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Analysis of Priority Queueing System with Working Breakdown,Vacation and Vacation Interruption under Random Environment

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ABSTRACT: We study priority queueing system consisting of working breakdown, repair, single vacation and vacation interruption. At a breakdown instant during the busy period, without stop the servie, the server provides service at slower rate for the current customer. After completion of priority service, depending on the environment, the server can choose type I or type II vacation. Both types of vacation are considered as a single vacation. On completion of type I vacation if the server finds thepriority queue is empty, he has the option to take type II vacation. During the type II vacation, when priority customers arrive the server has the option to interrupt the vacation. We use the established norm which is the corresponding steady state results for time dependent probability generating functions are obtained. Along with that, the expected waiting time for theexpected number of customers for both high and low priority queues are computed. Numerical results along with the graphical representations are shown elaborately.

1. INTRODUCTION

The queueing system has a lot of applications in communication networks, transportation networks, data communication systems etc.

An extensive study of the vacation models is done in Tagaki [15], who initially outlined about variant vacation for M/G/1 queueing system. Choudhury [2] has discussed an MX/G/1 queueing system with a single vacation. Li and Liu [8] have studied Geo/G/1 queue with vacations in random environment and Krishnamoorthy et.al [7] have studied an M/G/1 queue with vacation in random environment. A paper by Ke et. al.[6] provided a new outlook in vacation queueing models.

The study of queues with interruption was initiated by White and Christie [16] for a two priority system with preemption. Shan and Liu [12] presented an M/G/1 queue with single working vacation and vacation interruption under Bernoulli schedule, with the help of the supplementary variable technique and the matrix analytic method. Ayyappan and Thamizhselvi [1] obtained the transient analysis of $M^{X_1}, M^{X_2}/G_1, G_2/1$retrial queueing system with working vacations and vacation interruption under non-preemptive priority services, the steady state solution, moments under steady state, reliability indices, mean queue lengths and mean waiting time were calculated. Madhu and Anamika [9] considered a batch arrival priority queueing model with second optional service and server breakdown in which the inter arrival and service time are followed exponential and general distribution respectively.

The idea of working breakdown is different from working vacations. This concept was introduced by Kalidass and Kasturi [3]. Kim and Lee[5] presented M/G/1 queue with disaster and working breakdown and Rajadurai [11] studied M/G/1 retrial queueing system with disaster under working vacations and working breakdowns.

We consider a single server batch arrival priority based queuing system with working breakdown, repair, single vacations and vacation interruption. There are two kinds of queues in which customers arrive in batches according to two independent compound Poisson processes. At the end of priority service, depending on the environment, the server goes either for type I vacation with probability p1 or for type II vacation with probability p2 such that p1+p2=1. Both type I and type II vacation are considered as a single vacation. Further, at the end of a type I vacation server has an option to go for type II vacation, if there are no customers in the priority queue. During the type II vacation, priority customers arrival interrupt the server vacation. On completion of the type II vacation, the server provides the service for customers, if any. On account of that, the system may be subject to breakdowns; the breakdowns occur according to Poisson process. Once the system breaks down, the server will not stop the service immediately, he completes the current customer by giving the service at a slower rate which is known as working breakdown service which follows general distribution.

2. MATHEMATICAL DESCRIPTION

Priority and ordinary customers are arrive at the system in batches of variable size in a compound Poisson process. Let $\lambda 1 c_i dt$ (i=1,2,3,...) and $\lambda 2 c_j dt$ (j=1,2,3,...) be the first order probability that a batch of i and j customers arrive at the system during a short interval of time (t,t+dt), where $0 \le c_i \le 1, \sum_i C_i = 1$, $0 \le c_j \le 1, \sum_j C_j = 1$, and $\lambda 1>0, \lambda 2>0$ are the mean arrival rate for priority and ordinary customers entering into the system. Note that, the ordinary customers will be served only when there are no priority customers in the queue. Consequently,priority customers have non-preemptive priority over ordinary customers.

At the end of high priority service, depending on the environment, the server goes either type I or type II vacation.

On returning from type I vacation, there are no customers in priority queue then the server has an option to go for type II vacation with probability θ or provide the service to the odinary customers with probability (1-θ), if any.

Further, during type II vacation, the server can interrupt the vacation and transit to regular service without completing type II vacation because of the arrival of priority customers.

The system may breakdown during busy period and breakdowns are assumed to occur according to a Poisson stream with rate $\alpha>0$. But server work at a slower rate compared to a regular service rate for the current customer, after that it will go for repair.

The stochastic processes involved in the system are assumed to be independent of each other.

In the steady state, we assume that $B_i(0)=0, B_i(\infty)=1, V_I(0)=0,$ $V_I(\infty)=1, V_{II}(0)=0, V_{II}(\infty)=1, R(0)=0, R(\infty)=1$ are continues at x=0 (i=1,2,3,4).

3. DEFINITIONS AND NOTATIONS

Let

$N_1(t)$ and $N_2(t)$ be the priority and ordinary queue size at time t.

$B_i^0(t)$,i=1,2,3,4 be the elapsed service time of the priority and ordinary service and also working breakdown(WB)priority and ordinary service respectively.

$V_I^0(t)$ be the elapsed type I vacation time.

$V_{II}^0(t)$ be the elapsed type II vacation time.

$R^0(t)$ be the elapsed repair time.

Y(t) denote the server state at time t is given by

$$Y(t) = \begin{cases} 0, \text{ if the server is idle} \\ 1, \text{ if the server is busy with priority service} \\ 2, \text{ if the server is busy with ordinary service} \\ 3, \text{ if the server is busy with priority service during working breakdown period} \\ 4, \text{ if the server is busy with ordinary service during working breakdown period} \\ 5, \text{ if the server ison type I vacation ;} \\ 6, \text{ if the serveris on type II vacation;} \\ 7, \text{ if the serveris under repair;} \end{cases}$$

The priority and ordinary service time, working breakdown service time, vacation time and repair time follows general (arbitrary) distribution and the notions used for their cumulative distribution function (CDF), the probability density function (pdf) and the Laplace Transform (LT) are given in the table 1

Table 1Notations

Server state	CDF	pdf	LT	Hazard rate
High priority service for regular period	$B_1(t)$	$b_1(t)$	$\overline{B}_1(s)$	$\mu_1(\kappa)$
Low priority service for regular period	$B_2(t)$	$b_2(t)$	$\overline{B}_2(s)$	$\mu_2(\kappa)$
High priority service for WB period	$B_3(t)$	$b_3(t)$	$\overline{B}_3(s)$	$\mu_3(\kappa)$
Low priority service for WB period	$B_4(t)$	$b_4(t)$	$\overline{B}_4(s)$	$\mu_4(\kappa)$
Type I vacation	$V_I(t)$	$v_I(t)$	$\overline{V}_I(s)$	$\gamma_1(\kappa)$
Type II vacation	$V_{II}(t)$	$v_{II}(t)$	$\overline{V}_{II}(s)$	$\gamma_2(\kappa)$
Repair	$R(t)$	$r(t)$	$\overline{R}(s)$	$\eta(\kappa)$

Next, we define the probability $I_0(t)=\text{Prob}\{N_1(t)=0,N_2(t)=0,Y(t)=0\}$, and probability densities are

$$P_{m,n}^{(1)}(x,t)dx = \text{Prob}\{N_1(t)=m, N_2(t)=n, Y(t)=1; x \le B_1^0(t) < x+dx\},$$

$$P_{m,n}^{(2)}(x,t)dx = \text{Prob}\{N_1(t)=m, N_2(t)=n, Y(t)=2; x \le B_2^0(t) < x+dx\},$$

$$Q_{m,n}^{(1)}(x,t)dx = \text{Prob}\{N_1(t)=m, N_2(t)=n, Y(t)=3; x \le B_3^0(t) < x+dx\},$$

$$Q_{m,n}^{(2)}(x,t)dx = \text{Prob}\{N_1(t)=m, N_2(t)=n, Y(t)=4; x \le B_4^0(t) < x+dx\},$$

$$V_{I,m,n}(x,t)dx = \text{Prob}\{N_1(t)=m, N_2(t)=n, Y(t)=5; x \le V_I^0(t) < x+dx\},$$

$$V_{II,m,n}(x,t)dx = \text{Prob}\{N_1(t)=m, N_2(t)=n, Y(t)=6; x \le V_{II}^0(t) < x+dx\}$$

$$R_{m,n}(x,t)dx = \text{Prob}\{N_1(t)=m, N_2(t)=n, Y(t)=7; x \le R^0(t) < x+dx\},$$

for $x \ge 0$, $t \ge 0$, $m \ge 0$ and $n \ge 0$.

4. EQUATION GOVERNING THE SYSTEM

Here, we construct a set of Kolmogorov forward equations using supplementary variable technique as follows:

$$\frac{\partial}{\partial t}P_{m,n}^{(1)}(x,t) + \frac{\partial}{\partial x}P_{m,n}^{(1)}(x,t) = -(\lambda_1+\lambda_2+\alpha+\mu_1(\kappa))P_{m,n}^{(1)}(x,t) + (1-\delta_{m0})\lambda_1\sum_{i=1}^{m} c_i P_{m-i,n}^{(1)}(x,t) + (1-\delta_{n0})\lambda_2\sum_{j=1}^{n} c_j P_{m,n-j}^{(1)}(x,t) \quad (1)$$

$$\frac{\partial}{\partial t}P_{m,n}^{(2)}(x,t) + \frac{\partial}{\partial x}P_{m,n}^{(2)}(x,t) = -(\lambda_1+\lambda_2+\alpha+\mu_2(x))P_{m,n}^{(2)}(x,t) + (1-\delta_{m0})\lambda_1\sum_{i=1}^{m} c_i P_{m-i,n}^{(2)}(x,t) + (1-\delta_{n0})\lambda_2\sum_{j=1}^{n} c_j P_{m,n-j}^{(2)}(x,t) \quad (2)$$

$$\frac{\partial}{\partial t}V_{I,m,n}(x,t) + \frac{\partial}{\partial x}V_{I,m,n}(x,t) = -(\lambda_1+\lambda_2+\gamma_1(x))V_{I,m,n}(x,t) + (1-\delta_{m0})\lambda_1\sum_{i=1}^{m} c_i V_{I,m-i,n}(x,t) + (1-\delta_{n0})\lambda_2\sum_{j=1}^{n} c_j V_{I,m,n-j}(x,t) \quad (3)$$

$$\frac{\partial}{\partial t}V_{II,m,n}(x,t) + \frac{\partial}{\partial x}V_{II,m,n}(x,t) = -(\lambda_1+\lambda_2+\gamma_2(x))V_{II,m,n}(x,t) + (1-\delta_{m0})\lambda_1\sum_{i=1}^{m} c_i V_{II,m-i,n}(x,t) + (1-\delta_{n0})\lambda_2\sum_{j=1}^{n} c_j V_{II,m,n-j}(x,t)$$

(4)

$$\frac{\partial}{\partial t}Q_{m,n}^{(1)}(x,t)+\frac{\partial}{\partial x}Q_{m,n}^{(1)}(x,t) = -\big(\lambda_1+\lambda_2+\mu_3(x)\big)Q_{m,n}^{(1)}(x,t) + (1-\delta_{m0})\lambda_1\sum_{i=1}^{m}c_iQ_{m-i,n}^{(1)}(x,t) + (1-\delta_{n0})\lambda_2\sum_{j=1}^{n}c_jQ_{m,n-j}^{(1)}(x,t)$$

(5)

$$\frac{\partial}{\partial t}Q_{m,n}^{(2)}(x,t)+\frac{\partial}{\partial x}Q_{m,n}^{(2)}(x,t) = -(\lambda_1+\lambda_2+\mu_4(x))Q_{m,n}^{(2)}(x,t) + (1-\delta_{m0})\lambda_1\sum_{i=1}^{m}c_iQ_{m-i,n}^{(2)}(x,t) + (1-\delta_{n0})\lambda_2\sum_{j=1}^{n}c_jQ_{m,n-j}^{(2)}(x,t)$$

(6)

$$\frac{\partial}{\partial t}R_{m,n}(x,t)+\frac{\partial}{\partial x}R_{m,n}(x,t) = -\big(\lambda_1+\lambda_2+\eta(x)\big)R_{m,n}(x,t) + (1-\delta_{m0})\lambda_1\sum_{i=1}^{m}c_iR_{m-i,n}(x,t) + (1-\delta_{n0})\lambda_2\sum_{j=1}^{n}c_jR_{m,n-j}(x,t)$$

(7)

$$\frac{d}{dt}I_0(t) = -(\lambda_1+\lambda_2)I_0(t) + \int_0^\infty P_{0,0}^{(2)}(x,t)\mu_2(x)dx + \int_0^\infty R_{0,0}(x,t)\eta(x)dx + \int_0^\infty V_{II,0,0}(x,t)\gamma_2(x)dx + (1-\theta)\int_0^\infty V_{I,0,0}(x,t)\gamma_1(x)dx$$

(8)

To solve the equations (1)-(7), the boundary conditions at x = 0

$$P_{m,n}^{(1)}(0,t) = \delta_{n0}\lambda_1c_{m+1}I_0(t) + \int_0^\infty P_{m+1,n}^{(1)}(x,t)\mu_1(x)dx + \int_0^\infty P_{m+1,n}^{(2)}(\mathrm{x},t)\mu_2(x)dx + \int_0^\infty V_{I,m+1,n}(x,t)\gamma_1(x)dx + \int_0^\infty R_{m+1,n}(x,t)\eta(x)dx + \lambda_hc_{m+1}\varsigma\int_0^\infty V_{II,0,n}(\mathrm{x},t)dx$$

(9)

$$P_{0,n}^{(2)}(0,t) = \lambda_2c_{n+1}I_0(t) + \int_0^\infty P_{0,n+1}^{2}(x,t)\mu_2(x)dx + (1-\theta)\int_0^\infty V_{I,0,n+1}(x,t)\gamma_1(\mathrm{x})dx + \int_0^\infty V_{II,0,n+1}(x,t)\gamma_2(x)dx + \int_0^\infty R_{0,n+1}(x,t)\eta(x)dx$$

(10)

$$Q_{m,n}^{(i)}(0,t) = \alpha\int_0^\infty P_{m,n}^{(i)}(x,t)dxi = 1{,}2 \qquad (11)$$

$$R_{m,n}(0,t) = \int_0^\infty Q_{m,n}^{(1)}(x,t)\mu_3(x)dx + \int_0^\infty Q_{m,n}^{(2)}(x,t)\mu_4(x)dx$$

(12)

$$V_{I,0,n}(0,t) = p_1\int_0^\infty P_{0,n}^{(1)}(\kappa,t)\mu_1(\kappa)d\kappa \qquad (13)$$

$$V_{II,0,n}(0,t) = p_2\int_0^\infty P_{0,n}^{(1)}(x,t)\mu_1(x)dx + \theta\int_0^\infty V_{I,0,n}(x,t)\gamma_1(x)dx \qquad (14)$$

Initial conditions are

$$P_{m,n}^{(1)}(0) = P_{m,n}^{(2)}(0) = Q_{m,n}^{(1)}(0) = Q_{m,n}^{(2)}(0) = V_{I,m,n}(0) = V_{II,m,n}(0) = R_{m,n}(0) = 0$$

and $I_0(0) = 1; m,n \geq 0.$ (15)

The Probability Generating Function(PGF) of this model:

$$A(x,t,z_1) = \sum_{m=0}^{\infty} z_1^m A_m(x,t);$$

$$A(x,t,z_2) = \sum_{n=0}^{\infty} z_2^n A_n(x,t);$$

$$A(x,t,z_1,z_2) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} z_1^m z_2^n A_{m,n}(x,t)$$

where $A = P^{(1)}, P^{(2)}, Q^{(1)}, Q^{(2)}, V_I, V_{II}, R$. Which are convergent inside the circle given by $|z_1| \le 1, |z_2| \le 1$, and define the Laplace transform of a function f(t) as

$$\overline{f}(s) = \int_0^{\infty} f(t)e^{-st}dt$$

Now taking Laplace transforms from equation (1)-(14) and solving those equations, we get,

$$\overline{P}^{(1)}(x,s,z_1,z_2) = \overline{P}^{(1)}(0,s,z_1,z_2)[1 - \overline{B}_1(\phi_a(s,z_1,z_2))]e^{-\phi_a(s,z_1,z_2)x} \quad (17)$$

$$\overline{P}^{(2)}(x,s,z_1,z_2) = \overline{P}^{(2)}(0,s,z_1,z_2)[1 - \overline{B}_2(\phi_a(s,z_1,z_2))]\ e^{-\phi_a(s,z_1,z_2)x} \quad (18)$$

$$\overline{Q}^{(1)}(x,s,z_1,z_2) = \overline{Q}^{(1)}(0,s,z_1,z_2)[1 - \overline{B}_3(\phi_b(s,z_1,z_2))]e^{-\phi_b(s,z_1,z_2)x} \quad (19)$$

$$\overline{Q}^{(2)}(x,s,z_1,z_2) = \overline{Q}^{(2)}(0,s,z_1,z_2)[1 - \overline{B}_4(\phi_b(s,z_1,z_2))]e^{-\phi_b(s,z_1,z_2)x} \quad (20)$$

$$\overline{V}_I(x,s,z_1,z_2) = \overline{V}_I(0,s,z_1,z_2)[1 - \overline{V}_I(\phi_b(s,z_1,z_2))]e^{-\phi_b(s,z_1,z_2)x} \quad (21)$$

$$\overline{V}_{II}(x,s,z_1,z_2) = \overline{V}_{II}(0,s,z_1,z_2)[1 - \overline{V}_{II}(\phi_c(s,z_1,z_2))]e^{-\phi_c(s,z_1,z_2)x} \quad (22)$$

$$\overline{R}(x,s,z_1,z_2) = \overline{R}(0,s,z_1,z_2)[1 - \overline{R}(\phi_b(s,z_1,z_2))]e^{-\phi_b(s,z_1,z_2)x} \quad (23)$$

$$\overline{Q}^{(i)}(0,s,z_1,z_2) = \alpha\overline{P}^{(i)}(0,s,z_1,z_2)\left[\frac{1-\overline{B}_i(\phi_a(s,z_1,z_2))}{\phi_a(s,z_1,z_2)}\right] i = 1,2 \quad (24)$$

$$\overline{R}(0,s,z_1,z_2) = \alpha\overline{P}^{(1)}(0,s,z_1,z_2)[\frac{1-\overline{B}_1(\phi_a(s,z_1,z_2))}{\phi_a(s,z_1,z_2)}]\overline{B}_3(\phi_b(s,z_1,z_2)) + \alpha\overline{P}_0^{(2)}(0,s,z_2)[\frac{1-\overline{B}_2(\phi_a(s,z_1,z_2))}{\phi_a(s,z_1,z_2)}]\overline{B}_4(\phi_b(s,z_1,z_2)) \quad (25)$$

$$\overline{Q}_0^{(i)}(0,s,z_2) = \alpha\overline{P}_0^{(i)}(0,s,z_2)\left[\frac{1-\overline{B}_i(\psi_a(s,z_2))}{\psi_a(s,z_2)}\right] i = 1,2 \quad (26)$$

$$\overline{V}_{I,0}(0,s,z_2) = p_1\overline{P}_0^{(1)}(0,s,z_2)\overline{B}_1(\psi_a(s,z_2)) \quad (27)$$

$$\overline{V}_{II,0}(0,s,z_2) = [p_2 + p_1\overline{V}_I(\psi_b(z_2))]\overline{P}_0^{(1)}(0,s,z_2)\overline{B}_1(\psi_a(s,z_2) \quad (28)$$

$$\overline{R}_0(0,s,z_2) = \alpha\overline{P}_0^{(1)}(0,s,z_2)\left[\frac{1-\overline{B}_1(\psi_a(s,z_2))}{\psi_a(s,z_2)}\right]\overline{B}_3(\psi_b(s,z_2)) + \alpha\overline{P}_0^{(2)}(0,s,z_2)[\frac{1-\overline{B}_2(\psi_a(s,z_2))}{\psi_a(s,z_2)}]\overline{B}_4(\psi_b(s,z_2)) \quad (29)$$

where,

$$\phi_a(s,z_1,z_2) = s + \lambda_1[1 - C(z_1)] + \lambda_2[1 - C(z_2)] + \alpha$$

$$\phi_b(s,z_1,z_2) = s + \lambda_1[1 - C(z_1)] + \lambda_2[1 - C(z_2)]$$

$$\phi_c(s,z_1,z_2) = s + \lambda_1[1 - (1-\varsigma)C(z_1)] + \lambda_2[1 - C(z_2)]$$

$$\psi_a(s,z_2) = s + \lambda_1 + \lambda_2[1 - C(z_2)] + \alpha$$

$$\psi_b(s,z_2) = s + \lambda_1 + \lambda_2[1 - C(z_2)]$$

using equations (17)-(29) into (9) and (10), we get

$$\overline{P}^{(1)}(0,s,z_1,z_2)\{z_1 - \overline{B}_1(\phi_a(s,z_1,z_2)) - \alpha[\frac{1-\overline{B}_1(\phi_a(s,z_1,z_2))}{\phi_a(s,z_1,z_2)}]\overline{B}_3(\phi_b(s,z_1,z_2))\overline{R}(\phi_b(s,z_1,z_2))\} = \lambda_1 C(z_1)\overline{I}_0(s) + \overline{P}_0^{(2)}(0,s,z_2)\{\overline{B}_2(\phi_a(s,z_1,z_2)) - \overline{B}_2(\psi_a(s,z_2)) + \alpha[\frac{1-\overline{B}_2(\phi_a(s,z_1,z_2))}{\phi_a(s,z_1,z_2)}]\overline{B}_4(\phi_b(s,z_1,z_2))\overline{R}(\phi_b(s,z_1,z_2)) - \alpha[\frac{1-\overline{B}_2(\psi_a(s,z_2))}{\psi_a(s,z_2)}]\overline{B}_4(\psi_b(s,z_2))\overline{R}(\psi_b(s,z_2))\} - \overline{P}_0^{(1)}(0,s,z_2)\{\overline{B}_1(\psi_a(s,z_2))[1 + p_1[\overline{V}_I(\psi_b(s,z_2)) - \overline{V}_I(\phi_b(s,z_1,z_2))] - \lambda_1\varsigma C(z_1)[p_2 + \theta p_1\overline{V}_1(\psi_a(z_2))][\frac{1-\overline{V}_{II}(\psi_b(s,z_2))}{\psi_b(s,z_2)}]] + \alpha[\frac{1-\overline{B}_1(\psi_a(s,z_2))}{\psi_a(s,z_2)}]\overline{B}_3(\psi_b(s,z_2))\overline{R}(\psi_b(s,z_2))\} \quad (30)$$

$$\overline{P}_0^{(2)}(0,s,z_2)\left\{z_2-\overline{B}_2\big(\psi_a(s,z_2)\big)-\alpha\left[\frac{1-\overline{B}_2\big(\psi_a(s,z_2)\big)}{\psi_a(s,z_2)}\right]\overline{B}_4\big(\psi_b(s,z_2)\big)\overline{R}\big(\psi_b(s,z_2)\big)\right\}$$
$$=1-(s+\lambda_1+\lambda_2[1-C(z_2)])\overline{I}_0(s)+\overline{P}_0^{(1)}(0,s,z_2)\{\overline{B}_1\big(\psi_a(s,z_2)\big)\Big[p_1(1-\theta)\overline{V}_I\big(\psi_b(s,z_2)\big)+p_2\overline{V}_{II}\Big(\psi_b(s,z_2)\Big)\Big]+\alpha[\tfrac{1-\overline{B}_1(\psi_a(s,z_2))}{\psi_a(s,z_2)}]\overline{B}_3(\psi_b(s,z_2))\overline{R}(\psi_b(s,z_2))\}\quad(31)$$

We have to solve equations (30), (31). Letting $z_1=g(z_2)$ in (30) we get,

$$\overline{P}_0^{(1)}(0,s,z_2)=\frac{f_0(s,z_2)}{g_0(s,z_2)}\quad(32)$$

where,

$$f_0(s,z_l)=\lambda_1C[g(z_2)]\overline{I}_0(s)+\overline{P}_0^{(2)}(0,s,z_2)\{\overline{B}_2(\sigma_a(s,z_2))-\overline{B}_2(\psi_a(s,z_2))+\alpha[\frac{1-\overline{B}_2(\sigma_a(s,z_2))}{\sigma_a(s,z_2)}]\overline{B}_4(\sigma_b(s,z_2))\overline{R}(\sigma_b(s,z_2))-\alpha[\frac{1-\overline{B}_2(\psi_a(s,z_2))}{\psi_a(s,z_2)}]\overline{B}_4(\psi_b(s,z_2))\overline{R}(\psi_b(s,z_2))\}$$

$$g_0(s,z_2)=\overline{B}_1(\psi_a(s,z_2))[1+p_1[\overline{V}_I(\psi_b(s,z_2))-\overline{V}_I(\sigma_b(s,z_2))]-\lambda_1\varsigma C[g(z_2)][p_2+\theta p_1\overline{V}_1(\psi_a(s,z_2))][\frac{1-\overline{V}_{II}(\psi_b(s,z_2))}{\psi_b(s,z_2)}]]+\alpha[\frac{1-\overline{B}_1(\psi_a(s,z_2))}{\psi_a(s,z_2)}]\overline{B}_3(\psi_b(s,z_2))\overline{R}(\psi_b(s,z_2))$$

$$\overline{P}_0^{(2)}(0,s,z_2)=\frac{f_2(s,z_2)}{g_2(s,z_2)}\quad(33)$$

where,

$$f_2(s,z_2)=\{\{1-(s+\lambda_1+\lambda_2[1-C(z_2)])\overline{I}_0(s)\}\ \{\overline{B}_1(\psi_a(s,z_2))[1+p_1[\overline{V}_I(\psi_b(s,z_2))-\overline{V}_I(\sigma_b(s,z_2))]-\lambda_1\varsigma C(z_1)[p_2+\theta p_1\overline{V}_1(\psi_a(z_2))][\frac{1-\overline{V}_{II}(\psi_b(s,z_2))}{\psi_b(s,z_2)}]]+\alpha[\frac{1-\overline{B}_1(\psi_a(s,z_2))}{\psi_a(s,z_2)}]\{\overline{B}_1(\psi_a(s,z_2))[p_1(1-\theta)\overline{V}_I(\psi_b(s,z_2))+p_2\overline{V}_{II}(\psi_b(s,z_2))]+\alpha[\frac{1-\overline{B}_1(\psi_a(s,z_2))}{\psi_a(s,z_2)}](\psi_b(s,z_2))\overline{R}(\psi_b(s,z_2))\}$$

$$g_2(s,z_2)=\{\{z_2-\overline{B}_2(\psi_a(s,z_2))-\alpha[\frac{1-\overline{B}_2\big(\psi_a(s,z_2)\big)}{\psi_a(s,z_2)}]\overline{B}_4(\psi_b(s,z_2))\overline{R}(\psi_b(s,z_2))\}\ \{\overline{B}_1(\psi_a(s,z_2))[1+p_1[\overline{V}_I(\psi_b(s,z_2))-\overline{V}_I(\sigma_b(s,z_2))]-\lambda_1\varsigma C(z_1)[p_2+\theta p_1\overline{V}_1(\psi_a(s,z_2))]\ [\frac{1-\overline{V}_{II}(\psi_b(s,z_2))}{\psi_b(s,z_2)}]]+\alpha[\frac{1-\overline{B}_1(\psi_a(s,z_2))}{\psi_a(s,z_2)}]\overline{B}_3(\psi_b(s,z_2))\overline{R}(\psi_b(s,z_2))\}\}$$

$$\overline{P}^{(1)}(0,s,z_h,z_l)=\frac{f_1(s,z_1,z_2)}{g_1(s,z_1,z_2)}+\overline{P}_0^{(2)}(0,s,z_2)\frac{f_3(s,z_1,z_2)}{g_1(s,z_1,z_2)}\quad(34)$$

where,

$$f_1(s,z_1,z_2)=\overline{I}_0(s)\{\lambda_1C(z_1)\{\overline{B}_1(\psi_a(s,z_2))[1+p_1[\overline{V}_I(\psi_b(s,z_2))-\overline{V}_I(\sigma_b(s,z_2))]-\lambda_1\varsigma C[g(z_2)][p_2+\theta p_1\overline{V}_1(\psi_a(s,z_2))][\frac{1-\overline{V}_{II}(\psi_b(s,z_2))}{\psi_b(s,z_2)}]]+\alpha[\frac{1-\overline{B}_1(\psi_a(s,z_2))}{\psi_a(s,z_2)}]\overline{B}_3(\psi_b(s,z_2))\overline{R}(\psi_b(s,z_2))\}-\lambda_1C[g(z_2)]\{\overline{B}_1(\psi_a(s,z_2))[1+p_1[\overline{V}_I(\psi_b(s,z_2))-\overline{V}_I(\phi_b(s,z_1,z_2))]-\lambda_1\varsigma C(z_1)[p_2+\theta p_1\overline{V}_1(\psi_a(s,z_2))][\frac{1-\overline{V}_{II}(\psi_b(s,z_2))}{\psi_b(s,z_2)}]]+\alpha[\frac{1-\overline{B}_1(\psi_a(s,z_2))}{\psi_a(s,z_2)}]\overline{B}_3(\psi_b(s,z_2))\overline{R}(\psi_b(s,z_2))\}\}$$

$$f_3(s,z_1,z_2)=\left\{\overline{B}_1\big(\psi_a(s,z_2)\big)\left[1+p_1\big[\overline{V}_I\big(\psi_b(s,z_2)\big)-\overline{V}_I\big(\sigma_b(s,z_2)\big)\big]-\lambda_1\varsigma C[g(z_2)]\big[p_2+\theta p_1\overline{V}_1\big(\psi_a(s,z_2)\big)\big]\left[\frac{1-\overline{V}_{II}\big(\psi_b(s,z_2)\big)}{\psi_b(s,z_2)}\right]\right]+\alpha\left[\frac{1-\overline{B}_1\big(\psi_a(s,z_2)\big)}{\psi_a(s,z_2)}\right]\overline{B}_3\big(\psi_b(s,z_2)\big)\overline{R}\big(\psi_b(s,z_2)\big)\right\}$$

$$\{\overline{B}_2(\phi_a(s,z_1,z_2)) - \overline{B}_2(\psi_a(s,z_2)) + \alpha[\frac{1-\overline{B}_2(\phi_a(s,z_1,z_2))}{\phi_a(s,z_1,z_2)}]\overline{B}_4(\phi_b(s,z_1,z_2))\overline{R}(\phi_b(s,z_1,z_2)) - \alpha[\frac{1-\overline{B}_2(\psi_a(s,z_2))}{\psi_a(s,z_2)}]\overline{B}_4(\psi_b(s,z_2))\overline{R}(\psi_b(s,z_2))\} \quad (35)$$

$$g_1(s,z_1,z_2) = \{\overline{B}_1(\psi_a(s,z_2))[1 + p_1[\overline{V}_I(\psi_b(s,z_2)) - \overline{V}_I(\sigma_b(s,z_2))] - \lambda_1 \varsigma C[g(z_2)][p_2 + \theta p_1 \overline{V}_1(\psi_a(z_2))][\frac{1-\overline{V}_{II}(\psi_b(s,z_2))}{\psi_b(s,z_2)}]] + \alpha[\frac{1-\overline{B}_1(\psi_a(s,z_2))}{\psi_a(s,z_2)}]\overline{B}_3(\psi_b(s,z_2))\overline{R}(\psi_b(s,z_2))\}\{z_1 - \overline{B}_1(\phi_a(s,z_1,z_2)) - \alpha[\frac{1-\overline{B}_1(\phi_a(s,z_1,z_2))}{\phi_a(s,z_1,z_2)}]\overline{B}_3(\phi_b(s,z_1,z_2))\overline{R}(\phi_b(s,z_1,z_2))\}$$

Corollary3.1.The probability generating function of the Laplace transforms of the number of customers in the priority and ordinary queue while the system was in regular service, working breakdown service, repair and vacation are given by

$$\overline{P}^{(1)}(s,z_1,z_2) = \overline{P}^{(1)}(0,s,z_1,z_2)[\frac{1-\overline{B}_1(\phi_a(s,z_1,z_2))}{\phi_a(s,z_1,z_2)}] \quad (36)$$

$$\overline{P}^{(2)}(s,z_1,z_2) = \overline{P}_0^{(2)}(0,s,z_2)[\frac{1-\overline{B}_2(\phi_a(s,z_1,z_2))}{\phi_a(s,z_1,z_2)}] \quad (37)$$

$$\overline{Q}^{(1)}(s,z_1,z_2) = \alpha\overline{P}^{(1)}(0,s,z_1,z_2)\left[\frac{1-\overline{B}_1(\phi_a(s,z_1,z_2))}{\phi_a(s,z_1,z_2)}\right]\left[\frac{1-\overline{B}_3(\phi_b(s,z_1,z_2))}{\phi_b(s,z_1,z_2)}\right] \quad (38)$$

$$\overline{Q}^{(2)}(s,z_1,z_2) = \alpha\overline{P}_0^{(2)}(0,s,z_2)\left[\frac{1-\overline{B}_2(\phi_a(s,z_1,z_2))}{\phi_a(s,z_1,z_2)}\right]\left[\frac{1-\overline{B}_4(\phi_b(s,z_1,z_2))}{\phi_b(s,z_1,z_2)}\right] \quad (39)$$

$$\overline{V}_I(s,z_1,z_2) = p_1\overline{P}_0^{(1)}(0,s,z_2)\overline{B}_1(\psi_a(s,z_2))[\frac{1-\overline{V}_I(\phi_b(s,z_1,z_2))}{\phi_b(s,z_1,z_2)}] \quad (40)$$

$$\overline{V}_{II}(s,z_1,z_2) = \overline{P}_0^{(1)}(0,s,z_2)[p_2 + \theta p_1\overline{V}_I(\psi_b(s,z_2))]\overline{B}_1(\psi_a(s,z_2))[\frac{1-\overline{V}_{II}(\phi_c(s,z_1,z_2))}{\phi_c(s,z_1,z_2)}] \quad (41)$$

5. STEADY STATE ANALYSIS LIMITING BEHAVIOUR

Now, we study the steady state probability distribution for our queueing model. By applying the well-known Tauberian property,

$$\lim_{s\to 0} s\overline{f}(s) = \lim_{t\to\infty} f(t),$$

The normalizing condition is

$$P^{(1)}(1,1) + P^{(2)}(1,1) + Q^{(1)}(1,1) + Q^{(2)}(1,1) + R(1,1) + V_I(1,1) + V_{II}(1,1) + I_0 = 1$$

We get the probability generating function of the queue size irrespective of the state of the system.

$$W_q(z_1,z_2) = \frac{NR(z_1,z_2)}{Dr(z_1,z_2)} \quad (42)$$

where,

$$NR(z_1,z_2) = I_0 Nr_1(z_1,z_2) + P_0^{(2)}(0,z_2)Nr_2(z_1,z_2)$$

$$Nr_1(z_1,z_2) = \lambda_1\psi_a(z_2)\psi_b(z_2)S_3(z_1,z_3)\{C(z_1)\phi_a(z_1,z_2)H_1(z_2) - C[g(z_2)]\sigma_a(z_2)H_3(z_1,z_2)\} + \lambda_1 C[g(z_2)]\sigma_a(z_2)(\psi_a(z_2))^2 S_1(z_1,z_2)K(z_1,z_2)$$

$$Nr_2(z_1,z_2) = S_3(z_1,z_2)\{H_1(z_2)F_2(z_1,z_2) - F_1(z_2)H_3(z_1,z_2)\} + \psi_a(z_2)\psi_b(z_2)K(z_1,z_2)\{H_1(z_1)S_2(z_1,z_2) + F_1(z_1)S_1(z_1,z_2)\}$$

$$Dr(z_1,z_2) = \psi_a(z_2)\psi_b(z_2)\phi_a(z_1,z_2)\phi_b(z_1,z_2)\phi_c(z_1,z_2)H_1(z_2)K(z_1,z_2)$$

$$F_1(z_2) = \psi_a(z_2)\psi_b(z_2)\sigma_a(z_2)[\overline{B}_2(\sigma_a(z_2)) - \overline{B}_2(\psi_a(z_2))] + \alpha\psi_a(z_2)\psi_b(z_2)[1 - \overline{B}_2(\sigma_a(z_2))]\overline{B}_4(\sigma_b(z_2))\overline{R}(\sigma_b(z_2)) - \alpha\psi_b(z_2)\sigma_a(z_2)[1 - \overline{B}_2(\psi_a(z_2))]\overline{B}_4(\psi_b(z_2))\overline{R}(\psi_b(z_2))$$

$$F_2(z_1,z_2) = \psi_a(z_2)\psi_b(z_2)\phi_a(z_1,z_2)[\overline{B}_2(\phi_a(z_1,z_2)) - \overline{B}_2(\psi_a(z_1,z_2))] + \alpha\psi_a(z_2)\psi_b(z_2)[1 - \overline{B}_2(\phi_a(z_1,z_2))]\overline{B}_4(\phi_b(z_1,z_2))\overline{R}(\phi_b(z_1,z_2)) - \alpha\psi_b(z_2)\phi_a(z_1,z_2)[1 - \overline{B}_2(\psi_a(z_2))]\overline{B}_4(\psi_b(z_2))\overline{R}(\psi_b(z_2))$$

$$H_1(z_2) = \sigma_a(z_2)\overline{B}_1(\psi_a(z_2))\{1 + p_1[\overline{V}_I(\psi_b(z_2)) - \overline{V}_I(\sigma_b(z_2))]\} - \lambda_1\varsigma C[g(z_2)]\psi_a(z_2)\sigma_a(z_2)\overline{B}_1(\psi_a(z_2)) - \lambda_1\varsigma C[g(z_2)]\psi_a(z_2)\sigma_a(z_2)\overline{B}_1(\psi_a(z_2))[p_2 + \theta p_1\overline{V}_1(\psi_b(z_2))][1 - \overline{V}_{II}(\psi_b(z_2))] + \alpha\psi_b(z_2)\sigma_a(z_2)[1 - \overline{B}_1(\psi_a(z_2))]\overline{B}_3(\psi_b(z_2))\overline{R}(\psi_b(z_2))$$

$$H_3(z_1,z_2) = \psi_a(z_2)\psi_b(z_2)\phi_a(z_1,z_2)\overline{B}_1(\psi_a(z_2))\{1 + p_1[\overline{V}_I(\psi_b(z_2)) - \overline{V}_I(\phi_b(z_1,z_2))]\} - \lambda_1\varsigma C(z_1)\psi_a(z_2)\phi_a(z_1,z_2)\overline{B}_1(\psi_a(z_2))[p_2 + \theta p_1\overline{V}_1(\psi_b(z_2))][1 - \overline{V}_{II}(\psi_b(z_2))] + \alpha\psi_b(z_2)\phi_a(z_1,z_2)[1 - \overline{B}_1(\psi_a(z_2))]\overline{B}_3(\psi_b(z_2))\overline{R}(\psi_b(z_2))$$

$$S_1(z_1,z_{l2}) = \phi_a(z_1,z_2)\overline{B}_1(\psi_a(z_2))\{p_1\phi_c(z_1,z_2)[1 - \overline{V}_I(\phi_b(z_1,z_2))] + \phi_b(z_1,z_2)[p_2 + \theta p_1\overline{V}_I(\phi_b(z_1,z_2))][1 - \overline{V}_{II}(\phi_c(z_1,z_2))]\}$$

$$S_2(z_1,z_2) = \phi_c(z_1,z_2)[1 - \overline{B}_2(\phi_a(z_1,z_2))]\{\phi_b(z_1,z_2) + \alpha[1 - \overline{B}_4(\phi_b(z_1,z_2))\overline{R}(\phi_b(z_1,z_2))]\}$$

$$S_3(z_1,z_2) = \phi_c(z_1,z_2)[1 - \overline{B}_1(\phi_a(z_1,z_2))]\{\phi_b(z_1,z_2) + \alpha[1 - \overline{B}_3(\phi_b(z_1,z_2))\overline{R}(\phi_b(z_1,z_2))]\}$$

$$K(z_1,z_2) = \phi_a(z_1,z_2)\{z_1 - \overline{B}_1(\phi_a(z_1,z_2))\} - \alpha[1 - \overline{B}_1(\phi_a(z_1,z_2))]\overline{B}_3(\phi_b(z_1,z_2))\overline{R}(\phi_b(z_1,z_2))$$

To find the unknown probability I_0, by using the normalizing condition $W_q(1,1) + I_0 = 1$, we get,

$$I_0 = \frac{d_2(1)Dr''(1,1)}{d_2(1)Dr''(1,1) + d_2(1)Nr_1''(1,1) + n_2(1)Nr_2''(1,1)} \quad (43)$$

6. THE EXPECTED QUEUE LENGTH

The expected number of customers in the high priority queue is

$$L_{q_1} = \frac{d}{dz_1} W_{q_1}(z_1,1)|_{z_1=1} \quad (44)$$

and the expected number of customers in the low priority queue is

$$L_{q_2} = \frac{d}{dz_2} W_{q_2}(1,z_2)|_{z_2=1}$$

Then

$$L_{q_1} = \frac{DR''(1)NR'''(1) - DR'''(1)NR''(1)}{3(DR''(1))^2}$$

$$L_{q_2} = \frac{dr''(1)nr'''(1) - dr'''(1)nr''(1)}{3(dr''(1))^2}$$

7. THE EXPECTED WAITING TIME IN THE QUEUE

Expected waiting time of a customer in the high priority queue is

$$W_{q_1} = \frac{L_{q_1}}{\lambda_1} \quad (46)$$

Expected waiting time of a customer in the low priority queue is

$$W_{q_2} = \frac{L_{q_2}}{\lambda_2} \quad (47)$$

1.1. Particular Cases

Case 1: If there is no high priority queue, no breakdown, no immediate feedback. i.e $\lambda_1 = 0$, $\alpha = 0$. Then, our model can be reduced to $M^X/G/1$ queueing system with balking.

$$W(z_1) = [\frac{I_0\lambda_2(C(z_2)-1)}{z_2-\overline{B}_2(\lambda_2 b(1-C(z_2)))}][\frac{1-\overline{B}_2(\lambda_2(1-C(z_2)))}{\lambda_l b(1-C(z_l))}] \quad (48)$$

The above result coincides with the result of Charan et. al. [13] eq.(13).

Case 2: If there is no low priority queue, no breakdown, no immediate feedback, no vacation, no interruption. i.e $\lambda_l = 0$, $\alpha = 0$, $\theta = 0$, $\varsigma = 0$, $p_1 + p_2 = 0$ Then, our model can be reduced to $M^X/G/1$ queue.

$$W(z_1) = \frac{-I_0[1 - \overline{B}_1(\lambda_1(1 - C(z_1)))]}{z_1 - \overline{B}_1(\lambda_1(1 - C(z_1)))}$$

The above result coincides with the result of Medhi [10].

8. NUMERICAL RESULT

The above queueing model is analysed numerically with the following assumption. We consider that the service time in regular busy period, service time in working breakdown period, vacation time and repair time are to be exponentially distributed.

Table 2 shows that an increase in the high priority arrival rate, decreases the idle time and increases the expected queue length and waiting time of high priority and low priority queues for the arbitrary values, we choose $\lambda_2 = 4, \mu = 10, \mu_w = 8, \alpha = 2, \eta = 0.3, \gamma_1 = 0.7, \gamma_2 = 0.1, \theta = 0.1, \varsigma = 0.2, p_1 = 0.7,$ $p_2 = 0.3,\ E(X) = 1, E(X(X-1)) = 0$.While λ_1 varies from 1.6 to 1.9 such that the stability condition is satisfied.

Table 2 Impact of λ_1 on various queue characteristics

λ_1	I_0	L_{q_1}	W_{q_1}	L_{q_2}	W_{q_2}
1.6	0.0225	2.2456	1.4035	5.0587	1.2647
1.7	0.0221	2.9971	1.7630	6.4185	1.6046
1.8	0.0218	3.4232	1.9029	7.3162	1.8290
1.9	0.0216	3.7410	1.9689	7.7311	1.9328

Table 3 shows that an increase in the breakdown rate, decreases the idle time and increases the expected queue length and waiting time of high priority and low priority queues for the arbitrary values, we choose $\lambda_1 = 0.4, \lambda_2 = 2.5, \mu = 15, \mu_w = 14, \eta = 1, \gamma_1 = 5, \gamma_2 = 0.5, \theta = 0.5,\ \varsigma = 0.2, p_1 = 0.3, p_2 = 0.7, E(X) = 1, E(X(X-1)) = 0$. While α varies from 11 to 14 such that the stability condition is satisfied.

Table 3 Impact of α on various queue characteristics

α	I_0	L_{q_1}	W_{q_1}	L_{q_2}	W_{q_2}
11	0.0115	0.3242	0.8109	1.1721	0.4689
12	0.0114	0.3537	0.8842	1.8054	0.7222
13	0.0112	0.3730	0.9326	2.1025	0.8410
14	0.0110	0.3810	0.9524	2.1489	0.8596

Table 4 shows that an increase the working breakdown service rate, increases the idle time and decreases the expected queue length and waiting time of high priority and low priority queues for the arbitrary values, we choose $\lambda_1 = 5,\ \lambda_2 = 0.2,\ \mu = 12, \alpha = 7.8, \eta = 0.8,\ \gamma_1 = 15,\ \gamma_2 = 1, \theta = 0.6, \varsigma = 0.1, p_1 = 0.7, p_2 = 0.3,\ E(X) = 1, E(X(X-1)) = 0$.While μ_w, varies from 6 to 12 such that the stability condition is satisfied.

Table 4 Impact of μ_2 on various queue characteristics

μ_2	I_0	L_{q_1}	W_{q_1}	L_{q_2}	W_{q_2}
06	0.0642	3.0721	0.6144	18.2972	91.4860
08	0.0645	3.0220	0.6044	16.1125	80.5624
10	0.0646	2.9961	0.5992	14.5728	72.8639
12	0.0647	2.9808	0.5962	13.4203	67.1013

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