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GENERALIZED WEAK CONTRACTION FOR HYBRID PAIR OF MAPPINGS WITH APPLICATION

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Abstract. We establish some common coupled fixed point theorems for hybrid pair of mappings under generalized weak contraction on a non complete metric space, which is not partially ordered. As an application, we study the existence and uniqueness of the solution to an integral equation and also give an example to show the fruitfulness of our results. The results we obtain generalize, extend and improve several classical results in the literature in metric spaces.
Keywords: Coupled fixed point, generalized weak contraction, w-compatibility, T-weakly commuting, (CLRg) property, integral equation.

1. Introduction and Preliminaries

Let $(X,\ d)$ be a metric space. Throughout the paper, we denote by 2^X the class of all non empty subsets of X, by $CL(X)$ the class of all non empty closed subsets of X, by $CB(X)$ the class of all non empty closed bounded subsets of X and by $K(X)$ the class of all non empty compact subsets of X. A functional $H : CL(X)\times CL(X) \to \mathbb{R}_+ \cup \{+\infty\}$ is said to be the Pompeiu-Hausdorff generalized metric induced by d and is given by

$$H(U,\ V) = \begin{cases} \max\{\sup_{u\in U} D(u,\ V),\ \sup_{v\in V} D(v,\ U)\}, & \text{if maximum exists,} \\ +\infty, & \text{otherwise,} \end{cases}$$

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for all $U, V \in CL(X)$, where $D(u, V) = \inf_{v \in V} d(u, v)$ denotes the distance from u to $V \subset X$. For simplicity, if $x \in X$, we denote $B(x)$ by Bx.

The theory of multivalued mappings has applications in control theory, convex optimization, differential inclusions and economics.

The concepts related to coupled fixed point theory for multi valued mappings were introduced by Abbas et al. [2] and proved some common coupled fixed point theorems involving hybrid pair of mappings satisfying generalized contractive conditions in complete metric spaces. Very few researcher gave attention to coupled fixed point problems for hybrid pair of mappings including [1, 7 − 18, 21, 25].

Definition 1.1. [2] Let X be a non empty set, $T : X \times X \to 2^X$ and B be a self-mapping on X. An element $(x, y) \in X \times X$ is called

(1) a coupled fixed point of T if $x \in T(x, y)$ and $y \in T(y, x)$.

(2) a coupled coincidence point of hybrid pair $\{T, B\}$ if $Bx \in T(x, y)$ and $By \in T(y, x)$.

(3) a common coupled fixed point of hybrid pair $\{T, B\}$ if $x = Bx \in T(x, y)$ and $y = By \in T(y, x)$.

We denote the set of coupled coincidence points of mappings T and B by $C(T, B)$. Note that if $(x, y) \in C(T, B)$, then (y, x) is also in $C(T, B)$.

Definition 1.2. [2] Let $T : X \times X \to 2^X$ be a multivalued mapping and B be a self-mapping on X. The hybrid pair $\{T, B\}$ is called $w-$compatible if $BT(x, y) \subseteq T(Bx, By)$ whenever $(x, y) \in C(T, B)$.

Definition 1.3. [2] Let $T : X \times X \to 2^X$ be a multivalued mapping and B be a self-mapping on X. The mapping B is called $T-$weakly commuting at some point $(x, y) \in X \times X$ if $B^2x \in T(Bx, By)$ and $B^2y \in T(By, Bx)$.

Lemma 1.1. [22] *Let (X, d) be a metric space. Then, for each $u \in X$ and $V \in K(X)$, there is $v_0 \in V$ such that $D(u, V) = d(u, v_0)$, where $D(u, V) = \inf_{v \in V} d(u, v)$.*

Sintunavarat and Kumam [24] defined the notion of common limit in the range property in fuzzy metric space. Chauhan et al. [6] introduced the notion of the joint common limit in the range of mappings property called (JCLR) property and proved a common fixed point theorem for a pair of weakly compatible mappings using (JCLR) property in fuzzy metric space.

Definition 1.4. [24] Let (X, d) be a metric space and $T, B : X \to X$ be two mappings. Then T and B are said to satisfy the common limit in the range of B property (CLRB-property) if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Bx_n = Bx \text{ for some } x \in X.$$

Definition 1.5. [6] Let (X, d) be a metric space and $S, T, A, B : X \to X$ be four mappings. The pairs (S, A) and (T, B) are said to have (JCLR) property if there exist sequences $\{x_n\}$ and $\{y_n\}$ in X and $x, y \in X$ such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} Ax_n = \lim_{n\to\infty} By_n = Ax = By.$$

Khan and Sumitra [20] established the concept of (CLRB) property for mappings $T : X \times X \to X$ and $B : X \to X$.

Definition 1.6. [20] Let (X, d) be a metric space, $T : X\times X \to X$ and $B : X \to X$ be two mappings. Then T and B are said to satisfy the common limit in the range of B property (CLRB-property) if there exist sequences $\{x_n\}$ and $\{y_n\}$ in X, some x, y in X such that

$$\lim_{n\to\infty} T(x_n,\ y_n) = \lim_{n\to\infty} Bx_n = Bx \text{ and } \lim_{n\to\infty} T(y_n,\ x_n) = \lim_{n\to\infty} By_n = By.$$

In [3], Ahmed and Nafadi introduced the notion of common limit range property (CLR property) for two hybrid pairs of mappings in fuzzy metric spaces and proved common fixed point theorems using (CLR) property for these mappings with implicit relation.

Definition 1.7. [3] Mappings $T : X \to CB(X)$ and $B : X \to X$ are said to satisfy the common limit in the range of B property (CLRB-property) if there exist sequences $\{x_n\}$ in X, some x in X and G in $CB(X)$ such that

$$\lim_{n\to\infty} Bx_n = Bx \in G = \lim_{n\to\infty} Tx_n.$$

Definition 1.8. [3] Mappings $S, T : X \to CB(X)$ and $A, B : X \to X$ are said to satisfy the joint common limit in the range (JCLR) property if there exist sequences $\{x_n\}$ and $\{y_n\}$ in X, some x, y in X and G, H in $CB(X)$ such that

$$\begin{aligned} \lim_{n\to\infty} Ax_n &= Ax \in G = \lim_{n\to\infty} Sx_n, \\ \lim_{n\to\infty} By_n &= By \in H = \lim_{n\to\infty} Ty_n. \end{aligned}$$

Handa [18] introduced the notion of (CLRg) property for hybrid pair $F : X \times X \to 2^X$ and $g : X \to X$ and also defined the notion of joint common limit range (JCLR) property for two hybrid pairs $F, G : X \times X \to 2^X$ and $f, g : X \to X$. In [18], Handa proved some common coupled fixed point theorems for hybrid pair of mappings under generalized $(\psi, \theta, \varphi)-$contraction on a non complete metric space.

Definition 1.9. [18] Let (X, d) be a metric space. Mappings $F : X \times X \to 2^X$ and $g : X \to X$ are said to satisfy the common limit in the range of g property (CLRg-property) if there exist sequences $\{x_n\}$ and $\{y_n\}$ in X, some x, y in X and A, B in $CB(X)$ such that

$$\begin{aligned} \lim_{n\to\infty} gx_n &= gx \in A = \lim_{n\to\infty} F(x_n,\ y_n), \\ \lim_{n\to\infty} gy_n &= gy \in B = \lim_{n\to\infty} F(y_n,\ x_n). \end{aligned}$$

Definition 1.10. [18] Let (X, d) be a metric space and the mappings $f, g : X \to X$ and $F, G : X \times X \to 2^X$. The pairs (F, f) and (G, g) are said to have joint common limit range (JCLR) property if there exist sequences $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{v_n\}$ in X, some x, y, u, v in X and A, B, C, D in $CB(X)$ such that

$$\begin{aligned} \lim_{n\to\infty} F(x_n,\ y_n) &= A,\ \lim_{n\to\infty} G(u_n,\ v_n) = B, \\ \text{then } \lim_{n\to\infty} fx_n &= \lim_{n\to\infty} gu_n = fx = gu \in A \cap B, \\ \text{and } \lim_{n\to\infty} F(y_n,\ x_n) &= C,\ \lim_{n\to\infty} G(v_n,\ u_n) = D, \\ \text{then } \lim_{n\to\infty} fy_n &= \lim_{n\to\infty} gv_n = fy = gv \in C \cap D. \end{aligned}$$

Weak contraction was first studied in partially ordered metric spaces by Harjani and Sadarangani [19]. In [4], Choudhury and Kundu established some coincidence point results for generalized weak contractions with discontinuous control functions on a partially ordered metric spaces. Choudhury et al. [5] proved coincidence point results by assuming a weak contraction inequality with three control functions, two of which are not continuous. The results are obtained under two sets of additional conditions.

Definition 1.11. [23] An altering distance function is a function $\psi : [0, +\infty) \to [0, +\infty)$ which satisfy the following conditions:

(i_ψ) ψ is continuous and monotone-increasing,

(ii_ψ) $\psi(t) = 0$ if and only if $t = 0$.

Choudhury et al. [5] use the following classes of functions.

Let Ψ denote the set of all functions $\psi : [0, +\infty) \to [0, +\infty)$ satisfying

(i_ψ) ψ is continuous and monotone non-decreasing,
(ii_ψ) $\psi(t) = 0 \Leftrightarrow t = 0$.

Let Θ denote the set of all functions $\theta : [0, +\infty) \to [0, +\infty)$ satisfying

(i_θ) θ is bounded on any bounded interval in $[0, +\infty)$,
(ii_θ) θ is continuous at 0 and $\theta(0) = 0$.

In this paper, we prove some common coupled fixed point theorems for hybrid pair of mappings satisfying generalized weak contraction on a non complete metric space, which is not partially ordered. It is to be noted that to find coupled coincidence point, we do not employ the condition of continuity of any mapping involved therein. As an application, we study the existence and uniqueness of the solution to an integral equation. The effectiveness of our generalization is demonstrated with the help of an example. We modify, improve, sharpen, enrich and generalize the results of Choudhury et al. [5], Harjani and Sadarangani [19] and many other famous results in the literature.

2. Main results

Theorem 2.1. *Let $(X,\, d)$ be a metric space. Suppose $T : X \times X \to K(X)$ and $B : X \to X$ be two mappings for which there exist $\psi \in \Psi$, φ, $\theta \in \Theta$ such that*

$$\psi(x) \leq \varphi(y) \Rightarrow x \leq y, \tag{2.1}$$

for any sequence $\{x_n\}$ in $[0,\, +\infty)$ with $x_n \to t > 0$,

$$\psi(t) - \overline{\lim}\varphi(x_n) + \underline{\lim}\theta(x_n) > 0, \tag{2.2}$$

and

$$\begin{aligned} \psi(H(T(x,\ y),\ T(u,\ v))) \ \leq\ & \varphi(\max\{d(Bx,\ Bu),\ d(By,\ Bv)\}) \\ & -\theta(\max\{d(Bx,\ Bu),\ d(By,\ Bv)\}), \end{aligned} \tag{2.3}$$

for all x, y, u, $v \in X$. Furthermore assume that $T(X \times X) \subseteq B(X)$ and $B(X)$ is a complete subset of X. Then T and B have a coupled coincidence point. Moreover, T and B have a common coupled fixed point, if one of the following conditions holds:

(*a*) *T and B are $w-$compatible. $\lim_{n\to\infty} B^n x = u$ and $\lim_{n\to\infty} B^n y = v$ for some $(x,\, y) \in C(T,\, B)$ and for some u, $v \in X$ and B is continuous at u and v.*

(*b*) *B is $T-$weakly commuting for some $(x,\, y) \in C(T,\, B)$ and Bx and By are fixed points of B, that is, $B^2x = Bx$ and $B^2y = By$.*

(*c*) *B is continuous at x and y. $\lim_{n\to\infty} B^n u = x$ and $\lim_{n\to\infty} B^n v = y$ for some $(x,\, y) \in C(T,\, B)$ and for some u, $v \in X$.*

(*d*) *$B(C(T,\, B))$ is a singleton subset of $C(T,\, B)$.*

Proof. Let x_0, $y_0 \in X$ be arbitrary. Then $T(x_0,\, y_0)$ and $T(y_0,\, x_0)$ are well defined. Choose $Bx_1 \in T(x_0,\, y_0)$ and $By_1 \in T(y_0,\, x_0)$, because $T(X \times X) \subseteq B(X)$. Since $T : X \times X \to K(X)$, therefore by Lemma 1.1, there exist $z_1 \in T(x_1,\, y_1)$ and $z_2 \in T(y_1,\, x_1)$ such that

$$\begin{aligned} d(Bx_1,\ z_1) &\leq H(T(x_0,\ y_0),\ T(x_1,\ y_1)), \\ d(By_1,\ z_2) &\leq H(T(y_0,\ x_0),\ T(y_1,\ x_1)). \end{aligned}$$

Since $T(X \times X) \subseteq B(X)$, therefore $z_1 = Bx_2$ and $z_2 = By_2$ for some x_2, $y_2 \in X$. Thus

$$\begin{aligned} d(Bx_1,\ Bx_2) &\leq H(T(x_0,\ y_0),\ T(x_1,\ y_1)), \\ d(By_1,\ By_2) &\leq H(T(y_0,\ x_0),\ T(y_1,\ x_1)). \end{aligned}$$

Continuing this process, we obtain sequences $\{x_n\}$ and $\{y_n\}$ in X such that for all $n \geq 0$, we have $Bx_{n+1} \in T(x_n,\, y_n)$ and $By_{n+1} \in T(y_n,\, x_n)$ such that

$$\begin{aligned} d(Bx_{n+1},\ Bx_{n+2}) &\leq H(T(x_n,\ y_n),\ T(x_{n+1},\ y_{n+1})), \\ d(By_{n+1},\ By_{n+2}) &\leq H(T(y_n,\ x_n),\ T(y_{n+1},\ x_{n+1})). \end{aligned}$$

Let $\zeta_n = \max\{d(Bx_n,\ Bx_{n+1}),\ d(By_n,\ By_{n+1})\}$ for all $n \geq 0$. By (2.2), (2.3) and by the monotonicity of ψ, we have

$$\begin{aligned} \psi(d(Bx_{n+1},\ Bx_{n+2})) &\leq \psi(H(T(x_n,\ y_n),\ T(x_{n+1},\ y_{n+1}))) \\ &\leq \varphi(\max\{d(Bx_n,\ Bx_{n+1}),\ d(By_n,\ By_{n+1})\}) \\ &\quad -\theta(\max\{d(Bx_n,\ Bx_{n+1}),\ d(By_n,\ By_{n+1})\}). \end{aligned}$$

Similarly

$$\begin{aligned} \psi(d(By_{n+1},\ By_{n+2})) &\leq \varphi(\max\{d(Bx_n,\ Bx_{n+1}),\ d(By_n,\ By_{n+1})\}) \\ &\quad -\theta(\max\{d(Bx_n,\ Bx_{n+1}),\ d(By_n,\ By_{n+1}). \end{aligned}$$

Combining them, we get

$$\begin{aligned} &\max\{\psi(d(Bx_{n+1},\ Bx_{n+2})),\ \psi(d(By_{n+1},\ By_{n+2}))\} \\ \leq\ &\varphi(\max\{d(Bx_n,\ Bx_{n+1}),\ d(By_n,\ By_{n+1})\}) \\ &-\theta(\max\{d(Bx_n,\ Bx_{n+1}),\ d(By_n,\ By_{n+1}). \end{aligned}$$

It follows, from the monotonicity of ψ, that

$$\begin{aligned} &\psi(\max\{d(Bx_{n+1},\ Bx_{n+2}),\ d(By_{n+1},\ By_{n+2})\}) \\ \leq\ &\varphi(\max\{d(Bx_n,\ Bx_{n+1}),\ d(By_n,\ By_{n+1})\}) \\ &-\theta(\max\{d(Bx_n,\ Bx_{n+1}),\ d(By_n,\ By_{n+1}), \end{aligned}$$

which implies that

$$\psi(\zeta_{n+1}) \leq \varphi(\zeta_n) - \theta(\zeta_n), \tag{2.4}$$

it follows from $\theta \geq 0$ that $\psi(\zeta_{n+1}) \leq \varphi(\zeta_n)$, which, by (2.1), implies $\zeta_{n+1} \leq \zeta_n$ for all $n \geq 0$, that is, $\{\zeta_n\}$ is a monotone non-increasing sequence. Hence there exists an $\zeta \geq 0$ such that

$$\lim_{n\to\infty} \zeta_n = \lim_{n\to\infty} \max\{d(Bx_n,\ Bx_{n+1}),\ d(By_n,\ By_{n+1})\} = \zeta. \tag{2.5}$$

Taking limit supremum on both sides of (2.4), using (2.5) and the continuity of ψ, we get

$$\psi(\zeta) \leq \overline{\lim}\varphi(\zeta_n) - \underline{\lim}\theta(\zeta_n) \Rightarrow \psi(\zeta) - \overline{\lim}\varphi(\zeta_n) + \underline{\lim}\theta(\zeta_n) \leq 0.$$

It is a contradiction unless $\zeta = 0$. Therefore,

$$\lim_{n\to\infty} \zeta_n = \lim_{n\to\infty} \max\{d(Bx_n,\ Bx_{n+1}),\ d(By_n,\ By_{n+1})\} = 0. \tag{2.6}$$

We now claim that $\{Bx_n\}_{n\geq 0}$ and $\{By_n\}_{n\geq 0}$ are Cauchy sequences in X. Suppose, to the contrary, that at least one of the sequences $\{Bx_n\}_{n\geq 0}$ and $\{By_n\}_{n\geq 0}$ is not a Cauchy sequence. Then there exists an $\varepsilon > 0$ for which we can find subsequences

$\{Bx_{n(k)}\}$, $\{Bx_{m(k)}\}$ of $\{Bx_n\}_{n\geq 0}$ and $\{By_{n(k)}\}$, $\{By_{m(k)}\}$ of $\{By_n\}_{n\geq 0}$ such that, for $n(k) > m(k) > k$,

$$\max\{d(Bx_{n(k)},\ Bx_{m(k)}),\ d(By_{n(k)},\ By_{m(k)})\} \geq \varepsilon. \tag{2.7}$$

Let $n(k)$ be the smallest positive integer satisfying (2.7), then

$$\max\{d(Bx_{n(k)-1},\ Bx_{m(k)}),\ d(By_{n(k)-1},\ By_{m(k)})\}) < \varepsilon. \tag{2.8}$$

Now, by triangle inequality, we have

$$\begin{aligned} \varepsilon &\leq \omega_k = \max\{d(Bx_{n(k)},\ Bx_{m(k)}),\ d(By_{n(k)},\ By_{m(k)})\} \\ &\leq \max\{d(Bx_{n(k)},\ Bx_{n(k)-1}),\ d(By_{n(k)},\ By_{n(k)-1})\} \\ &\quad + \max\{d(Bx_{n(k)-1},\ Bx_{m(k)}),\ d(By_{n(k)-1},\ By_{m(k)})\} \\ &< \max\{d(Bx_{n(k)},\ Bx_{n(k)-1}),\ d(By_{n(k)},\ By_{n(k)-1})\} + \varepsilon. \end{aligned}$$

Letting $k \to \infty$ in the above inequality and by using (2.6), we get

$$\lim_{k\to\infty} \omega_k = \lim_{k\to\infty} \max\{d(Bx_{n(k)},\ Bx_{m(k)}),\ d(By_{n(k)},\ By_{m(k)})\} = \varepsilon. \tag{2.9}$$

By the triangle inequality, we have

$$\begin{aligned} &\max\{d(Bx_{n(k)+1},\ Bx_{m(k)+1}),\ d(By_{n(k)+1},\ By_{m(k)+1})\} \\ \leq\ &\max\{d(Bx_{n(k)+1},\ Bx_{n(k)}),\ d(By_{n(k)+1},\ By_{n(k)})\} \\ &+ \max\{d(Bx_{n(k)},\ Bx_{m(k)}),\ d(By_{n(k)},\ By_{m(k)})\} \\ &+ \max\{d(Bx_{m(k)},\ Bx_{m(k)+1}),\ d(By_{m(k)},\ By_{m(k)+1})\}. \end{aligned}$$

Letting $k \to \infty$ in the above inequalities, using (2.6) and (2.9), we have

$$\lim_{k\to\infty} \max\{d(Bx_{n(k)+1},\ Bx_{m(k)+1}),\ d(By_{n(k)+1},\ By_{m(k)+1})\} = \varepsilon. \tag{2.10}$$

Now, by the monotonicity of ψ and (2.3), implies

$$\begin{aligned} &\psi(d(Bx_{n(k)+1},\ Bx_{m(k)+1})) \\ \leq\ &\psi(H(T(x_{n(k)},\ y_{n(k)}),\ T(x_{m(k)},\ y_{m(k)}))) \\ \leq\ &\varphi(\max\{d(Bx_{n(k)},\ Bx_{m(k)}),\ d(By_{n(k)},\ By_{m(k)})\}) \\ &-\theta(\max\{d(Bx_{n(k)},\ Bx_{m(k)}),\ d(By_{n(k)},\ By_{m(k)})\}). \end{aligned}$$

Similarly

$$\begin{aligned} &\psi(d(By_{n(k)+1},\ By_{m(k)+1})) \\ \leq\ &\varphi(\max\{d(Bx_{n(k)},\ Bx_{m(k)}),\ d(By_{n(k)},\ By_{m(k)})\}) \\ &-\theta(\max\{d(Bx_{n(k)},\ Bx_{m(k)}),\ d(By_{n(k)},\ By_{m(k)})\}). \end{aligned}$$

Combining them, we get

$$\begin{aligned} & \max\{\psi(d(Bx_{n(k)+1},\ Bx_{m(k)+1})),\ \psi(d(By_{n(k)+1},\ By_{m(k)+1}))\} \\ \leq\ & \varphi(\max\{d(Bx_{n(k)},\ Bx_{m(k)}),\ d(By_{n(k)},\ By_{m(k)})\}) \\ & -\theta(\max\{d(Bx_{n(k)},\ Bx_{m(k)}),\ d(By_{n(k)},\ By_{m(k)})\}). \end{aligned}$$

It follows, from the monotonicity of ψ, that

$$\begin{aligned} & \psi(\max\{d(Bx_{n(k)+1},\ Bx_{m(k)+1}),\ d(By_{n(k)+1},\ By_{m(k)+1})\}) \\ \leq\ & \varphi(\max\{d(Bx_{n(k)},\ Bx_{m(k)}),\ d(By_{n(k)},\ By_{m(k)})\}) \\ & -\theta(\max\{d(Bx_{n(k)},\ Bx_{m(k)}),\ d(By_{n(k)},\ By_{m(k)})\}). \end{aligned}$$

Taking limit supremum on both sides of the above inequality, using (2.9), (2.10) and the continuity of ψ, we obtain

$$\psi\,(\varepsilon) \leq \overline{\lim}\varphi(\omega_k) - \underline{\lim}\theta(\omega_k) \Rightarrow \psi\,(\varepsilon) - \overline{\lim}\varphi(\omega_k) + \underline{\lim}\theta(\omega_k) \leq 0.$$

It is a contradiction. Therefore, $\{Bx_n\}_{n\geq 0}$ and $\{By_n\}_{n\geq 0}$ are Cauchy sequences in $B(X)$. Since $B(X)$ is complete, therefore there exist $x,\ y \in X$ such that

$$\lim_{n\to\infty} Bx_n = Bx \text{ and } \lim_{n\to\infty} By_n = By. \tag{2.11}$$

Now, since $Bx_{n+1} \in T(x_n, y_n)$ and $By_{n+1} \in T(y_n, x_n)$, therefore by using condition (2.3) and by the monotonicity of ψ, we get

$$\begin{aligned} \psi(D(Bx_{n+1},\ T(x,\ y))) &\leq \psi(H(T(x_n,\ y_n),\ T(x,\ y))) \\ &\leq \varphi(\max\{d(Bx_n,\ Bx),\ d(By_n,\ By)\}) \\ &\quad -\theta(\max\{d(Bx_n,\ Bx),\ d(By_n,\ By)\}). \end{aligned}$$

On taking $n \to \infty$ in the above inequality and by using (ii_θ) of θ, φ, by the continuity of ψ and (2.11), we get

$$\psi(D(Bx,\ T(x,\ y))) = 0,$$

which implies, by (ii_ψ), that

$$D(Bx,\ T(x,\ y)) = 0, \text{ similarly } D(By,\ T(y,\ x)) = 0.$$

It follows that

$$Bx \in T(x,\ y) \text{ and } By \in T(y,\ x),$$

that is, $(x,\ y)$ is a coupled coincidence point of T and B. Hence $C(T,\ B)$ is non empty.

Suppose now that (a) holds. Assume that for some $(x,\ y) \in C(T,\ B)$,

$$\lim_{n\to\infty} B^n x = u \text{ and } \lim_{n\to\infty} B^n y = v, \tag{2.12}$$

where $u,\ v \in X$. Since B is continuous at u and v, we have, by (2.12), that u and v are fixed points of B, that is,

$$Bu = u \text{ and } Bv = v. \tag{2.13}$$

As T and B are $w-$compatible, so

$$(B^n x,\ B^n y) \in C(T,\ B),\ \text{for all } n \geq 1,$$

that is, for all $n \geq 1$,

$$B^n x \in T(B^{n-1}x,\ B^{n-1}y) \text{ and } B^n y \in T(B^{n-1}y,\ B^{n-1}x). \tag{2.14}$$

Now, by using (2.3), (2.14) and by the monotonicity of ψ, we obtain

$$\begin{aligned}\psi(D(B^n x,\ T(u,\ v))) &\leq \psi(H(T(B^{n-1}x,\ B^{n-1}y),\ T(u,\ v))) \\ &\leq \varphi(\max\{d(B^n x,\ Bu),\ d(B^n y,\ Bv)\}) \\ &\quad -\theta(\max\{d(B^n x,\ Bu),\ d(B^n y,\ Bv)\}).\end{aligned}$$

On taking limit as $n \to \infty$ in the above inequality, by using (ii_θ) of θ, φ, by the continuity of ψ, (2.12) and (2.13), we get

$$\psi(D(Bu,\ T(u,\ v))) = 0,$$

which implies, by (ii_ψ), that

$$D(Bu,\ T(u,\ v)) = 0,\ \text{similarly } D(Bv,\ T(v,\ u)) = 0.$$

It follows that

$$Bu \in T(u,\ v) \text{ and } Bv \in T(v,\ u). \tag{2.15}$$

Now, from (2.13) and (2.15), we have

$$u = \ Bu \in T(u,\ v) \text{ and } v = Bv \in T(v,\ u),$$

that is, $(u,\ v)$ is a common coupled fixed point of T and B.

Suppose now that (b) holds. Assume that for some $(x,\ y) \in C(T,\ B)$, B is $T-$weakly commuting, that is $B^2x \in T(Bx,\ By)$ and $B^2y \in T(By,\ Bx)$ and $B^2x = Bx$ and $B^2y = By$. Thus $Bx = B^2x \in T(Bx,\ By)$ and $By = B^2y \in T(By,\ Bx)$, that is, $(Bx,\ By)$ is a common coupled fixed point of T and B.

Suppose now that (c) holds. Assume that for some $(x,\ y) \in C(T,\ B)$ and for some $u,\ v \in X$, $\lim_{n\to\infty} B^n u = x$ and $\lim_{n\to\infty} B^n v = y$. Since B is continuous at x and y, then x and y are fixed points of B, that is, $Bx = x$ and $By = y$. As $(x,\ y) \in C(T,\ B)$ and so we obtain $x = Bx \in T(x,\ y)$ and $y = By \in T(y,\ x)$, that is, $(x,\ y)$ is a common coupled fixed point of T and B.

Finally, suppose that (d) holds. Let $B(C(T,\ B)) = \{(x,\ x)\}$. Then $\{x\} = \{Bx\} = T(x,\ x)$. Hence $(x,\ x)$ is a common coupled fixed point of T and B. $\square$

If we put $B = I$ (the identity mapping) in the Theorem 2.1, we get the following result:

Corollary 2.1. *Let (X, d) be a complete metric space, $T : X \times X \to K(X)$ be a mapping for which there exist $\psi \in \Psi$, φ, $\theta \in \Theta$ satisfying* (2.1), (2.2) *and*

$$\psi(H(T(x,\ y),\ T(u,\ v)) \leq \varphi(\max\{d(x,\ u),\ d(y,\ v)\}) - \theta(\max\{d(x,\ u),\ d(y,\ v)\}),$$

for all x, y, u, $v \in X$. Then T has a coupled fixed point.

If we take $\psi(t) = \varphi(t)$ in Theorem 2.1, we obtain the following corollary.

Corollary 2.2. *Let (X, d) be a metric space. Suppose $T : X \times X \to K(X)$ and $B : X \to X$ are two mappings for which there exist $\psi \in \Psi$, $\theta \in \Theta$ satisfying* (2.1), (2.2) *and*

$$\begin{aligned}(2.16) \qquad & \psi(H(T(x,\ y),\ T(u,\ v))) \\ \leq\ & \psi(\max\{d(Bx,\ Bu),\ d(By,\ Bv)\}) \\ & -\theta(\max\{d(Bx,\ Bu),\ d(By,\ Bv)\}),\end{aligned}$$

for all x, y, u, $v \in X$. Furthermore assume that $T(X \times X) \subseteq B(X)$ and $B(X)$ is a complete subset of X. Then T and B have a coupled coincidence point. Moreover, T and B have a common coupled fixed point, if one of the conditions $(a) - (d)$ of Theorem 2.1 holds.

If we put $B = I$ (the identity mapping) in the Corollary 2.3, we get the following result:

Corollary 2.3. *Let (X, d) be a complete metric space and $T : X \times X \to K(X)$ be a mapping for which there exist $\psi \in \Psi$, $\theta \in \Theta$ satisfying* (2.1), (2.2) *and*

$$\psi(H(T(x,\ y),\ T(u,\ v))) \leq \psi(\max\{d(x,\ u),\ d(y,\ v)\}) - \theta(\max\{d(x,\ u),\ d(y,\ v)\}),$$

for all x, y, u, $v \in X$. Then T has a coupled fixed point.

If we take $\psi(t) = \varphi(t) = t$ and $\theta(t) = (1 - k)t$ with $k < 1$ in Theorem 2.1, we get the following corollary.

Corollary 2.4. *Let (X, d) be a metric space. Suppose $T : X \times X \to K(X)$ and $B : X \to X$ are two mappings satisfying*

$$(2.17) \qquad H(T(x,\ y),\ T(u,\ v)) \leq k \max\{d(Bx,\ Bu),\ d(By,\ Bv)\}),$$

for all x, y, u, $v \in X$, where $k < 1$. Furthermore assume that $T(X \times X) \subseteq B(X)$ and $B(X)$ is a complete subset of X. Then T and B have a coupled coincidence point. Moreover, T and B have a common coupled fixed point, if one of the conditions $(a) - (d)$ of Theorem 2.1 holds.

If we put $B = I$ (the identity mapping) in the Corollary 2.5, we get the following result:

Corollary 2.5. *Let* $(X,\, d)$ *be a complete metric space and* $T : X \times X \to K(X)$ *be a mapping satisfying*

$$\psi(H(T(x,\ y),\ T(u,\ v))) \leq k \max\{d(x,\ u),\ d(y,\ v)\},$$

for all $x,\, y,\, u,\, v \in X$, *where* $k < 1$. *Then* T *has a coupled fixed point.*

If we take T to be a singleton set in Theorem 2.1, then we get the following result:

Corollary 2.6. *Let* $(X,\, d)$ *be a metric space. Suppose* $T : X \times X \to X$ *and* $B : X \to X$ *are two mappings for which there exist* $\psi \in \Psi$, $\varphi,\, \theta \in \Theta$ *satisfying* (2.1), (2.2) *and*

$$\begin{aligned}\psi(d(T(x,\ y),\ T(u,\ v))) \quad \leq \quad & \varphi(\max\{d(Bx,\ Bu),\ d(By,\ Bv)\}) \\ & -\theta(\max\{d(Bx,\ Bu),\ d(By,\ Bv)\}),\end{aligned}$$

for all $x,\, y,\, u,\, v \in X$. *Furthermore* $T(X \times X) \subseteq B(X)$ *and* $B(X)$ *is a complete subset of* X. *Then* T *and* B *have a coupled coincidence point.*

Put $B = I$ (the identity mapping) and $T = F$ in Corollary 2.7, we get the following result:

Corollary 2.7. *Let* $(X,\, d)$ *be a complete metric space. Assume* $F : X \times X \to X$ *is a mapping for which there exist* $\psi \in \Psi$, $\varphi,\, \theta \in \Theta$ *satisfying* (2.1), (2.2) *and*

$$\psi(d(F(x,\ y),\ F(u,\ v))) \leq \varphi(\max\{d(x,\ u),\ d(y,\ v)\}) - \theta(\max\{d(x,\ u),\ d(y,\ v)\}),$$

for all $x,\, y,\, u,\, v \in X$. *Then* F *has a coupled fixed point.*

Theorem 2.2. *Let* $(X,\, d)$ *be a metric space. Suppose* $T : X \times X \to CB(X)$ *and* $B : X \to X$ *are two mappings for which there exist* $\psi \in \Psi$, $\varphi,\, \theta \in \Theta$ *satisfying* (2.1), (2.2), (2.3) *and* $(T,\, B)$ *satisfies* $(CLRB)$ *property. Then* T *and* B *have a coupled coincidence point. Moreover,* T *and* B *have a common coupled fixed point, if one of the conditions* $(a) - (d)$ *of Theorem 2.1 holds.*

Proof. Since $(T,\, B)$ satisfies $(CLRB)$ property, therefore there exist sequences $\{x_n\}$ and $\{y_n\}$ in X, some $x,\, y$ in X and $E,\, F$ in $CB(X)$ such that

$$\begin{aligned}\lim_{n\to\infty} Bx_n &= Bx \in E = \lim_{n\to\infty} T(x_n,\ y_n), \\ \lim_{n\to\infty} By_n &= By \in F = \lim_{n\to\infty} T(y_n,\ x_n).\end{aligned} \tag{2.18}$$

Now, by using contractive condition (2.3), we have

$$\begin{aligned}\psi(H(T(x_n,\ y_n),\ T(x,\ y))) &\leq \varphi(\max\{d(Bx_n,\ Bx),\ d(By_n,\ By)\}) \\ &\quad -\theta(\max\{d(Bx_n,\ Bx),\ d(By_n,\ By)\}).\end{aligned}$$

On taking $n \to \infty$ in the above inequality and by using the property of ψ, θ, φ and (2.18), we get

$$\psi(H(E,\ T(x,\ y))) \leq \varphi(0) - \theta(0) = 0 - 0 = 0,$$

which, by (ii_ψ), implies

$$H(E,\ T(x,\ y)) = 0, \text{ similarly } H(F,\ T(y,\ x)) = 0.$$

Since $Bx \in E$ and $By \in F$, therefore

$$Bx \in T(x,\ y) \text{ and } By \in T(y,\ x),$$

that is, $(x,\ y)$ is a coupled coincidence point of T and B. Hence $C(T,\ B)$ is non empty.

Suppose now that (a) holds. Assume that for some $(x,\ y) \in C(T,\ B)$,

$$\lim_{n\to\infty} B^n x = u \text{ and } \lim_{n\to\infty} B^n y = v, \tag{2.19}$$

where $u,\ v \in X$. Since B is continuous at u and v, we have, by (2.19), that u and v are fixed points of B, that is,

$$Bu = u \text{ and } Bv = v. \tag{2.20}$$

As T and B are $w-$compatible and so

$$(B^n x,\ B^n y) \in C(T,\ B), \text{ for all } n \geq 1,$$

that is, for all $n \geq 1$,

$$B^n x \in T(B^{n-1}x,\ B^{n-1}y) \text{ and } B^n y \in T(B^{n-1}y,\ B^{n-1}x). \tag{2.21}$$

Now, by using contractive condition (2.3), (2.21) and by the monotonicity of ψ, we obtain

$$\begin{aligned}\psi(D(B^n x,\ T(u,\ v))) &\leq \psi(H(T(B^{n-1}x,\ B^{n-1}y),\ T(u,\ v))) \\ &\leq \varphi(\max\{d(B^n x,\ Bu),\ d(B^n y,\ Bv)\}) \\ &\quad -\theta(\max\{d(B^n x,\ Bu),\ d(B^n y,\ Bv)\}).\end{aligned}$$

On taking limit as $n \to \infty$ in the above inequality, by using the property of ψ, θ, φ and (2.19), (2.20), we get

$$\psi(D(Bu,\ T(u,\ v))) \leq \varphi(0) - \theta(0) = 0 - 0 = 0,$$

which, by (ii_ψ), implies

$$D(Bu,\ T(u,\ v)) = 0,\ \text{similarly } D(Bv,\ T(v,\ u)) = 0,$$

which implies that

$$Bu \in T(u,\ v) \text{ and } Bv \in T(v,\ u), \tag{2.22}$$

Now, from (2.20) and (2.22), we have

$$u = \ Bu \in T(u,\ v) \text{ and } v = Bv \in T(v,\ u),$$

that is, $(u,\, v)$ is a common coupled fixed point of T and B.

Suppose now that (b) holds. Assume that for some $(x,\, y) \in C(T,\, B)$, B is $T-$weakly commuting, that is $B^2x \in T(Bx,\, By)$ and $B^2y \in T(By,\, Bx)$ and $B^2x = Bx$ and $B^2y = By$. Thus $Bx = B^2x \in T(Bx,\, By)$ and $By = B^2y \in T(By,\, Bx)$, that is, $(Bx,\, By)$ is a common coupled fixed point of T and B.

Suppose now that (c) holds. Assume that for some $(x,\, y) \in C(T,\, B)$ and for some $u,\, v \in X$, $\lim_{n\to\infty} B^n u = x$ and $\lim_{n\to\infty} B^n v = y$. Since B is continuous at x and y, then x and y are fixed points of B, that is, $Bx = x$ and $By = y$. As $(x,\, y) \in C(T,\, B)$ and so we obtain $x = Bx \in T(x,\, y)$ and $y = By \in T(y,\, x)$, that is, $(x,\, y)$ is a common coupled fixed point of T and B.

Finally, suppose that (d) holds. Let $B(C(T,\, B)) = \{(x,\, x)\}$. Then $\{x\} = \{Bx\} = T(x,\, x)$. Hence $(x,\, x)$ is a common coupled fixed point of T and B. □

If we take $\psi(t) = \varphi(t)$ in Theorem 2.2, we obtain the following corollary.

Corollary 2.8. *Let $(X,\, d)$ be a metric space. Suppose $T : X \times X \to CB(X)$ and $B : X \to X$ are two mappings for which there exist $\psi \in \Psi$, $\theta \in \Theta$ satisfying* (2.1), (2.2), (2.16) *and $(T,\, B)$ satisfies $(CLRB)$ property. Then T and B have a coupled coincidence point. Moreover, T and B have a common coupled fixed point, if one of the conditions $(a) - (d)$ of Theorem* 2.1 *holds.*

If we take $\psi(t) = \theta(t) = t$ and $\varphi(t) = (1-k)t$ with $k < 1$ in Theorem 2.2, we get the following corollary.

Corollary 2.9. *Let $(X,\, d)$ be a metric space. Suppose $T : X \times X \to CB(X)$ and $B : X \to X$ are two mappings satisfying* (2.17) *and $(T,\, B)$ satisfies $(CLRB)$ property. Then T and B have a coupled coincidence point. Moreover, T and B have a common coupled fixed point, if one of the conditions $(a) - (d)$ of Theorem* 2.1 *holds.*

Example 2.1. Suppose that $X = [0,\, 1]$, furnished with the usual metric $d : X \times X \to [0,\, +\infty)$. Let $T : X \times X \to K(X)$ be defined as

$$T(x,\ y) = \left[0,\ \frac{x^2+y^2}{3}\right],\ \text{for all } x,\ y \in X,$$

and $B : X \to X$ be defined as

$$Bx = x^2, \text{ for all } x \in X.$$

Define φ, ψ, $\theta : [0, +\infty) \to [0, +\infty)$ as follows

$$\psi(t) = t^2, \ \varphi(t) = \begin{cases} \frac{2}{3}[t]^2, & \text{if } 3 < t < 4, \\ \frac{4}{9}t^2, & \text{otherwise,} \end{cases} \quad \text{and } \theta(t) = \begin{cases} \frac{1}{9}[t]^2, & \text{if } 3 < t < 4, \\ 0, & \text{otherwise.} \end{cases}$$

Then ψ, φ and θ have all the properties mentioned in Theorem 2.1. Now, for all x, y, u, $v \in X$ we have

$$\begin{aligned} & \psi(H(T(x,\ y),\ T(u,\ v))) \\ = \ & (H(T(x,\ y),\ T(u,\ v)))^2 \\ = \ & |T(x,\ y) - T(u,\ v)|^2 \\ = \ & \left| \frac{x^2+y^2}{3} - \frac{u^2+v^2}{3} \right|^2 \\ = \ & \frac{1}{9} \left| (x^2+y^2) - (u^2+v^2) \right|^2 \\ \leq \ & \frac{1}{9} (|x^2-u^2| + |y^2-v^2|)^2 \\ \leq \ & \frac{1}{9} (d(Bx,\ Bu) + d(By,\ Bv))^2 \\ \leq \ & \frac{4}{9} (\max\{d(Bx,\ Bu),\ d(By,\ Bv)\})^2 \\ \leq \ & \varphi(\max\{d(Bx,\ Bu),\ d(By,\ Bv)\}) - \theta(\max\{d(Bx,\ Bu),\ d(By,\ Bv)\}). \end{aligned}$$

Hence, the hybrid pair $\{T, B\}$ satisfies the contractive condition (2.1), for all x, y, u, $v \in X$. In addition, all the other conditions of Theorem 2.1 and Theorem 2.2 are satisfied and $z = (0, 0)$ is a common coupled fixed point of hybrid pair $\{T, B\}$.

3. Applications

In this section, we provide an application to our results. Consider the integral equation

$$x(t) = \int_0^T K(t,\ s,\ x(s))ds + h(t),\ t \in [0,\ T], \tag{3.1}$$

where $T > 0$. We introduce the following space:

$$C[0,\ T] = \{u : [0,\ T] \to \mathbb{R} : u \text{ is continuous on } [0,\ T]\},$$

furnished with the metric

$$d(x,\ y) = \sup_{t \in [0,\ T]} |x(t) - y(t)|\,, \text{ for each } x,\ y \in C[0,\ T].$$

It is clear that $(C[0, T], d)$ is a complete metric space.

Theorem 3.1. *We imagine that the following hypotheses hold:*

(i) $K : [0,\ T] \times [0,\ T] \times \mathbb{R} \to \mathbb{R}$ *and* $h : [0,\ T] \to \mathbb{R}$ *are continuous,*

(ii) there exists a continuous function $G : [0,\ T] \times [0,\ T] \to [0,\ +\infty)$ *such that*

$$|K(t,\ s,\ x) - K(t,\ s,\ y)| \leq G(t,\ s) \cdot \frac{|x - y|}{3},$$

for all $s,\ t \in C[0,\ T]$ *and* $x,\ y \in \mathbb{R}$,

(iii) $\sup_{t\in[0,\ T]} \int_0^T G(t,\ s)^2 ds \leq \frac{1}{T}$.

Then the integral (3.1) *has a solution* $(u,\ v) \in C[0,\ T] \times C[0,\ T]$.

Proof. We first define $\varphi,\ \psi,\ \theta : [0,\ +\infty) \to [0,\ +\infty)$ as follows

$$\psi(t) = t^2,\ \varphi(t) = \begin{cases} \frac{2}{3}[t]^2, & \text{if } 3 < t < 4, \\ \frac{4}{9}t^2, & \text{otherwise,} \end{cases} \quad \text{and } \theta(t) = \begin{cases} \frac{1}{9}[t]^2, & \text{if } 3 < t < 4, \\ 0, & \text{otherwise.} \end{cases}$$

Define $F : C[0,\ T] \times C[0,\ T] \to C[0,\ T]$ by

$$F(x,\ y)(t) = \int_0^T [K(t,\ s,\ x(s)) + K(t,\ s,\ y(s))]ds + h(t),$$

for all $t \in [0,\ T]$ and $x,\ y \in C[0,\ T]$. Now, for all $x,\ y,\ u,\ v \in C[0,\ T]$, due to *(ii)* and by using Cauchy-Schwarz inequality, we get

$$\begin{aligned} & |F(x,\ y)(t) - F(u,\ v)(t)| \\ \leq\ & \int_0^T |K(t,\ s,\ x(s)) - K(t,\ s,\ u(s))|\, ds + \int_0^T |K(t,\ s,\ y(s)) - K(t,\ s,\ v(s))|\, ds \\ \leq\ & \int_0^T G(t,\ s) \cdot \left(\frac{|x(s) - u(s)| + |y(s) - v(s)|}{3} \right) ds \\ \leq\ & \left(\int_0^T G(t,\ s)^2 ds \right)^{\frac{1}{2}} \left(\int_0^T \left(\frac{|x(s) - u(s)| + |y(s) - v(s)|}{3} \right)^2 ds \right)^{\frac{1}{2}}. \end{aligned}$$

Thus

$$\begin{aligned} & |F(x,\ y)(t) - F(u,\ v)(t)| \\ \leq\ & \left[\int_0^T G(t,\ s)^2 ds \right]^{\frac{1}{2}} \left[\int_0^T \left(\frac{|x(s) - u(s)| + |y(s) - v(s)|}{3} \right)^2 ds \right]^{\frac{1}{2}}. \end{aligned} \tag{3.2}$$

Taking (iii) into account, we estimate the first integral in (3.2) as follows:

$$\left[\int_0^T G(t,\ s)^2 ds\right]^{\frac{1}{2}} \leq \frac{1}{\sqrt{T}}. \tag{3.3}$$

For the second integral in (3.2) we proceed in the following way:

$$\left[\int_0^T \left(\frac{|x(s)-u(s)|+|y(s)-v(s)|}{3}\right)^2 ds\right]^{\frac{1}{2}} \leq \sqrt{T}\cdot\frac{d(x,\ u)+d(y,\ v)}{3}. \tag{3.4}$$

Combininig (3.2), (3.3) and (3.4), we conclude that

$$|F(x,\ y)(t)-F(u,\ v)(t)| \leq \frac{d(x,\ u)+d(y,\ v)}{3} \leq \frac{2}{3}\max\{d(x,\ u),\ d(y,\ v)\}.$$

It yields

$$\begin{aligned}
\psi(d(F(x,\ y),\ F(u,\ v))) &= (d(F(x,\ y),\ F(u,\ v)))^2\\
&= |F(x,\ y)(t)-F(u,\ v)(t)|^2\\
&\leq \frac{4}{9}(\max\{d(x,\ u),\ d(y,\ v)\})^2\\
&\leq \varphi(\max\{d(x,\ u),\ d(y,\ v)\})\\
&\quad -\theta(\max\{d(x,\ u),\ d(y,\ v)\}),
\end{aligned}$$

for all $x,\ y,\ u,\ v \in C[0,\ T]$. Hence, all hypotheses of Corollary 2.8 are satisfied. Thus, F has a coupled fixed point $(u,\ v) \in C[0,\ T] \times C[0,\ T]$ which is a solution of (3.1). $\square$

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