MODIFIED HOUSEHÖLDER METHOD OF FIFTH ORDER OF CONVERGENCE AND ITS DYNAMICS ON COMPLEX PLANE

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Abstract. In this paper, a modified Househölder method of fifth order is proposed for solving nonlinear equations. The modification is done by adapting a cubic interpolation polynomial to approximate the second derivative in the Househölder method. We provide a theorem to prove the order of convergence of the proposed method. The simulations and basins of attraction exhibit that our method is advantageous.

Keywords: Iterative methods, Househölder method, order of convergence, basins of attraction

I. INTRODUCTION

There is an abundant phenomena in nature that can be modeled via nonlinear equations. However finding the solution through analytical means do not always work. Meanwhile, the development of computational technology has been a trigger for the exponential progress in applied mathematics. This resulting in one of the most important and popular research domain in mathematics which is to find the best possible solution of nonlinear equations by exercising numerical approach via computational tools. The goal is to find the solution of

$$\xi(x) = 0 \tag{1}$$

by employing an efficient iterative method. According to Traub [1], there are two measures of an efficient iterative method. One of which is called computational efficiency. If a method is convergent to a simple root of (1), say α, with order of convergence d, then the computational efficiency of the method is given by $I = d^{1/p}$ where p is the number of function evaluations required in each iteration.

There has been a vast research in the topic of root-finding. The classic one is Newton's method which convergence quadratically with $I = 1.414$. Another popular method was proposed by Househölder [2] as follows:

$$x_{i+1} = x_i - \frac{\xi(x_i)}{\xi(x_{i+1})} - \frac{\xi(x_i)^2\xi''(x_i)}{2\xi'(x_i)}, \quad i = 0, 1, 2, \ldots. \tag{2}$$

This method is convergent cubically and has efficiency index $I = 1.442$. The drawback of this method is that it involves a second derivative. The appearance of higher order convergence tend to cause a problem due to the cost of calculating it and difficulty in its application. Hence many researchers seek to find free second derivatives iterative methods with various approaches such as in [3], [4], [5], [6], [7], and [8]. In this paper, we propose a new approximation to the second derivative in Househölder method by means of interpolating polynomial to gain a novel iterative method without second derivative.

Another course of observing the behavior of an iterative method is thorough its basins of attraction which was introduced first by [1]. Researchers have been trying to explore this subject such as in [9],[10], [11], [12], [13], [14], [15], [16], and [3]. An extensive analysis regarding the performance of iterative methods through their basins of attraction and its relation to several efficiency measures is given in [17]. We present the basins of attraction of our proposed method as well as the comparisons with several methods in order to affirm that our method is preferable in term of speed, efficiency index and number of convergent points.

The organization of this paper is as follows: The derivation of modified Householder method of fifth order of convergence by approximating the second derivative using a cubic interpolating polynomial is presented in the next section. The third section discusses the behavior of the proposed method through some simulations on several transcendental functions. We also display the dynamics of the discussed method on complex plain in section fourth. The conclusion of the paper is given in the last section of this article.

II. MODIFIED HOUSEHÖLDER METHOD FIFTH ORDER OF CONVERGENCE

In this section, we present a modified Householder method. The modification is done by approximating the second derivative in (2) using polynomial interpolation. Consider an interpolating polynomial

$$\phi_2(x) = a + b(x - x_i) + c(x - x_i)^2 + d(t - x_i)^3 \tag{3}$$

that satisfies the interpolation conditions $\xi(x_i) = \phi_2(x_i), \xi(y_i) = \phi_2(y_i), \xi'(x_i) = \phi_2'(x_i)$ and $\xi'(y_i) = \phi_2'(y_i)$. Imposing these conditions onto (3) and simplifying, we obtain

$$\xi''(y_i) = \frac{2}{y_i - x_i}\left(\frac{\xi(y_i) - \xi(x_i)}{y_i - x_i} - \xi'(x_i)\right) = \phi_2(x_i, y_i) \tag{4}$$

Applying (4) to (2), we attain a new fifth order Householder method free from second derivative as follows:

$$y_i = x_i - \frac{\xi(x_i)}{\xi'(x_i)} \tag{5}$$

$$x_{i+1} = y_i - \frac{\xi(y_i)}{\xi'(y_i)} - \frac{\xi(y_i)^3\xi'(x_i)^2}{\xi(x_i)^2\xi'(y_i)^3} \tag{6}$$

Equations (5) and (6) will be called MHM for the rest of this paper. The convergence analysis of the method is given by the following theorem:

Theorem 1 *Suppose $\xi : X \longrightarrow \mathbb{R}$ where $X \subseteq \mathbb{R}$ is an open interval. Let $\alpha \in X$ be the simple root of* (1) *where ξ is sufficiently differentiable around α. Then the method described by* (5) *and* (6) *(MHM) is of fifth order.*

Proof. Let α be a simple root of $\xi(x) = 0$. By expanding $\xi(x)$ around $x = \alpha$ through Taylor series, one attains

$$\begin{aligned}\xi(x) = \xi(\alpha) + \xi'(\alpha)(x-\alpha) + \frac{1}{2!}\xi''(\alpha)(x-\alpha)^2 + \frac{1}{3!}\xi'''(x)(x-\alpha)^3 \\ + \frac{1}{4!}\xi^{(4)}(x)(x-\alpha)^4 + \frac{1}{5!}\xi^{(5)}(x)(x-\alpha)^5 + \frac{1}{6!}\xi^{(6)}(x)(x-\alpha)^6 \\ + O(x-\alpha)^7.\end{aligned} \tag{7}$$

By evaluating $\xi(x)$ at x_i, we have

$$\xi(x_i) = \xi'(\alpha)\big(e_i + C_2e_i^2 + C_3e_i^3 + C_4e_i^4 + C_5e_i^5 + C_6e_i^6 + O(e_i^7)\big), \tag{8}$$

where $e_i = x_i - \alpha$ denotes error at i-th iteration and $C_i = (1/i!)(\xi^{(i)}(\alpha)/\xi'(\alpha)), i = 1, 2, 3, \cdots$. Differentiating (7) and evaluating it at x_i, we have

$$\xi'(x_i) = \xi'(\alpha)\big(1 + 2C_2e_i^2 + 3C_3e_i^2 + 4C_4e_i^3 + 5C_5e_i^4 + 6C_6e_ki^5 + O(e_i^6)\big). \tag{9}$$

Substituting (8) and (9) into (5) resulting in

$$\begin{aligned}y_i = \alpha + C_2e_i^2 + (-2C_2^2 + 2C_3)e_i^3 + (-4C_2^3 - 7C_2C_3 + 3C_4)e_i^4 + (-16C_2^2C_3 \\ - 10C_2C_4 - 6C_3^2 + 4C_5)e_i^5 + (-20C_2^2C_4 - 21C_2C_3^2 - 13C - 2C_5 \\ - 17C_3C_4 + 5C_6)e_i^6 + O(e_i^7)\end{aligned} \tag{10}$$

Again by expanding $\xi(y_i)$ around α, we obtain

$$\begin{aligned}\xi(y_i) = \xi'(\alpha)\big(C_2e_i^2 + (-2C_2^2 + 2C_3)e_i^3 + (-3C_2^3 - 7C - 2C_3 + 3C - 4)e_i^4 \\ + (-4C_2^4 - 12C_2^2C_3 - 10C_2C_4 - 6C_3^2 + 4C_5)e_i^5 + (-4C_2^5 - 22C_2^3C_3 \\ - 14C_2^2C_4 - 17C_2C_3^2 - 13C_2C_5 - 17C_3C_4 + 5C_6)e_i^6 + O(e_i^7)\big)\end{aligned} \tag{11}$$

Once again, by differentiating (7) and evaluating it at y_i, one has

$$\begin{aligned}\xi'(y_i) = \xi'(\alpha)\Big(1 + 2C_2\big(C_2e_i^2 + (-2C_2^2 + 2C - 3)e_i^3 + (-4C_2^3 - 7C_2C_3 + 3C_4)e_i^4 \\ (-16C_2^2C_3 - 10C_2C_4 - 6C_3^2 + 4C_5)e_i^5 + (-20C_2^2C_4 - 21C_2C_3 \\ - 13C_2C_5 - 17C_3C_4 + 5C_6)e_i^6 + O(e_i)^7\big)\Big)\end{aligned} \tag{12}$$

Now, inserting (8), (9), (11) and (12) into (6) and simplifying yields

$$e_{i+1} = -2C_2^2C_3e_i^5 + (10C_2^5 + 7C_2^3C_3 - 3C_2^2C_4 - 8C_2C_3^2)e_i^6 + O(e_i^7) \tag{13}$$

Hence, the method depicted by (5) and (6) is proven to be of fifth order. □

Number of functions evaluations for each iteration of this method is four. Hence, the efficiency index of the method is $I = 1.495$.

III. NUMERICAL SIMULATIONS

In this section, we test MHM method for eight transcendental functions. We also compare our proposed method with several modified Househölder methods such as Househölder method (HM3) with $I = 1.442$, Househölder method of fourth order (NHM4) from [3] with $I = 1.414$ and Househölder method of fifth order (NHM5) from [4] with $I = 1.495$. The followings are the testing functions:

- $\xi_1(x) = x^2 - \exp(x) - 3x + 2$
- $\xi_2(x) = \cos(x) - x$
- $\xi_3(x) = (x-1)^3 - 1$
- $\xi_4(x) = x^3 + x^2 - 10$
- $\xi_5(x) = x^2 - x\exp(x) + \cos(x)$
- $\xi_6(x) = \sin(x)^2 - x^2 + 1$
- $\xi_7(x) = x\exp(x^2) - \sin(x)^2 + 3\cos(x) + 5$
- $\xi_8(x) = \ln(x\exp(x) + 1)$

The stopping criteria for the iterations are : maximum iterations ≤ 100, $|x_{i+1} - x_i| \leq 10^{-15}$ or $|x_{i+1} - \alpha| \leq 10^{-15}$. The comparison of the discussed methods is given by the following table.

Table 1: Comparison of discussed iterative methods for functions $\xi_1(x)$ through $\xi_8(x)$

function	root	x_0	HM	NHM	ANM	MHM
$\xi_1(x)$	0.257530285439861	-2.0	4	19	5	3
		2.0	3	3	5	3
$\xi_2(x)$	0.739085133215161	1.0	3	2	4	2
		2.0	NaN	2	5	3
$\xi_3(x)$	2.00000000000000	-0.9	13	12	6	8
		3.5	5	13	7	3
$\xi_4(x)$	1.86746002460432	-1.9	6	24	10	20
		5.0	7	$*$	7	3
$\xi_5(x)$	0.639154096332008	3.5	8	5	5	4
		0.1	4	3	4	3
$\xi_6(x)$	1.40449164821534	4.5	5	$*$	5	3
		0.01	36	$*$	13	11
$\xi_7(x)$	-1.20764782713092	-2.5	8	NaN	6	4
		1.0	74	1	6	4
$\xi_8(x)$	0.00000000000000	0.9	4	3	5	3
		3.4	4	3	5	3

In Table 1 we provide two initial guesses for our observations. There is a case where a method exceeds the fixed maximum iteration, hence we denote this case with $*$. Another case is when the iterations diverge and this case we mark with NaN.

It is evident from the table that MHM relatively performs better than the rest of the methods. Given the two initial guesses, our proposed method approximates the root smoothly while several methods do not succeed in doing so. The method also needs fewer iterations compared to the rest of the methods. Based on the simulations, we can conclude that MHM is favorable in terms of efficiency index, computational cost and number of iterations.

IV. THE DYNAMICS OF THE METHOD ON COMPLEX PLANE

In order to get a better view on the behavior of the discussed method, we provide the basins of attraction. The analysis on the basins of attraction gives a clear information on the convergence and stability of the tested function when it is applied to an iterative method.

In this test, we consider some function $\xi(z) = 0$ where $\xi : \mathbb{C} \longrightarrow \mathbb{C}$ is a complex plane. The figure of basins of attraction of the tested function is generated from a uniform grid of $[-1, 1] \times [1, 1] \subset \mathbb{C}$. This gives us 1000000 initial points to be tested. Each point will be assigned to a fixed color that marks its convergence. In this work, we fixed error tolerance to be 10^{-15}. In order to see the speed of convergence of iterative method, we set the maximum iterations to be just 10. Time of generating basins of attraction on our computer is in measured in seconds.

In this work, we generate basins of attraction from four methods in the simulation section, namely HM, NHM4, NHM5, and MHM. There are four test functions :

1. $\xi_1(z) = z^2 - z + 1$
2. $\xi_2(z) = z^3 - 1$
3. $\xi_3(z) = z^4 - 10z^2 + 9$
4. $\xi_4(z) = z^5 - 5z^3 + 4z$

Table 2: Comparison of number of divergent points of iterative methods in solving $\xi(z) = 0$ in complex plane

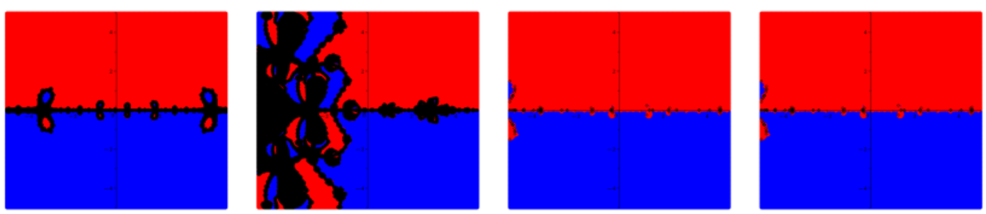
Function	Roots	HM	NHM4	NHM5	MHM
$\xi_1(z)$	$0.5000000000 - 0.8660254038i$	489257	412064	312902	499943
	$0.5000000000 - 0.8660254038i$	489257	412064	312902	499943
	divergent	175872	21486	374196	114
	time	1969.402	5323.196	5206.946	1875.426
$\xi_2(z)$	$-0.5000000000 - 0.8660254038i$	291887	301676	256200	316599
	$-0.5000000000 + 0.8660254038i$	291887	301676	256200	316599
	1	151726	207802	302188	345648
	divergent	114038	244922	279798	21154
	time	3314.902	9621.241	6483.919	2408.467
$\xi_3(z)$	-3	112754	96730	41600	123678
	-1	348898	363250	394448	372550
	1	348898	360012	394448	372550
	3	112754	91132	41600	123678
	divergent	76696	88876	127904	7544
	time	3857.295	11295.855	7009.016	3917.205

Table 3: Comparison of number of divergent points of iterative methods in solving $\xi(z) = 0$ in complex plane

Function	Roots	HM	NHM4	NHM5	MHM
$\xi_4(z)$	$-2.$	99554	52654	13382	111266
	-1	246996	145286	293654	266420
	0	199896	134560	285604	230708
	1	246996	145286	293654	266420
	2	99554	52654	13382	111266
	divergent	107004	469560	100324	13920
	time	5306.896	15930.315	7979.46	4113.141

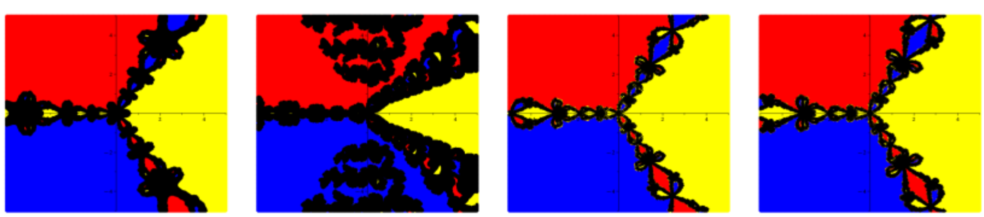
It can be seen from Table 2 and Table 3 that MHM succeeds in sending great percentage of the initial points than the rest of the methods. In details, in function $\xi_1(z)$ through $\xi_3(z)$, MHM has the least divergent points namely 0.01%, 2.1%, and 0.8% of consecutively and NHM5 has the most divergent points in all three cases. In $\xi_4(z)$ MHM is still favorable with only 1.4% divergent points while NHM4 fails to send almost 47% initial points to converge. It is evident as well that MHM generates basins of attraction faster than the rest of the discussed methods.

Figures below display the basins of attraction of the aforementioned functions.



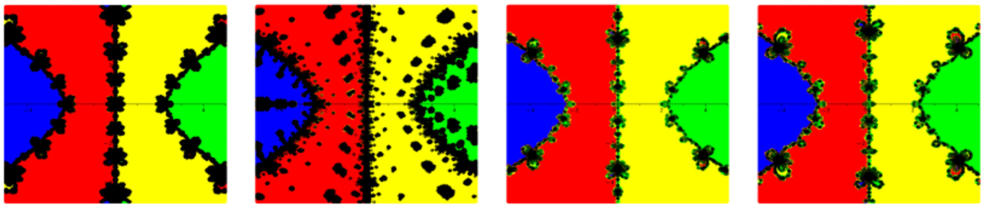
(a) HM (b) NHM4 (c) NHM5 (d) MHM

Figure 1: Basins of attraction of iterative methods for $\xi(z) = z^2 - z + 1$



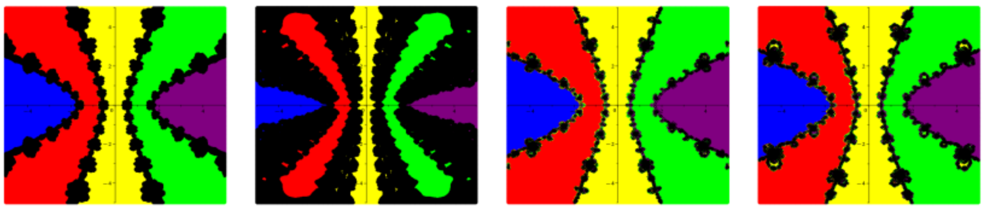
(a) HM (b) NHM4 (c) NHM5 (d) MHM

Figure 2: Basins of attraction of iterative methods for $\xi(z) = z^3 - 1$



(a) HM (b) NHM4 (c) NHM5 (d) MHM

Figure 3: Basins of attraction of iterative methods for $\xi(z) = z^4 - 10z^2 + 9$



(a) HM (b) NHM4 (c) NHM5 (d) MHM

Figure 4: Basins of attraction of iterative methods for $\xi(z) = z^5 - 5z^3 + 4z$

V. CONCLUSIONS AND FUTURE RESEARCH DIRECTION

In this work we have proposed a modified Househölder method where the second derivative is approximated by cubic polynomial interpolation. We have done simulation on the proposed method and given comparisons with several Househölder methods with different order of convergence. We have provided basins of attraction and calculated the number of initial guesses that converge and diverge. We also have timed the basins of attraction generation process. In terms of efficiency index, our method can compete with a method of higher order. Based on the simulations, the method needs fewer iterations method, succeeds in sending the most initial points and converges rapidly. We conclude that our proposed method is favorable than the rest of the tested methods.

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