Applications of the homotopy perturbation method for some linear and non-linear partial differential equations

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ARTICLE INFO

ABSTRACT

In this study, some linear PDEs and nonlinear PDEs are investigated using the homotopy perturbation method (HPM). The primary objective of this research is to employ the HPM as a tool for investigating a range of PDEs and extracting their analytical solutions. To clarify the practicality and efficacy of this method, we present illustrative examples of linear PDEs encompassing the classical heat, wave, and Laplace equations. Subsequently, a comparative analysis is performed, contrasting the outcomes derived from the HPM with established accurate solutions. Through this comparative approach, we aim to provide a comprehensive understanding of the HPM's applicability, robustness, and precision in solving a spectrum of PDEs. Our study contributes to the broader exploration of innovative mathematical techniques for tackling complex PDEs, while also shedding light on the potential advantages and limitations of the homotopy perturbation method in practical applications.

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INTRODUCTION

The Partial Differential Equation (PDE) is a commonly used mathematical tool to describe natural phenomena. PDEs can be classified into different types, including elliptic, parabolic, and hyperbolic. Elliptic PDEs describe stationary events, while parabolic PDEs are used to model time-dependent processes such as heat conduction and particle diffusion. The Homotopy Perturbation Method (HPM) is a popular method used to solve PDEs and was developed by a Chinese researcher named He (1999). The HPM has been used by many scholars to solve these types of PDEs. The mathematical study of the biological population model by HPM has been studied in Roul (2010). The mathematical study of diabetes and its complications by HPM is studied in Enagi, Bawa, & Sani (2017). Nonlinear Schrodinger equations have been studied by HPM in Biazar & Ghazvini (2007). The HPM has been applied to the mathematical study of the SIR Mumps Modelin (Ayoade, Peter, Abioye, Aminu, & Uwaheren, 2020).

In Ganji (2006), nonlinear Burger equations and nonlinear equations arising in heat transforms are studied. Hoseinzadeh, Heyns, Chamkha, & Shirkhani (2019) compare analytical and numerical methods to analyze the thermal analysis of a porous fin enclosure. Nonlinear Oscillators are studied in Anjum & He (2020), the non-linear K-dV equation is studied in Rahman, Murshed, & Akhter (2022), and many other types of nonlinear problems have been studied by this method.

The purpose of this paper is to evaluate the performance of the Homotopy Perturbation Method (HPM) for solving different types of partial differential equations (PDEs), including elliptic, parabolic, and hyperbolic PDEs. Specifically, the Laplace equation, heat equation, and wave equation are considered, and the results obtained using the HPM are compared with their corresponding exact solutions (Burden & Faires, 1978).

METHOD

To complete this study, the following research approach was used:

(i) The literature has been reviewed to establish the justification and history of this research.

(ii) The Homotopy Perturbation Method (HPM), which is found in Rahman et al. (2022), has been investigated.

(iii) Creating a numerical strategy for a certain parabolic PDE.

(iv) Creating a numerical strategy for a certain elliptic PDE.

(v) Creating a numerical strategy for a certain hyperbolic PDE.

(vi) Compare all the estimated outcomes with Burden & Faires (1978).

(vii) Finally, the results have been analyzed.

RESULTS AND DISCUSSION

The numerical schemes of HPM for elliptic, parabolic, and hyperbolic PDEs are described below:

Example 1: Consider the following linear **elliptic PDEs (Laplace equation)**:
(1) $u_{xx} + u_{tt} = 4$, $0 < x < 1$; $0 < t < 2$, subject to the initial conditions $u(x, 0) = x^2$; $u_t(x, 0) = -2x$.

For solving equation (1) by the homotopy perturbation method, first we construct a homotopy $w: \Omega \times [0,1] \rightarrow \mathbb{R}^2$ that satisfies the homotopy equation $H(w, p) = \mathcal{L}(w) - \mathcal{L}(u_0) + p[\mathcal{L}(u_0) + \mathcal{N}(w) - f(t)] = 0$, where $t \in \Omega$, $p \in [0, 1]$, $\mathcal{L} = \frac{\partial^2}{\partial t^2}$, $\mathcal{N}(w) = \frac{\partial^2 w}{\partial x^2}$, $f(t) = 4$.
Then we have:
(2) $w_{tt} - (u_0)_{tt} + p(u_0)_{tt} + p[w_{xx} - 4] = 0$

Substituting the initial condition in equation (2), we have,
$w_{tt} - (x^2)_{tt} + p(x^2)_{tt} + p[w_{xx} - 4] = 0$
i.e., **(3)** $w_{tt} - 0 + p.0 + p[w_{xx} - 4] = 0$,
i.e., $w_{tt} + p[w_{xx} - 4] = 0$
Substituting $w = w_0 + pw_1 + p^2 w_2 + p^3 w_3 + \cdots$ in equation (3), we have
(4) $(w_0 + pw_1 + p^2 w_2 + p^3 w_3 + \cdots)_{tt} + p[(w_0 + pw_1 + p^2 w_2 + p^3 w_3 + \cdots)_{xx} - 4] = 0$.
Considering $u(x, 0) = w(x, 0) = x^2$, we get $(w_0 + pw_1 + p^2 w_2 + p^3 w_3 + \cdots)(x, 0) = x^2$
i.e., $(w_0(x, 0) = x^2, w_1(x, 0) = w_2(x, 0) = w_3(x, 0) = \cdots = w_n(x, 0) = \cdots = 0$,
and $u_t(x, 0) = w_t(x, 0) = -2x$,
$i.e., (w_0 + pw_1 + p^2 w_2 + p^3 w_3 + \cdots)_t(x, 0) = -2x$
i.e.,. $(w_0)_t(x, 0) = -2x$, $(w_1)_t(x, 0) = (w_2)_t(x, 0) = (w_3)_t(x, 0) = \cdots = (w_n)_t(x, 0) = \cdots = 0$.
Now equation (4) can be written as:
$p^0(w_0)_{tt} + p^1[(w_1)_{tt} + (w_0)_{xx} - 4] + p^2[(w_2)_{tt} + (w_1)_{xx}] + p^3[(w_3)_{tt} + (w_2)_{xx}] + \cdots + p^n[(w_n)_{tt} + (w_{n-1})_{xx}] + \cdots = 0$.

Which can be written as:

p^0: $(w_0)_{tt}$, $(w_0(x,0) = x^2$, $(w_0)_t(x,0) = -2x$

p^1: $(w_1)_{tt} + (w_0)_{xx} - 4 = 0$, $w_1(x,0) = 0$, $(w_1)_t(x,0) = 0$

p^2: $(w_2)_{tt} + (w_1)_{xx} = 0$, $w_2(x,0) = 0$, $(w_2)_t(x,0) = 0$

p^3: $(w_3)_{tt} + (w_2)_{xx} = 0$, $w_3(x,0) = 0$, $(w_3)_t(x,0) = 0$

$\vdots$

p^n: $(w_n)_{tt} + (w_{n-1})_{xx} = 0$, $w_n(x,0) = 0$, $(w_n)_t(x,0) = 0$

$\vdots$

Solving the above equations, we have $w_0 = -2xt + x^2, w_1 = t^2, w_2 = 0, w_3 = 0$ Continuing this process, we have $w_4 = w_5 = \cdots = w_n = \cdots = 0$

Therefore, the solution series is given by $u(x,t) = \lim_{p \to 1} w(x,t) = w_0 + w_1 + w_2 + w_3 + \cdots$

i.e., $u(x,t) = -2xt + x^2 + t^2 + 0 + 0 + 0 + \cdots$

i.e., $u(x,t) = (x-t)^2$.

Example 2: Consider the following parabolic PDEs (heat equation):

(5) $u_t = \frac{4}{\pi^2} u_{xx}$, $0 < x < 4; 0 < t$;

Subject to the initial conditions, $u(x,0) = sin(\frac{\pi x}{4})(1 + 2cos(\frac{\pi x}{4}))$, $0 \le x \le 4$.

For solving equation (5) by the homotopy perturbation method, first we construct a homotopy $w: \Omega \times [0,1] \to \mathbb{R}^2$ that satisfies the homotopy equation

$H(w,p) = \mathcal{L}(w) - \mathcal{L}(u_0) + p[\mathcal{L}(u_0) + \mathcal{N}(w) - f(r)] = 0$, where $r \in \Omega$, $p \in [0,1], \mathcal{L} = \frac{\partial}{\partial t}, \mathcal{N}(w) = \frac{\partial^2 w}{\partial x^2}, f(t) = 0$.

Then we have,

(6) $w_t - (u_0)_t + p(u_0)_t - \frac{4}{\pi^2} p w_{xx} = 0$.

Substituting $w = w_0 + pw_1 + p^2 w_2 + p^3 w_3 + \cdots$ in equation (6), we have

(7) $(w_0 + pw_1 + p^2 w_2 + p^3 w_3 + \cdots)_t - (u_0)_t + p(u_0)_t - \frac{4}{\pi^2} p(w_0 + pw_1 + p^2 w_2 + p^3 w_3 + \cdots)_{xx} = 0$,

For simplifying we consider $u(x,0) = w(x,0) = sin(\frac{\pi x}{4})(1 + 2cos(\frac{\pi x}{4}))$

i.e.,$(w_0 + pw_1 + p^2 w_2 + p^3 w_3 + \cdots)(x,0) = sin(\frac{\pi x}{4})(1 + 2cos(\frac{\pi x}{4}))$

i.e., $(w_0)(x,0) = sin(\frac{\pi x}{4})(1 + 2cos(\frac{\pi x}{4}))$, $(w_1)(x,0) = (w_2)(x,0) = (w_3)(x,0) = \cdots = 0$.

Now equation (7) can be written as

$p^0[(w_0)_t - (u_0)_t] + p^1\left[(w_1)_t + (u_0)_t - \frac{4}{\pi^2}(w_0)_{xx}\right] + p^2\left[(w_2)_t - \frac{4}{\pi^2}(w_1)_{xx}\right] + p^3\left[(w_3)_t - \frac{4}{\pi^2}(w_2)_{xx}\right] + \cdots + p^n\left[(w_n)_t - \frac{4}{\pi^2}(w_{n-1})_{xx}\right] + \cdots = 0$.

Which can be written as:

p^0: $(w_0)_t - (u_0)_t = 0$, $(w_0)(x,0) = sin(\frac{\pi x}{4})(1 + 2cos(\frac{\pi x}{4}))$,

p^1: $(w_1)_t + (u_0)_t - \frac{4}{\pi^2}(w_0)_{xx} = 0$, $(w_1)(x,0) = 0$

p^2: $(w_2)_t - \frac{4}{\pi^2}(w_1)_{xx} = 0$, $(w_2)(x,0) = 0$

p^3: $(w_3)_t - \frac{4}{\pi^2}(w_2)_{xx} = 0$, $(w_3)(x,0) = 0$

$\vdots$

p^n: $(w_n)_t - \frac{4}{\pi^2}(w_{n-1})_{xx} = 0$, $(w_n)(x,0) = 0$

$\vdots$

Solving the above equations, we get

$w_0 = u_0 = sin(\frac{\pi x}{4}) + sin(\frac{\pi x}{2})$,

$w_1 = -\frac{t}{4}sin(\frac{\pi x}{4}) - tsin(\frac{\pi x}{2})$,

$w_2 = \frac{1}{4^2}\frac{t^2}{2!}sin(\frac{\pi x}{4}) + \frac{t^2}{2!}sin(\frac{\pi x}{2})$,

$w_3 = -\frac{1}{4^3}\frac{t^3}{3!}sin(\frac{\pi x}{4}) - \frac{t^3}{3!}sin(\frac{\pi x}{2})$,

Continuing in this process, we have $w_n = (-1)^n\frac{1}{4^n}\frac{t^n}{n!}sin(\frac{\pi x}{4}) + (-1)^n\frac{t^n}{n!}sin(\frac{\pi x}{2})$

Therefore, the solution of the series is given by $u(x,t) = \lim_{p \to 1} w(x,t) = w_0 + w_1 + w_2 + \cdots$

i.e., $u(x,t) = sin\left(\frac{\pi x}{4}\right) + sin\left(\frac{\pi x}{2}\right) - \frac{t}{4}sin\left(\frac{\pi x}{4}\right) - tsin\left(\frac{\pi x}{2}\right) + \frac{1}{4^2}\frac{t^2}{2!}sin\left(\frac{\pi x}{4}\right) + \frac{t^2}{2!}sin\left(\frac{\pi x}{2}\right) - \frac{1}{4^3}\frac{t^3}{3!}sin\left(\frac{\pi x}{4}\right) - \frac{t^3}{3!}$

$$sin\left(\frac{\pi x}{2}\right) + \cdots + (-1)^n \frac{1}{4^n} \frac{t^n}{n!} sin(\frac{\pi x}{4})$$
$$+ (-1)^n \frac{t^n}{n!} sin(\frac{\pi x}{2}) + \cdots$$

i.e., $u(x,t) = sin\left(\frac{\pi x}{4}\right)[1 - \frac{t}{4} + \frac{1}{4^2} \cdot \frac{t^2}{2!} -$
$\frac{1}{4^3} \cdot \frac{t^3}{3!} + \cdots + (-1)^n \frac{1}{4^n} \cdot \frac{t^n}{n!} + \cdots] +$
$sin\left(\frac{\pi x}{2}\right)$
$[1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots + (-1)^n \cdot \frac{t^n}{n!} + \cdots]$

i.e., $u(x,t) = sin\left(\frac{\pi x}{4}\right) e^{-\frac{t}{4}} + sin\left(\frac{\pi x}{2}\right) e^{-t}$.

Example 3: Consider the following hyperbolic PDEs (wave equation):
(8) $u_{tt} - 4u_{xx} = 0, \ 0 < x < 1; \ 0 < t,$

Subject to the initial conditions $u(x,0) = sin(\pi x); \ 0 \le x \le 1, u_t(x,0) = 0;$.

For solving equation (8) by the homotopy perturbation method, first we construct a homotopy $w: \Omega \times [0,1] \to \mathbb{R}^2$ that satisfies the homotopy equation
$H(w,p) = \mathcal{L}(w) - \mathcal{L}(u_0) + p[\mathcal{L}(u_0) + \mathcal{N}(w) - f(t)] = 0$, where $t \in \Omega$, $p \in [0,1], \mathcal{L} = \frac{\partial^2}{\partial t^2}$, $\mathcal{N}(w) = \frac{\partial^2 w}{\partial x^2}$, $f(t) = 0$.
Then we have,
(9) $w_{tt} - (u_0)_{tt} + p(u_0)_{tt} - 4pw_{xx} = 0$.
Substituting $w = w_0 + pw_1 + p^2 w_2 + p^3 w_3 + \cdots$ in equation (9), we have
(10) $(w_0 + pw_1 + p^2 w_2 + p^3 w_3 + \cdots)_{tt} - (u_0)_{tt} + p(u_0)_{tt} - 4p(w_0 + pw_1 + p^2 w_2 + p^3 w_3 + \cdots)_{xx} = 0,$
For simplifying we consider $u(x,0) = w(x,0) = sin(\pi x)$
i.e., $(w_0 + pw_1 + p^2 w_2 + p^3 w_3 + \cdots)(x,0) = sin(\pi x)$
i.e., $w_0(x,0) = sin(\pi x)$,
$w_1(x,0) = w_2(x,0) = w_3(x,0) = \cdots = w_n(x,0) = \cdots = 0$.
Also, $u_t(x,0) = w_t(x,0) = 0$
i.e., $(w_0 + pw_1 + p^2 w_2 + p^3 w_3 + \cdots)_t(x,0) = 0$
i.e., $(w_0)_t(x,0) = (w_1)_t(x,0) = (w_2)_t(x,0) = \cdots = (w_n)_t(x,0) = \cdots = 0$
Now equation (10) can be written as
$p^0[(w_0)_{tt} - (u_0)_{tt}] + p^1[(w_1)_{tt} + (u_0)_{tt} -$

$4(w_0)_{xx}] + p^2[(w_2)_{tt} - 4(w_1)_{xx}] + p^3[(w_3)_{tt} - 4(w_2)_{xx}] + \cdots + p^n[(w_n)_{tt} - 4(w_{n-1})_{xx}] + \cdots = 0$.
Which can be written as:
$p^0 : (w_0)_{tt} - (u_0)_{tt} = 0, (w_0)(x,0) = sin(\pi x), \ (w_0)_t(x,0) = 0$
$p^1 : (w_1)_{tt} + (u_0)_{tt} - 4(w_0)_{xx} = 0, \ (w_1)(x,0) = 0, (w_1)_t(x,0) = 0$
$p^2 : (w_2)_{tt} - 4(w_1)_{xx} = 0, \ (w_2)(x,0) = 0, \ (w_2)_t(x,0) = 0$
$p^3 : (w_3)_{tt} - 4(w_2)_{xx} = 0, \ (w_3)(x,0) = 0, \ (w_3)_t(x,0) = 0$
$\vdots$
$p^n : (w_n)_t - 4(w_{n-1})_{xx} = 0, \ (w_n)(x,0) = 0, (w_0)_t(x,0) = 0$
$\vdots$

Solving the above equations, we get
$w_0 = u_0 = sin(\pi x), \ w_1 = -(2\pi)^2 \frac{t^2}{2!} sin(\pi x), w_2 = (2\pi)^4 \frac{t^4}{4!} sin(\pi x), \quad w_3 = -(2\pi)^6 \frac{t^6}{6!} sin(\pi x), w_4 = (2\pi)^8 \frac{t^8}{8!} sin(\pi x),$

Continuing in this process, we can find
$w_n = (-1)^n (2\pi)^{2n} \frac{t^{2n}}{(2n)!} sin(\pi x)$,
Therefore, the solution series is given by
$u(x,t) = \lim_{p \to 1} w(x,t) = w_0 + w_1 + w_2 + w_3 + \cdots + w_n + \cdots$
i.e., $u(x,t) = sin(\pi x) - (2\pi)^2 \frac{t^2}{2!} sin(\pi x)$
$+ (2\pi)^4 \frac{t^4}{4!} sin(\pi x) - (2\pi)^6 \frac{t^6}{6!} sin(\pi x)$
$+ (2\pi)^8 \frac{t^8}{8!} \cdot sin(\pi x) + \cdots + (-1)^n (2\pi)^{2n} \frac{t^{2n}}{(2n)!} sin(\pi x) + \cdots$
i.e., $u(x,t) = sin(\pi x)[1 - \frac{(2\pi t)^2}{2!} + \frac{(2\pi t)^4}{4!} - \frac{(2\pi t)^6}{6!} + \frac{(2\pi t)^8}{8!} + \cdots + (-1)^n \frac{(2\pi t)^{2n}}{(2n)!} + \cdots$
i.e., $u(x,t) = sin(\pi x) \, cos(2\pi t)$.

Example 4: Consider the following nonlinear PDEs
(11) $u_t + uu_x = 0, 0 \le x \le 1; \ 0 < t,$
Subject to the initial conditions
$$u(x,0) = 2 - x$$
For solving equation (11) by the homotopy perturbation method, first we

construct a homotopy $w: \Omega \times [0,1] \rightarrow \mathbb{R}^2$ that satisfies the homotopy equation $H(w, p) = \mathcal{L}(w) - \mathcal{L}(u_0) + p[\mathcal{L}(u_0) + \mathcal{N}(w) - f(t)] = 0$, where $t \in \Omega$, $p \in [0,1], \mathcal{L} = \frac{\partial}{\partial t}, f(t) = 0$.

Then we have,

(12) $w_t - (u_0)_t + p(u_0)_t + p\, ww_x = 0$.

Substituting $w = w_0 + pw_1 + p^2 w_2 + p^3 w_3 + \cdots$ in equation (12), we have

(13) $(w_0 + pw_1 + p^2 w_2 + p^3 w_3 + \cdots)_t - (u_0)_t + p(u_0)_t + p\,(w_0 + pw_1 + p^2 w_2 + p^3 w_3 + \cdots).(w_0 + pw_1 + p^2 w_2 + p^3 w_3 + \cdots)_x = 0$.

For simplifying we consider $u(x, 0) = w(x, 0) = 2 - x$

i.e., $(w_0 + pw_1 + p^2 w_2 + p^3 w_3 + \cdots)(x, 0) = 2 - x$

i.e., $w_0(x, 0) = 2 - x$,

$w_1(x, 0) = w_2(x, 0) = w_3(x, 0) = \cdots = w_n(x, 0) = \cdots = 0$.

Now equation (13) can be written as

$p^0[(w_0)_t - (u_0)_t] + p^1[(w_1)_t + (u_0)_t + w_0(w_0)_x] + p^2[(w_2)_t + w_0(w_1)_x + w_1(w_0)_x] + p^3[(w_3)_t + w_0(w_2)_x + w_1(w_1)_x + w_2(w_0)_x] + \cdots + p^n[(w_n)_t + \sum_{i=0}^{n-1} w_i(w_{n-1-i})_x] + \cdots = 0$.

Which can be written as:

$p^0: (w_0)_t - (u_0)_t = 0, (w_0)(x, 0) = 2 - x$

$p^1: (w_1)_t + (u_0)_t + w_0(w_0)_x = 0, (w_1)(x, 0) = 0$

$p^2: (w_2)_t + w_0(w_1)_x + w_1(w_0)_x = 0, (w_2)(x, 0) = 0$

$p^3: (w_3)_t + w_0(w_2)_x + w_1(w_1)_x + w_2(w_0)_x = 0, (w_3)(x, 0) = 0$

$\vdots$

$p^n: (w_n)_t + \sum_{i=0}^{n-1} w_i(w_{n-1-i})_x = 0, (w_n)(x, 0) = 0$

$\vdots$

Solving the above equations, we get

$w_0 = u_0 = 2 - x; \quad w_1 = (2 - x)t, \ w_2 = (2 - x)t^2; w_3 = (2 - x)t^3$,

Continuing in this process, we can find $w_n = (2 - x)t^n$,

Therefore, the solution series is given by

$u(x, t) = \lim_{p \to 1} w(x, t)$

$= w_0 + w_1 + w_2 + w_3 + \cdots + w_n + \cdots$

i.e., $u(x, t) = (2 - x) + (2 - x)t + (2 - x)t^2 + (2 - x)t^3 + \cdots + (2 - x)t^n + \cdots$

i.e., $u(x, t) = (2 - x)[1 + t + t^2 + t^3 + \cdots + t^n + \cdots]$

i.e., $u(x, t) = (2 - x)\sum_{n=0}^{\infty} t^n$

From Example 1, 2, 3, and 4, it has been shown that HPM gives an accurate solution for all the cases.

CONCLUSIONS AND SUGGESTIONS

In this paper, HPM has been applied to solve some PDEs, such as the Laplace equation, heat equation, and wave equation, and the obtained results have been compared with the exact solutions of these equations. From the results, we see that the solution obtained by HPM is the same as the exact solution of all three equations.

We hope the study will be helpful for further studies on the HPM for managing nonlinear differential equations in fields like structural engineering and biology, making it easier to develop analytical solutions for issues like nonlinear vibrations and population dynamics. An efficient approach for analyzing complex nonlinear phenomena and systems is HPM's iterative method.

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