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SOLVING BURGER'S AND COUPLED BURGER'S EQUATIONS WITH CAPUTO-FABRIZIO FRACTIONAL OPERATOR

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Abstract. In this paper, we apply Daftardar-Jafari method (DJM) to obtain approximate solutions of the nonlinear Burgers (NBE) and coupled nonlinear Burger's equations (CNBEs) with Caputo-Fabrizio fractional operator (CFFO). The efficiency of the considered method is illustrated by some examples. Graphical results are utilized and discussed quantitatively to illustrate the solution. The results reveal that the suggested algorithm is very effective and simple and can be applied for other problems in sciences and engineering.
Keywords: nonlinear equations, fractional operator, approximate solutions.

1. Introduction

The fractional-order derivatives and integrals have numerous applications in physics and mathematics; for example, modeling nonlinear oscillations of earthquakes, electrodynamics, signal processing phenomena, the fractional-order fluid

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dynamic traffic model, fractional model of cancer chemotherapy, fractional diabetes model, and other areas [1–17].

Several analytical and numerical methods were successfully applied to FPDEs, such us the HPM [18], SEM [19], ADM [20], VIM [21], RDTM [22, 23], HPTM [24, 25], and other approaches [26–35]. In this paper, we use the DJM to solve nonlinear FPDEs involving CFFO. The paper has been organized as follows: The basic definitions of FC are given in Section 2, analysis of the method used is given in Section 3, several test problems that show the effectiveness of the proposed method are given in Section 5, and finally the conclusion is given in Section 6.

2. Preliminaries of fractional calculus

Definition 2.1. [25, 36, 37] The CFFO is defined for $n-1<\alpha\leq n$ as:

$$ {}^{CF}D_t^{\alpha}u(t)=\frac{(2-\alpha)M(\alpha)}{2(1-\alpha)}\int_0^t exp\left[-\frac{\alpha(t-s)}{1-\alpha}\right]u^{(n)}(s)ds,\ t\geq 0, \tag{2.1}$$

where $M(\alpha)$ is a normalization function such that $M(0)=M(1)=1$.

The properties of the operator ${}^{CF}D_t^{\alpha}$:

1. ${}^{CF}D_t^{\alpha}u(t)=u(t)$, where $\alpha=0$.
2. ${}^{CF}D_t^{\alpha}[u(t)+v(t)]={}^{CF}D_t^{\alpha}u(t)+{}^{CF}D_t^{\alpha}v(t)$.
3. ${}^{CF}D_t^{\alpha}(c)=0$, where c is constant.

Definition 2.2. [23, 28, 29] The CFFIO of order $0<\alpha\leq 1$ and $t>0$ is given by:

$$ {}^{CF}I_t^{\alpha}u(t)=\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}u(t)+\frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t u(s)ds. \tag{2.2}$$

The properties of the operator ${}^{CF}I_t^{\alpha}$:

1. ${}^{CF}I_t^{\alpha}u(t)=u(t)$, where $\alpha=0$.
2. ${}^{CF}I_t^{\alpha}[u(t)+v(t)]={}^{CF}I_t^{\alpha}u(t)+{}^{CF}I_t^{\alpha}v(t)$.
3. ${}^{CF}I_t^{\alpha}\ \left[{}^{CF}D_t^{\alpha}u(t)\right]=u(t)-u(0)$.

3. Analysis of DJM

Let us consider the nonlinear PDE in the CF sense:

$$ {}^{CF}D_t^{n+\alpha}u(x,t)+R\left[u(x,t)\right]+N\left[u(x,t)\right]=g(x,t). \tag{3.1}$$

with

$$\frac{\partial^k u(x,0)}{\partial t^k} = \phi_k(x), k = 0, 1, ..., m-1, \tag{3.2}$$

where ${}_0^{CF}D_t^{n+\alpha}u(x,t)$ is CFFO of $u(x,t)$, $m-1 < n+\alpha \leq m, m \in N$.

Taking CFFIO of (3.1):

$$\begin{aligned} & {}^{CF}I_t^{n+\alpha}\ {}^{CF}\left[D_t^{n+\alpha}u(x,t)\right] + {}^{CF}I_t^{n+\alpha}R\left[u(x,t)\right] + {}^{CF}I_t^{n+\alpha}N\left[u(x,t)\right] \\ = & {}^{CF}I_t^{n+\alpha}\left[g(x,t)\right]. \end{aligned} \tag{3.3}$$

Then, we obtain

$$\begin{aligned} u(x,t) &= \sum_{k=0}^{m-1} u^{(k)}(x,0)\frac{x^k}{k!} + {}^{CF}I_t^{n+\alpha}\left[g(x,t)\right] \\ & -{}^{CF}I_t^{n+\alpha}R\left[u(x,t)\right] - {}^{CF}I_t^{n+\alpha}N\left[u(x,t)\right]. \end{aligned} \tag{3.4}$$

Let

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \tag{3.5}$$

and

$$N\left[\sum_{n=0}^{\infty} u_n(x,t)\right] = N\left[u_0\right] + \sum_{n=1}^{\infty}\left(N\left[\sum_{i=0}^{n} u_i\right] - N\left[\sum_{i=0}^{n-1} u_i\right]\right). \tag{3.6}$$

In view of (3.5) and (3.9), Eq. (3.4) is equivalent to

$$\begin{aligned} & \sum_{n=0}^{\infty} u_n(x,t) \\ = & \sum_{k=0}^{m-1} u^{(k)}(x,0)\frac{x^k}{k!} + {}^{CF}I_t^{n+\alpha}\left[g(x,t)\right] - {}^{CF}I_t^{n+\alpha}R\left[\sum_{n=0}^{\infty} u_n\right] \\ & -{}^{CF}I_t^{n+\alpha}N\left[u_o(x,t)\right] - {}^{CF}I_t^{n+\alpha}\left(\sum_{n=1}^{\infty} N\left[\sum_{i=0}^{n} u_i\right] - N\left[\sum_{i=0}^{n-1} u_i\right]\right). \end{aligned} \tag{3.7}$$

Moreover, we have

$$\begin{aligned} u_0(x,t) &= \sum_{k=0}^{m-1} u^{(k)}(x,0)\frac{x^k}{k!} + {}^{CF}I_t^{n+\alpha}\left[g(x,t)\right], \\ u_1(x,t) &= -{}^{CF}I_t^{n+\alpha}R\left[u_0\right] - {}^{CF}I_t^{n+\alpha}N\left[u_o(x,t)\right], \\ u_{n+1}(x,t) &= -{}^{CF}I_t^{n+\alpha}R\left[u_n\right] - {}^{CF}I_t^{n+\alpha}\left(N\left[\sum_{i=0}^{n} u_i\right] - N\left[\sum_{i=0}^{n-1} u_i\right]\right), \\ & \quad n = 1, 2, \dots \end{aligned} \tag{3.8}$$

Then approximate solution of Eq. (3.1) is:

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \cdots \tag{3.9}$$

4. Illustrative examples

Example 4.1. Let us consider the NBE with CFFO:

$$^{CF}D_t^{\alpha} u(x,t) + u\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2}, 0 < \alpha \leq 1, \tag{4.1}$$

with

$$u(x,0) = x. \tag{4.2}$$

Taking CFFIO of (4.1), we obtain

$$u(x,t) = u(,0)x +^{CF} I_t^{\alpha}\left[\frac{\partial^2 u}{\partial x^2}\right] -^{CF} I_t^{\alpha}\left[u\frac{\partial u}{\partial x}\right]. \tag{4.3}$$

Thus according to Eq. (3.8):

$$\begin{aligned}
u_0(x,t) &= u(x,0),\\
u_1(x,t) &= {}^{CF}I_t^{\alpha}\left[\frac{\partial^2 u_0(x,t)}{\partial x^2}\right] -^{CF} I_t^{\alpha}\left[u_0(x,t)\frac{\partial u_0(x,t)}{\partial x}\right],\\
u_2(x,t) &= {}^{CF}I_t^{\alpha}\left[\frac{\partial^2 u_1(x,t)}{\partial x^2}\right] -^{CF} I_t^{\alpha}\left[(u_0+u_1)\frac{\partial (u_0+u_1)}{\partial x} - u_0\frac{\partial u_0}{\partial x}\right].\\
&\vdots
\end{aligned} \tag{4.4}$$

Then, we obtain:

$$\begin{aligned}
u_0(x,t) &= x,\\
u_1(x,t) &= -x(1-\alpha+\alpha t),\\
u_2(x,t) &= x\left(2\alpha^2 - 4\alpha + \alpha^2 t^2 - 4\alpha^2 t + 4\alpha t + 2\right),\\
&\vdots
\end{aligned} \tag{4.5}$$

and so on.

Therefore, the $u(x,t)$ of (4.1) is

$$u(x,t) = x - x(1-\alpha+\alpha t) + x\left(2\alpha^2 - 4\alpha + \alpha^2 t^2 - 4\alpha^2 t + 4\alpha t + 2\right) - \cdots \tag{4.6}$$

If we put $\alpha \longrightarrow 1$ in (4.6) , we get

$$\begin{aligned}
u(x,t) &= x - xt + xt^2 - \cdots\\
&= x\sum_{k=0}^{\infty}(-t)^k,\\
&= \frac{x}{1+t}.
\end{aligned} \tag{4.7}$$

Remark 4.1. Fig. 5.1 show the graphs of the approximate and exact solutions among different values of x and t when $\alpha = 0.8, 0.9, 1$ for NBE in the CFFO. In Figure 5.2, we plotted the graphs of the approximate and exact solutions among different values of t and α when x is fixed for the NBE (4.1).

Example 4.2. Consider the CNBEs with CFFO:

$$\begin{aligned} {}^{CF}D_t^{\alpha}u(x,t) - u_{xx} - 2uu_x + (uv)_x = 0, \\ {}^{CF}D_t^{\beta}v(x,t) - v_{xx} - 2vv_x + (uv)_x = 0, \end{aligned} \tag{4.8}$$

where $0 < \alpha, \beta \leq 1$ and

$$\begin{aligned} u(x,0) = sin(x), \\ v(x,0) = sin(x). \end{aligned} \tag{4.9}$$

Taking ${}^{CF}I_t^{\alpha}$ and ${}^{CF}I_t^{\beta}$ of (4.8) respectively, we get

$$\begin{aligned} u(x,t) &= u(x,0) + {}^{CF}I_t^{\alpha}\left[\frac{\partial^2 u}{\partial x^2} + 2u\frac{\partial u}{\partial x} - \frac{\partial}{\partial x}(uv)\right], \\ v(x,t) &= v(x,0) + {}^{CF}I_t^{\beta}\left[\frac{\partial^2 v}{\partial x^2} + 2v\frac{\partial v}{\partial x} - \frac{\partial}{\partial x}(uv)\right]. \end{aligned} \tag{4.10}$$

From (3.8), we obtain:

$$\begin{aligned} u_0(x,t) &= u(x,0), \\ v_0(x,t) &= v(x,0), \end{aligned} \tag{4.11}$$

$$\begin{aligned} u_1(x,t) &= {}^{CF}I_t^{\alpha}\left[\frac{\partial^2 u_0}{\partial x^2} + 2u_0\frac{\partial u_0}{\partial x} - \frac{\partial}{\partial x}(u_0v_0)\right], \\ v_1(x,t) &= {}^{CF}I_t^{\beta}\left[\frac{\partial^2 v_0}{\partial x^2} + 2v_0\frac{\partial v_0}{\partial x} - \frac{\partial}{\partial x}(u_0v_0)\right], \end{aligned} \tag{4.12}$$

$$\begin{aligned} u_2(x,t) &= {}^{CF}I_t^{\alpha}\left[\frac{\partial^2 u_1}{\partial x^2} + 2(u_0+u_1)\frac{\partial(u_0+u_1)}{\partial x} - \frac{\partial}{\partial x}((u_0+u_1)(v_0+v_1))\right] \\ &\quad - {}^{CF}I_t^{\alpha}\left[2u_0\frac{\partial u_0}{\partial x} - \frac{\partial}{\partial x}(u_0v_0)\right], \\ v_2(x,t) &= {}^{CF}I_t^{\beta}\left[\frac{\partial^2 v_1}{\partial x^2} + 2(v_0+v_1)\frac{\partial(v_0+v_1)}{\partial x} - \frac{\partial}{\partial x}((u_0+u_1)(v_0+v_1))\right], \\ &\quad - {}^{CF}I_t^{\alpha}\left[2v_0\frac{\partial v_0}{\partial x} - \frac{\partial}{\partial x}(u_0v_0)\right]. \\ &\vdots \end{aligned} \tag{4.13}$$

By the above algorithms, we have:

$$\begin{aligned} u_0(x,t) &= sin(x), \\ v_0(x,t) &= sin(x). \end{aligned} \tag{4.14}$$

$$\begin{aligned} u_1(x,t) &= {}^{CF}I_t^{\alpha}\left[-sin(x) + 2\ sinx\ cosx - 2\ sinx\ cosx\right], \\ &= -sinx(1-\alpha+\alpha t), \\ v_1(x,t) &= {}^{CF}I_t^{\beta}\left[-sin(x) + 2\ sinx\ cosx - 2\ sinx\ cosx\right], \\ &= -sinx(1-\beta+\beta t), \end{aligned} \tag{4.15}$$

$$\begin{aligned} u_2(x,t) &= {}^{CF}I_t^{\alpha}\left[sin(x)(1-\alpha+\alpha t)\right] \\ &\quad + {}^{CF}I_t^{\alpha}\left[2\alpha^2 sin(x)\ cos(x)\ (1-t)^2 - 2\alpha^2 sin(x)\ cos(x)\ (1-t)^2\right] \\ &\quad - {}^{CF}I_t^{\alpha}\left[2\ sin(x)\ cos(x) - 2\ sin(x)\ cos(x)\right], \\ &= sin(x)\left[(1-\alpha)(1-\alpha+\alpha t) + \alpha(t-\alpha t + \frac{1}{2}\alpha^2 t^2)\right], \\ &= sin(x)\left[(1-\alpha)^2 + (2\alpha - 2\alpha^2)t + \frac{1}{2}\alpha^2 t^2\right], \\ v_2(x,t) &= {}^{CF}I_t^{\beta}\left[sin(x)(1-\beta+\beta t)\right] \\ &\quad + {}^{CF}I_t^{\beta}\left[2\beta^2 sin(x)\ cos(x)\ (1-t)^2 - 2\beta^2 sin(x)\ cos(x)\ (1-t)^2\right] \\ &\quad - {}^{CF}I_t^{\beta}\left[2\ sin(x)\ cos(x) - 2\ sin(x)\ cos(x)\right], \\ &= sin(x)\left[(1-\beta)(1-\beta+\beta t) + \beta(t-\beta t + \frac{1}{2}\beta^2 t^2)\right], \\ &= sin(x)\left[(1-\beta)^2 + (2\beta - 2\beta^2)t + \frac{1}{2}\beta^2 t^2\right]. \\ &\ \vdots \end{aligned} \tag{4.16}$$

Then the approximate solution of (4.8) is

$$\begin{aligned} u(x,t) = sin(x)\left[(1-\alpha+\alpha^2) + (\alpha - 2\alpha^2)t + \frac{1}{2}\alpha^2 t^2 + \cdots\right], \\ v(x,t) = sin(x)\left[(1-\beta+\beta^2) + (\beta - 2\beta^2)t + \frac{1}{2}\beta^2 t^2 + \cdots\right]. \end{aligned} \tag{4.17}$$

If we put $\alpha \longrightarrow 1$ and $\beta \longrightarrow 1$ in (4.17), we have

$$\begin{aligned} u(x,y,t) &= sin(x)\left(1 - t + \frac{t^2}{2!} - \cdots\right), \\ v(x,y,t) &= sin(x)\left(1 + t + \frac{t^2}{2!} + \cdots\right), \\ &= sin(x)e^{t}, \\ &= sin(x)e^{-t}. \end{aligned} \tag{4.18}$$

Remark 4.2. Figure 5.3 and Figure 5.4 show the graphs of the approximate and the exact solutions $u(x,t)$ and $v(x,t)$ among different values of x and t when $\alpha = 0.8, 0.9, 1$ respectively for CNBEs. In Figure 5.5 and Figure 5.6, we plotted the graphs of the approximate and exact solutions $u(x,t)$ and $v(x,t)$ among different values of t and α when x is fixed for the same problem (4.8).

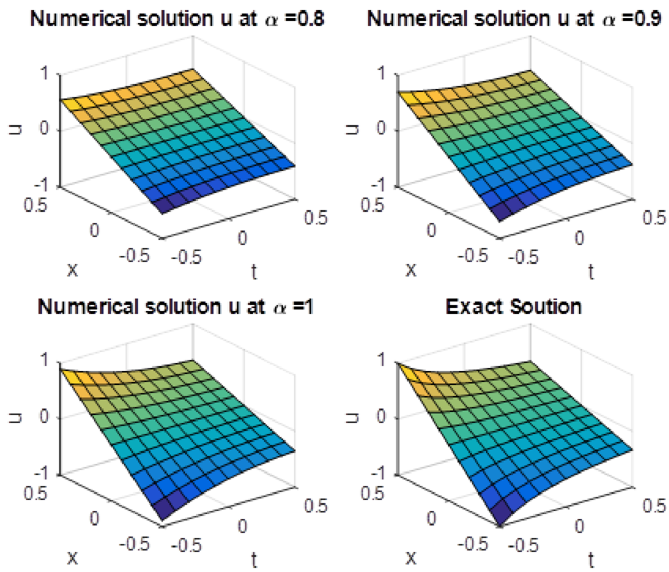

FIG. 5.1: The graph of the approximate and exact solutions among different values of x and t for NBE.

5. Conclusions

In this article, we presented two applications with their graphs of the fractional DJM and we discussed solutions at multiple values for α and β. The results showed that the solutions approach the exact solution whenever α and β approach the correct one.

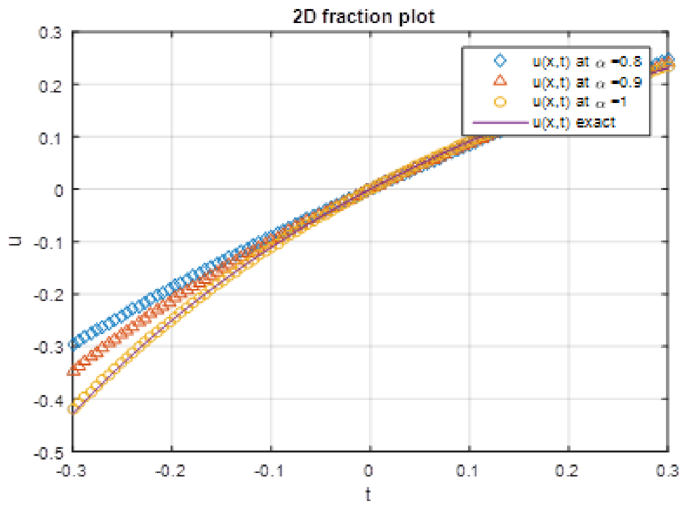

FIG. 5.2: The graph of the approximate and exact solutions among different values of t and α when x is fixed for nonlinear Burger equation.

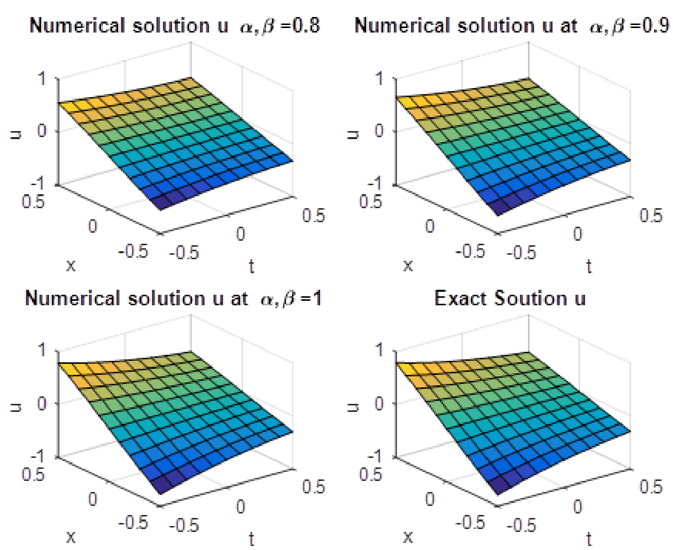

FIG. 5.3: The graph of the approximate and exact solutions $u(x,t)$ among different values of x and t for the system (42).

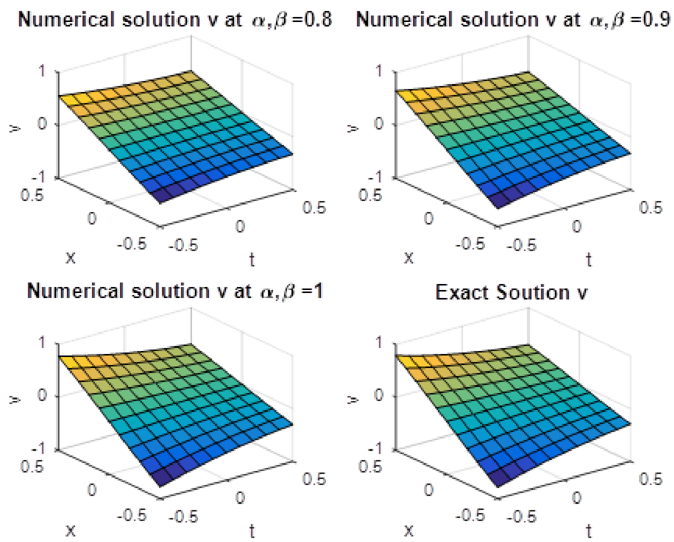

FIG. 5.4: The graph of the approximate and exact solutions $v(x,t)$ among different values of x and t for the system (42).

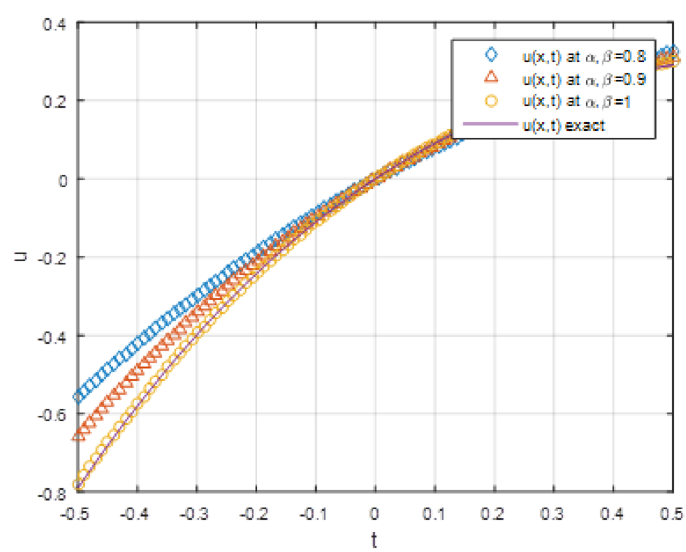

FIG. 5.5: The graph of the approximate and exact solutions $u(x,t)$ among different values of t and α when x is fixed for the system (42).

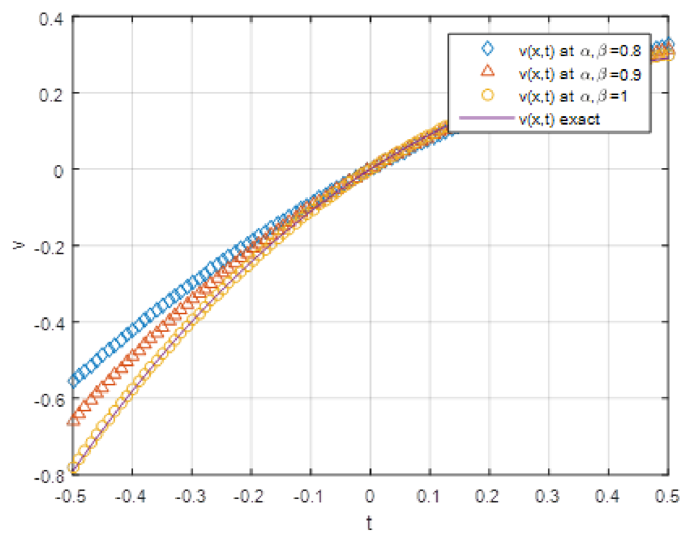

Fig. 5.6: The graph of the approximate and exact solutions $v(x,t)$ among different values of t and α when x is fixed for the system (42).

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