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THE SPIRALS ON THE OBLATE AND PROLATE SPHEROIDS OF LORENTZ-MINKOWSKI 3− SPACE $\mathbb{R}^3_1$

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Abstract. Spirials are differentiable curves that meet all meridians of a rotational surface at a constant angle. In this study, we obtain differential equations of all spirals on hyperbolic oblate and Lorentzian prolate spheroids. Then we define the general parametrizations of spirals which are solutions of differential equations.
Keywords: Unit hyperbolic sphere, unit Lorentzian sphere, hyperbolic oblate spiral, Lorentzian prolate spiral, Lorentz Minkowski 3−space.

1. Introduction

Spherical coordinates are one of them most used curvilinear coordinate systems in such fields as Earth science, cartography and physics(in particular quantum mechanics, relativity) and engineering (in particular, electric and electronic) [4].

The oblate spheroidal coordinate system is generated by taking an orthogonal family of confocal ellipses and hyperbolas and rotating it about the minor axis of the ellipses. The resulting coordinate surfaces are oblate spheroids, hyperboloids of one sheet and half planes. Oblate spheroidal coordinates are often useful in solving partial differential equations when the boundary conditions are defined on an oblate

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spheroid or a hyperboloid of revolution. We can use these coordinates to solve the electrostatics problems [4].

The prolate spheroidal coordinates are generated by rotating an orthogonal family of confocal ellipses and hyperbolas about the major axis of the ellipses. Prolate spheroidal coordinates are a three-dimensional orthogonal coordinate system that results from rotating the two-dimensional elliptic coordinate system about the focal axis of the ellipse i.e., the symmetry axis on which the foci are located. Rotation about the other axis produces oblate spheroidal coordinates. Prolate spheroidal coordinates can also be considered as a limiting case of ellipsoidal coordinates. The resulting coordinate surfaces are prolate spheroids, hyperboloids of two shhets and meridian planes. The boundary value problems involving prolate spheroid bodies may be treated in prolate spheroidal coordinates [4].

Lorentzian and hyperbolic spherical coordinates are two of used curvilinear coordinate systems in such fields as Lorentzian geometry (in particular, relativity), non-Euclidean geometry (in particular, hyperbolic geometry), Lorentzian mechanism, Lorentzian field theory. Ugurlu and Gurdal [8] defined the Lorentzian and hyperbolic coordinate systems in the Lorentz Minkowski 3-space R_1^3 . Two of them are Lorentzian prolate and hyperbolic oblate coordinate systems.

The Lorentzian prolate coordinate system is generated by taking an orthogonal family of confocal Lorentzian circles and hyperbolic circles, and rotating it about the time axis of the timelike hyperbol. The resulting coordinate surfaces are Lorentzian spheroids, hyperbolic spheroids and timelike half planes. The coordinates on these spheroids are used in solving partial differential equations when the boundary conditions are defined on an prolate or oblate spheroids, respectively.

The hyperbolic oblate coordinate system is also generated by taking an orthogonal family of confocal spacelike hyperbols and timelike hyperbolas and rotating it about the time axis of the spacelike hyperbol. The resulting coordinate surfaces are hyperbolic oblate spheroids, Lorentzian oblate spheroids and timelike half planes. Hyperbolic oblate coordinates are often useful in solving partial differential equations when the boundary conditions are defined on an Lorentzian or hyperbolic oblate spheroid. (For these coordinate systems, see [8]). These spheroids are newly introduced surface families in the space $\mathbb{R}_1^3$.The presence of some special curves on Lorentzian and hyperbolic spheroids is an important research topic.

In this paper, we obtain the differential equations of spirals on the surfaces of Lorentzian prolate and hyperbolic oblate spheroids in the space $\mathbb{R}_1^3$, and define the general parametrizations of all spirals which are the solutions of the differential equations.

2. Preliminaries

The Minkowski 3-space $\mathbb{R}_1^3$ is the real vector space endowed with the natural Lorentzian metric given by:

$$\langle .,. \rangle = dx_1^2 + dx_2^2 - dx_3^2 \tag{2.1}$$

where (x_1, x_2, x_3) is a standard rectangular coordinate system of $\mathbb{R}^3$. An arbitrary vector $\boldsymbol{v} = (v_1, v_2, v_3)$ in $\mathbb{R}^3_1$ is said to be spacelike, timelike or lightlike (null) if $\langle \boldsymbol{v}, \boldsymbol{v} \rangle > 0$ or $\boldsymbol{v} = 0$, $\langle \boldsymbol{v}, \boldsymbol{v} \rangle < 0$, $\langle \boldsymbol{v}, \boldsymbol{v} \rangle = 0$ and $\boldsymbol{v} \neq 0$, respectively. Thus, a spacelike (timelike) vector $\boldsymbol{v}$ is unit if $\langle \boldsymbol{v}, \boldsymbol{v} \rangle = 1$ ($\langle \boldsymbol{v}, \boldsymbol{v} \rangle = -1$). Two non zero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are said to be orthogonal if $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = 0$. A set of $\{\boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3}\}$ of vectors in $\mathbb{R}^3_1$ is called an orthonormal frame if it satisfies that

$$\langle \boldsymbol{e_1}, \boldsymbol{e_1} \rangle = \langle \boldsymbol{e_2}, \boldsymbol{e_2} \rangle = 1 \ \ \langle \boldsymbol{e_3}, \boldsymbol{e_3} \rangle = -1 \text{ and } \ \langle \boldsymbol{e_i}, \boldsymbol{e_j} \rangle = 0, \ i \neq j.$$

The norm of a vector $\boldsymbol{v} \in \mathbb{R}^3_1$ is defined by $\|\boldsymbol{v}\| = \sqrt{|\langle \boldsymbol{v}, \boldsymbol{v} \rangle|}$. The spheres of the space $\mathbb{R}^3_1$ are defined as follows: The set of all spacelike vectors of radius $r > 0$ with origin-centered is called *Lorentzian sphere* of radius r and denoted by

$$S^2_1(r) = \{ \boldsymbol{v} \in \mathbb{R}^3_1 \big| \langle \boldsymbol{v}, \boldsymbol{v} \rangle = r^2 \}.$$

The sphere $S^2_1(r)$ is a Lorentzian 2-manifold of constant sectional curvature $1/r^2$. On the other hand, for $r > 0$, the quadric

$$H^2_0(r) = \{ \boldsymbol{v} \in \mathbb{R}^3_1 \big| \langle \boldsymbol{v}, \boldsymbol{v} \rangle = -r^2 \}.$$

is called *hyperbolic sphere* of radius r. The sphere $H^2_0(r)$ is also a Riemannian 2-manifold of constant sectional curvature $-1/r^2$. This quadric has two connected components given by

$$H^+(r) = \{ \boldsymbol{v} \in H^2_0(r) \big| \langle \boldsymbol{v}, \boldsymbol{e_3} \rangle < 0 \},$$

$$H^-(r) = \{ \boldsymbol{v} \in H^2_0(r) \big| \langle \boldsymbol{v}, \boldsymbol{e_3} \rangle > 0 \}.$$

The surface $H^2_0(r)$ is the hyperboloid model of hyperbolic geometry from non-Euclidean geometries in yhe space $\mathbb{R}^3_1$. The set Λ^2

$$\Lambda^2 \ = \{ \boldsymbol{v} \in \mathbb{R}^3_1 - \{0\} \big| \langle \boldsymbol{v}, \boldsymbol{v} \rangle = 0 \}$$

off all lightlike vectors with length $r = 0$ is called *lightlike cone* of $\mathbb{R}^3_1$. This quadric has also two components given by

$$\begin{aligned} \Lambda^+ \quad &= \{ \boldsymbol{v} \in \Lambda^2 \big| \langle \boldsymbol{v}, \boldsymbol{e_3} \rangle < 0 \}, \\ \Lambda^- \quad &= \{ \boldsymbol{v} \in \Lambda^2 \big| \langle \boldsymbol{v}, \boldsymbol{e_3} \rangle > 0 \}. \end{aligned} \tag{2.2}$$

The components Λ^+ and Λ^- are called *future cone* and *past cone*, respectively. Then a ray in Λ^+ stating at the origin corresponds to a point on boundary of H^3. The set of such rays form the sphere at infinity $S^2_\infty = \partial H^3$. For the two non-zero vectors $\boldsymbol{u} = (u_1, u_2, u_3)$ and $\boldsymbol{v} = (v_1, v_2, v_3)$ in $\mathbb{R}^3_1$, the Lorentzian cross product of $\boldsymbol{u}$ and $\boldsymbol{v}$ is defined as the following:

$$\boldsymbol{u} \times \boldsymbol{v} = (u_3 v_2 - u_2 v_3,\, u_1 v_3 - u_3 v_1,\, u_1 v_2 - u_2 v_1). \tag{2.3}$$

The equation (2.3) is the reflection of Euclidean cross product with respect to the plane of equation $x = 0$. For standard base vectors, we have

$$\boldsymbol{e_1} \times \boldsymbol{e_2} = \boldsymbol{e_3},\ \boldsymbol{e_2} \times \boldsymbol{e_3} = -\boldsymbol{e_1},\ \boldsymbol{e_3} \times \boldsymbol{e_1} = -\boldsymbol{e_2}.$$

Let V be a 2-dimensional linear subspace $\mathbb{R}^3_1$. Then, there are three mutually exclutive possibilities for V :

(i) V is said to be *spacelike* if the restriction $\langle .,. \rangle |_V$ of the Lorentzian metric on V is positive definite.

(ii) V is said to be *timelike* if the restriction $\langle .,. \rangle |_V$ of the Lorentzian metric on V is Lorentzian, i.e, non-degenerate and of the signature (1). Then, V is a timelike plane.

(iii) V is *lightlike* (or *null*) if $\langle .,. \rangle |_V$ is degenerate.

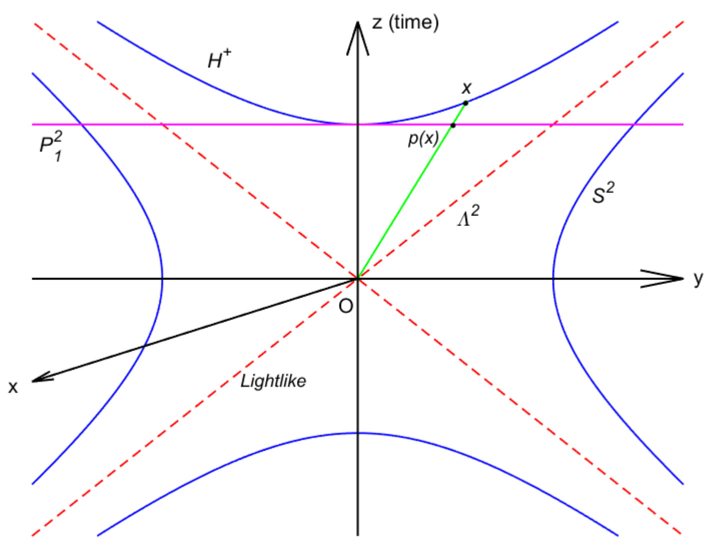
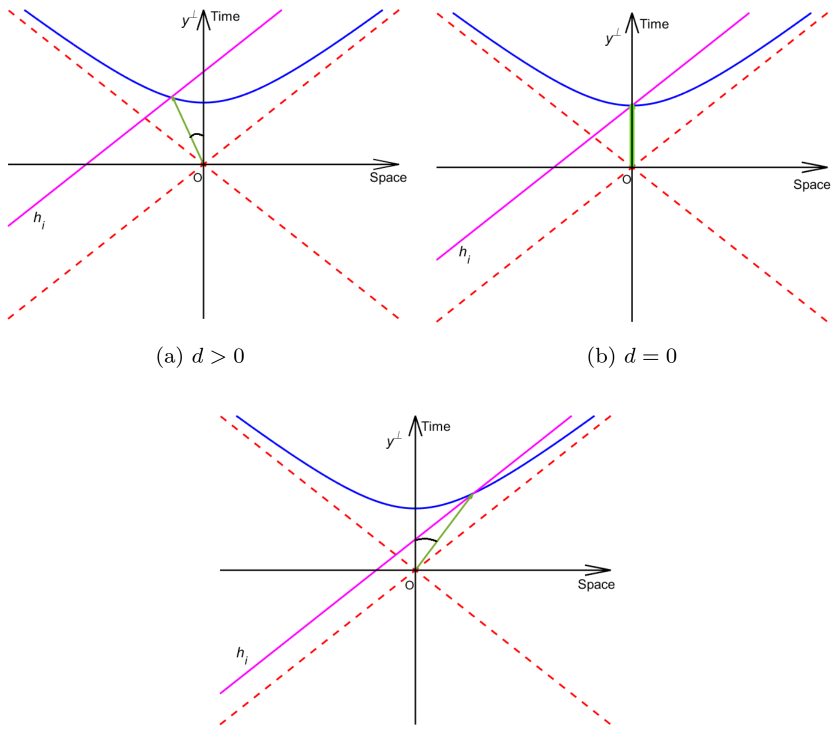


FIG. 2.1: The space $\mathbb{R}^3_1$ equipped with the Lorentzian inner product $\langle\, ,\, \rangle$ is 3-dimensional Lorentzian space [9].

For an arbitrary point $\boldsymbol{u}$ in Λ^+ the *horosphere* is defined as

$$h_u = \{\boldsymbol{x} \in H^+ \big| \, \langle \boldsymbol{x}, \boldsymbol{u} \rangle = -1\},$$



(a) $d > 0$ (b) $d = 0$

(c) $d < 0$

FIG. 2.2: The signed distance from a plane to a horosphere is the distance d by which the horosphere extends to the plane.

which inherits an Euclidean structure [1].

Assume that p denotes the radial projection. Then the projection is given by

$$p\{x \in \mathbb{R}^3_1 \big| \, x_3 \neq 0\} \to P^3_1 = \{x \in \mathbb{R}^3_1 \big| \, x_3 = 1\},$$

where P^3_1 is affine plane along the rays through the origin. The projection p is a homeomorphism from H^+ onto the 3-dimensional open unit ball B^3 in P^3_1 centered at the origin $(0, 0, 1)$ of P^3_1, which yields the projective model of H^3, The affine plane P^3_1 contains B^3 and its see theoretic boundary ∂B^3 in P^3_1, which is identified with S^2_∞. In this case, we have $B^3 = B^3 \cup \partial B^3$. Now we define the *geodesic plane* $u^\perp$ for an arbitrary point u in S^2_1 as

$$\boldsymbol{u}^\perp = \{\boldsymbol{x} \in H^3 \big| \, \langle \boldsymbol{x}, \boldsymbol{u} \rangle = 0\}.$$

A point u also defines a $half - space$ in H^3 given by

$$\Pi_M = \{\boldsymbol{x} \in H^3 \big| \, \langle \boldsymbol{x}, \boldsymbol{u} \rangle \leq 0\},$$

where $\boldsymbol{u}$ is the position vector field of the point u.

Definition 2.1. The *signed dis* tan *ce* d between horosphere and a plane (respectively, point, horosphere) is the distance by which the horosphere extends past the plane (respectively, point, horosphere). The distance d may be positive, negative, or zero, as shown in (Fig. 2.2). (For hyperbolic distances, see [1]).

3. The angles in the Minkowski Space $\mathbb{R}^3_1$

Let $\boldsymbol{u}$, $\boldsymbol{v}$ be two vectors in the space $\mathbb{R}^3_1$. The angle between $\boldsymbol{u}$ and $\boldsymbol{v}$ is defined with respect to causal characters of these vectors as follows [1], [4], [6]:

(i) Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be two positive (negative) timelike vectors. Then $|\langle \boldsymbol{u}, \boldsymbol{v}\rangle| \geq \|\boldsymbol{u}\| \, \|\boldsymbol{v}\|$ and equality holds if and only if $\boldsymbol{u}$ and $\boldsymbol{v}$ are proportional. Thus, there exists a unique non-negative real number $\theta \geq 0$ such that

$$\langle \boldsymbol{u}, \boldsymbol{v}\rangle = -\|\boldsymbol{u}\| \, \|\boldsymbol{v}\| \cosh\theta. \tag{3.1}$$

This number is called the *hyperbolic angle* between the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$.

(ii) Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be spacelike vectors and they span a timelike vector subspace. Thus, there is a unique non negative real number $\theta \geq 0$ such that

$$\langle \boldsymbol{u}, \boldsymbol{v}\rangle = \|\boldsymbol{u}\| \, \|\boldsymbol{v}\| \cosh\theta. \tag{3.2}$$

This number is called the *central angle* between the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$.

(iii) Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be spacelike vectors in $\mathbb{R}^3_1$ and they span a spacelike vector subspace. Then $|\langle \boldsymbol{u}, \boldsymbol{v}\rangle| \leq \|\boldsymbol{u}\| \, \|\boldsymbol{v}\|$ and equality holds if and only if $\boldsymbol{u}$ and $\boldsymbol{v}$ are proportional. Thus, there exists a unique real number $\theta \geq 0$ such that

$$|\langle \boldsymbol{u}, \boldsymbol{v}\rangle| = \|\boldsymbol{u}\| \, \|\boldsymbol{v}\| \cos\theta. \tag{3.3}$$

This number is called the *spacelike angle* between the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$.

(iv) Let $\boldsymbol{u}$ be spacelike vector and $\boldsymbol{v}$ be a timelike vectors in $\mathbb{R}^3_1$. Thus, there is a unique real number $\theta \geq 0$ such that

$$|\langle \boldsymbol{u}, \boldsymbol{v}\rangle| = \|\boldsymbol{u}\| \, \|\boldsymbol{v}\| \sinh\theta. \tag{3.4}$$

This number is called the *timelike angle* between the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$.

(v) Two geodesic planes $\boldsymbol{u}^\perp$ and $\boldsymbol{v}^\perp$ do not intersect in H^2 but intersect in ∂H^2 if and only if

$$|\langle \boldsymbol{u}, \boldsymbol{v}\rangle| = \cos 0 = 1. \tag{3.5}$$

In this case the *Lorentzian angle* between them is zero. Note that the Lorentzian angles correspond to hyperbolic distances between two points on the hyperboloid model $H^+(r)$.

(vi) Let $\boldsymbol{u} \in \Lambda^{+}$ and $\boldsymbol{v} \in S_1^2$. The *lightlike angle* between them is signed distance d between $\boldsymbol{v}$ and h_u. This angle gives the equality as

$$|\langle \boldsymbol{u}, \boldsymbol{v} \rangle| = e^{-d}. \tag{3.6}$$

(vii) Let $\boldsymbol{u} \in \Lambda^{+}$ and $\boldsymbol{v} \in H^{+}$. The *lightlike angle* between them is signed distance d between $\boldsymbol{v}^{\perp}$ and h_u. This angle gives the equality as

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = -e^{-d}. \tag{3.7}$$

(viii) Let $\boldsymbol{u}, \boldsymbol{v} \in \Lambda^{+}$.The *lightlike angle* between them is signed distance d between h_u and h_v. This angle gives the equality as

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = -2e^{-d}. \tag{3.8}$$

(For the geometrical interpretations of angles in the space $\mathbb{R}_1^3$, see [3], [7]).

4. The Spirals on the Oblate Forms of Hyperboloid Model $H^{+}(r)$

We know that the hyperbolic sphere of center O and radius r is the surface given by

$$H_0^2(r) = \left\{(x, y, z) \in \mathbb{R}_1^3 \mid x^2 + y^2 - z^2 = -r^2\right\}.$$

We express that the set $H_0^2(r)$ has exactly two connected components. Therefore, the hyperboloid model $H^{+}(r)$ is obtained by taking $z > 0$. This model is a spacelike surface and called a hyperbolic plane [3], [5], [7]. The oblate form of hyperbolic sphere $H_0^2(r)$ which is also called "hyperbolic oblate spheroid" is given by

$$H_0^2(\lambda, r) = \left\{ (x, y, z) \in \mathbb{R}_1^3 \Big| \, \frac{x^2 + y^2}{1+\lambda^2} - \frac{z^2}{\lambda^2} = -r^2 \right\}, \tag{4.1}$$

where $\lambda \in \mathbb{R}\backslash\{0\}$ and $r > \lambda$. Similarly, there exists an oblate form $H^{+}(\lambda, r)$ for $z > 0$, which is a spacelike surface.

Now, we compute the first fundamental from of $H^{+}(\lambda, r)$ at a point of the coordinate neightborhood given by parametrization

$$\boldsymbol{x}(u, v) = \left(r\sqrt{1+\lambda^2}\sinh u \cos v,\, r\sqrt{1+\lambda^2}\sinh u \sin v,\, r\lambda \cosh u\right), \tag{4.2}$$

where $0 \leq v \leq 2\pi$ and $u \in \mathbb{R}$. The partial derivatives of the surface in (4.2) yields

$$\left.\begin{aligned} \boldsymbol{x_u}(u, v) &= \left(r\sqrt{1+\lambda^2}\cosh u \cos v,\, r\sqrt{1+\lambda^2}\cosh u \sin v,\, r\lambda \sinh u\right), \\ \boldsymbol{x_v}(u, v) &= \left(-r\sqrt{1+\lambda^2}\sinh u \sin v,\, r\sqrt{1+\lambda^2}\sinh u \cos v,\, 0\right). \end{aligned}\right\} \tag{4.3}$$

Hence, the coefficients of the first fundamental form are

$$
\left.\begin{aligned}
E(u,v) &= \langle \boldsymbol{x_u}, \boldsymbol{x_u} \rangle = r^2\left(\cosh^2 u + \lambda^2\right), \\
F(u,v) &= \langle \boldsymbol{x_u}, \boldsymbol{x_v} \rangle = 0, \\
G(u,v) &= \langle \boldsymbol{x_v}, \boldsymbol{x_v} \rangle = r^2\left(\lambda^2 + 1\right)\sinh^2 u.
\end{aligned}\right\} \tag{4.4}
$$

Thus, if $\boldsymbol{w}$ is a tangent vector to the spheroid at the point $\boldsymbol{x}(u,v)$, given in the basis associated to $\boldsymbol{x}(u,v)$ by

$$\boldsymbol{w} = a\boldsymbol{x_u} + b\boldsymbol{x_v}.$$

Then, the square of the length of $\boldsymbol{w}$ is given by

$$|\boldsymbol{w}|^2 = I(\boldsymbol{w}) = a^2E + 2abF + b^2G = a^2r^2\left(\cosh^2 u + \lambda^2\right) + b^2r^2\left(1+\lambda^2\right)\sinh^2 u.$$

Let us determine the curves in this coordinate neighborhood of the $H^+(\lambda, r)$ which make a constant spacelike angle θ with meridians ($v = const.$).

Definition 4.1. Let $\boldsymbol{\alpha_\lambda}(t)$ be a differentiable curve on the oblate form of hyperboloid model $H^+(r)$ denoted by $H^+(\lambda, r)$ for $\lambda \in \mathbb{R}$. Then, the curve $\boldsymbol{\alpha_\lambda}(t)$ is called *hyperbolic spheroidal spiral* if $\boldsymbol{\alpha_\lambda}(t)$ cuts all meridians of $H^+(\lambda, r)$ with a constant spacelike angle.

Theorem 4.1. *Let θ be constant spacelike angle between the meridians of the hyperbolic oblate spheroid and the spacelike curve $\boldsymbol{\alpha_\lambda}(t)$ and assume that $\lambda = \tan\theta$. Then the parametrization of all spirals on the oblate form the spheroid $H^+(\lambda, r)$ is given by*

$$\boldsymbol{\alpha_\lambda}(t) = (r\sqrt{1+\lambda^2}\sinh t\cos v(t), r\sqrt{1+\lambda^2}\sinh t\sin v(t), r\lambda\cosh t), \tag{4.5}$$

where

$$
\left.\begin{aligned}
v(t) = {} & \pm\frac{\tan\theta}{\sqrt{1+\lambda^2}}\left[-\sqrt{1+\lambda^2}\tanh^{-1}\left(\tfrac{\sqrt{2+2\lambda^2}\cosh t}{\sqrt{1+2\lambda^2+\cosh 2t}}\right)\right. \\
& \left. + \ln\left(\sqrt{2}\cosh t + \sqrt{1+2\lambda^2+\cosh 2t}\right)\right] + k
\end{aligned}\right\} \tag{4.6}
$$

where k is the constant of integration.

Proof. We may assume that the required curve

$$
\begin{aligned}
\boldsymbol{\alpha_\lambda}(t) = \boldsymbol{x}\left(u(t), v(t)\right) = {} & \left(r\sqrt{1+\lambda^2}\sinh u(t)\cos v(t),\right. \\
& \left. r\sqrt{1+\lambda^2}\sinh u(t)\sin v(t),\ r\lambda\cosh u(t)\right).
\end{aligned}
$$

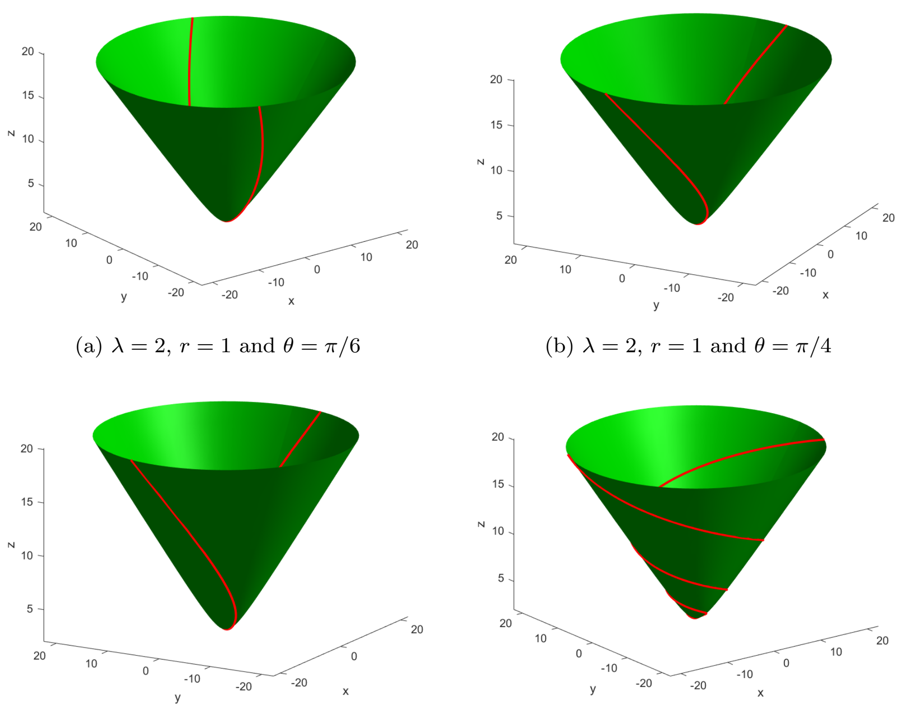
is the image by $\boldsymbol{x}$ of a curve $(u(t), v(t))$ of the $uv-$ plane. In this case we have

$$\begin{aligned}\boldsymbol{\alpha}'_\lambda = \quad & \big(r\sqrt{1+\lambda^2}u'(t)\cosh u(t)\cos v(t) - r\sqrt{1+\lambda^2}v'(t)\sinh u(t)\sin v(t), \\ & r\sqrt{1+\lambda^2}u'(t)\cosh u(t)\sin v(t) + r\sqrt{1+\lambda^2}v'(t)\sinh u(t)\cos v(t), \\ & \lambda u'(t)\sinh u(t)\big).\end{aligned}$$

At the point $\boldsymbol{x}(u, v)$ where the curve meets the meridians ($v = const.$), we have

$$\begin{aligned}\cos\theta &= \frac{|\langle \boldsymbol{x_u}, \boldsymbol{\alpha}'_\lambda(t)\rangle|}{\|\boldsymbol{x_u}\|\,\|\boldsymbol{\alpha}'_\lambda(t)\|} \\ &= \frac{u'\left(\lambda^2+\cosh^2 u\right)}{\sqrt{\lambda^2+\cosh^2 u}\sqrt{u'^2\left(\lambda^2+\cosh^2 u\right)+\left(1+\lambda^2\right)v'^2\sinh^2 u}}\end{aligned} \tag{4.7}$$

since in the basis $\{\boldsymbol{x_u}, \boldsymbol{x_u}\}$ the vector $\boldsymbol{\alpha}'_\lambda(t)$ has coordinates (u', v') and the vector



(a) $\lambda = 2$, $r = 1$ and $\theta = \pi/6$

(b) $\lambda = 2$, $r = 1$ and $\theta = \pi/4$

(c) $\lambda = 2$, $r = 1$ and $\theta = \pi/3$

(d) $\lambda = 2$, $r = 1$ and $\theta = 4\pi/9$

FIG. 4.1: Spirals on $H^+(r, \lambda)$ for different values of θ

$\boldsymbol{x_u}$ has coordinates $(1,0)$. By (4.7) we obtain the differantial equation

$$u'^2\left(\lambda^2+\cosh^2 u\right)\left(\cos^2\theta-1\right)+\left(1+\lambda^2\right)v'^2\sinh^2 u\cos^2\theta=0$$

or

$$\frac{dv}{du}=\pm\tan\theta\frac{\sqrt{\lambda^2+\cosh^2 u}}{\sqrt{1+\lambda^2}\sinh u} \tag{4.8}$$

The solution of the differential equation (4.8) gives (4.6). If we take the signature $+$, the constant of integration as zero and $u(t)=t$, then we obtain the equations of the hyperbolic spheroidal spirals and this completes the proof (See, Figure 4.1). □

5. The spirals on the Prolate Forms of the Lorentzian Sphere $S_1^2(r)$

We know that there exist three tupes of curves on the surface of the unit Lorentzian sphere $S_1^2(r)$; namely spacelike, timelike and lightlike (null). Since the induced metric on a null curve is degenerate, the null curves are different from the timelike and spacelike curves. Therefore, there should be three types of curves on the prolate Lorentzian spheroids. Are there three types of spirals on these spheroids? If there is any, what are the differential equations of these spirals and the parameterizations of the casual spirals that are solutions of these differential equations. In this section, we will look for answers to these questions.

Let's consider Lorentzian sphere S_1^2 with radius r. this sphere is defined as

$$S_1^2\left(r\right)=\left\{x\in R_1^3 \middle| \left\langle x,x\right\rangle=r^2\right\}.$$

From an Euclidean viewpoint, this surface is a ruled surface. It is a timelike surface since $T_pS_1^2\left(r\right)=s_p\left\{\boldsymbol{p}\right\}^{\perp}$ and $\boldsymbol{p}$ is a spacelike vector.

The equation of the prolate Lorentzian spheroids in cartesian coordinates is given as

$$\frac{x^2+y^2}{\lambda^2-1}-\frac{z^2}{\lambda^2}=r^2,\quad 0\leq\lambda\leq\infty. \tag{5.1}$$

A parameterization of the equation (5.1) is

$$\boldsymbol{X}\left(u,v\right)=\left(r\sqrt{\lambda^2-1}\cosh u\cos v,\ r\sqrt{\lambda^2-1}\cosh u\sin v,\ r\lambda\sinh u\right), \tag{5.2}$$

where $0\leq v\leq 2\pi$, $u\in R$. Using the partial derivatives of the equation (5.2) as

$$\left.\begin{aligned}\boldsymbol{X}_u= &\left(r\sqrt{\lambda^2-1}\sinh u\cos v,\ r\sqrt{\lambda^2-1}\sinh u\sin v,\ r\lambda\cosh u\right),\\ \boldsymbol{X}_v= &\left(-r\sqrt{\lambda^2-1}\cosh u\sin v,\ r\sqrt{\lambda^2-1}\cosh u\cos v,\ 0\right),\end{aligned}\right\} \tag{5.3}$$

the coefficients of first fundamental form are found as

$$
\left.\begin{aligned}
E &= \langle \boldsymbol{X}_u, \boldsymbol{X}_u \rangle = -r^2 \left(\lambda^2 + \sinh^2 u\right), \\
F &= 0, \\
G &= \langle \boldsymbol{X}_v, \boldsymbol{X}_v \rangle = r^2 \left(\lambda^2 - 1\right) \cosh^2 u.
\end{aligned}\right\} \tag{5.4}
$$

Theorem 5.1. *Let θ be constant hyperbolic angle between the meridians of the Lorentzian prolate spheroid $S_1^2(\lambda, r)$ and the timelike curve $\boldsymbol{\beta}_\lambda(t)$. Then the parametrizations of all spirals on the spheroid $S_1^2(\lambda, r)$ is given by*

$$
\boldsymbol{\beta}_\lambda(t) = \left(r\sqrt{\lambda^2-1}\cosh t \cos v(t), r\sqrt{\lambda^2-1}\cosh t \sin v(t), r\lambda \sinh t\right), \tag{5.5}
$$

where

$$
\left.\begin{aligned}
v(t) = {} & \frac{\tanh\theta}{\sqrt{\lambda^2-1}} \left(\sqrt{\lambda^2-1} \tan^{-1}\left(\frac{\sqrt{2\lambda^2-2}\sinh t}{\sqrt{\cosh 2t + 2\lambda^2 - 1}} \right) \right. \\
& \left. + \sinh^{-1}\left(\frac{\sinh t}{|\lambda|}\right)\right) + k, \quad (k: \mathit{int.\ constant})
\end{aligned}\right\} \tag{5.6}
$$

Proof. Let a timelike curve that cuts all meridians of the prolate Lorentzian spheroids with a hiperbolic angle θ be as follows:

$$
\begin{aligned}
\boldsymbol{\beta}_\lambda(t) = {} & \left(r\sqrt{\lambda^2-1}\cosh u(t) \cos v(t),\right. \\
& \left. r\sqrt{\lambda^2-1}\cosh u(t) \sin v(t),\ r\lambda \sinh u(t)\right).
\end{aligned} \tag{5.7}
$$

In this case, we obtain

$$
\begin{aligned}
\boldsymbol{\beta}_\lambda'(t) = {} & \left(r\sqrt{\lambda^2-1}u' \sinh u \cos v - r\sqrt{\lambda^2-1}v' \sin v \cosh u,\right. \\
& r\sqrt{\lambda^2-1}u' \sinh u \sin v + r\sqrt{\lambda^2-1}v' \cos v \cosh u, \\
& \left. r\lambda u' \cosh u\right).
\end{aligned}
$$

Hence, if the values

$$
\left.\begin{aligned}
|\boldsymbol{X}_u| &= r\sqrt{\lambda^2 + \sinh^2 u} \\
\left|\boldsymbol{\beta}_\lambda'(t)\right| &= r\sqrt{(\lambda^2-1)\, v'^2 \cosh^2 u - \left(\lambda^2 + \sinh^2 u\right) u'^2} \\
\left\langle \boldsymbol{X}_u, \boldsymbol{\beta}_\lambda'(t) \right\rangle &= -r^2 \left(\lambda^2 + \sinh^2 u\right) u',
\end{aligned}\right\} \tag{5.8}
$$

are substituted into the equation

$$
\left\langle \boldsymbol{X}_u, \boldsymbol{\beta}_\lambda'(t) \right\rangle = -|\boldsymbol{X}_u| \left|\boldsymbol{\beta}_\lambda'(t)\right| \cosh\theta, \tag{5.9}
$$

we find the differential equation

$$\frac{dv}{du} = \pm\frac{\sqrt{\lambda^2+\sinh^2 u}}{\sqrt{\lambda^2-1}\cosh u}\tanh\theta. \tag{5.10}$$

The solution of the differential equation (5.10) gives (5.6). If we take the signature $+$, the constant of integration as zero and $u(t) = t$, we obtain the equations of Lorentzian prolate spheroidal spirals. This completes the proof (see Figure 5.1). $\square$

Theorem 5.2. *Let θ be constant timelike angle between the meridians of the Lorentzian prolate spheroid $S_1^2(\lambda, r)$ and the spacelike curve $\boldsymbol{\beta}_\lambda(t)$. Then the parametrization of all spirals on the spheroid $S_1^2(\lambda, r)$ is given by*

$$\boldsymbol{\beta}_\lambda(t) = \left(r\sqrt{\lambda^2-1}\cosh t\cos v(t), r\sqrt{\lambda^2-1}\cosh t\sin v(t), r\lambda\sinh t\right), \tag{5.11}$$

where

$$\begin{aligned} v(t) = \; & \pm\frac{\coth\theta}{\sqrt{\lambda^2-1}}\left[\sqrt{\lambda^2-1}\tan^{-1}\left(\frac{\sqrt{2\lambda^2-2}\sinh t}{\sqrt{\cosh 2t+2\lambda^2-1}}\right)\right. \\ & \left. + \sinh^{-1}\left(\frac{\sinh t}{|\lambda|}\right)\right] + k, \quad (k: \mathit{int.\ constant}) \end{aligned} \tag{5.12}$$

Proof. Assume that, the curve (5.11) be a spacelike curve that cuts all meridians of the prolate Lorentzian spheroids with a timelike angle θ. In this case, we can write

$$\left|\left\langle \boldsymbol{X}_u, \boldsymbol{\beta}'_\lambda(t)\right\rangle\right| = |\boldsymbol{X}_u|\left|\boldsymbol{\beta}'_\lambda(t)\right|\sinh\theta. \tag{5.13}$$

If the equations in (5.8) are substituted into the equation (5.13), we obtain the differential equation.

$$\frac{dv}{du} = \pm\frac{\sqrt{\lambda^2+\sinh^2 u}}{\sqrt{\lambda^2-1}\cosh u}\coth\theta. \tag{5.14}$$

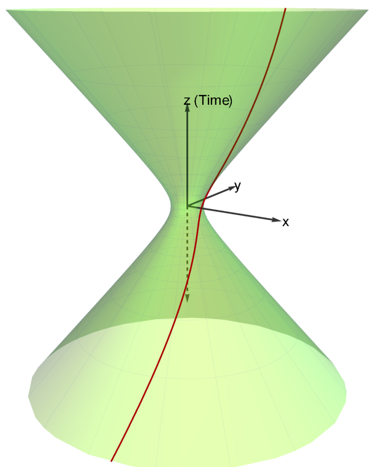
The solution of the equation (5.14) is

$$\begin{aligned} v(t) = \; & \pm\frac{\coth\theta}{\sqrt{\lambda^2-1}}\left[\sqrt{\lambda^2-1}\tan^{-1}\left(\frac{\sqrt{2\lambda^2-2}\sinh t}{\sqrt{\cosh 2t+2\lambda^2-1}}\right)\right. \\ & \left. + \sinh^{-1}\left(\frac{\sinh t}{|\lambda|}\right)\right] + k, \quad (k: \mathit{int.\ constant}) \end{aligned} \tag{5.15}$$

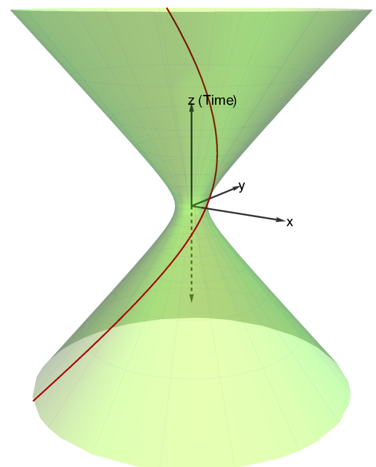
$\square$

It is presented some Lorentzian prolate spheroidal spacelike spirals for different values of λ in Figure 5.2.

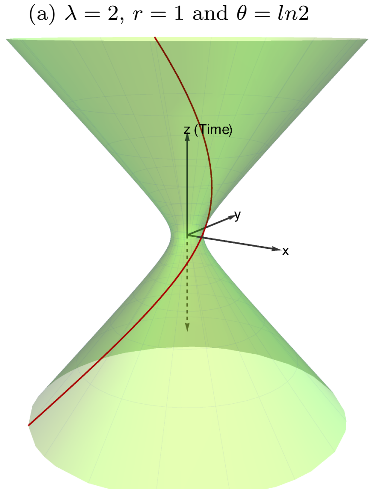
For the spirals and special curves on the surfaces of the Euclidean space E^3, see [2], [6], [7].



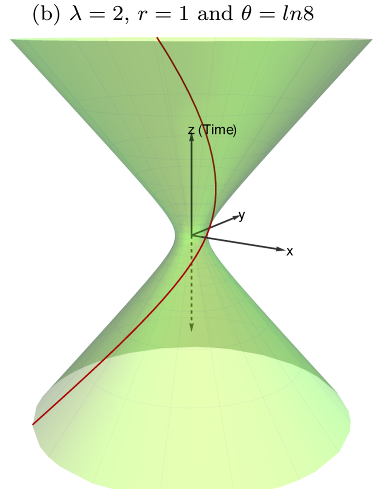
(a) $\lambda = 2$, $r = 1$ and $\theta = ln2$



(b) $\lambda = 2$, $r = 1$ and $\theta = ln8$

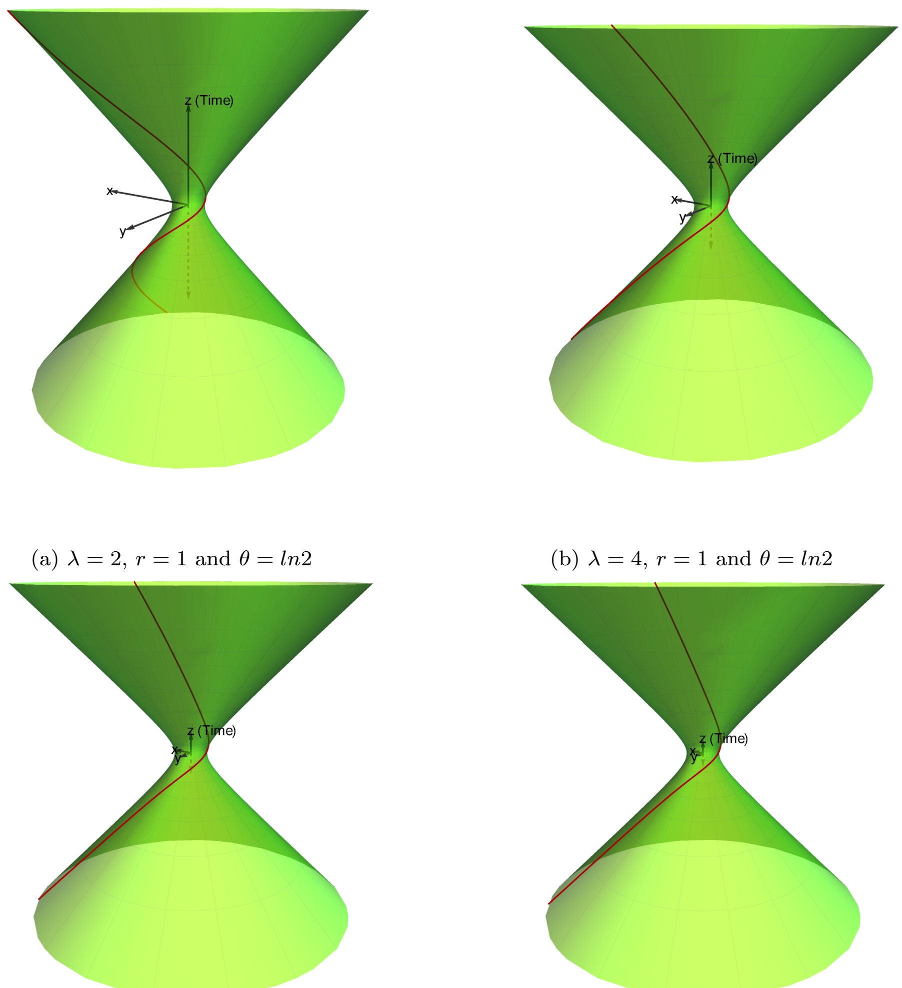


(c) $\lambda = 2$, $r = 1$ and $\theta = ln16$



(d) $\lambda = 2$, $r = 1$ and $\theta = ln32$

FIG. 5.1: Lorentzian prolate spheroidal timelike spirals for different values of θ



(a) $\lambda = 2$, $r = 1$ and $\theta = ln2$

(b) $\lambda = 4$, $r = 1$ and $\theta = ln2$

(c) $\lambda = 8$, $r = 1$ and $\theta = ln2$

(d) $\lambda = 12$, $r = 1$ and $\theta = ln2$

FIG. 5.2: Lorentzian prolate spheroidal spacelike spirals for different values of λ

6. Conclusion

In this paper, we defined the differential equations of causal spirals on the surfaces of the hyperboloid model $H^+(r)$, the set of positive timelike vectors of the Lorentzian sphere $S_1^2(r)$, the set of spacelike vectors with length r and of the lightlike cone Λ^2 gave the general parametrizations of casual spirals which are the solutions of their differential equations. In the next study, we will examine the geometries of the causal spirals on the spheres of the Minkowski $3-$ Space $\mathbb{R}_1^3$ provided with Lorentz metric $\langle , \rangle$ with signature $(+,+,-)$.

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