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APPROXIMATING COMMON ELEMENTS OF FIXED POINTS OF BREGMAN TOTALLY QUASI-ASYMPTOTICALLY NONEXPANSIVE MAPPINGS AND SOLUTIONS OF A SYSTEM OF GENERALIZED MIXED EQUILIBRIUM PROBLEMS IN REFLEXIVE BANACH SPACES

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Abstract. In this paper, we introduce a hybrid iterative method for approximating common elements of common fixed points of a finite family of Bregman totally quasi-asymptotically nonexpansive mappings and solutions of a finite system of generalized mixed equilibrium problems. After that, a strong convergence result for the proposed iterative method is established and proved in reflexive Banach spaces. By this result, we get some convergence results for generalized mixed equilibrium problems in reflexive Banach spaces. Furthermore, we give a numerical example to illustrate the obtained results.
Keywords: Bregman totally quasi-asymptotically nonexpansive mapping, hybrid iterative method, generalized mixed equilibrium problem, reflexive Banach space.

1. Introduction

Let W be a real reflexive Banach space, U be a nonempty, closed and convex subset of W, W^* be the dual space of W. We denote the value of the function of $u^* \in W^*$ at $x \in W$ by $\langle u^*, x\rangle$. Let $F : U \times U \longrightarrow \mathbb{R}$ be a function, $A : U \longrightarrow \mathbb{R}$ be a real valued function and $B : U \longrightarrow W^*$ be a nonlinear mapping. The generalized

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mixed equilibrium problem ($GMEP$) was introduced by Peng and Yao [24] which is to find $u \in U$ such that

$$F(u, v) + \langle B(u), v - u\rangle + A(v) - A(u) \geq 0, \forall v \in U.$$

The set of solutions of ($GMEP$) is denoted by

$$GMEP(F, A, B) = \big\{u \in U : F(u, v) + \langle B(u), v - u\rangle + A(v) - A(u) \geq 0, \forall v \in U\big\}.$$

In particular, if $B \equiv 0$, ($GMEP$) is reduced to the mixed equilibrium problem (MEP) which is to find $u \in U$ such that

$$F(u, v) + A(v) \geqslant A(u), \forall v \in U.$$

If $A \equiv 0$, ($GMEP$) is reduced to the generalized equilibrium problem (GEP) which is to find $u \in U$ such that

$$F(u, v) + \langle B(u), v - u\rangle \geqslant 0, \forall v \in U.$$

If $f \equiv 0$, ($GMEP$) is reduced to the mixed variational inequality (MVI) of Browder type which is to find $u \in U$ such that

$$\langle B(u), v - u\rangle + A(v) \geqslant A(u), \forall v \in U.$$

If $A \equiv 0$ and $B \equiv 0$ ($GMEP$) is reduced to the equilibrium problem (EP) which is to find $u \in U$ such that

$$F(u, v) \geqslant 0, \forall v \in U.$$

The set of solutions of (EP) is denoted by $EP(F) = \big\{u \in U : F(u, v) \geqslant 0, \forall v \in U\big\}$.

In this paper, we consider the following problem:

$$\text{Find } u \in \Big(\bigcap_{i\in I} F(H_i)\Big) \bigcap \Big(\bigcap_{k\in K} GMEP(F_k, A_k, B_k)\Big), \tag{1.1}$$

where $I := \{1, 2, \dots, N\}$ and $K := \{1, 2, \dots, M\}$ for some $M, N \in \mathbb{N}$, and for each $i \in I$, $F(H_i) = \{u \in U : H_i(u) = u\}$ is the set of fixed points of the mapping $H_i : U \longrightarrow U$, and for each $k \in K$,

$$\begin{aligned} & GMEP(F_k, A_k, B_k) \\ = \ & \{u \in U : F_k(u, v) + \langle B_k(u), v - u\rangle + A_k(v) - A_k(u) \geq 0, \forall v \in U\}. \end{aligned}$$

In the case $I = \{1\}$ and $K = \{1\}$, the problem (1.1) becomes the following problem:

$$\text{Find } \ u \in F(H) \cap \ GMEP(F, A, B). \tag{1.2}$$

In recent times, some authors have tried to propose certain iterative methods for approximating the solutions of the problem (1.1) and the problem (1.2). Recently, by using the Bregman distance and the Bregman projection, Darvish [12, 13, 14] introduced some iterative methods for solving the problem (1.1) where H_i are mappings with respect to the Bregman distance in reflexive Banach spaces. After that, some authors extended and improved the existing convergence results to solutions of the above problems from Hilbert spaces to reflexive Banach spaces [22, 23, 39]. In 2017, by basing on a parallel iterative method which is proposed by Anh and Chung [2], Tuyen [36] introduced some parallel iterative methods for a finite family of Bregman strongly nonexpansive mappings in reflexive Banach spaces. Similarly, Tuyen [35] introduced some parallel iterative methods for solving a system of generalized mixed equilibrium problems. One of iterative processes in [35] is defined by

$$\tag{1.3} \begin{cases} u_1 \in W, U_1 = W \\ z_n^{(k)} = \mathrm{Res}_{F_k,A_k,B_k}(u_n) \\ k_n = \mathrm{argmax}\{D_g(z_n^{(k)}, u_n) : k \in I\} \\ U_{n+1} = \big\{u \in U_n : D_g(u, z_n^{(k_n)}) \leq D_g(u, u_n)\big\} \\ u_{n+1} = P^g_{U_{n+1}}(u_1) \text{ for all } \ n \geq 2. \end{cases}$$

Furthermore, there were many methods for constructing new iterative processes which generalize some previous ones. In 2008, Mainge [19] proposed the inertial Mann iteration by combining the Mann iterative process and the inertial extrapolation as follows.

$$\begin{cases} v_n = u_n + \eta_n(u_n - u_{n-1}) \\ u_{n+1} = (1 - a_n)v_n + a_n T v_n. \end{cases}$$

After that, some iterative process es with the inertial extrapolation were introduced [15, 26]. In 2018, Chidume *et al.* [11] introduced an inertial algorithm for approximating a common fixed point for a countable family of relatively nonexpansive mappings in uniformly convex and uniformly smooth Banach spaces as follows.

$$\begin{cases} u_1, u_2 \in W, U_1 = U_2 = W \\ w_n = u_n + \eta_n(u_n - u_{n-1}) \\ v_n = J^{-1}\big((1 - \mu)Jw_n + \mu JTw_n\big) \\ U_{n+1} = \big\{u \in U_n : \phi(u, v_n) \leq \phi(u, w_n)\big\} \\ u_{n+1} = P^g_{U_{n+1}}(u_1) \text{ for all } \ n \geq 2. \end{cases}$$

Motivated by the above mentioned works, we introduce a new inertial iterative method for solving the problem (1.1) where H_i is a Bregman totally quasi-asymptotically nonexpansive mapping for each $i \in I$. After that, we prove a strong convergence theorem for the proposed iteration in reflexive Banach spaces. In addition, we give a numerical example to illustrate the obtained results.

2. Preliminaries

Let W be a real reflexive Banach space, U be a nonempty, closed and convex subset of W, W^* be the dual space of W. Let $g : W \longrightarrow (-\infty, +\infty]$ be a proper, lower semi-continuous and convex function. We denote by

$$\mathrm{dom}g = \{u \in W : g(u) < +\infty\}$$

the domain of g. For any $u \in \mathrm{int}(\mathrm{dom}g)$ and $v \in W$, we denote by $g'(u, v)$ *the right-hand derivative of* g *at* u *in the direction* v, that is

$$g'(u, v) = \lim_{\lambda \downarrow 0} \frac{g(u + \lambda v) - g(u)}{\lambda}. \tag{2.1}$$

The function g is said to be *Gâteaux differentiable at* u if the limit (2.1) exists for any v. In this case, the gradient of g at u is the function $\nabla g(u)$, which is defined by $\langle \nabla g(u), v\rangle = g'(u, v)$ for all $v \in W$. The function g is said to be *Gâteaux differentiable* on $\mathrm{int}(\mathrm{dom}g)$ if it is Gâteaux differentiable at each $u \in \mathrm{int}(\mathrm{dom}g)$. The function g is said to be *Fréchet differentiable* at u if the limt (2.1) is attained uniformly in $\|v\| = 1$. The function g is said to be *uniformly Fréchet differentiable on a subset* U of W if the limit (2.1) is attained uniformly for $u \in U$ and $\|v\| = 1$.

Remark 2.1. ([1], Theorem 1) Let W be a real reflexive Banach and $g : W \longrightarrow (-\infty, +\infty]$ be uniformly Fréchet differentiable on W. Then g is uniformly continuous on W.

Definition 2.1. ([18], p.509) Let W be a Banach space. The function $g : W \longrightarrow (-\infty, +\infty]$ is said to be *bounded on bounded subsets* of W if for any bounded subset U of W, then $g(U)$ is a bounded set.

By combining [7, Proposition 1.1.10] and [7, Proposition 1.1.11], we get the following result.

Proposition 2.1. ([7], Proposition 1.1.10 and Proposition 1.1.11) *Let* $g : W \longrightarrow \mathbb{R}$ *be a Gâteaux differentiable and lower semi-continuous convex function. Then* g *is bounded on bounded sets if and only if* ∇g *is bounded on bounded sets.*

Proposition 2.2. ([32], Proposition 1) *Let* W *be a real reflexive Banach space, and* $g : W \longrightarrow (-\infty, +\infty]$ *be uniformly Fréchet differentiable and bounded on bounded subsets of* W*. Then* ∇g *is uniformly continuous on bounded subsets of* W *from the strong topology of* W *to the strong topology of* W^**.*

Let $u \in \mathrm{int}(\mathrm{dom}g)$, the subdifferential g at $u \in W$ is defined by

$$\partial g(u) = \{u^* \in W^* : g(u) + \langle u^*, v - u\rangle \le g(v) \ \text{ for all } \ v \in W\},$$

and the Fenchel conjugate of g is the function $g^* : W^* \longrightarrow (-\infty, +\infty]$ defined by

$$g^*(u^*) = \sup\{\langle u^*, u\rangle - g(u) : u \in W\}$$

for all $u^* \in W^*$. Note that if $g : W \longrightarrow (-\infty, +\infty]$ is a proper, lower semi-continuous function, then $g^* : W^* \longrightarrow (-\infty, +\infty]$ is a proper weak* lower semi-continuous and convex function, then $g^* : W^* \longrightarrow (-\infty, +\infty]$ is a proper weak* lower semi-continuous and convex function. In addition, $g(u) + g^*(u^*) \leq \langle u^*, u\rangle$ for all $(u, u^*) \in W \times W^*$. Furthermore, it follows from [18] that $(u, u^*) \in \partial g$ if and only if $g(u) + g^*(u^*) = \langle u^*, u\rangle$.

Next, we recall some basic notions and results concerning a Legendre function function for our main results. More information on Legendre functions can be found in the references, for example [28].

Definition 2.2. ([10], Definition 2.2) Let W be a real reflexive Banach and $g : W \longrightarrow (-\infty, +\infty]$ be a function. Then g is said to be *Legendre* if the following two conditions are satisfied.

(1) $\mathrm{Int}(\mathrm{dom}g) \neq \emptyset$, g is Gâteaux differentiable on $\mathrm{int}(\mathrm{dom}g)$ and $\mathrm{dom}(\nabla g) = \mathrm{int}(\mathrm{dom}g)$.

(2) $\mathrm{Int}(\mathrm{dom}g^*) \neq \emptyset$, g^* is Gâteaux differentiable on $\mathrm{int}(\mathrm{dom}g^*)$ and $\mathrm{dom}(\nabla g^*) = \mathrm{int}(\mathrm{dom}g^*)$.

Remark 2.2. ([4]) Let W be a real reflexive Banach space and $g : W \longrightarrow (-\infty, +\infty]$ be a Legendre function. Then

(1) g is a Legendre function if and only if g^* is a Legendre function.

(2) $(\partial f)^{-1} = \partial g^*$.

(3) $\nabla g = (\nabla g^*)^{-1}$, $\mathrm{ran}(\nabla g) = \mathrm{dom}(\nabla g^*) = \mathrm{int}(\mathrm{dom}g^*)$ and $\mathrm{ran}(\nabla g^*) = \mathrm{dom}(\nabla g) = \mathrm{int}(\mathrm{dom}g)$, where $\mathrm{ran}(\nabla g)$ denotes the range of ∇g.

(4) g and g^* are strictly convex on the interior of their respective domains.

Definition 2.3. ([9], p.324) Let W be a real reflexive Banach space, $g : W \longrightarrow (-\infty, +\infty]$ be a Gâteaux differentiable function. Then the function $D_g : \mathrm{dom}g \times \mathrm{int}(\mathrm{dom}g) \longrightarrow [0, +\infty)$, defined by

$$D_g(u, v) = g(u) - g(v) - \langle \nabla g(v), u - v\rangle$$

is said to be the *Bregman distance* with respect to g.

Notice that the Bregman distance is not a distance in the usual sense of the term. In general, $D_g(u, u) = 0$, but $D_g(u, v) = 0$ may not imply $u = v$; D_g is not symmetric and does not satisfy the triangle inequality. By the definition of the Bregman distance, the Bregman distance has the following properties. Note that more information on Bregman functions and distances can be found in the references, for example [29].

(1) For any $u, v \in \mathrm{int}(\mathrm{dom}g)$, we have

$$D_g(u, v) + D_g(v, u) = \langle \nabla g(u) - \nabla g(v), u - v\rangle.$$

(2) For any $u \in \text{dom}g$ and $v, w \in \text{int}(\text{dom}g)$, we have

$$D_g(u,v) + D_g(v,w) - D_g(u,w) = \langle \nabla g(w) - \nabla g(v), u - v.\rangle.$$

(3) For any $u, w \in \text{dom}g$ and $v, z \in \text{int}(\text{dom}g)$, we have

$$D_g(u,v) - D_g(u,z) - D_g(w,v) + D_g(w,z) = \langle \nabla g(z) - \nabla g(v), u - w\rangle.$$

Let $g : W \longrightarrow \mathbb{R}$ be a Gâteaux differentiable function. Consider $V_g : W \times W^* \longrightarrow [0, +\infty]$ defined by

$$V_g(u, u^*) = g(u) - \langle u^*, u\rangle + g^*(u^*)$$

for all $u \in W$ and $u^* \in W^*$. The following result presents some properties of the function V_g.

Remark 2.3. Let W be a real reflexive Banach space, $g : W \longrightarrow \mathbb{R}$ be a Gâteaux differentiable function. Then

(1) ([18], Lemma 3.2) V_g is nonnegative and $V_g(u, u^*) = D_g(u, \nabla g^*(u^*))$ for all $u \in W$ and $u^* \in W^*$.

(2) ([18], Lemma 3.3) For any $u \in W$ and $u^*, v^* \in W^*$, we have

$$V_g(u, u^*) + \langle \nabla g^*(u^*) - u, v^*\rangle \leq V_g(u, u^* + v^*).$$

(3) ([17], p.7) V_g is convex in the second variable. Therefore, for all $u \in W$, we have

$$D_g\Big(u, \nabla g^*\big(\sum_{n=1}^{m} \lambda_n \nabla g(u_n)\big)\Big) \leq \sum_{n=1}^{m} \lambda_n D_g(u, u_n),$$

where $\{u_n\}_{n=1}^m \subset W$ and $\{\lambda_n\}_{n=1}^m \subset [0,1]$ with $\sum\limits_{n=1}^{m} \lambda_n = 1$.

Definition 2.4. ([7], p.69) Let W be a real reflexive Banach space, $g : W \longrightarrow (-\infty, +\infty]$ is a convex and Gâteaux differentiable function, and U be a nonempty, closed and convex subset of $\text{int}(\text{dom}g)$. The *Bregman projection* of $u \in \text{int}(\text{dom}g)$ onto U is the unique vector $P_U^g(u) \in U$ such that

$$D_g\big(P_U^g(u), u\big) = \inf \big\{D_g(v, u) : v \in U\big\}.$$

Remark 2.4. ([23], Remark 2.2) Let W be a smooth, strictly convex Banach space and $g(u) = \|u\|^2$ for all $u \in W$. Then $\nabla g(u) = 2Ju$ for all $u \in W$ and J is the normalized duality mapping which is defined by $J(u) = \{u^* \in W^* : \langle u, u^*\rangle = \|u\|^2 = \|u\|^*\}$ for all $u \in W$. Therefore, Bregman distance $D_g(u, v)$ is reduced to $\phi(u, v)$, where $\phi(u, v)$ is a Lyapunov function which is defined by $\phi(u, v) = \|u\|^2 - 2\langle u, Jv\rangle + \|v\|^2$. Thus, the Bregman projection $P_U^g(u)$ is reduced to the generalized projection $\Pi_U(u)$ in smooth Banach which is defined by

$$\phi\big(\Pi_U(u), u\big) = \min \big\{\phi(v, u) : v \in U\}.$$

If W is a Hilbert space and $g(u) = \|u\|^2$ for all $u \in W$, then $D_g(u, v) = \|u - v\|^2$ for all $u, v \in W$, and J is the identity mapping. Therefore, the Bregman projection $P_U^g(u)$ is reduced to the metric projection from W onto U.

Next, we recall some basic notions and results concerning a totally convex function for our main results. More information on totally convex functions can be found in the references, for example [6].

Definition 2.5. ([33], p.1) Let W be a real reflexive Banach space, $g : W \longrightarrow (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. Then

(1) g is said to be *totally convex* at $u \in \operatorname{int}(\operatorname{dom} g)$ if any $t > 0$, we have

$$v_g(u,t) := \inf\left\{D_g(v,u) : v \in \operatorname{dom} g, \|v-u\| = t\right\} > 0.$$

(2) g is said to be *totally convex* if g is totally convex at every point $u \in \operatorname{int}(\operatorname{dom} g)$.

(3) g is said to be *totally convex on bounded subsets of* W if any nonempty bounded subset B of W and $t > 0$, we have

$$v_g(B,t) := \inf\left\{v_g(u,t) : u \in B \cap \operatorname{dom} g\right\} > 0.$$

Proposition 2.3. ([33], Proposition 2.2) *Let W be a real reflexive Banach space, $g : W \longrightarrow (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. Then g is totally convex at $u \in \operatorname{int}(\operatorname{dom} g)$ if and only if any $\{v_n\} \subset \operatorname{dom} g$ such that $\lim\limits_{n\to\infty} D_g(v_n,u) = 0$, we have $\lim\limits_{n\to\infty} \|v_n - u\| = 0$.*

Proposition 2.4. ([7], Lemma 2.1.2) *Let W be a real reflexive Banach space, $g : W \longrightarrow (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. Then g is totally convex on bounded subsets of W if and only if any sequence $\{u_n\} \subset \operatorname{int}(\operatorname{dom} g)$ and $\{v_n\} \subset \operatorname{dom} g$ such that $\{u_n\}$ is bounded and $\lim\limits_{n\to\infty} D_g(v_n,u_n) = 0$, we have $\lim\limits_{n\to\infty} \|v_n - u_n\| = 0$.*

Proposition 2.5. ([31], Lemma 1) *Let W be a real Banach space, $g : W \longrightarrow (-\infty, +\infty]$ be Gâteaux differentiable and totally convex, $u_0 \in \operatorname{int}(\operatorname{dom} g)$ and the sequence $\{u_n\} \subset \operatorname{dom} g$ satisfying $\{D_g(u_n,u_0)\}$ is bounded. Then the sequence $\{u_n\}$ is bounded.*

Proposition 2.6. ([34], Proposition 2.3) *Let W be a real Banach space, $g : W \longrightarrow \mathbb{R}$ be Legendre such that ∇g^* is bounded on bounded subsets of $\operatorname{int}(\operatorname{dom} g^*)$, $u_0 \in W$ and $\{u_n\} \subset W$ satisfying $\{D_g(u_0,u_n)\}$ is bounded. Then the sequence $\{u_n\}$ is bounded.*

Proposition 2.7. ([8], Corollary 4.4) *Let W be a real reflexive Banach space, $g : W \longrightarrow (-\infty, +\infty]$ be Gâteaux differentiable and totally convex on $\operatorname{int}(\operatorname{dom} g)$, U be a nonempty, closed and convex subset of $\operatorname{int}(\operatorname{dom} g)$ and $u \in \operatorname{int}(\operatorname{dom} g)$. Then the following statements are equivalent.*

(1) $w = P_U^g(u)$.

(2) w is the unique vector such that $\langle \nabla g(u) - \nabla g(w), w - v\rangle \geq 0$ for all $v \in U$.

(3) w is the unique vector such that $D_g(v, w) + D_g(w, u) \leq D_g(v, u)$ for all $v \in U$.

Definition 2.6. ([37], p.203, p.207, p.221) Let W be a Banach space and denote by $S_1 = \{u \in W : \|u\| < 1\}$ and $B_\varepsilon = \{u \in W : \|u\| \leq r\}$ for some $r > 0$. Then

(1) $g : W \longrightarrow \mathbb{R}$ is said to be *uniformly convex on bounded subsets of* W if $\rho_\varepsilon(t) > 0$ for all $t, \varepsilon > 0$, where the function $\rho_\varepsilon : [0, +\infty) \longrightarrow [0, +\infty)$ is defined by

$$\rho_\varepsilon(t) = \inf_{u,v\in B_\varepsilon, \|u-v\|=t, \eta\in(0,1)} \frac{\eta g(u) + (1-\eta)g(v) - g(\eta u + (1-\eta)v)}{\eta(1-\eta)}.$$

(2) $g : W \longrightarrow \mathbb{R}$ is said to be *uniformly smooth on bounded subsets of* W if $\lim\limits_{t\to 0} \frac{\sigma_\varepsilon(t)}{t} = 0$ for all $\varepsilon > 0$, where the function $\sigma_\varepsilon : [0, +\infty) \longrightarrow [0, +\infty)$ is defined by

$$\sigma_\varepsilon(t) = \sup_{u\in B_\varepsilon, v\in S_1, \eta\in(0,1)} \frac{\eta g(u + (1-\eta)tv) + (1-\eta)g(u - \eta tv) - g(u)}{\eta(1-\eta)}.$$

Note that if g is uniformly convex, then the function ρ_ε is nondecreasing mapping. Furthermore, $\rho_\varepsilon(t) = 0$ if and only if $t = 0$ (see [37, page 203]).

Remark 2.5. ([21], p.6) The function g is totally convex on bounded subsets of W if and only if g is uniformly convex on bounded subsets of W.

Definition 2.7. ([16], Definition 1.3.7) Let W be a Banach space and $g : W \longrightarrow (-\infty, +\infty]$ be a function. Then

(1) g is said to be *coercive* if $\lim\limits_{\|u\|\to+\infty} g(u) = +\infty$.

(2) g is said to be *strongly coercive* if $\lim\limits_{\|u\|\to+\infty} \dfrac{g(u)}{\|u\|} = +\infty$.

Proposition 2.8. ([37], Proposition 3.6.3) *Let W be a real reflexive Banach space and $g : W \longrightarrow \mathbb{R}$ be a convex function which is strongly coercive. Then the following statements are equivalent.*

(1) g is bounded on bounded subsets of W and uniformly smooth on bounded subsets of W.

(2) g is Fréchet differentiable and ∇g is uniformly continuous on bounded subsets of W.

(3) $\mathrm{Dom}(g^*) = W^*$, *g^* is strongly coercive and uniformly convex on bounded subsets of W^*.*

Proposition 2.9. ([37], Proposition 3.6.4) *Let W be a real reflexive Banach space and $g : W \longrightarrow \mathbb{R}$ be a convex function which is bounded on bounded subsets of W. Then the following statements are equivalent.*

(1) g is strongly coercive and uniformly convex on bounded subsets of W.

(2) $\mathrm{Dom}(g^*) = W^*$, g^* is bounded on bounded subsets of W^* and uniformly smooth on bounded subsets of W^*.

(3) $\mathrm{Dom}(g^*) = W^*$, g^* is Fréchet differentiable and ∇g^* is uniformly continuous on bounded subsets of W^*.

Lemma 2.1. ([21], Lemma 2.2) *Let W be a Banach space, $\varepsilon > 0$ and $g : W \longrightarrow \mathbb{R}$ be convex on W and uniformly convex on bounded subsets of W. Then*

$$g\Big(\sum_{n=1}^{m} a_n u_n\Big) \leq \sum_{n=1}^{m} a_n g(u_n) - a_i a_j \rho_\varepsilon\big(\|u_i - u_j\|\big)$$

where $i, j \in \{1, 2, \ldots, m\}$, $u_n \in B_\varepsilon = \{u \in W : \|u\| \leq \varepsilon\}$ and $a_n \in (0, 1)$ such that $\sum\limits_{n=1}^{m} a_n = 1$, and the ρ_ε is defined as in Definition 2.6.

By using Lemma 2.1, we get the following result.

Lemma 2.2. *Let W be a real reflexive Banach space, $g : W \longrightarrow \mathbb{R}$ be a Legendre, strongly coercive function which is uniformly Fréchet differentiable and bounded on bounded subsets of W. Then*

$$D_g\Big(u, \nabla g^*\big(\sum_{n=1}^{m} a_n \nabla g(u_n)\big)\Big) \leq \sum_{n=1}^{m} a_n D_g(u, u_n) - a_i a_j \rho^*_\varepsilon\big(\|\nabla g(u_i) - \nabla g(u_j)\|\big),$$

where $i, j \in \{1, 2, \ldots, m\}$, $\nabla g(u_n) \in B^*_\varepsilon = \{u \in W^* : \|u\| \leq \varepsilon\}$ and $a_n \in [0, 1]$ such that $\sum\limits_{n=1}^{m} a_n = 1$, and the ρ^*_ε is defined as in Definition 2.6.

We denote by $F(H) = \{u \in W : Hu = u\}$ the set of fixed points of the mapping $H : W \longrightarrow W$. Next, we recall some notions of the mappings with respect to the Bregman distance for our main results. More information on various classes of Bregman nonexpansive operators can be found in the references, for example [20].

Definition 2.8. Let W be a real reflexive Banach space, $g : W \longrightarrow \mathbb{R}$ be a Gâteaux differentiable function and $H : W \longrightarrow W$ be a mapping. Then

(1) ([5], Definition 2) H is said to be a *Bregman quasi-nonexpansive mapping* if $F(H) \neq \emptyset$ and for all $u \in W$ and $p \in F(H)$, we have $D_g(p, Hu) \leq D_g(p, u)$.

(2) ([38], Definition 2.10) H is said to be a *Bregman quasi-asymptotically nonexpansive mapping* if $F(H) \neq \emptyset$ and there exists a real sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that $D_g(p, H^n u) \leq k_n D_g(p, u)$ for all $u \in W$ and $p \in F(H)$.

(3) ([10], Definition 2.10) H is said to be a *Bregman totally quasi-asymptotically nonexpansive mapping* if $F(H) \neq \emptyset$ and there exist nonnegative real sequences $\{\eta_n\}$, $\{\mu_n\}$ with $\lim_{n\to\infty} \eta_n = \lim_{n\to\infty} \mu_n = 0$ and a strictly increasing continuous function $\xi : [0, \infty) \longrightarrow [0, \infty)$ with $\xi(0) = 0$ such that

$$D_g(u, H^n x) \leq D_g(u, x) + \eta_n \xi(D_g(u, x)) + \mu_n \tag{2.2}$$

for all $u \in W$ and $p \in F(H)$.

(4) ([5], Definition 2) H is said to be a *Bregman firmly nonexpansive mapping* if for all $u, v \in W$, we have

$$\langle \nabla g(Hu) - \nabla g(Hv), Hu - Hv \rangle \leq \langle \nabla g(u) - \nabla g(v), Hu - Hv \rangle.$$

(5) H is said to be *closed* if any sequence $\{u_n\}$ in W such that $\lim_{n\to\infty} u_n = u \in W$ and $\lim_{n\to\infty} Hu_n = v \in W$, we have $Hu = v$.

(6) ([27], p.3877) H is said to be *uniformly asymptotically regular* on W if for any bounded subset U of W, we have $\lim_{n\to\infty} \sup_{u\in U} \|H^{n+1}u - H^n u\| = 0$.

Remark 2.6. (1) Every Bregman quasi-asymptotically nonexpansive mapping is a Bregman totally quasi-asymptotically nonexpansive mapping with $\xi(t) = t$ for all $t \geq 0$, $\eta_n = k_n - 1$ with $k_n \geq 1$ satisfying $\lim_{n\to\infty} k_n = 1$, and $\mu_n = 0$, but the converse is not true.

(2) Every Bregman firmly nonexpansive mapping is a Bregman quasi-nonexpansive mapping.

Lemma 2.3. ([10], Lemma 2.16) *Let W be a real reflexive Banach space, $g : W \longrightarrow (-\infty, +\infty]$ be a Legendre function which is totally convex on bounded subsets of W, U be a nonempty, closed and convex subset of* $\operatorname{int}(\operatorname{dom} g)$*, $H : U \longrightarrow U$ be a closed and Bregman totally quasi-asymptotically nonexpansive mapping. Then $F(H)$ is a closed and convex subset of U.*

For solving the problem $(GMEP)$, let us assume that F satisfies the following conditions.

(C1) $F(u, u) = 0$ for all $u \in U$.

(C2) F is monotone, that is, $F(u, v) + F(v, u) \leq 0$ for all $u, v \in U$.

(C3) For all $u, v, w \in U$, we have $\limsup_{t\downarrow 0} F(tw + (1-t)u, v) \leq F(u, v)$.

(C4) For each $u \in U$, $v \longmapsto F(u,v)$ is convex and lower semi-continuous.

In order to find the solution of the problem $(GMEP)$, Darvish [12] introduced the notion of mixed resolvent of F. In addition, this notion was studied in [13, 14, 35].

Definition 2.9. ([12], Definition 2.4) Let W be a real reflexive Banach space, U be a nonempty, closed and convex subset of W, $g : W \longrightarrow (-\infty,+\infty]$ be a Gâteaux differentiable function, $F : U \times U \longrightarrow \mathbb{R}$ be a bifunction satisfying the conditions (C_1) - (C_4), $A : U \longrightarrow \mathbb{R}$ be a lower semi-continuous and convex function, $B : U \longrightarrow W^*$ be a continuous monotone mapping. *The mixed resolvent of* F is the operator $\mathrm{Res}^g_{F,A,B} : W \longrightarrow 2^U$ defined by

$$\begin{aligned}\mathrm{Res}^g_{F,A,B}(u) \quad = \quad & \Big\{ w \in U : F(w,v) + A(v) + \langle B(u), v - w\rangle \\ & \quad + \langle \nabla g(w) - \nabla g(u), v - w\rangle \geq A(w) \text{ for all } v \in U \Big\}.\end{aligned}$$

By using a similar idea of [30, Lemma 1], the author of [12, 13] proved that if $g : W \longrightarrow (-\infty,+\infty]$ is a strongly coercive and Gâteaux differentiable function, then $\mathrm{dom}\big(\mathrm{Res}^g_{F,A,B}\big) = W$. We find that the formula of the function $\mathrm{Res}^g_{F,A,B}$ contains the term $B(u)$ for all $u \in W$. Since $\mathrm{dom}B = U \subset W$, the value $B(u)$ does not exist for all $u \in W \backslash U$. Motivated by this confusion, we revise the formula of the function $\mathrm{Res}^g_{F,A,B}$ by replacing the term $B(u)$, $u \in W$ by $B(w)$, $w \in U$. This formula has been stated in [23, Lemm 2.5] as follows

$$\begin{aligned}\mathrm{Res}^g_{F,A,B}(u) \quad = \quad & \Big\{ w \in U : F(w,v) + A(v) + \langle B(w), v - w\rangle \\ & \quad + \langle \nabla g(w) - \nabla g(u), v - w\rangle \geq A(w) \text{ for all } v \in U \Big\}. \end{aligned} \tag{2.3}$$

Next, by using the idea of [30, Lemma 1], we also prove that $\mathrm{dom}\big(\mathrm{Res}^g_{F,A,B}\big) = W$ under some suitable conditions, where the function $\mathrm{Res}^g_{F,A,B}$ is defined by (2.3).

The following lemma presents some properties of the mixed resolvent $\mathrm{Res}^g_{F,A,B}$ which is defined by (2.3). The proof of this lemma is similar to the proof [12, Lemma 2.8]. Furthermore, these results have been studied in [23, Lemm 2.5].

Lemma 2.4. ([12], Lemma 2.8) *Let W be a real reflexive Banach space, U be a nonempty, closed and convex subset of W, $g : W \longrightarrow (-\infty,+\infty]$ be a Legendre function and $F : U \times U \longrightarrow \mathbb{R}$ be a bifunction satisfying the conditions* (C_1) - (C_4). *Then*

(1) $\mathrm{Res}^g_{F,A,B}$ *is a single-valued.*

(2) $\mathrm{Res}^g_{F,A,B}$ *is a Bregman firmly nonexpansive mapping.*

(3) $F(\mathrm{Res}^g_{F,A,B}) = GMEP(F, A, B)$ *with*

$$F(\mathrm{Res}^g_{F,A,B}) = \{u \in U : \mathrm{Res}^g_{F,A,B}(u) = u\}.$$

(4) $GMEP(F, A, B)$ *is a closed and convex subset of* W*.*

(5) For all $p \in F(\mathrm{Res}^g_{F,A,B})$ *and* $u \in W$*, we have*

$$D_g(p, \mathrm{Res}^g_{F,A,B}(u)) + D_g(\mathrm{Res}^g_{F,A,B}(u), u) \leq D_g(p, u).$$

3. Main results

Let $H_i : W \longrightarrow W$ be Bregman totally quasi-asymptotically nonexpansive mapping with nonnegative real sequences $\{\eta_n^{(i)}\}$ and $\{\mu_n^{(i)}\}$ satisfying

$$\lim_{n\to\infty} \eta_n^{(i)} = \lim_{n\to\infty} \mu_n^{(i)} = 0$$

and strictly increasing continuous functions $\xi^{(i)} : [0, \infty) \longrightarrow [0, \infty)$ with $\xi^{(i)}(0) = 0$ for each $i \in I := \{1, 2, ..., N\}$ with $N \in \mathbb{N}$. Put

$$\eta_n = \max\{\eta_n^{(i)} : i \in I\}, \mu_n = \max\{\mu_n^{(i)} : i \in I\}, \text{and } \xi(t) = \max\{\xi^{(i)}(t) : i \in I\}$$

for all $t \geq 0$. Then $\lim\limits_{n\to\infty} \eta_n = \lim\limits_{n\to\infty} \mu_n = 0$, $\xi(0) = 0$, and by (2.2), we obtain

$$D_g(p, H_i^n u) \leq D_g(p, u) + \eta_n \xi(D_g(p, u)) + \mu_n$$

for all $u \in W$ and $p \in \bigcap\limits_{i\in I} F(H_i)$, and for all $i \in I$.

Theorem 3.1. *Let* W *be a real reflexive Banach space, and* U *is a nonempty, closed and convex subset of* W*, and* $g : W \longrightarrow \mathbb{R}$ *be Legendre, strongly coercive on* W*, and* g *be bounded, totally convex, uniformly Fréchet differentiable on bounded subsets of* W*. For each* $k \in K := \{1, 2, \ldots, M\}$ *with* $M \in \mathbb{N}$*,* $F_k : U \times U \longrightarrow \mathbb{R}$ *satisfies the conditions* (C_1) *-* (C_4)*,* $A_k : U \longrightarrow \mathbb{R}$ *is a lower semicontinuous and convex function,* $B_k : U \longrightarrow W^*$ *is a continuous monotone mapping. For each* $i \in I$*,* $H_i : W \longrightarrow W$ *is a closed, uniformly asymptotically regular and Bregman totally quasi-asymptotically nonexpansive mapping with nonnegative real sequences* $\{\eta_n^{(i)}\}$ *and* $\{\mu_n^{(i)}\}$ *satisfying* $\lim\limits_{n\to\infty} \eta_n^{(i)} = \lim\limits_{n\to\infty} \mu_n^{(i)} = 0$ *and strictly increasing continuous function* $\xi^{(i)} : [0, \infty) \longrightarrow [0, \infty)$ *with* $\xi^{(i)}(0) = 0$ *such that* $\mathcal{F} = \Big(\bigcap\limits_{i\in I} F(H_i)\Big) \bigcap \Big(\bigcap\limits_{k\in K} GMEP(F_k, A_k, B_k)\Big)$ *is nonempty and bounded. Let*

$\{u_n\}$ be a sequence generated by

$$\begin{cases} u_1, u_2 \in U, U_1 = U_2 = U \\ v_n = u_n + b_n(u_n - u_{n-1}), n \geq 2 \\ w_n = \nabla g^*\Big(a_{n,0}\nabla g(v_n) + \sum\limits_{i=1}^{N} a_{n,i}\nabla g(H_i^n v_n)\Big) \\ z_n^{(k)} \in U \text{ such that } F_k(z_n^{(k)}, y) + A_k(y) + \langle B_k(z_n^{(k)}), y - z_n^{(k)}\rangle \\ \qquad\qquad + \langle \nabla g(z_n^{(k)}) - \nabla g(w_n), y - z_n^{(k)}\rangle \geq A_k(z_n^{(k)}), \forall y \in U. \\ k_n = \operatorname{argmax}\{D_g(z_n^{(k)}, v_n) : k \in K\} \\ U_{n+1} = \big\{u \in U_n : D_g(u, z_n^{(k_n)}) \leq D_g(u, v_n) + \theta_n\big\} \\ u_{n+1} = P^g_{U_{n+1}}(u_1), \end{cases} \tag{3.1}$$

where $\theta_n = \eta_n \sup\big\{\xi\big(D_g(u, v_n)\big) : u \in \mathcal{F}\big\} + \mu_n$, $\{b_n\} \subset [0,1]$, and $\{a_{n,i}\} \subset [0,1]$ for all $i \in I$ such that $\sum\limits_{i=0}^{N} a_{n,i} = 1$ and $\liminf\limits_{n\to\infty} a_{n,0}a_{n,i} > 0$ for all $i \in I$.

Then the sequence $\{u_n\}$ strongly converges to $p = P^g_{\mathcal{F}}(u_1)$.

Proof. The proof of Theorem 3.1 is divided into following six steps.

Step 1. *We claim that $P^g_{\mathcal{F}}(u_1)$ is well-defined.* Indeed, we conclude from Lemma 2.3 and Lemma 2.4 that $F(H_i)$ and $GMEP(F_k, A_k, B_k)$ are closed and convex sets for all $i \in I$ and $k \in K$. This proves that

$$\mathcal{F} = \Big(\bigcap_{i\in I} F(H_i)\Big) \bigcap \Big(\bigcap_{k\in K} GMEP(F_k, A_k, B_k)\Big)$$

is a closed and convex subset of U. Since $\mathcal{F}$ is a nonempty set, we find that $\mathcal{F}$ is a nonempty, closed and convex subset of U. This fact ensures that $P^g_{\mathcal{F}}(u_1)$ is well-defined.

Step 2. *We claim that $P^g_{U_{n+1}}(u_1)$ is well-defined.* Indeed, we first claim that U_n is closed and convex for all $n \geq 2$ by mathematical induction. Obviously, we have $U_2 = U$ is closed and convex. Now, we assume that U_m is closed and convex for some $m \geq 2$. It follows from the definition of U_{m+1}, we get that

$$\begin{aligned} U_{m+1} \quad = \quad & \big\{u \in U_m : \langle \nabla g(v_m), u - v_m\rangle - \langle \nabla g(z_m^{(k_m)}), u - z_m^{(k_m)}\rangle \\ & \qquad\qquad \leq g(z_m^{(k_m)}) - g(v_m) + \theta_m\big\}. \end{aligned} \tag{3.2}$$

Then by directly checking, we find that U_{m+1} is convex. Furthermore, we conclude from (3.2) and the continuity of $\nabla g(.)$ that U_{m+1} is closed. Therefore, we find that U_{m+1} is closed and convex, and hence U_n is closed and convex for all $n \geq 2$. Combining this with $U_1 = U_2$ is closed and convex, we get that U_n is closed and convex for all $n \in \mathbb{N}$.

Next, we prove by mathematical induction that $\mathcal{F} \subset U_n$ for all $n \geq 2$. Obviously, we obtain $\mathcal{F} \subset U = U_2$. Suppose that $\mathcal{F} \subset U_m$ for some $m \geq 2$. Now, we prove that $\mathcal{F} \subset U_{m+1}$. Assume that $u \in \mathcal{F}$. It follows from $\mathcal{F} \subset U_m$ that $u \in U_m$. By using Remark 2.3(3) and the fact that H_i is a Bregman totally quasi-asymptotically nonexpansive mapping, we get

$$\begin{aligned} D_g(u, w_m) &= D_g\Big(u, \nabla g^*\big(a_{m,0}\nabla g(v_m) + \sum_{i=1}^{\infty} a_{m,i}\nabla g(H_i^m v_m)\big)\Big) \\ &\leq a_{m,0}D_g(u, v_m) + \sum_{i=1}^{\infty} a_{m,i}D_g(u, H_i^m v_m) \\ &\leq a_{m,0}D_g(u, v_m) + \sum_{i=1}^{\infty} a_{m,i}[D_g(u, v_m) + \eta_n\zeta(D_g(u, v_m)) + \mu_m] \\ &= a_{m,0}D_g(u, v_m) + (1 - a_{m,0})[D_g(u, v_m) + \eta_n\zeta(D_g(u, v_m)) + \mu_m] \\ &\leq D_g(u, v_m) + \theta_m. \qquad (3.3) \end{aligned}$$

By definition of the function $\mathrm{Res}^g_{F_k,A_k,B_k}$ as in (2.3), we get that

$$z_m^{(k_m)} = \mathrm{Res}^g_{f_{k_m},A_{k_m},B_{k_m}}(w_m).$$

From Lemma 2.4, we find that $\mathrm{Res}^g_{f_{k_m},A_{k_m},B_{k_m}}$ is a Bregman firmly nonexpansive mapping and hence it is a Bregman quasi-nonexpansive mapping for each $k_m \in J$. Then, by Remark 2.6(2), we conclude that $\mathrm{Res}^g_{f_{k_m},A_{k_m},B_{k_m}}$ is a Bregman quasi nonexpansive mapping. It follows from (3.3) that

$$\begin{aligned} D_g(u, z_m^{(k_m)}) &= D_g(u, \mathrm{Res}^g_{f_{k_m},A_{k_m},B_{k_m}}(w_m) \\ &\leq D_g(u, w_m) \\ &\leq D_g(u, v_m) + \theta_m. \qquad (3.4) \end{aligned}$$

This leads to $u \in U_{m+1}$. It means $\mathcal{F} \subset U_{m+1}$. This imples that $\mathcal{F} \subset U_n$ for all $n \geq 2$. Then, we conclude from $U_1 = U_2$ that $\mathcal{F} \subset U_n$ for all $n \in \mathbb{N}$. Since $\mathcal{F}$ is nonempty, we conclude that U_n is nonempty. By the above, we obtain that U_n is nonempty, closed and convex. Therefore, we find that $P^g_{U_{n+1}}(u_1)$ is well-defined.

Step 3. *We claim that* $\{u_n\}$ *is bounded and* $\lim_{n\to\infty} D_g(u_n, u_1)$ *exists.* Indeed, we conclude from $u_n = P^g_{U_n}(u_1)$ and Proposition 2.7 that

$$D_g(v, u_n) + D_g(u_n, u_1) \leq D_g(v, u_1) \qquad (3.5)$$

for all $v \in U_n$. Suppose $u \in \mathcal{F}$. It follows from $\mathcal{F} \subset U_n$ that $u \in U_n$. By taking $v = u$ in (3.5), we get

$$D_g(u, u_n) + D_g(u_n, u_1) \leq D_g(u, u_1). \qquad (3.6)$$

This leads to $D_g(u_n, u_1) \leq D_g(u, u_1) - D_g(u, u_n) \leq D_g(u, u_1)$, and hence $\{D_g(u_n, u_1)\}$ is bounded. By Proposition 2.5, we find that the sequence $\{u_n\}$ is bounded.

It follows from the definition of U_n that $u_{n+1} = P^g_{U_{n+1}}(u_1) \in U_{n+1} \subset U_n$. By choosing $v = u_{n+1}$ in (3.5), we obtain $D_g(u_{n+1}, u_n) + D_g(u_n, u_1) \leq D_g(u_{n+1}, u_1)$, and hence $D_g(u_n, u_1) \leq D_g(u_{n+1}, u_1) - D_g(u_{n+1}, u_n) \leq D_g(u_{n+1}, u_1)$. This implies that the sequence $\{D_g(u_n, u_1)\}$ is nondecreasing. It follows from the boundedness of the sequence $\{D_g(u_n, u_1)\}$ that the limit $\lim_{n\to\infty} D_g(u_n, u_1)$ exists.

Step 4. *We claim that* $\lim_{n\to\infty} u_n = p \in U$. Indeed, for $m > n$, it follows from the definition of U_n that $u_m = P^g_{U_m}(u_1) \in U_m \subset U_n$. Therefore, by taking $v = u_m$ in (3.5), we obtain $D_g(u_m, u_n) + D_g(u_n, u_1) \leq D_g(u_m, u_1)$. This implies that

$$0 \leq D_g(u_m, u_n) \leq D_g(u_m, u_1) - D_g(u_n, u_1). \tag{3.7}$$

Letting the limit (3.7) as $m, n \to \infty$, and using the existence of the limit $\lim_{n\to\infty} D_g(u_n, u_1)$, we find that

$$\lim_{m,n\to\infty} D_g(u_m, u_n) = 0. \tag{3.8}$$

Then, we conclude from (3.8), the boundedness of $\{u_n\}$ and Proposition 2.4 that

$$\lim_{m,n\to\infty} \|u_m - u_n\| = 0. \tag{3.9}$$

This implies that the sequence $\{u_n\}$ is a Cauchy sequence in U. Since W is a Banach space and U is a closed subset of W, there exists $p \in U$ such that $\lim_{n\to\infty} u_n = p$.

Step 5. *We claim that* $p \in \mathcal{F}$. First, we prove that $p \in \bigcap_{i\in I} F(H_i)$. Indeed, by taking $m = n + 1$ in (3.8) and (3.9), we obtain

$$\lim_{n\to\infty} D_g(u_{n+1}, u_n) = \lim_{n\to\infty} \|u_{n+1} - u_n\| = 0. \tag{3.10}$$

It follows from $u_{n+1} = P^g_{U_{n+1}}(u_1) \in U_{n+1} \subset U_n$ that

$$D_g(u_{n+1}, z_n^{(k_n)}) \leq D_g(u_{n+1}, v_n) + \theta_n. \tag{3.11}$$

We have $\|v_n - u_n\| = b_n\|u_n - u_{n-1}\|$. By combining this with (3.10) and the boundedness of $\{b_n\}$, we obtain $\lim_{n\to\infty} \|v_n - u_n\| = 0$. Since $\lim_{n\to\infty} u_n = p$, we find that $\lim_{n\to\infty} v_n = p$. Therefore, we conclude from $\lim_{n\to\infty} v_n = p$ and $\lim_{n\to\infty} u_{n+1} = p$ that $\lim_{n\to\infty} \|u_{n+1} - v_n\| = 0$. It follows from the definition of D_g that

$$\begin{aligned} |D_g(u_{n+1}, v_n)| &= |g(u_{n+1}) - g(v_n) - \langle \nabla g(v_n), u_{n+1} - v_n\rangle| \\ &\leq |g(u_{n+1}) - g(v_n)| + \|u_{n+1} - v_n\|.\|\nabla g(v_n)\|. \end{aligned} \tag{3.12}$$

Furthermore, by Remark 2.1, we obtain that g is uniformly continuous on W. By using Proposition 2.1, we find that ∇g is bounded on bounded subsets of W. Then, by combining this with the boundedness of $\{v_n\}$, $\lim_{n\to\infty} \|u_{n+1} - v_n\| = 0$ and (3.12), we find that

$$\lim_{n\to\infty} D_g(u_{n+1}, v_n) = 0. \tag{3.13}$$

Suppose that $u \in \mathcal{F}$. By the definition of D_g, we have

$$\begin{aligned} |D_g(u, v_n)| &= |g(u) - g(v_n) - \langle \nabla g(v_n), u - v_n \rangle| \\ &\leq |g(u) - g(v_n)| + \|u - v_n\|.\|\nabla g(v_n)\| \\ &\leq |g(u)| + |g(v_n)| + (\|u\| + \|v_n\|).\|\nabla g(v_n)\|. \end{aligned} \tag{3.14}$$

Then, we conclude from (3.14), the boundedness of $\mathcal{F}$ and $\{v_n\}$, the uniform continuity of g and the boundedness on bounded subsets of ∇g that $|D_g(u, v_n)| < \infty$, and hence the sequence $\{D_g(u, v_n)\}$ is bounded. It follows from $\lim_{n\to\infty} \eta_n = \lim_{n\to\infty} \mu_n = 0$ that

$$\lim_{n\to\infty} \theta_n = \lim_{n\to\infty} \Big(\eta_n \sup \big\{ \xi\big(D_g(u, v_n)\big) : u \in \mathcal{F} \big\} + \mu_n \Big) = 0.$$

By combining (3.11), (3.13) and $\lim_{n\to\infty} \theta_n = 0$, we find that $\lim_{n\to\infty} D_g(u_{n+1}, z_n^{(k_n)}) = 0$. By using the same proof as in that of (3.4), we conclude that

$$D_g(u, z_n^{(k_n)}) \leq D_g(u, v_n) + \theta_n. \tag{3.15}$$

Then, we conclude from the boundedness of $\{D_g(u, v_n)\}$, $\{\theta_n\}$ and (3.15) that the sequence $\{D_g(u, z_n^{(k_n)})\}$ is bounded. By Proposition 2.9, we find that g^* is bounded on bounded subsets of W^*.This implies that ∇g^* is bounded on bounded subsets of W^*. By combining this with the boundedness of $\{D_g(u, z_n^{(k_n)})\}$ and using Proposition 2.6, we find that the sequence $\{z_n^{(k_n)}\}$ is bounded. By combining this with $\lim_{n\to\infty} D_g(u_{n+1}, z_n^{(k_n)}) = 0$, and using Proposition 2.4, we have

$$\lim_{n\to\infty} \|u_{n+1} - z_n^{(k_n)}\| = 0.$$

Then, it follows from (3.10) and $\lim_{n\to\infty} \|u_{n+1} - v_n\| = 0$ that

$$\lim_{n\to\infty} \|u_n - z_n^{(k_n)}\| = \lim_{n\to\infty} \|v_n - z_n^{(k_n)}\| = 0.$$

Since ∇g is uniformly continuous on bounded subsets, we get that

$$\lim_{n\to\infty} \|\nabla g(v_n) - \nabla g(z_n^{(k_n)})\| = 0. \tag{3.16}$$

Furthermore, by using similar arguments as in the proof of (3.13), from

$$\lim_{n\to\infty} \|z_n^{(k_n)} - v_n\| = 0,$$

we obtain

$$\lim_{n\to\infty} D_g(z_n^{(k_n)}, v_n) = 0. \tag{3.17}$$

By the definition of k_n, we find that $\lim_{n\to\infty} D_g(z_n^{(k)}, v_n) = 0$ for each $k \in I$. By combining this with the boundedness of $\{v_n\}$ and using Proposition 2.4, we get that $\lim_{n\to\infty} \|z_n^{(k)} - v_n\| = 0$. Then, it follows from $\lim_{n\to\infty} v_n = p$ that $\lim_{n\to\infty} z_n^{(k)} = p$.

Next, by the definition of $\mathrm{Res}^g_{f_k,A_k,B_k}$, we obtain $z_n^{(k_n)} = \mathrm{Res}^g_{f_{k_n},A_{k_n},B_{k_n}}(w_n)$. Then, by Lemma 2.4(3)&(5), we find that $D_g(u, z_n^{(k_n)}) + D_g(z_n^{(k_n)}, w_n) \leq D_g(u, w_n)$. This leads to

$$D_g(z_n^{(k_n)}, w_n) \leq D_g(u, w_n) - D_g(u, z_n^{(k_n)}). \tag{3.18}$$

By using the same proof as in that of (3.3), we obtain

$$D_g(u, w_n) \leq D_g(u, v_n) + \theta_n. \tag{3.19}$$

It follows from (3.18) and (3.19) that

$$D_g(z_n^{(k_n)}, w_n) \leq D_g(u, v_n) - D_g(u, z_n^{(k_n)}) + \theta_n. \tag{3.20}$$

From the property of the Bregman distance D_g, we obtain

$$\begin{aligned}
& |D_g(u, z_n^{(k_n)}) - D_g(u, v_n)| \\
= \; & | - D_g(z_n^{(k_n)}, v_n) + \langle \nabla g(v_n) - \nabla g(z_n^{(k_n)}), u - z_n^{(k_n)} \rangle | \\
\leq \; & |D_g(z_n^{(k_n)}, v_n)| + \|u - z_n^{(k_n)}\|.\|\nabla g(v_n) - \nabla g(z_n^{(k_n)})\|.
\end{aligned} \tag{3.21}$$

Therefore, we conclude from (3.16), (3.17), (3.21) that

$$\lim_{n\to\infty} |D_g(u, z_n^{(k_n)}) - D_g(u, v_n)| = 0.$$

By using (3.20) and $\lim_{n\to\infty} \theta_n = 0$, we find that $\lim_{n\to\infty} D_g(z_n^{(k_n)}, w_n) = 0$. Moreover, by using (3.19) and the boundedness of $\{D_g(u, v_n)\}$, we get that $\{D_g(u, w_n)\}$ is bounded. It follows from the boundedness on bounded subsets of ∇g^* and Proposition 2.6 that $\{w_n\}$ is bounded. Then, we conclude from Proposition 2.4 and $\lim_{n\to\infty} D_g(z_n^{(k_n)}, w_n) = 0$ that $\lim_{n\to\infty} \|z_n^{(k_n)} - w_n\| = 0$. Then, by $\lim_{n\to\infty} \|z_n^{(k_n)} - v_n\| = 0$, we have $\lim_{n\to\infty} \|w_n - v_n\| = 0$. By using similar arguments as in the proof of (3.13), we get that $\lim_{n\to\infty} D_g(w_n, v_n) = 0$. By combining this with the boundedness of $\{v_n\}$ and using Proposition 2.4, we obtain that $\lim_{n\to\infty} \|w_n - v_n\| = 0$. It follows from the uniform continuous on bounded subsets of ∇g that

$$\lim_{n\to\infty} \|\nabla g(w_n) - \nabla g(v_n)\| = 0. \tag{3.22}$$

Since H_i is a Bregman totally quasi-asymptotically nonexpansive mapping, we obtain

$$D_g(u, H_i^n v_n) \leq D_g(u, v_n) + \eta_n \xi(D_g(u, v_n)) + \mu_n.$$

Then, it follows from the boundedness of $\{D_g(u, v_n)\}$ that $\{D_g(u, H_i^n v_n)\}$ is bounded. By using the boundedness on bounded subsets of ∇g^* and Proposition 2.6, we find that $\{H_i^n v_n\}$ is bounded. Then, we conclude from the boundedness of $\{v_n\}$,

$\{H_i^n v_n\}$ and the uniform continuity on bounded subsets of ∇g that $\{\nabla g(v_n)\}$ and $\{\nabla g(H_i^n v_n)\}$ are bounded in W^*. Put

$$\varepsilon = \max\{\sup_{n\in\mathbb{N}} \|\nabla g(v_n)\|, \sup_{n\in\mathbb{N}} \|\nabla g(H_i^n v_n)\|\}.$$

This leads to $\nabla g(v_n), \nabla g(H_i^n v_n) \in B_\varepsilon^*$. Therefore, by Lemma 2.2, we obtain

$$\begin{aligned} D_g(u, w_n) &= D_g\Big(u, \nabla g^*\big(a_n \nabla g(v_n) + (1-a_n)\nabla g(H_i^n v_n)\big)\Big) \\ &\leq a_n D_g(u, v_n) + (1-a_n) D_g(u, H_i^n v_n) \\ &\quad - a_n(1-a_n)\rho_\varepsilon^*(\|\nabla g(v_n) - \nabla g(H_i^n v_n)\|) \\ &\leq a_n D_g(u, v_n) + (1-a_n)[D_g(u, v_n) + \eta_n \xi(D_g(u, v_n)) + \mu_n] \\ &\quad - a_n(1-a_n)\rho_\varepsilon^*(\|\nabla g(v_n) - \nabla g(H_i^n v_n)\|) \\ &\leq D_g(u, v_n) + \theta_n - a_n(1-a_n)\rho_\varepsilon^*(\|\nabla g(v_n) - \nabla g(H_i^n v_n)\|). \end{aligned}$$

This proves that

$$a_n(1-a_n)\rho_\varepsilon^*(\|\nabla g(v_n) - \nabla g(H_i^n v_n)\|) \leq D_g(u, v_n) - D_g(u, w_n) + \theta_n. \tag{3.23}$$

By the property of the Bregman distance D_g, we have

$$\begin{aligned} |D_g(u, w_n) - D_g(u, v_n)| &= |-D_g(w_n, v_n) + \langle \nabla g(v_n) - \nabla g(w_n), u - w_n\rangle| \\ &\leq |D_g(w_n, v_n)| + \|u - w_n\|.\|\nabla g(v_n) - \nabla g(w_n)\|. \end{aligned} \tag{3.24}$$

Therefore, we conclude from $\lim_{n\to\infty} D_g(w_n, v_n) = 0$, (3.22) and (3.24) that

$$\lim_{n\to\infty} |D_g(u, w_n) - D_g(u, v_n)| = 0.$$

It follows from (3.23) and $\liminf_{n\to\infty} a_n(1-a_n) > 0$ that

$$\lim_{n\to\infty} \rho_\varepsilon^*(\|\nabla g(v_n) - \nabla g(H_i^n v_n)\|) = 0.$$

Now, we claim that $\lim_{n\to\infty} \|\nabla g(v_n) - \nabla g(H_i^n v_n)\| = 0$. Suppose the assertion is false. Then we find that there exist $\varepsilon > 0$ and a subsequence $\{k(n)\}$ of n such that

$$\|\nabla g(v_{k(n)}) - \nabla g(H_i^{k(n)} v_{k(n)})\| \geq \varepsilon.$$

By using the nondecreasing property of ρ_ε^*, we find that

$$\rho_\varepsilon^*(\|\nabla g(v_{k(n)}) - \nabla g(H_i^{k(n)} v_{k(n)})\|) \geq \rho_\varepsilon^*(\varepsilon)$$

for all $n \in \mathbb{N}$. By letting the limit as $n \to \infty$, we have $0 \geq \rho_\varepsilon^*(\varepsilon)$. This contradicts the fact that $\rho_\varepsilon^*(\varepsilon) > 0$. Hence, $\lim_{n\to\infty} \|\nabla g(v_n) - \nabla g(H_i^n v_n)\| = 0$. Since $\nabla g^* = (\nabla g)^{-1}$ is uniformly continuous on bounded subsets, we have $\lim_{n\to\infty} \|v_n - H_i^n v_n\| = 0$. It follows from $\lim_{n\to\infty} v_n = p$ that $\lim_{n\to\infty} H_i^n v_n = p$. We also have

$$\|H_i^{n+1} v_n - p\| \leq \|H_i^{n+1} v_n - H_i^n v_n\| + \|H_i^n v_n - p\|. \tag{3.25}$$

Therefore, since H_i is uniformly asymptotically regular and using (3.25), we find that $\lim_{n\to\infty} H_i^{n+1} v_n = p$. This proves that $\lim_{n\to\infty} H_i(H_i^n v_n) = p$. By using the closedness of H_i, we find that $H_i p = p$, and hence $p \in \bigcap_{i\in I} F(H_i)$.

Next, we claim that $p \in \bigcap_{k\in K} GMEP(F_k, A_k, B_k)$. Indeed, for each $k \in K = \{1, 2, \ldots, M\}$, we have $z_n^{(k)} = \mathrm{Res}^g_{F_k,A_k,B_k}(w_n)$. It follows from (2.3) that

$$F_k(z_n^{(k)}, y)+A_k(y)+\langle B_k(z_n^{(k)}), y-z_n^{(k)}\rangle+\langle \nabla g(z_n^{(k)})-\nabla g(w_n), y-z_n^{(k)}\rangle \geq A_k(z_n^{(k)}), \forall v \in U.$$

By using the condition (C_2), we get

$$\begin{aligned}(3.26)\quad & \langle B_k(z_n^{(k)}), v - z_n^{(k)}\rangle + \langle \nabla g(z_n^{(k)}) - \nabla g(w_n), v - z_n^{(k)}\rangle + A_k(v) - A_k(z_n^{(k)}) \\ \geq\ & -F_k(z_n^{(k)}, v) \geq F_k(v, z_n^{(k)}).\end{aligned}$$

Now, by $\lim_{n\to\infty} \|v_n - w_n\| = 0$ and $\lim_{n\to\infty} v_n = p$, we find that $\lim_{n\to\infty} w_n = p$. Then, from $\lim_{n\to\infty} \|z_n^{(k)} - v_n\| = 0$ and $\lim_{n\to\infty} \|v_n - w_n\| = 0$, we conclude that $\lim_{n\to\infty} \|z_n^{(k)} - w_n\| = 0$. Since ∇g is uniformly continuous on bounded subsets, we obtain

$$\lim_{n\to\infty} \|\nabla g(z_n^{(k)}) - \nabla g(w_n)\| = 0.$$

This implies that

$$(3.27)\qquad \lim_{n\to\infty} |\langle \nabla g(z_n^{(k)}) - \nabla g(w_n), v - z_n^{(k)}\rangle| = 0.$$

Since A_k is lower semi-continuous and $\lim_{n\to\infty} z_n^{(k)} = p$, we find that

$$(3.28)\qquad \liminf_{n\to\infty} A_k(z_n^{(k)}) \geq A_k(p).$$

By the condition (C_4), we get that F_k is lower semi-continuous in the second variable. It follows from $\lim_{n\to\infty} z_n^{(k)} = p$ that

$$(3.29)\qquad \liminf_{n\to\infty} F_k(v, z_n^{(k)}) \geq F_k(v, p).$$

We also have

$$\begin{aligned}(3.30)\quad & |\langle B_k(z_n^{(k)}), v - z_n^{(k)}\rangle - \langle B_k(p), v - p\rangle| \\ =\ & |\langle B_k(z_n^{(k)}) - B_k(p), v\rangle - \langle B_k(z_n^{(k)}), z_n^{(k)}\rangle + \langle B_k(p), p\rangle| \\ \leq\ & |\langle B_k(z_n^{(k)}) - B_k(p), v\rangle| + |\langle B_k(z_n^{(k)}), z_n^{(k)} - p\rangle| + |\langle B_k(z_n^{(k)}) - B_k(p), p\rangle| \\ \leq\ & |\langle B_k(z_n^{(k)}) - B_k(p), v\rangle| + \|B_k(z_n^{(k)})\|.\|z_n^{(k)} - p\| + |\langle B_k(z_n^{(k)}) - B_k(p), p\rangle|.\end{aligned}$$

It follows from (3.30), the continuity of B_k, $B_k(z_n^{(k)}) \in W^*$ and $\lim_{n\to\infty} z_n^{(k)} = p$ that

$$(3.31)\qquad \lim_{n\to\infty} \langle B_k(z_n^{(k)}), v - z_n^{(k)}\rangle = \langle B_k(p), v - p\rangle.$$

Then, by (3.26), (3.27), (3.28), (3.29) and (3.31), we find that

$$\langle B_k(p), v-p\rangle + A_k(v) - A_k(p) \geq F_k(v,p) \tag{3.32}$$

for all $v \in U$. For all $t \in (0,1]$, put $v_t = tv + (1-t)p$. Due to $y, p \in U$ and U is convex, we have $v_t \in U$. Then, by replacing y by v_t in (3.32), we conclude that

$$F_k(v_t, p) + \langle B_k(p), p - v_t\rangle + A_k(p) - A_k(v_t) \leq 0. \tag{3.33}$$

By using the condition (C_1), the convexity in the second variable of F_k and the convexity of A_k and (3.33), we conclude that

$$\begin{aligned} 0 &= F_k(v_t,v_t) = F_k(v_t,v_t) + \langle B_k(p), v_t - v_t\rangle + A_k(v_t) - A_k(v_t) \\ &\leq tF_k(v_t,y) + (1-t)F_k(v_t,p) + t\langle B_k(p), y - v_t\rangle \\ &\qquad +(1-t)\langle B_k(p), p-v_t\rangle + tA_k(y) + (1-t)A_k(p) - A_k(v_t) \\ &= t\big[F_k(v_t,v) + \langle B_k(p), v-v_t\rangle + A_k(v) - A_k(v_t)\big] \\ &\qquad +(1-t)\big[F_k(v_t,p) + \langle B_k(p), p-v_t\rangle + A_k(p) - A_k(v_t)\big] \\ &\leq t\big[F_k(v_t,y) + \langle B_k(p), v-v_t\rangle + A_k(v) - A_k(v_t)\big]. \end{aligned} \tag{3.34}$$

It follows from (3.34) and $t > 0$ that

$$F_k(v_t, v) + \langle B_k(p), v - v_t\rangle + A_k(v) - A_k(v_t) \geq 0. \tag{3.35}$$

Therefore, by the condition (C_3), we have

$$\limsup_{t\downarrow 0} F_k(v_t, v) = \limsup_{t\downarrow 0} F_k(tv + (1-t)p, v) \leq F_k(p,v). \tag{3.36}$$

Since A_k is lower semi-continuous, we get that $-A_k$ is upper semi-continuous. From $\lim\limits_{t\to 0} v_t = \lim\limits_{t\to 0}(tv + (1-t)p) = p$, we find that

$$\limsup_{t\to 0}[-A_k(v_t)] \leq -A_k(p). \tag{3.37}$$

By (3.35), (3.36), (3.37) and $\lim\limits_{t\to 0} v_t = p$, we find that

$$F_k(p,v) + \langle B_k(p), v-p\rangle + A_k(v) - A_k(p) \geq 0.$$

This implies that $p \in \bigcap\limits_{k\in K} GMEP(F_k, A_k, B_k)$. By the above, we conclude that

$$p \in \mathcal{F} = \Big(\bigcap_{i\in I} F(H_i)\Big) \bigcap \Big(\bigcap_{k\in K} GMEP(F_k, A_k, B_k)\Big).$$

Step 6. *We claim that* $p = P^g_{\mathcal{F}}(u_1)$. Indeed, we put $u = P^g_{\mathcal{F}}(u_1)$. We will prove that $u = p$. By $u_n = P^g_{U_n}(u_1)$ and Definition 2.4, we find that

$$D_g(u_n, u_1) \leq D_g(v, u_1) \tag{3.38}$$

for all $v \in U_n$. It follows $u = P^g_{\mathcal{F}}(u_1) \in \mathcal{F}$ and $\mathcal{F} \subset U_n$ that $u \in U_n$. Therefore, by choosing $v = u$ in (3.38), we conclude that

$$D_g(u_n, u_1) \leq D_g(u, u_1) \tag{3.39}$$

We also have

$$\begin{aligned} |D_g(u_n, u_1) - D_g(p, u_1)| &= |g(u_n) - g(p) + \langle \nabla g(u_1), p - u_n \rangle| \\ &\leq |g(u_n) - g(p)| + \|\nabla g(u_1)\|.\|p - u_n\|. \end{aligned} \tag{3.40}$$

It follows from (3.40) as $n \longrightarrow \infty$, $\lim_{n\to\infty} u_n = p$, the uniform continuity of g and the boundedness on bounded subsets of ∇g that $\lim_{n\to\infty} D_g(u_n, u_1) = D_g(p, u_1)$. Therefore, we conclude from (3.39) that $D_g(p, u_1) \leq D_g(u, u_1)$. By definition of u and $p \in \mathcal{F}$, we conclude that $p = u = P^g_{\mathcal{F}}(u_1)$. $\square$

In Theorem 3.1, by choosing $I = \{1\}$, $H_1 = H$, $F_k = F$, $A_k = A$ and $B_k = B$ for all $k \in K = \{1, 2, \ldots, M\}$, we get the following result.

Corollary 3.1. *Let W be a real reflexive Banach space, and U is a nonempty, closed and convex subset of W, and $g : W \longrightarrow \mathbb{R}$ is Legendre, strongly coercive on W, and g is bounded, totally convex, uniformly Fréchet differentiable on bounded subsets of W. Suppose that $F : U \times U \longrightarrow \mathbb{R}$ satisfies the conditions (C_1) - (C_4), $A : U \longrightarrow \mathbb{R}$ is a lower semi-continuous and convex function, $B : U \longrightarrow W^*$ is a continuous monotone mapping. Let $H : W \longrightarrow W$ be a closed, uniformly asymptotically regular and Bregman totally quasi-asymptotically nonexpansive mapping with nonnegative real sequences $\{\eta_n\}$ and $\{\mu_n\}$ satisfying $\lim_{n\to\infty} \eta_n = \lim_{n\to\infty} \mu_n = 0$ and strictly increasing continuous function $\xi : [0, \infty) \longrightarrow [0, \infty)$ with $\xi(0) = 0$ such that $\mathcal{F} = F(H) \cap\ GMEP(F, A, B)$ is nonempty and bounded. Let $\{u_n\}$ be a sequence generated by*

$$\begin{cases} u_1, u_2 \in U, U_1 = U_2 = U \\ v_n = u_n + b_n(u_n - u_{n-1}) \text{ for all } n \geq 2 \\ w_n = \nabla g^*\Big(a_n \nabla g(v_n) + (1 - a_n)\nabla g(H^n v_n)\Big) \\ z_n = \mathrm{Res}^g_{F,A,B}(w_n) \\ U_{n+1} = \big\{u \in U_n : D_g(u, z_n) \leq D_g(u, v_n) + \theta_n\big\} \\ u_{n+1} = P^g_{U_{n+1}}(u_1), \end{cases}$$

where $\theta_n = \eta_n \sup\big\{\xi\big(D_g(u, v_n)\big) : u \in \mathcal{F}\big\} + \mu_n$, $\{b_n\} \subset [0, 1]$, and $\{a_n\} \subset [0, 1]$ such that $\liminf_{n\to\infty} a_n(1 - a_n) > 0$, and the function $\mathrm{Res}^g_{F,A,B}$ is defined as in (2.3). Then the sequence $\{u_n\}$ strongly converges to $p = P^g_{\mathcal{F}}(u_1)$.

Remark 3.1. (1) Since every Bregman quasi-asymptotically nonexpansive mapping is a Bregman totally quasi-asymptotically nonexpansive mapping with $\xi(t) = t$ for all $t \geq 0$, $\eta_n = k_n - 1$ with $k_n \geq 1$ satisfying $\lim_{n\to\infty} k_n = 1$, and $\mu_n = 0$, the conclusions of Theorem 3.1 and Corollary 3.1 hold when H_i is a Bregman quasi-asymptotically nonexpansive mapping for all $i \in I$ and $\theta_n = (k_n - 1)\sup\big\{D_g(u, v_n) : u \in \mathcal{F}\big\}$.

(2) The conclusions of Theorem 3.1 and Corollary 3.1 are satisfied when $(GMEP)$ is replaced by (GEP), (GMP), (MVI) and (EP)

In Theorem 3.1 and Corollary 3.1, when H_i is an identity mapping for all $i \in I$, we obtain the following two corollaries, respectively. Note that the iterative process (3.41) is an improvement of the the iterative process (1.3) in the sense of adding the inertial extrapolation. Therefore, the following result is a generalization of the main result in [35].

Corollary 3.2. *Suppose that W is a real reflexive Banach space, and U is a nonempty, closed and convex subset of W, and $g : W \longrightarrow \mathbb{R}$ is Legendre, strongly coercive on W, and g is bounded, totally convex, uniformly Fréchet differentiable on bounded subsets of W. For each $k \in K := \{1, 2, \ldots, M\}$ with $M \in \mathbb{N}$, $F_k : U \times U \longrightarrow \mathbb{R}$ satisfies the conditions (C_1) - (C_4), $A_k : U \longrightarrow \mathbb{R}$ is a lower semi-continuous and convex function, $B_k : U \longrightarrow W^*$ is a continuous monotone mapping such that $\mathcal{F}_1 = \bigcap\limits_{k \in K} GMEP(F_k, A_k, B_k)$ is nonempty and bounded. Let $\{u_n\}$ be a sequence generated by*

$$
\begin{cases}
u_1, u_2 \in U, U_1 = U_2 = U \\
v_n = u_n + b_n(u_n - u_{n-1}) \text{ for all } n \geq 2 \\
z_n^{(k)} = \mathrm{Res}^g_{F_k, A_k, B_k}(v_n) \\
k_n = \mathrm{argmax}\{D_g(z_n^{(k)}, v_n) : k \in K\} \\
U_{n+1} = \big\{u \in U_n : D_g(u, z_n^{(k_n)}) \leq D_g(u, v_n)\big\} \\
u_{n+1} = P^g_{U_{n+1}}(u_1),
\end{cases}
\tag{3.41}
$$

where $\{b_n\} \subset [0, 1]$ and the function $\mathrm{Res}^g_{F_k, A_k, B_k}$ is defined as in (2.3). Then the sequence $\{u_n\}$ strongly converges to $p = P^g_{\mathcal{F}_1}(u_1)$.

Corollary 3.3. *Let W be a real reflexive Banach space, and U is a nonempty, closed and convex subset of W, and $g : W \longrightarrow \mathbb{R}$ be Legendre, strongly coercive on W, and g be bounded, totally convex, uniformly Fréchet differentiable on bounded subsets of W. Assume that $F : U \times U \longrightarrow \mathbb{R}$ satisfies the conditions (C_1) - (C_4), $A : U \longrightarrow \mathbb{R}$ is a lower semi-continuous and convex function, $B : U \longrightarrow W^*$ is a continuous monotone mapping such that $\mathcal{F}_2 = GMEP(F, A, B)$ is nonempty and bounded. Let $\{u_n\}$ be a sequence generated by*

$$
\begin{cases}
u_1, u_2 \in U, U_1 = U_2 = U \\
v_n = u_n + b_n(u_n - u_{n-1}) \text{ for all } n \geq 2 \\
z_n = \mathrm{Res}^g_{F, A, B}(v_n) \\
U_{n+1} = \big\{u \in U_n : D_g(u, z_n) \leq D_g(u, v_n)\big\} \\
u_{n+1} = P^g_{U_{n+1}}(u_1),
\end{cases}
$$

where $\{b_n\} \subset [0, 1]$ and the function $\mathrm{Res}^g_{F, A, B}$ is defined as in (2.3). Then the sequence $\{u_n\}$ strongly converges to $p = P^g_{\mathcal{F}_2}(u_1)$.

Finally, we give a numerical example to illustrate for the convergence of the mentioned iterations.

Example 3.1. Let $W=\mathbb{R}$, $U=[0,1]$, $g(u)=u^4$, $H_i(u)=\dfrac{u}{2^i}$ for all $u\in W$ and $i=1,2$. Let $B_k(u)=ku$, $A_k(u)=ku^2$ and $F_k(u,v)=k(-u^2+uv)$ for all $u,v\in U$ and $k=1,2$. Then

(1) By directly calculating, we have $\nabla g(u)=4u^3$ for all $u\in W$, $g^*(w)=3\sqrt[3]{\left(\dfrac{w}{4}\right)^4}$ and $\nabla g^*(w)=\sqrt[3]{\dfrac{w}{4}}$ for all $w\in W$.

(2) For all $u,v\in W$, we have

$$\begin{aligned} D_g(u,v) &= F(u)-F(v)-\langle \nabla g(v),u-v\rangle \\ &= u^4-v^4-4v^3(u-v)=u^4+3v^4-4uv^3. \end{aligned}$$

(3) For each $i=1,2$, we obtain $F(H_i)=\{0\}$. Therefore, for all $p\in F(H_i)$ and $u\in U$, we find that

$$D_g(p,H_i^n u)=3(H_i^n u)^4=3\Big(\frac{u}{2^{ni}}\Big)^4\le 3(u)^4=D_g(0,u)=D_g(p,u).$$

This proves that H_i is a Bregman totally quasi-asymptotically nonexpansive mapping with $\eta_n^{(i)}=\mu_n^{(i)}=0$ for all $n\in\mathbb{N}$.

(4) By directly checking, for each $k=1,2$, we find that F_k satisfies the conditions (C_1) - (C_4), and A_k is a lower semi-continuous and convex function, and B_k is a continuous monotone mapping.

(5) Now, we will find the formula of $\mathrm{Res}^g_{F_k,A_k,B_k}$ as in (2.3). Indeed, $w=\mathrm{Res}^g_{F_k,A_k,B_k}(u)$ for all $u\in W$ if and only if

$$F_k(w,v)+A_k(v)+\langle B_k(w),v-w\rangle+\langle \nabla g(w)-\nabla g(u),v-w\rangle\ge A_k(w) \tag{3.42}$$

for all $v\in U$. By substituting F_k,A_k,B_k into (3.42) and by directly calculating, we find that

$$kv^2+(2kw+4w^3-4u^3)v+4u^3w-4w^4-3kw^2\ge 0.$$

Put $h(v)=kv^2+(2kw+4w^3-4u^3)v+4u^3w-4w^4-3kw^2$. We have

$$\Delta=(4kw+4w^3-4u^3)^2.$$

We consider the following two cases.

Case 1. $\Delta>0$. Then the quadratic equation $h(v)=0$ have two solutions as follows.

$$v_1=w \text{ and } v_2=\frac{4u^3-4w^3-3kw}{k}.$$

In oder to $h(v)\ge 0$ for all $v\in U$, we have the following cases.

Case 1.1. $v_1=1$ and $v_2>v_1$. Then $w=v_1=1$, and $v_2=\dfrac{4u^3-3k-4}{k}>1$ and hence $u>\sqrt[3]{k+1}$.

Case 1.2. $v_1=0$ and $v_2<v_1$. Then $w=v_1=0$, and $v_2=\dfrac{4u^3}{k}<0$ and hence $u<0$.

Case 2. $\Delta \le 0$. Then $kw + w^3 = u^3$ and $h(v) \ge 0$ for all $v \in U$. Note that $kw + w^3 = u^3$ if and only if $w = \dfrac{\left(\sqrt[3]{\sqrt{81u^6 + 12k^3} + 9u^3}\right)^2 - \sqrt[3]{12}k}{\sqrt[3]{18}\sqrt[3]{\sqrt{81u^6 + 12k^3} + 9u^3}}$. Since $w \in U$, we have $0 \le kw + w^3 = u^3 \le k + 1$ and hence $0 \le u \le \sqrt[3]{k+1}$. Therefore,

$$\mathrm{Res}^g_{F_k, A_k, B_k}(u) = w = \begin{cases} 0 & \text{if } u < 0 \\ \dfrac{\left(\sqrt[3]{\sqrt{81u^6 + 12k^3} + 9u^3}\right)^2 - \sqrt[3]{12}k}{\sqrt[3]{18}\sqrt[3]{\sqrt{81u^6 + 12k^3} + 9u^3}} & \text{if } 0 \le u \le \sqrt[3]{k+1} \\ 1 & \text{if } u > \sqrt[3]{k+1}. \end{cases}$$

By the above, all assumptions in Theorem 3.1 are satisfied with the given functions

n	Iteration (1.3)	Iteration (3.41) with $b_n = \frac{1}{n}$	Iteration (3.41) with $b_n = \frac{1}{2}$	Iteration (3.41) with $b_n = \frac{9n+2}{10n+2}$
1	1.000000	1.000000	1.000000	1.000000
2	0.792136	0.800000	0.800000	0.800000
3	0.606144	0.530425	0.530425	0.121742
4	0.456147	0.330499	0.296756	0.121742
5	0.342213	0.210390	0.134941	0.091306
6	0.256668	0.139776	0.040525	0.047862
7	0.192502	0.096005	0.040525	0.006481
8	0.144376	0.067314	0.030394	0.006481
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
17	0.010840	0.003690	0.000087	0.000014
18	0.008130	0.002708	0.000018	0.000007
19	0.006097	0.001990	0.000018	0.000001
20	0.004573	0.001464	0.000013	0.000001
21	0.003430	0.001078	0.000008	0.
22	0.002572	0.000795	0.000004	0.
23	0.001929	0.000586	0.000001	0.
24	0.001447	0.000433	0.	0.
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
45	0.000004	0.000001	0.	0.
46	0.000003	0.	0.	0.
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
49	0.000001	0.	0.	0.
50	0.	0.	0.	0.

Table 3.1: Number of iterations of the iterative processes (1.3) and (3.41).

F_k, A_k, B_k, T_i. Therefore, by Theorem 3.1, the sequence $\{u_n\}$ which is defined by (3.1) converges to $0 \in \Big(\bigcap_{i=1}^{2} F(H_i)\Big) \bigcap \Big(\bigcap_{k=1}^{2} GMEP(F_k, A_k, B_k)\Big)$.

Now, we compare the rate of convergence of the iterative process (1.3) and the iterative process (3.41) to 0 which is a solution of the system of $(GMEP)$. Numerical results of the mentioned iterative process es with the initial point $u_1 = 1, u_2 = 0.8$ and the different choices of b_n are presented in Table 3.1.

The above table shows that for given mappings, the iterative process (3.41) has a better convergence rate and requires a smaller number of iterations than the iterative process (1.3).

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