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ON ROTATIONAL INVARIANCE OF HIGHER-ORDER MOMENTS OF REGULAR POLYHEDRA

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Abstract. *This paper covers aspects of establishing a relationship between the highest-order rotation-invariant moments of inertia and the order of symmetry of Platonic polyhedra. Moments of inertia about arbitrary, but centroidal axes are considered. After an introductory part which summarizes the possible applications of higher-order moments of area and inertia, the revision of the already solved two-dimensional version of this problem is presented, in other words the highest-order rotation-invariant moments of area about the origin of regular m-gons are studied. As a continuation, some aspects of the possible decomposition of spatial finite rotations into a sequence of rotations about given axes will be covered which are of great importance by the extension of the two-dimensional problem into three dimensions. Finally, the solution of the 3D problem will be presented, emphasizing the differences between the behavior of the even- and odd-order moments which is present in the 2D case, too, and also between the behaviour of the invariant moments of the tetrahedral and the octahedral or icosahedral solids. As a last part, possible applications will be presented.*

Key words: *moments of area and inertia, higher-order moments, rotation-invariant moments, rotation of inhomogeneous bodies, functionally graded materials.*

1. INTRODUCTION

Application of moments of area and inertia up to the second-order is common in the everyday engineering practice. Applying higher-order moments of area and inertia, however, is not as frequent, even though they might appear in basic engineering problems, such as rotation of objects with inhomogeneous density distribution, stability analysis or bending problem of beams with varying material properties inside their cross-sections etc.

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This paper covers aspects of the rotation-invariance of higher-order moments of inertia of regular polyhedra based on the research of Domokos [1] who established the relationship between the highest-order moment of area about the centroid of a regular polygon and the number of its sides. The main goal of this paper is to extend these results to three dimensions, in other words, to derive the relationship between the highest-order rotation-invariant moment of inertia about arbitrary but centroidal axis and the order of symmetry of the Platonic polyhedra.

First of all, it will be clarified what the authors understand under the concept of rotation-invariant moments. In two dimensions, moments of area are defined as an integral of a polynomial expression over a planar domain and their rotation-invariance is understood in this paper as the value of this integral does not change under any rotation about the centroid of the domain. The axis of rotation is trivial in two dimensions, it has to pass through the centroid and be normal to the domain. Turning to 3D cases, moments of inertia have to be analyzed which are integrals over a spatial domain of polynomial expressions. According to the terminology of this paper, they will be denoted invariant if their value does not change under any rotation about arbitrary but centroidal axes. Note that in 3D the axis of rotation is not as trivial as in planar cases. From now on, when it is clearly specified whether a 2D or 3D case is analyzed, instead of the denominations “moments of area” and “moments of inertia” simply “moments” will be used for short. Similarly, since only the rotational invariance of the moments is examined, “rotational” might be omitted and just the expression “invariant” or “invariance” will be applied.

As a next part of the introduction, some concrete applications of higher-order moments are presented.

Let us start by the most direct application, so the analysis of the rotational movement of objects. Sulikashvili [2] analyzed the rotation of a cube, a cone and a cylinder in a central Newtonian field. The conclusion of the paper is that the third- and fourth-order moments of area have influence on the stability of the rotation.

It was mentioned at the very beginning of the introduction that if material properties, such as Young’s modulus, change inside the cross-section of a structure according to some kind of function, then even by the simple bending of a Euler-Bernoulli beam higher-order moments of area may show up. Materials of this kind are called functionally graded materials. Horgan and Chan [3] analyzed a circular cross-section under torsion, where the shear modulus varied along the radius. If the variation can be described as a polynomial function, by the calculation of the shear stiffness higher-order polar moments will appear. Examples of functionally graded materials will be presented later in this paper, too.

The outline of the paper is as follows: after the short revision of the 2D case and its some basic aspects of extending it to three dimensions (Section 2), highest-order invariant moments of regular polyhedra will be presented in Section 3 and finally examples (Section 4) and conclusion (Section 5) close the paper.

2. THE NECESSARY THEORETICAL BACKGROUNDS

2.1 About the invariance of moments of area

An n-th moment of area in the xy plane about the centroid of A can be defined as follows:

$$\mu_{j,n-j} = \iint_A x^j\, y^{n-j}\, dxdy, \tag{1}$$

where n and j are nonnegative integers, j ≤ n. Since regular m-gons are examined, let A be a symmetric domain in the xy plane having C_m symmetry (in other words, m-fold rotational symmetry). To simplify the problem, instead of integrals over the domain A, a set of points having also C_m symmetry will be analyzed, so instead of the integration we will have a discrete summation as follows:

$$M_{j,n-j} = \sum_{i=0}^{m-1} x_i^j\, y_i^{n-j} = r^n \sum_{i=0}^{m-1} cos^j\left(\varphi_0 + i\frac{2\pi}{m}\right) sin^{n-j}\left(\varphi_0 + i\frac{2\pi}{m}\right). \tag{2}$$

Note that in (2) a change to a polar coordinate system has been made, where r is the distance between each point of the discrete set and the centroid of the set, φ_0 is an initial angle enclosed by the first member of the discrete set and the horizontal x axis. Also note that for this discrete case, instead of μ, the letter M has been introduced. For invariance, it has to be proven that (2) cannot depend on φ_0.

Without getting much into the details, extending the trigonometric power expressions into a weighted sum of either sines or cosines of multiple angles and applying trigonometric product-to-sum formulae, depending on the parity of j and n-j we obtain the following:

$$M_{j,n-j} = r^n \left(c_0 + \sum_{i=0}^{m-1} \sum_{k=1}^{\left[\frac{n+1}{2}\right]} c_k\, cos\left(\left(2k - (n\text{mod}2)\right)\left(\varphi_0 + i\frac{2\pi}{m}\right)\right)\right), \tag{3}$$

for n – j even and

$$M_{j,n-j} = r^n \left(\sum_{i=0}^{m-1} \sum_{k=1}^{\left[\frac{n+1}{2}\right]} c_k\, sin\left(\left(2k - (n\text{mod}2)\right)\left(\varphi_0 + i\frac{2\pi}{m}\right)\right)\right), \tag{4}$$

for n – j odd. In the expressions c_0 and the coefficients ck can be determined based on [4], the square brackets mean the floor function. A very important remark is that we will always get a series of either pure sines or pure cosines.

Note that the order of the summations in (3) or (4) can be interchanged and once the summation with respect to i becomes the inner one, closed formulae for both of the expressions with cosines and sines are provided in [5]. Once we have closed formulae for the summations with respect to i in (3) and (4), we are able to evaluate them and what we get is either constant zero (in this case obviously the summation with respect to k vanishes, too) or m (in this case the summation with respect to k will not vanish and dependence on φ_0 remains. These latter cases are the non-invariant moments, while the former ones are the desired invariant moments.

It can be proven that the non-invariant moments are obtained if n is such that either n itself or positive integers n-2, n-4,… are divisible by m. At this point we claim that nth-order moments are only considered invariant if all of the moments of order n are invariant. Note that this approach verifies the well-known fact that invariant odd-order moments are constant zero (note that in (4) we do not have c_0), however, even-order invariant moments can be nonzero as well.

Let the following table summarize the relationship between m and n (which is now the highest-order invariant moment).

Table 1 The invariance of moments of area of regular polygons

Polygon (m)	Order of the highest invariant moments (n)
Regular triangle (m = 3)	4
Square (m = 4)	any odd-order (highest even: 2)
Regular pentagon (m = 5)	8
Regular hexagon (m = 6)	any odd-order (highest even: 4)
⋮	⋮

2.2 The decomposition of finite spatial rotations

Platonic polyhedra belong to symmetry groups Td, Oh or Ih (tetrahedral, octahedral and icosahedral groups) which groups contain cyclic subgroups of maximum order 3, 4 or 5, respectively. This means that in order to determine the highest-order invariant moment of inertia of a tetrahedron, octahedron (or its dual, the cube) or icosahedron (or its dual, the dodecahedron), the planar cases with three-, four- or fivefold cyclic symmetry, respectively, are good starting points, these case are to be extended into three dimensions.

Without getting much into the details, based on references [6] and [7] it can be proven that an arbitrary finite spatial rotation can be decomposed into a sequence of three consecutive rotations about three different three-, four- or fivefold axes of symmetry of point groups Td, Oh or Ih, which are attached to the rotating body. This means that we are always able to handle a rotation about an arbitrary but centroidal axis as a sequence of rotations about current axes of symmetry. This statement is of key importance when extending the invariance of planar moments into three dimensions.

3. The Invariance of Moments of Inertia

Given a volume V in the xyz coordinate system, its nth-order moments of inertia about any of its centroidal axes can be calculated as follows:

$$\mu_{j,k,n-j-k} = \iiint_V x^j\, y^k z^{n-j-k} dxdydz. \tag{5}$$

As we did before in the planar case, instead of integrating over the continuous domain V, discrete sets of points will be analyzed, in which the points are located at the vertices of a regular polyhedron. Changing the integration to summation leads:

$$M_{j,k,n-j-k} = \sum_{i=0}^{p-1} x_i^j \, y_i^k z_i^{n-j-k}, \tag{6}$$

where p is the number of points in the set, namely, p = 4, 6 (8), 12 (20) for tetrahedral, octahedral or icosahedral symmetry, respectively. Note that here a coordinate transformation also could be performed, i.e., to spherical coordinates. However, developing a solution similar to the 2D case based on spherical coordinates would be much more complicated, so the problem will be analysed from a different approach.

We emphasize that rotational invariance means here that the axis of rotation can be arbitrary but centroidal. As we mentioned before (see previous section), instead of an arbitrary axis of rotation, we are able always to consider a rotation about an axis of symmetry (see Fig. 1).

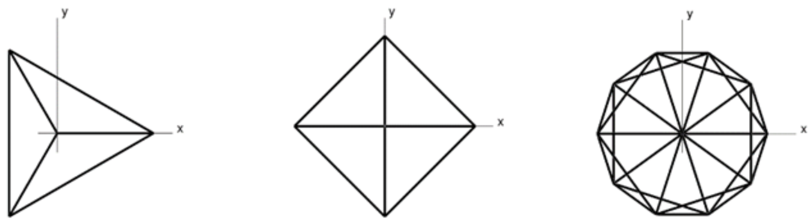

Fig. 1 View of a tetrahedron, an octahedron and an icosahedron from a three-, four- and fivefold axis of symmetry

Let us consider first the case of tetrahedra. In this case our set contains four points and we are able to interpret any rotation about an arbitrary centroidal axis as a sequence of rotations about three out of its four threefold axes of symmetry. Letting the z axis coincide with the current axis of rotation, what we get in the xy plane is nothing else, than a regular triangle, so the planar case corresponding to m = 3 is restored (see Fig. 1). According to Table 1, the highest-order invariant moments are the fourth-order moments, however, the third-order moments (n = m, n divides m) are not invariant, while for n < m, the moments are always invariant. Let us consider the case when the sum of the exponents of x and y is 3. In this case, in the xy plane we obtain a non-invariant moment and since the fourth point on the z axis has no contribution to the moment (x = y = 0), rotation about the z axis with arbitrary exponent of z results in non-invariant moments. So the highest-order rotation-invariant moments of inertia of tetrahedra are the second-order.

Turning to the other two cases, to the octahedral and icosahedral solids, the planar cases m = 4 and m = 5, respectively, can be applied, so what follows immediately from the previous section is that the order of the highest even-order moments of inertia will be the highest even number less that 4 or 5, respectively. Those numbers are 2 and 4 for the two cases (n = 4 is not invariant for m = 4 according to Table 1 and even though n = 6 is invariant for m = 5, with the same way of thinking as we excluded the fourth-order moments from the invariant ones in case of tetrahedra, it can be proven that sixth-order moments are not invariant for icosahedral solids).

Turning to the odd-order moments, according to Table 1 and for m = 4, any odd-order moment will be invariant, this will be true in three dimensions, too. It is enough to consider Fig. 1, with the axis of rotation (now considered to be axis z) pointing towards the reader. In that view there are four nodes at the z = 0 level and other two nodes at the z axis, so evaluating (6) what we get is undoubtedly 0, if the sum of the exponents is odd.

Even though for the case m = 5 the order of the highest invariant moments is finite, we can state that in 3D all odd moments are invariant zero. To see this, consider now the right panel of Fig 1, with the axis z as the axis of rotation pointing towards the reader. We see two quintets of points forming two regular pentagons, at heights -z and z, respectively. If the sum of the exponents of x and y is odd, then an odd-order moment implies the exponent of z to be even which means that the minus sign of z in case of the lower pentagon is not of any importance. Thus, a 10-gon can be analyzed, and for even m all of the odd-order moments are zero. If the exponent of z is odd, however, then the sum of the exponents of x and y must be even. In this case we have a "positive" and a "negative" quintet of points, but since the icosahedron possesses twofold axes of symmetry perpendicular to fivefold ones, there are such twofold axes in the xy plane, too. Hence, in this case the positive and the negative quintets will cancel each other out, so a constant zero moment is obtained.

The following table summarizes the highest-order invariant moments corresponding to each Platonic polyhedra:

Table 2 The highest-order invariant spatial moments corresponding to Platonic polyhedra

Polyhedron	Order of the highest invariant moments
Tetrahedron	2
Octahedron (cube)	any odd-order (highest even: 2)
Icosahedron (Dodecahedron)	any odd-order (highest even: 4)

4. EXAMPLES

In this part some possible engineering applications will be presented. As a first example let us consider simple bending problems of Euler-Bernoulli beams. For the sake of transparency, the formula of the normal stresses from the bending moment will be assumed to have a polynomial distribution of the modulus of elasticity. Let us suppose that the bending moment acts around the y axis and the cross-section is in the yz plane.

$$\sum M_{iy} = M_y = \iint_A \sigma_x \, z dA = \iint_A E\,(y,z)\varepsilon_x z dA = \kappa_y \iint_A E\,(y,z) z^2 dA. \tag{7}$$

Note that here M_y stands for the bending moment and not for the moments, κ_y is the curvature about the y axis. Note also that the Young's modulus E can be taken out from the integral if and only if it is constant inside the cross-section (in this case we obtain the basic formula $M_y = \kappa_y E I_y$), with polynomial distributions of E, higher-order moments appear.

Let us consider a distribution of E (the origin is at the centroid, the y axis points leftwards, the z axis downwards) given by the formula below:

$$E(y,z) = E_1 + E_2 z^2. \tag{8}$$

In this case (7) takes the following form (E_1 and E_2 are known values):

$$M_y = \kappa_y \iint_A (E_1 + E_2 z^2)\, z^2 dA = \kappa_y \big(E_1 I_y + E_2 \mu_{0,0,4}\big), \tag{9}$$

so a fourth-order moment appears as well. Now let us examine two different cross-sections which are a square and a regular hexagon with the above introduced distribution of E (see Fig. 2). It is clear that, based on just the geometry, these cross-sections can only undergo simple bending, they have infinitely many principal directions. However, according to Table 1 we can state that fourth-order moments are not invariant for squares, but it is so for regular hexagons, so the hexagonal cross-section can be bent only simply, the other one, however, can undergo either simple or composite bending, depending on the orientation of the coordinate system. In other words, in case of the hexagonal cross-section, despite the fact that the modulus of elasticity varies along one axis, the bending moment vector should not necessarily be parallel to this axis to generate simple bending.

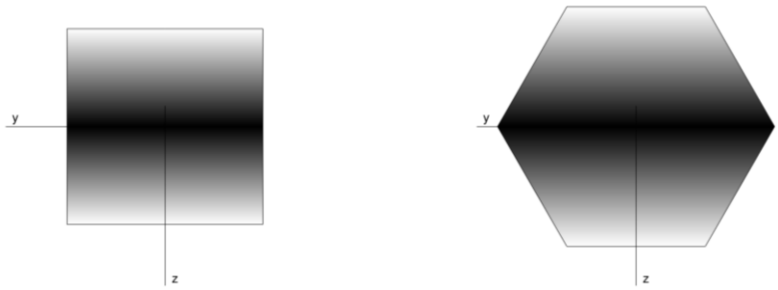

Fig. 2 A square and a hexagonal cross-section with distribution of Young's modulus shown in (8)

Similar examples in 3D also can be presented. Let us consider now rotating solid bodies with the following density distribution:

$$\varrho(x, y, z) = \varrho_0 (x^2 + y^2), \tag{10}$$

where ϱ_0 is a known value. It can be seen that this type of density distribution will cause mathematically fourth-order moments of inertia which, according to Table 2, are non-invariant for octahedra but invariant for icosahedra. It is to mention that the axis of symmetry of the distribution of the density (which is now the z axis) does not have to coincide with an axis of symmetry of the icosahedral body itself, the only requirement of getting an invariant moment is that the centroid must remain at its original position.

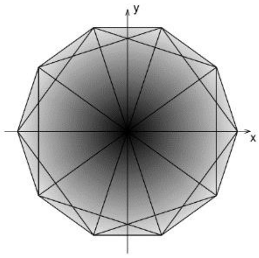

Fig. 3 The density distribution (Eq. (10)) inside an icosahedron

5. CONCLUSION

The main goal of this paper was to establish a relationship between the highest-order rotation-invariant moments of inertia about arbitrary but centroidal axes of Platonic polyhedra and their order of symmetry based on the invariance of planar moments. We can conclude that the order of the invariant spatial moments is only bounded in the case of the tetrahedron (they are invariant up to the second-order), while odd-order moments of inertia of the octahedron or icosahedron (and their duals) are always invariant, namely zero. The order of the invariant even moments, however, are bounded for the octahedron and icosahedron, too (2 and 4, respectively).

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O INVARIJANTNOSTI ROTACIJA MOMENATA VIŠEG REDA KOD PRAVILNIH POLIEDARA

Ovaj rad pokriva aspekte uspostavljanja odnosa između momenata inercije najvišeg reda invarijantnih rotacija i reda simetrije Platonovih poliedara. Razmatraju se momenti inercije oko proizvoljnih, ali centralnih osa. Nakon uvodnog dela koji sumira moguće primene statičkih momenata površine i inercije višeg reda, predstavljena je revizija već rešene dvodimenzionalne *verzije ovog problema, odnosno proučavaju se nepromenjivi momenti invarijantinh rotacija najvišeg reda površine oko koordinantnog početka pravilnih m-uglova. U nastavku će biti obrađeni neki aspekti moguće dekompozicije prostornih konačnih rotacija u niz rotacija oko datih osa koji su od velike važnosti proširenjem dvodimenzionalnog problema na tri dimenzije. Na kraju će biti predstavljeno rešenje 3D problema, naglašavajući razlike između ponašanja momenata parnog i neparnog reda koje je prisutno i u 2D slučaju, kao i između ponašanja invarijantnih momenata tetraedarskih i oktaedarskih ili ikosaedarskih tela. Kao poslednji deo biće predstavljene moguće primene.*

Ključne reči: *moment površine i inercije, moment višeg reda, invarijantni rotacijski moment, rotacija nehomogenih tela, funkcionalno gradirani materijali.*