show that in every bilateral trading scenario, in which those five principles jointly hold, two laws of bilateral trading follow:

Proportionality: The payoff distribution in a buyer-seller pair when committing to a trade is expected to occur in proportion to their respective opportunity costs.

Stability: Trading pairs whose opportunity cost is lowest will agree to a trade before all others, which could still be formed by uncommitted participants.

3.3.2 Formal definitions and terminology

We model bilateral trading with multiple buyers and sellers as a strategic situation, and apply tools from cooperative game theory based on some “natural” conditions or axioms. Specifically, the term *game* shall be used to refer to a formal description of a strategic situation in the form of a transferable utility (TU) game, namely, a game of coalition formation in which unrestricted side payments are allowed.

For a given nonempty set of players N , a TU game is completely defined by its characteristic function v , a function that assigns a worth (collective payoff) to each non-empty subset (coalition) of players $S \subset N$. If enlarging a coalition never decreases its worth the game is said to be monotonic.

We represent a general bilateral trading scenario with multiple buyers and sellers as a game, where buyers and sellers are regarded as players. Each player needs a compatible partner of the other side of the market to create a positive surplus. Any subset of buyers and sellers shall be referred to as a coalition.

Let K denote the set of all sellers and M denote the set of all buyers on the market. The worth $v(S)$ of every coalition S is determined as the smallest super-additive set function on $K \cup M$ satisfying

$$v(S) = 0 \text{ if } |S| = 0 \text{ or } 1 \tag{15}$$

$$v(\overline{ij}) = \max(0, h_{ij} - c_i) \text{ if } i \in K \text{ and } j \in M \quad (16)$$

where c_i is the value of the seller i of the item and h_{ij} is the value of buyer j the same item.

In case all buyers on the market have identical values of 1 of the traded items, and all sellers on the market have identical values of 0, the surplus generated by any traded pair amounts to 1. This defines a simple cooperative market game.

The five informal principles introduced in Section 3.3.1 can now be formally restated as axioms.

- **Budget balance:**

$$\sum_{i \in S} \varphi_i(v, S) = v(S) \quad (17)$$

If a coalition S is formed, the sum of payoffs expected by its members must equal its worth $v(S)$.

- **Symmetry:** If i, j are symmetric in v , then

$$\varphi_i(v, S) = \varphi_j(v, S) \quad (18)$$

Two players whose contributions to any coalition are always identical receive the same payoff.

- **Marginality:** If i is null in v , then

$$\varphi_i(v, S) = 0 \quad (19)$$

A player that, joining any coalition, always leaves its worth unchanged receives zero payoff.

- **Absence of interference:**

$$\varphi_i(v + w) = \varphi_i(v) + \varphi_i(w) \quad (20)$$

The payoff a player receives by jointly playing two different games equals the sum of payoffs obtained by playing them separately.

- **Consistency of expectations:**

$$\varphi_i(v, S) \cdot \varphi_j(v, T) = \varphi_j(v, S) \cdot \varphi_i(v, T) \quad (21)$$

If a coalition S is already formed, and a smaller coalition $T \subset S$ is announced, the expected payoffs of players in T must be in the same proportion before and after the announcement.

Shapley (1953) showed that there exists a unique rule to allocate payoffs to players in a TU game, which always satisfies four of the five principles defined above: budget balance, symmetry, marginality, and absence of interference. The Shapley value, is given by

$$Sh_i(v) := \sum_{S \subset N \setminus \{i\}} p(S)(v(S \cup \{i\}) - v(S)), \quad (22)$$

where $p(S) = \frac{|S|!(|N| - |S| - 1)!}{N!}$ is the probability that player i gets to join coalition S under a random sequential formation process.

According to the definition, the Shapley value can be interpreted as the opportunity cost of a player. The measure assumes that the grand coalition always forms, and, therefore, cannot be applied to the situations when non-trivial coalition structure is expected to emerge.

Let $\mathbb{M}(N)$ denote the set of all monotonic games and $\mathfrak{P}(N)$ denote the set of all coalition structures for N . A coalition structure for N is a partition \mathcal{P} of N .

Casajus and La Mura (2020) show that for all $v \in \mathbb{M}(N)$, $\mathcal{P} \in \mathfrak{P}(N)$, and $i \in N$ there exists a unique conditional solution for monotonic TU games that always satisfies budget balance, symmetry, marginality, absence of interference, and consistency of expectations. The conditional Shapley value, is given by:

$$cSh_i(v, \mathcal{P}) := \begin{cases} \frac{Sh_i(v) \cdot v(\mathcal{P}(i))}{\sum_{l \in \mathcal{P}(i)} Sh_l(v)}, & v(\mathcal{P}(i)) > 0 \\ 0, & v(\mathcal{P}(i)) = 0 \end{cases} \quad (23)$$

where $Sh_l(v)$ represents the ordinary (unconditional) Shapley value for player i in game v , and $cSh_i(v, \mathcal{P})$ denotes i 's expected payoff given that coalition S has

already formed.

Unlike the original notion the conditional Shapley value reflects an interim perspective, in which value is expressed conditionally to all currently available information. In particular, they interpret the announcement that a certain coalition S forms as a commitment of players in S not to cooperate with any players outside of S . This does not mean, however, that all players within S will necessarily all cooperate with each other: it may still happen that a smaller coalition $T \subset S$ is further announced, whereas players in T make a commitment not to cooperate with any player outside. Hence, the setting allows for new information to be incorporated, in a way that refines previous information without contradicting it.

The Consistency of expectations property describes situations in which players i and j are both members of a given coalition S , which was already announced, and are both also members of a smaller coalition $T \subset S$. In this case, if T is announced, their expected payoff will change in the same direction and in equal proportion.

A coalition structure is a partition of the players into disjoint subsets, or components. The components represent the productive units or coalitions, which the players join to generate the worth v . A coalition structure is (cSh-) *stable* if there is no coalition that can deviate from it and make all its members strictly better off, when all players formulate their expectations according to the conditional Shapley value.

Casajus and La Mura show that, for any monotonic TU game, there exists at least one stable coalition structure. Moreover, they show that all stable coalition structures can be constructed in the following way. One first assigns a rank $\Pi(S)$ to every (non-null) coalition S , defined by the ratio between its worth and the sum of expectations of its members:

$$\Pi(S) := \frac{v(S)}{\sum_{l \in S} Sh_l(v)}. \quad (24)$$

All stable coalition structures can then be iteratively generated by selecting at each step a coalition with highest rank, among those which only involve unassigned players, and assigning its members to it. The process continues until there are no more unassigned players.

3.3.3 Laws of stable matching and surplus distribution: a formal derivation

If $X_i(S)$ denotes opportunity cost of a player i in coalition S , and $\varphi_i(S)$ the payoff expected by i when coalition S forms, then two laws of bilateral trading that we introduced before can be formally stated as follows.

Proportionality: The payoff distribution in a buyer-seller pair when committing to a trade is expected to occur in proportion to their respective opportunity costs.

$$\varphi_i(S) = \frac{X_i(S) \cdot v_{ij}(S)}{X_i(S) + X_j(S)}. \quad (25)$$

Stability: Trading pairs whose opportunity cost is lowest will agree to a trade before all others, which could still be formed by uncommitted participants.

A coalition structure is expected to form consisting of trading pairs iteratively generated by selecting at each step a trading pair with the lowest opportunity cost, among those which only involve unassigned players.

For each iteration, the realized coalition S^* must have the highest worth relative to the sum of the expected payoffs of the members:

$$S^* \in \arg \max_{S \subset W} \frac{v(S)}{\sum_{l \in S} X_l(S)}. \quad (26)$$

where $W \subset N$ is a set of players unassigned to a coalition on previous iterations.

Under the assumptions introduced in Section 3.3.1 (budget balance, symmetry, marginality, absence of interference, and consistency of expectations) and if contributions of the players are identified with opportunity cost, as measured by the (unconditional) Shapley value (i.e., in which $X_i(S) = Sh_i$), the outcome of the bilateral trading process is correctly predicted by the conditional Shapley value.

In the described scenario, any realized coalition S within the stable coalition structure consists of one buyer and one seller and has a worth of $v(S)$. Hence, the payoff predicted by the conditional Shapley value coincides with the one in Proportionality law (i.e., $\varphi_i(S) = cSh_i(S)$).

Next, observe that in order for a coalition structure to be stable in the sense of Section 3.2.1 the realized coalitions must form iteratively, where on each step a

realized coalition S^* must be an element of $\arg \max_{S \subseteq W} \Pi(S)$. Hence, the Stability Law immediately follows.

In the case of simple market games, stability reduces to the condition that all participants of the scarce side of the market are part of a trading pair.

3.3.4 Market games with different valuations

Auctions

For markets with one seller or one buyer, any cSh-stable coalition structure is efficient in line with the prediction from the auction theory under independent private values. The surplus distribution depends on the number and quality of buyers present on the market.

Consider the following example with four buyers:

	Valuation	Shapley value		Valuation	Shapley value
b_1	5.000	0.700	s_1	0.000	3.7
b_2	4.000	0.200			
b_3	4.000	0.200			
b_4	4.000	0.200			

Table 11: Auction market: an example

Both auction theory and conditional Shapley value predict that seller 1 will trade with buyer 1 generating a surplus of 5. Seller 1 can expect a higher payoff based on their opportunity cost than predicted by auction theory (see Table 12).

	Second-price auction	Conditional Shapley value
Surplus	5.000	5.000
Payoff of seller 1	4.000	4.205
Payoff of buyer 1	1.000	0.795

Table 12: Example: surplus distribution

Markets with multiple buyers and sellers

We also consider a scenario of a market for a homogeneous good with multiple buyers and sellers with different valuations. Böhm-Bawerk's horse market Böhm-

Bawerk (1891) is a classic example of such a market. A buyer and a seller will trade if a positive surplus can be generated from the trade, otherwise, they don't trade and generate a zero surplus. We would like to discuss whether efficient allocation (matching generating the greatest sum of trading surpluses) can be expected to emerge in this scenario.

Without providing proofs, we mention the following fact: for games with at most five traders, any cSh-stable coalition structure is efficient. For general markets with six or seven traders, straightforward but tedious calculations would show whether this holds true.

In Böhm-Bawerk horse market games with at least 8 players, there exist configurations where no cSh-stable coalition structure is efficient. Consider the following configuration. There are two buyers with valuations b_1 and b_2 and six sellers with valuations s_1, s_2, s_3, s_4, s_5 , and s_6 such that

$$b_1 > s_6 = s_5 = s_4 = s_3 = s_2 > b_2 > s_1.$$

Set

$$\alpha := b_1 - s_2 > 0, \quad \beta := s_2 - b_2 > 0, \quad \gamma := b_2 - s_1 > 0$$

(See general setup in Figure 5).

Non-cooperative game theory predicts an efficient outcome, where only one trade takes place (between buyer 1 and seller 1) and no further positive surplus can be generated on the market.

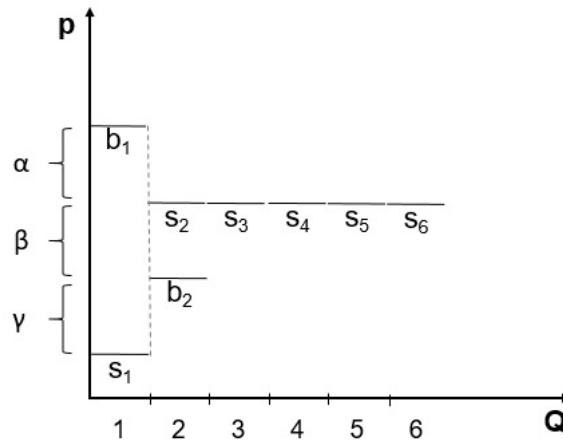


Figure 5: Tension between stability and efficiency: an example

Source: Own illustration

Consider a market with eight participants and the following valuations (see Table 13).

	Valuation	Shapley value		Valuation	Shapley value
b_1	5.000	1.348	s_1	1.000	2.036
b_2	4.000	0.500	s_2	4.025	0.023
			s_3	4.025	0.023
			s_4	4.025	0.023
			s_5	4.025	0.023
			s_6	4.025	0.023

Table 13: Tension between stability and efficiency: an example

In this example, the efficient outcome includes one trade between seller 1 and buyer 1, who jointly generate a surplus of 4, which, however, is not the most stable coalition according to Equation 24. The highest stability rank is generated by the trading pair of seller 1 and buyer 2, which allows seller 1 to receive a slightly higher payoff (see Table 14).

	$\{s_1, b_1\}$	$\{s_1, b_2\}$
Stability rank	1.182	1.183
Payoff of seller 1	2.406	2.408

Table 14: Tension between stability and efficiency: stability rank and surplus distribution

The trading pair of seller 1 and buyer 2 generates a surplus of 3, while buyer 1 will trade with any of the remaining sellers generating a surplus of 0.75. The overall surplus generated from all trades on the market amounts to 3.75, which is lower than the surplus with the efficient allocation.

More generally, straightforward calculations give the following Shapley payoffs:

$$\begin{aligned}
\text{Sh}_{b_1} &= \frac{6}{7} \cdot \alpha + \frac{1}{2} \cdot \beta + \frac{1}{6} \cdot \gamma \\
\text{Sh}_{s_2} = \text{Sh}_{s_3} = \text{Sh}_{s_4} = \text{Sh}_{s_5} = \text{Sh}_{s_6} &= \frac{1}{42} \cdot \alpha \\
\text{Sh}_{b_2} &= \frac{1}{6} \cdot \gamma \\
\text{Sh}_{s_1} &= \frac{1}{42} \cdot \alpha + \frac{1}{2} \cdot \beta + \frac{2}{3} \cdot \gamma
\end{aligned}$$

Therefore, we obtain the stability coefficients

$$\Pi(\{b_1, s_1\}) = \frac{\alpha + \beta + \gamma}{\frac{6}{7} \cdot \alpha + \frac{1}{2} \cdot \beta + \frac{1}{6} \cdot \gamma + \frac{1}{42} \cdot \alpha + \frac{1}{2} \cdot \beta + \frac{2}{3} \cdot \gamma}$$

and

$$\Pi(\{b_2, s_1\}) = \frac{\gamma}{\frac{1}{6} \cdot \gamma + \frac{1}{42} \cdot \alpha + \frac{1}{2} \cdot \beta + \frac{2}{3} \cdot \gamma}.$$

This gives

$$\Pi(\{b_1, s_1\}) - \Pi(\{b_2, s_1\}) = \frac{42(\alpha(\alpha - \gamma) + \beta(22\alpha + 21\beta + 14\gamma))}{37\alpha^2 + 819\alpha\beta + 1330\alpha\gamma + 882\beta^2 + 2205\beta\gamma + 1225\gamma^2}$$

which is strictly below zero for $\gamma > \alpha$ and sufficiently small β . For such α , β , and γ , this implies that any cSh-stable coalition structure contains the component $\{b_2, s_1\}$, whereas buyer b_1 forms a component together with one of the sellers s_2 , s_3 , s_4 , s_5 , s_6 . That is, in any cSh-stable coalition structure, a surplus of $\alpha + \gamma$ is generated. Yet, any efficient coalition structure contains a component that contains

buyer b_1 and seller s_1 implying that a surplus of $\alpha + \beta + \gamma > a + \gamma$ is generated. That is, in this game/market, any cSh-stable coalition structure is not efficient. This example can easily be generalized to larger player sets by increasing the number of sellers with valuation s_2 .

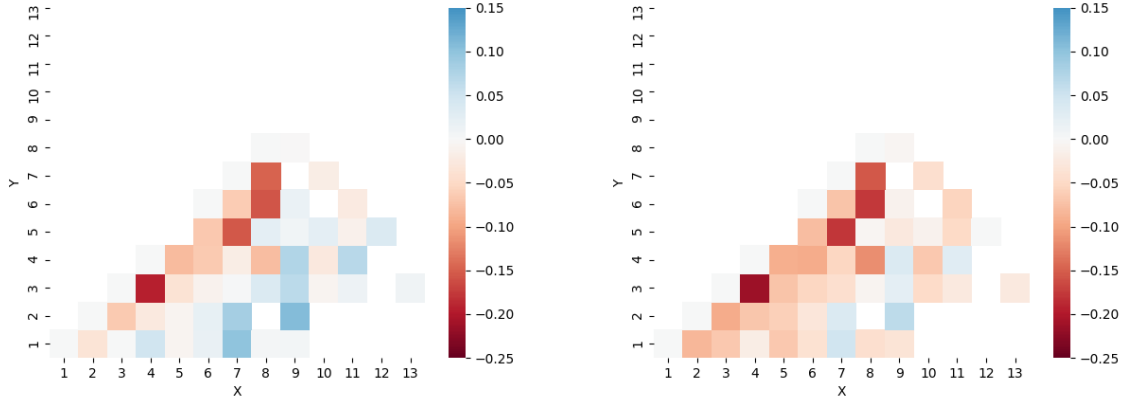
3.4 Bilateral trading with unit surpluses: empirical evaluation in the gloves game

The gloves game described by Shapley and Shubik (1969) is an example of a simple market game. In this game, two sides, owners of right and left gloves, complement each other and can only achieve positive worth of a coalition with a partner from the other side of the market. This game can also be interpreted as a simple bilateral trading scenario, where buyers are seen as owners of left gloves, and sellers are seen as owners of the right gloves. The players in each trading pair need to reach an agreement on how to divide a surplus of one.

Several solution concepts can be identified as applicable to address the surplus division problem: nucleolus (Schmeidler, 1969), AD value (Aumann & Dreze, 1974), Wiese value (Wiese, 2007), and chi-value (Casajus, 2009). The nucleolus and AD value represent two extremes: the former assigns 0 payoff to the more abundant side, the latter always predicts an equal split. Predictions based on Wiese and chi-value fall between those based on the nucleolus and the AD value. Tutic et al. (2011) tested those four approaches against experimental data in a simple market game setup and found limited predictive accuracy in all cases.

Using the data from a gloves game experiment conducted by Tutic et al. (2011), we evaluate the accuracy of the predictions of the surplus distribution based on the conditional Shapley value and chi-value (Casajus, 2009), the best performing measure of surplus distribution as found by Tutic et al. (2011). We use the data from the experiment consisting of 1217 agreed deals in 12 negotiation rounds with varying number of participants on the two sides of the market.

Figure 6 depicts the difference between the average payoff distribution of the stipulated number of buyers and sellers and the predictions based on the conditional Shapley value and the chi-value.



Payoffs are shown for the number of players on the scarce side of the market.

Figure 6: Difference between the average empirical value and the conditional Shapley value and chi-value respectively

Source: Own illustration

We find that the conditional Shapley value ($M=0.025$, $SD=0.003$) provides a significantly more accurate forecast of the bilateral trading outcomes compared to the chi-value ($M=0.028$, $S=0.002$), the best-performing method among other approaches. The difference is significant at the 99% confidence level in the case of both one- and two-tailed t-tests ($t(1217) = 12.873$).

The patterns observed on Figure 6 can be further discussed. We observe that, when the number of players on both sides of the market is close to equal, the players of the scarce side on average receive less than predicted by the conditional Shapley value. This indicates that in these situations, market participants may fail to fully realize the potential represented by their opportunity cost. This effect may be connected with behavioral aspects of negotiation that cannot be easily captured by formal methods.

3.5 Discussion

We provided axiomatic foundations for an empirically accurate theory of bilateral trading and stated two laws of stable matching and surplus distribution that hold for a broad class of bilateral trading scenarios. Our approach does not require a detailed specification of the market process, nor the assumption of complete information about other participants' types. In scenarios with heterogeneous buyers

and sellers' values, the theory accounts for a tension between efficiency and stability that in other approaches is only clearly identified in the presence of incomplete information about other participants' types. Future research could provide characterization of the relationship between stability and efficiency.

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4 Resource Allocation and the Strategic Prioritization of Swing States in the US Presidential Campaign

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Abstract

We model resource allocation during US presidential campaigns in a two-party system, with a specific emphasis on swing states where the election outcome remains uncertain. We provide axiomatic foundations for a model of resource allocation to swing states based on their strategic contribution of the number of the electoral votes towards attaining a majority. Unlike previous contributions, the model assumes interim perspective on resource allocation and allows for consistent sequential adjustments. Moreover, the models provides different recommendations for the two parties based on their unique strategic positions.

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4.1 Introduction

The US president is not elected by popular vote but through a system employing the institute of Electoral College. Each of the fifty states and the District of Columbia are allocated a number of electoral votes, proportional to their population with a minimum of three votes. In all of these states, with the exception of Maine and Nebraska, the votes are allocated to the candidate leading in the popular vote within the state. A president is elected with a majority of 270 votes out of 538. The natural questions arising in this setting are:

- Which states are more important with a view to securing a majority?
- How should the campaign resources be allocated?

Several contributions focus on the *ex ante* approaches to calculating the voting power of the states including Banzhaf measure (Penrose, 1946; Banzhaf III, 1968) and the Shapley value (Shapley, 1953), applied to the setting by Owen (1975). These approaches have limited applicability to clarifying the importance of each state towards securing a majority, as they do not take into account the interim information about the winning candidates in each state and the historic preferences in the so-called “red” and “blue” states. Several empirical studies proceed on this premise and discuss the matters of representation of the Electoral College (Katz et al., 2002; Warf, 2009; De Mouzon et al., 2021) but cannot be used to answer the questions posed.

Some studies concentrate on resource allocation in the US presidential election campaign. Brams and Davis (1974) model US presidential election as a two-person zero-sum infinite game and conclude that the candidates should allocate “campaign resources roughly in proportion to the $3/2$ ’s power of the electoral votes of each state” known as the $3/2$ ’s rule in presidential campaigning. The authors assume that candidates maximize their expected electoral vote and that there is an equal share of undecided voters in each state. Their analysis leads to the conclusion that resource allocation should prioritize larger states.

Colantoni et al. (1975) propose a modified proportional rule aiming at improving on the $3/2$ ’s rule. The authors conclude that while there is unlikely to be one single formula accounting for the consequences of the Electoral College institute

for campaign resource allocation, they expect the competitiveness of states and sequential campaign adjustments to play a role.

Strömberg (2008) builds on the contribution of Brams and Davis (1974) but extends the approach to allow for different partisan leanings, uncertainty regarding the election outcome at both state and national levels, and by setting the goal of maximizing the probability of winning the election instead of the number of electoral votes received.

While these models enhance our understanding of the drivers of campaign resource allocation, they suffer from several shortcomings. Brams and Davis (1974) and Colantoni et al. (1975) assume that presidential candidates maximize the expected electoral vote received in an election, which does not seem a realistic premise. Rather, we may assume that the candidates aim at securing a majority (similar to (Strömberg, 2008)). A further argument can be made for an allocation maximizing the resources spent per state needed to secure a majority.

Moreover, the models by Brams and Davis (1974) and Colantoni et al. (1975) assume an *ex ante* approach and, as a consequence, prescribe an identical share of campaign resources to both parties of the two-party system regardless of the strategic positions of these parties. The approach by Strömberg (2008) incorporates *ex post* perspective but requires large amounts of empirical data to make a prediction (data on twenty variables was used).

Finally, these studies (Brams & Davis, 1974; Colantoni et al., 1975; Strömberg, 2008) aim at predicting the actual campaign spending creating a circular argument. This paper would like to take a different angle and provide recommendations for campaign spending depending on the strategic role of a state in securing a majority for a particular candidate based on clear principles and allowing for sequential campaign adjustments (as advocated by Colantoni et al. (1975)). The proposed model is parsimonious in terms of data required to make a recommendation.

We would like to focus on the role of so-called swing states, as several contributions provide evidence that they have a higher importance in the electoral race than the states with known and stable preferences. In particular, Ma and McLaren (2018) estimate empirically that the US political process treats a voter living in a non-swing state as being worth 77% as much as a voter in a swing state. McLean et al. (2018, p.2) take a step further and assert that “a very limited number of states - the swing states - susceptible of changing their allegiance, are the ones

ultimately deciding who the leader of the United States will be for years”. The authors use four criteria: bellwether status (the winning state candidate also wins the presidency), competitiveness, “flippability”, and perception as a battleground, to identify swing states.

The paper is organized as follows: In Section 4.2 we develop a theory for a class of resource allocation scenarios obeying five natural principles (budget balance, equivalence, redundancy, absence of interference, and consistency). In Section 4.3 we provide an example of recommended campaign resource allocation based on the five principles discussed. Next, we compare our recommendation to other approaches to campaign resource allocation. In the last Section, we provide a discussion of our findings.

4.2 Setup and results

4.2.1 Resource allocation scenario

We consider a resource allocation scenario, where campaign resources (financial, time, or other resources) are allocated to a group of states with a view to securing a majority. Each state disposes of a certain number of electoral votes. A group of states will be referred to as a coalition. A majority coalition of states is such that the sum of electoral votes exceeds the designated majority quota.

We propose several natural conditions for a resource allocation scenario:

- **Budget balance:** Campaign resources are fully and exclusively distributed among the coalition of states.
- **Equivalence:** Equal amount of resources is allocated to two states with an equal level of voting power towards a majority.
- **Redundancy:** No resources are allocated to a state that can enable no majority coalition.
- **Absence of interference:** The resources allocated to a state in the course of a presidential election campaign are (additively) independent of the resources allocated to any other parallel activity.

The Absence of interference property can be interpreted in the context of the legal requirements towards presidential campaign funding, including disclosure re-

quirements and mandatory audits (Federal Election Commission, n.d.), which provide monitoring of sources and use of campaign funds.

- **Consistency:** The resources expected to be allocated in the presence of new information are affected in equal proportion for all members of a new coalition.

The Consistency property describes situations in which two states are both members of a given coalition S , which was already announced as the target for resource allocation, and are both also members of a smaller coalition $T \subset S$. In this case, if T is announced, the amount of resources allocated will change in the same direction and in equal proportion. In the context of the campaign allocation scenario, the Consistency property can be understood as a property that allows consistent sequential adjustments of the allocated resources in light of the interim information obtained in the course of the election campaign.

We will show that in every resource allocation scenario, in which those five principles jointly hold, the campaign resources are allocated to states in proportion to their strategic contribution with a view to securing the election majority.

We will refer to resource allocations such that no subset of included states can receive a higher amount of resources as stable allocations. An allocation that ensures winning and no larger (provided victory in all included states) will be referred to as minimally sufficient. Such minimally sufficient allocations allow for maximizing the share of allocated resources per state.

4.2.2 Formal definitions and terminology

We model resource allocation scenario in the presidential campaign in a two-party system as a strategic situation and apply tools from cooperative game theory based on some “natural” conditions or axioms. Specifically, the term *game* shall be used to refer to a formal description of a strategic situation in the form of a transferable utility (TU) game, namely, a game of coalition formation in which unrestricted side payments are allowed.

For a given nonempty set of players N , a TU game is completely defined by its characteristic function v , a function that assigns a worth (collective payoff) to each non-empty subset (coalition) of players $S \subset N$. If enlarging a coalition never decreases its worth the game is said to be monotonic.

We represent the resource allocation scenario as a game, where states are regarded as players. Each political party (represented by a candidate) needs to secure a majority of electoral votes with a view to a quota Q . The states are divided into those with a known preference towards one of the parties and swing states, where the outcome of the election is uncertain. Any subset of states shall be referred to as a coalition.

Let e_i denote the number of electoral votes of swing state i ; p_j denote the number of electoral votes in states j with known preferences towards one of the two parties; D denote the set of all states with known preference towards the Democratic party (“blue” states); R denote the set of all states with known preference towards the Republican party (“red” states). In this case,

$$P_D = \sum_{j \in D} p_j \quad (27)$$

and

$$P_R = \sum_{j \in R} p_j \quad (28)$$

denote the sum of the electoral votes of the “blue” and “red” states, respectively.

For the Democratic party, the worth $v(S)$ of every coalition S is determined as

$$\begin{cases} v(S) = 1 \text{ if } \sum_{i \in S} e_i \geq Q - P_D \\ v(S) = 0 \text{ if } \sum_{i \in S} e_i < Q - P_D \end{cases} \quad (29)$$

For the Republican party, the worth $v(S)$ of every coalition S is determined similarly:

$$\begin{cases} v(S) = 1 \text{ if } \sum_{i \in S} e_i \geq Q - P_R \\ v(S) = 0 \text{ if } \sum_{i \in S} e_i < Q - P_R \end{cases} \quad (30)$$

The five informal principles introduced in Section 4.2.1 can now be formally restated as axioms.

- **Budget balance:**

$$\sum_{i \in S} \varphi_i(v, S) = v(S) \quad (31)$$

If a coalition S is formed, the sum of payoffs expected by its members must equal its worth $v(S)$.

- **Equivalence:** If i, j are symmetric in v , then

$$\varphi_i(v, S) = \varphi_j(v, S) \quad (32)$$

Two players whose contributions to any coalition are always identical receive the same payoff.

- **Redundancy:** If i is null in v , then

$$\varphi_i(v, S) = 0 \quad (33)$$

A player that, joining any coalition, always leaves its worth unchanged receives zero payoff.

- **Absence of interference:**

$$\varphi_i(v + w) = \varphi_i(v) + \varphi_i(w) \quad (34)$$

The payoff a player receives by jointly playing two different games equals the sum of payoffs obtained by playing them separately.

- **Consistency:**

$$\varphi_i(v, S) \cdot \varphi_j(v, T) = \varphi_j(v, S) \cdot \varphi_i(v, T) \quad (35)$$

Shapley (1953) showed that there exists a unique rule to allocate payoffs to players in a TU game, which always satisfies four desirable properties:

- **Efficiency:** The players share exactly the worth of the grand coalition $v(N)$.
- **Symmetry:** Two players whose contributions to the worth of any coalition are always identical expect the same payoff.
- **Null Player:** A player that, joining any coalition, always leaves its worth unchanged expects zero payoff.

- **Additivity:** The payoff a player receives by jointly playing two different games equals the sum of payoffs obtained by playing them separately.

The unique payoff allocation rule that jointly satisfies the above conditions, known as the Shapley value, is given by

$$Sh_i(v) := \sum_{S \subset N \setminus \{i\}} p(S)(v(S \cup \{i\}) - v(S)), \quad (36)$$

where $Sh_i(v)$ is the payoff expected by player i in game v ; $p(S)$ is the probability that player i gets to join coalition S under a random sequential formation process.

Casajus and La Mura (2020) introduced the conditional Shapley value, a conditional extension of the original Shapley value for monotonic TU games. Unlike the original notion the conditional Shapley value reflects an interim perspective, in which value is expressed conditionally to all currently available information. In particular, they interpret the announcement that a certain coalition S forms as a commitment of players in S not to cooperate with any players outside of S . This does not mean, however, that all players within S will necessarily all cooperate with each other: it may still happen that a smaller coalition $T \subset S$ is further announced, whereas players in T make a commitment not to cooperate with any player outside. Hence, the setting allows for new information to be incorporated, in a way that refines previous information without contradicting it.

Casajus and La Mura show that there exists a unique conditional solution for monotonic TU games that always satisfies, besides the Null Player, Symmetry and Additivity conditions, the following conditional extension of the Efficiency condition, and an additional Consistency condition which reflects the role of new information.

- **Conditional Efficiency:** If a coalition S is formed, the sum of payoffs expected by its members must equal its worth $v(S)$.
- **Consistency:** If a coalition S is already formed, and a smaller coalition $T \subset S$ is announced, the expected payoffs of players in T must be in the same proportion before and after the announcement.

The resulting solution, namely, the conditional Shapley value, is given by:

$$cSh_i(v, S) := \begin{cases} \frac{Sh_i(v) \cdot v(S)}{\sum_{l \in S} Sh_l(v)}, & v(S) > 0 \\ 0, & v(S) = 0 \end{cases} \quad (37)$$

where $Sh_i(v)$ represents the ordinary (unconditional) Shapley value for player i in game v , and $cSh_i(v, S)$ denotes i 's expected payoff given that coalition S has already formed.

A coalition structure is a partition of the players into disjoint subsets, or components. The components represent the productive units or coalitions, which the players join to generate the worth v . A coalition structure is (cSh-) *stable* if there is no coalition that can deviate from it and make all its members strictly better off, when all players formulate their expectations according to the conditional Shapley value.

Casajus and La Mura show that, for any monotonic TU game, there exists at least one stable coalition structure. Moreover, they show that all stable coalition structures (up to participation of null players) can be constructed in the following way. One first assigns a rank $\Pi(S)$ to every non-null coalition S , defined by the ratio between its worth and the sum of expectations of its members:

$$\Pi(S) := \frac{v(S)}{\sum_{l \in S} Sh_l(v)}. \quad (38)$$

All stable coalition structures can then be iteratively generated by selecting at each step a coalition with highest rank, among those which only involve unassigned players, and assigning its members to it. The process continues until there are no more unassigned players.

4.2.3 Resource allocation scenario

Let us now propose a way of resource allocation to states based on the five formal assumptions (Budget Balance, Equivalence, Redundancy, Absence of Interference, and Consistency). If X_i denotes the strategic contribution by state i to coalition S with a view to securing a majority, and $\varphi_i(S)$ the share of resources allocated to i when coalition S forms, then the following can be stated about resulting resource allocation:

- The resource allocation occurs in proportion to the overall resource contributions by the included states:

$$\varphi_i(S) = \frac{X_i}{\sum_{l \in S} X_l}. \quad (39)$$

- The minimally sufficient allocation in terms of overall resource contributions is defined as follows:

$$S^* \in \arg \min_{S \subset N} \sum_{l \in S} X_l. \quad (40)$$

We shall consider the resource allocation in which the critical resource is identified with strategic influence, as measured by the (unconditional) Shapley value (i.e., in which $X_i = Sh_i$). We can now establish our main result: a resource allocation in which the critical resource is identified with the (unconditional) Shapley value holds under the above conditions (Budget balance, Equivalence, Redundancy, Absence of interference, and Consistency).

Notice that, for the game associated with the resource allocation scenario, the conditions stated in Section 4.2.1 reduce to five conditions above (Budget balance, Equivalence, Redundancy, Absence of Interference, and Consistency). Therefore, the conditional expected payoffs (share of allocated resources) for all states are identified by conditional Shapley value.

Note that in simple coalition formation scenarios, equation 2 reduces to the following form:

$$cSh_i(S) = \frac{Sh_i}{\sum_{l \in S} Sh_l}. \quad (41)$$

It follows that $\varphi_i(S) = cSh_i(S)$ identifies proportional resource allocation for $X_i = Sh_i$.

Next, observe that a coalition structure is stable in the sense of Section 4.2.1 just in case the realized winning coalition S^* is an element of $\arg \max_{S \subset N} \Pi(S)$.

Furthermore, observe that in simple games $v(S) = 1$ for every winning coalition, and hence the stability rank defined in Section 4.2.1 reduces to

$$\Pi(S) = \frac{1}{\sum_{l \in S} Sh_l}. \quad (42)$$

Considering that $\arg \max_{S \subset N} \Pi(S) = \arg \min_{S \subset N} \sum_{l \in S} Sh_l$, stable resource allocations are identified by means of stability analysis.

4.3 An example of the application of the model

We provide an example of the model application. We define a state as “blue” (a Democratic candidate is expected to win) or “red” (a Republican candidate is expected to win) if a candidate of a respective party has won in that state in the last four presidential elections. The states where candidates from both parties have won over the last election cycles are referred to as swing states (See Table 15). Other definitions of red, blue, and swing states can be used for analysis. In this scenario, a candidate needs to secure 270 electoral votes to win the election.

Blue states	El. Votes	Red states	El. Votes	Swing states	El. Votes
California	55	Alabama	9	Arizona	11
Colorado	9	Alaska	3	Florida	29
Connecticut	7	Arkansas	6	Georgia	16
Delaware	3	Idaho	4	Indiana	11
Dist. of Columbia	3	Kansas	6	Iowa	6
Hawaii	4	Kentucky	8	Michigan	16
Illinois	20	Louisiana	8	N. Carolina	15
Maine	4	Mississippi	6	Ohio	18
Maryland	10	Missouri	10	Pennsylvania	20
Massachusetts	11	Montana	3	Wisconsin	10
Minnesota	10	Nebraska	5		
Nevada	6	N. Dakota	3		
New Hampshire	4	Oklahoma	7		
New Jersey	14	S. Carolina	9		
New Mexico	5	S. Dakota	3		
New York	29	Tennessee	11		
Oregon	7	Texas	38		
Rhode Island	4	Utah	6		
Vermont	3	W. Virginia	5		
Virginia	13	Wyoming	3		
Washington	12				
	233		153		152

Table 15: Number of electoral votes in blue, red, and swing states

The approach produces recommendations for campaign resource allocation, as summarized in Table 16. The recommended allocation redistributes 9.1 and 7.3% for the Democratic and the Republican party, respectively, if compared to allocation proportional to electoral votes (these shares are calculated as the sum of absolute difference).

State	El. Votes	Share of El. Votes	Recommended allocation Dem. Party	Recommended allocation Rep. Party
Arizona	11	7.237	7.460	7.063
Florida	29	19.079	21.944	20.556
Georgia	16	10.526	9.841	9.643
Indiana	11	7.237	7.460	7.063
Iowa	6	3.947	2.897	4.286
Michigan	16	10.526	9.841	9.643
North Carolina	15	9.868	9.048	9.643
Ohio	18	11.842	12.421	10.437
Pennsylvania	20	13.158	12.421	15.000
Wisconsin	10	6.579	6.667	6.667

Table 16: Recommended allocation of campaign resources

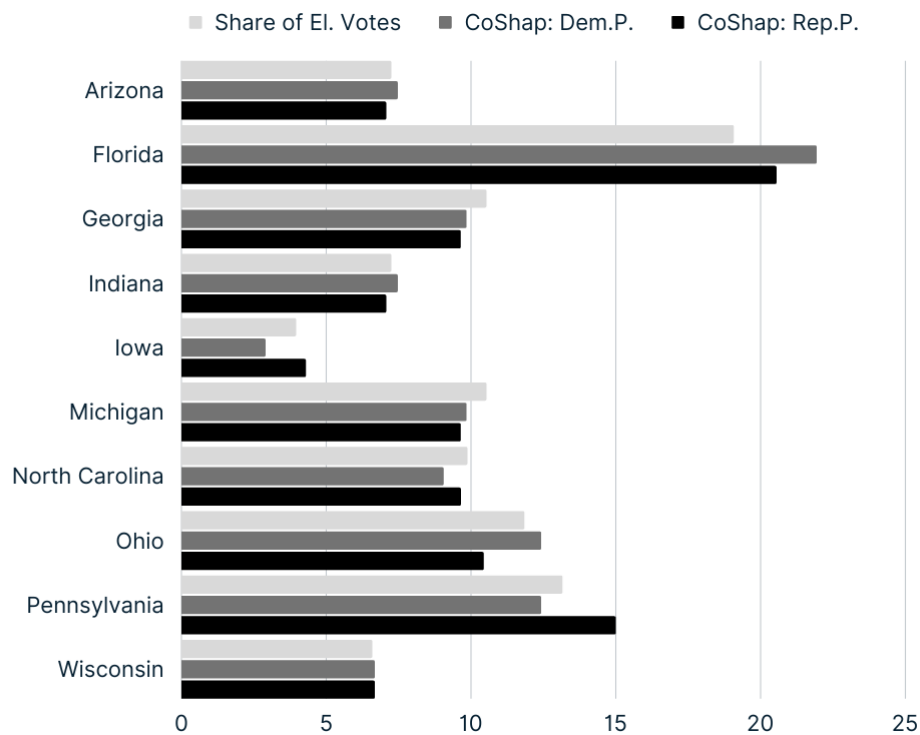


Figure 7: Recommended allocation of campaign resources

Source: Own illustration

As shown in Figure 7, the model recommends a different distribution of resources to the Democratic and Republican parties based on their strategic situation in terms of support already gained in non-swing states. For instance, Florida is relatively more important strategically for the Democratic party, while Pennsylvania is relatively more important for the Republican party.

The stability analysis prioritizes the allocations which are minimally sufficient and allow for the included states to receive the maximum share of resources. The examples of such minimally sufficient allocations (with the highest stability coefficients) for the considered case are summarized in Table 17.

Democratic Party	$\Pi(S)$	El. Votes	Republican Party	$\Pi(S)$	El. Votes
Georgia, Iowa, Michigan	4.43	38	Arizona, Florida, Georgia Iowa, Michigan, North Carolina, Ohio, Wisconsin	1.28	121
Iowa, North Carolina, Ohio	4.10	39	Florida, Georgia, Indiana Iowa, Michigan, North Carolina, Ohio, Wisconsin	1.28	121
Iowa, North Carolina, Pennsylvania	4.10	41	Arizona, Florida, Georgia Indiana, Iowa, Michigan, North Carolina, Ohio	1.28	122

Table 17: Minimally sufficient allocations

In practice, it is expected that the candidates do not start their campaign by allocating resources to minimally sufficient coalitions of states but rather from a grand coalition. However, as new information comes in, the investment of resources can be channeled to prioritized states - a narrower coalition of states. Thanks to the Consistency property these campaign adjustments can be done consistently.

4.4 Comparison with other approaches

We compare the recommended resource allocation based on the conditional Shapley value with other approaches. We do not provide a statistical comparison due to the absence of a valid benchmark but rather discuss the implications of the different approaches to resource allocation.

State	Share of El. Votes	CoShap Dem. P.	CoShap Rep. P.	3/2's rule	Following Strömberg (2008) ²
Arizona	7.237	7.460	7.063	5.776	2.857
Florida	19.079	21.944	20.556	24.958	27.143
Georgia	10.526	9.841	9.643	10.173	1.429
Indiana	7.237	7.460	7.063	5.776	0.476
Iowa	3.947	2.897	4.286	2.303	6.190
Michigan	10.526	9.841	9.643	10.173	12.381
North Carolina	9.868	9.048	9.643	9.313	4.286
Ohio	11.842	12.421	10.437	12.119	12.381
Pennsylvania	13.158	12.421	15.000	14.339	22.857
Wisconsin	6.579	6.667	6.667	5.070	10.000

Table 18: Comparison with other approaches

Both the 3/2's rule and the model by Strömberg (2008) favor Florida in terms of resource allocation as the state with the biggest amount of electoral votes, which exceeds its strategic contribution towards a majority. Georgia, Indiana, and North Carolina are allocated a disproportionately small amount of resources by Strömberg (2008) if compared to the strategic influence of those states.

4.5 Discussion

We provided a model of resource allocation within US presidential campaigns in a two-party system that provides recommendations based on clear natural principles and the strategic contribution of the number of electoral votes of a given state towards attaining a majority. Unlike previous contributions, the model assumes an interim perspective on resource allocation and allows for consistent sequential adjustments. Moreover, the model provides different recommendations for the two parties and is parsimonious in terms of the data input and computation requirements. While we assumed a two-party system, with smaller modifications the model can incorporate the impact of minority candidates.

This approach to resource allocation is generally appropriate, as electoral votes

²Data for 2004 normalized for the identified swing states

are assigned to states in rough proportion to their population with a minimum value for sparsely populated states, which reflects a higher cost of reaching the electorate there.

The paper does not address the issue of the resource allocation between states with known preferences and the swing states, as well as among the states with known preferences. The following practical steps can be considered. The resource allocation to “red” and “blue” states for the Republican and the Democratic party respectively can be executed following Owen (1975), who also provides an approximation algorithm to address the issue of computation complexity. The resource division between states with known preferences and swing states can be executed in proportion to the empirical findings of Ma and McLaren (2018). We also do not take into account secondary strategic issues, such as how exactly the campaign resources should be spent within each state.

Future work can consider further extensions of the model. A game with varying payoff that attributes higher stability depending on the level of support of a candidate can be instrumental in identifying minimally sufficient allocations with a greater likelihood of winning. A version of a model involving constraints may allow the inclusion of all states in the analysis but requires an approximation solution.

4.6 References

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5 List of Appendices

1. Computational Tools

6 Appendix

Table 19: Links to Computational Tools

Gamson-Shapley Laws: a Formal Approach to Parliamentary Coalition Formation	https://github.com/NataliyaDemyanenko/Goldenapp-28-05-2021
Bilateral trading: predicting matching and payoff distribution in markets for indivisible goods	https://github.com/NataliyaDemyanenko/Bilateral-trading
Resource Allocation and the Strategic Prioritization of Swing States in the US Presidential Campaign	https://github.com/NataliyaDemyanenko/Resource-Allocation