EULER CHARACTERISTIC OF TANGO BUNDLES

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Abstract

We are interested in a vector bundle constructed by **T**. *The Tango bundle is an indecomposable vector bundle of rank $n-1$ on the complex projective space $\mathbb{P}^n$. In particular, we show that the Euler characteristic of the Tango bundle on $\mathbb{P}^n$ is equal to $2n-1$.*

Keywords: Euler characteristic; Tango bundle.

1.   INTRODUCTION

In this paper, we will use the following notations. Let $\mathbb{P}^n$ be the n-dimensional complex projective space. $\mathscr{O}_{\mathbb{P}^n}(m)$ is the line bundle (vector bundle of rank 1) on $\mathbb{P}^n$ associated with the divisor mh, where h is the hyperplane class on $\mathbb{P}^n$ and m is an integer. $T_{\mathbb{P}^n}$ is the tangent bundle on $\mathbb{P}^n$. For a vector bundle E, we denote by $c(E)$ the total Chern class, by $\mathrm{ch}(E)$ the Chern character, and by $\mathrm{td}(E)$ the Todd class of E.

A vector bundle on $\mathbb{P}^n$ is said to be *indecomposable* if it is not a direct sum of vector bundles of smaller rank. It is difficult to construct indecomposable vector bundles of small rank on the projective space $\mathbb{P}^n$. In fact, the Hartshorne conjecture (see Hartshorne, 1979) asserts that we can not construct such bundles for large dimension n and small rank. More precisely, every bundle of rank 2 on $\mathbb{P}^n$, where $n \geq 7$, splits as a direct sum of line bundles.

An interesting example of an indecomposable vector bundle on $\mathbb{P}^n$ was constructed by Tango (1976). Consider the following exact sequence of vector bundles

$$0 \longrightarrow T_{\mathbb{P}^n}(-2) \longrightarrow F_1 \longrightarrow E_n \longrightarrow 0, \tag{1}$$

where F_1 is the trivial vector bundle of rank $\frac{1}{2}n(n+1)$ and $T_{\mathbb{P}^n}(-2)$ is the vector bundle $T_{\mathbb{P}^n} \otimes \mathscr{O}_{\mathbb{P}^n}(-2)$, and where $\mathscr{O}_{\mathbb{P}^n}(-2)$ is the line bundle on $\mathbb{P}^n$ with the first Chern class $c_1 = -2h$.

It was proved by Tango that E_n is generated by its global sections and that $c_n(E_n) = 0$ (see Tango, 1976). It follows that E_n has a trivial vector bundle F_2 of rank $\frac{1}{2}n(n-1) - n + 1$ as a subbundle. Let T_n be the quotient bundle of E_n by F_2, then T_n is a vector bundle of rank $n-1$. More precisely, the Tango bundle T_n is determined by the following exact sequence

$$0 \longrightarrow F_2 \longrightarrow E_n \longrightarrow T_n \longrightarrow 0. \tag{2}$$

The geometric aspects of Tango bundles were studied by Jaczewski et al. (1986). The aim of this paper is to establish a general formula for the Euler characteristic of Tango bundles. Our main result is the following theorem.

Theorem 1. *Let $n \geq 3$ be an integer. The Euler characteristic of the Tango bundle T_n on $\mathbb{P}^n$ is equal to $2n-1$.*

Our approach is based on the Hirzebruch-Riemann-Roch theorem and the computation of Chern characters of Tango bundles and Todd classes of tangent bundles on projective spaces.
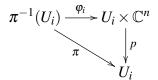
2.   PRELIMINARIES

In this section, we review some basic notions, such as vector bundles, Chern classes, Chern characters, and Todd classes. All definitions and results in this section,

along with the detail explanations and proofs, can be found in Eisenbud and Harris (2016), Fulton (1998), Dang (2014), Hirzebruch (1978), and Okonek et al. (1980).

2.1. Vector bundles and Chern classes

Let X be a variety. A *vector bundle* of rank n on X is a variety E together with a morphism $\pi : E \to X$, such that there is an open covering $\{U_i\}$ of X satisfying the following conditions:

(i) For each $i \in I$, there is an isomorphism $\varphi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}^n$ such that the following diagram commutes,

$$
\begin{array}{ccc}
\pi^{-1}(U_i) & \xrightarrow{\varphi_i} & U_i \times \mathbb{C}^n \\
& \searrow{\scriptstyle \pi} & \downarrow{\scriptstyle p} \\
& & U_i
\end{array}
$$

where $p : U_i \times \mathbb{C}^n \to U_i$ is the natural projection onto the first factor.

(ii) For each $i, j \in I$, there is an $n \times n$-matrix (A_{ij}), whose entries are regular functions on $U_i \cap U_j$, such that the composition

$$
\psi_{ij} = \varphi_j \circ \varphi_i^{-1} : (U_i \cap U_j) \times \mathbb{C}^n \longrightarrow (U_i \cap U_j) \times \mathbb{C}^n
$$

is given by $\psi_{ij}(x, v) = (x, A_{ij}(x)v)$.

The variety E is called the total space, but we often denote the entire vector bundle by the notation for the total space. The vector bundles of rank 1 are called *line bundles*.

Let E be a vector bundle on a variety X. A section of E over an open set $U \subseteq X$ is a morphism $s : U \to E$ such that $\pi \circ s$ is the identity on U. The set of sections of E over U is denoted by $\mathscr{E}(U)$. A section over X is called global.

We say that a vector bundle is globally generated if it generated by its golbal sections. Let X be a smooth variety of dimension n, and E be a globally generated vector bundle of rank r on X. Given a section $s \in \mathscr{E}(X)$ of E, one can show that the zero locus of s is a cycle of codimension r and its cycle class does not depend on the choice of s. We denote

$$
Z(s) = \{x \in X \mid s(x) = 0\}.
$$

The top Chern class of E is defined to be

$$
c_r(E) = [Z(s)].
$$

More generally, given a collection of $r - k + 1$ sections $s_1, \ldots, s_{r-k+1} \in \mathscr{E}(X)$ of E, one can show that the locus where the s_i fail to be independent is a cycle of codimension k and its cycle class does not depend on the choice of $s_1, \ldots, s_{r-k+1}$. We denote

$$
Z(s_1 \wedge \cdots \wedge s_{r-k+1}) = \{x \in X \mid s_1(x), \ldots, s_{r-k+1}(x) \text{ are linearly dependent}\}.
$$

The k-th Chern class of E is defined to be

$$c_k(E) = [Z(s_1 \wedge \cdots \wedge s_{r-k+1})].$$

The total Chern class of E is the sum

$$c(E) = 1 + c_1(E) + \cdots + c_r(E).$$

Here are the basic properties of Chern classes:

(1) $c_0(E) = 1$.

(2) $c_i(E) = 0$ whenever $i > \text{rank}(E)$.

(3) If $E^\vee = \text{Hom}(E, \mathbb{C})$ is the dual of E, then

$$c_i(E^\vee) = (-1)^i c_i(E) \text{ for all } i.$$

(4) If E and F are two vector bundles, then

$$c(E \oplus F) = c(E)c(F).$$

Equivalently, we have

$$c_k(E \oplus F) = \sum_{i=0}^{k} c_i(E)c_{k-i}(F).$$

2.2. Splitting principle

The splitting principle is a useful technique for reducing questions concerning vector bundles to questions concerning line bundles.

Let E be a vector bundle of rank r on a variety X. The splitting principle says that we can regard the Chern classes of E as the elementary symmetric polynomials in r variables α_i for all $i = 1, \ldots, r$, which are called the Chern roots of E. More precisely, we have

$$\begin{aligned}
c_0(E) &= 1, \\
c_1(E) &= \sum_{1 \le i \le r} \alpha_i, \\
c_2(E) &= \sum_{1 \le i < j \le r} \alpha_i \alpha_j, \\
&\vdots \\
c_r(E) &= \alpha_1 \alpha_2 \ldots \alpha_r.
\end{aligned}$$

2.3. Chern characters and Todd classes

Let E be a vector bundle of rank r on a variety X. We define formally the Chern character and the Todd class of E as follows:

$$\mathrm{ch}(E) = \sum_{i=1}^{r} \exp(\alpha_i),$$

and

$$\mathrm{td}(E) = \prod_{i=1}^{r} \frac{\alpha_i}{1 - \exp(-\alpha_i)},$$

where $\alpha_1, \ldots, \alpha_r$ are the Chern roots of E and the expressions in the α_i are understood as formal power series, that is

$$\exp(\alpha_i) = 1 + \alpha_i + \frac{1}{2}\alpha_i^2 + \frac{1}{6}\alpha_i^3 + \cdots,$$

and

$$\frac{\alpha_i}{1 - \exp(-\alpha_i)} = 1 + \frac{1}{2}\alpha_i + \frac{1}{12}\alpha_i^2 + \cdots.$$

Here are some basic properties of Chern characters and Todd classes:

1. If E and F are two vector bundles, then we have the following formulas:

    (a) $\mathrm{ch}(E \otimes F) = \mathrm{ch}(E) \cdot \mathrm{ch}(F)$,

    (b) $\mathrm{ch}(E \oplus F) = \mathrm{ch}(E) + \mathrm{ch}(F)$,

    (c) $\mathrm{td}(E \oplus F) = \mathrm{td}(E) \cdot \mathrm{td}(F)$.

2. The Chern character and the Chern classes of E are related by the following formal identity

$$\mathrm{ch}(E) = r + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \frac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3) + \cdots$$
$$= \sum_{k \geq 0} \frac{p_k}{k!},$$

where $c_i = c_i(E)$ and p_k is the determinant of matrix

$$\begin{pmatrix} c_1 & 1 & 0 & \cdots & 0 \\ 2c_2 & c_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 1 \\ kc_k & c_{k-1} & \cdots & \cdots & c_1 \end{pmatrix}.$$

3. The Chern character and the Todd class of a vector bundle E are related by the following formula:

$$\sum_{i=0}^{r}(-1)^i \operatorname{ch}(\Lambda^i E^{\vee}) = c_r(E) \cdot \operatorname{td}(E)^{-1},$$

where r is the rank of E. This immediately follows using the splitting principle (see Fulton, 1998, Example 3.2.5).

For example, if X is a variety of dimension 3 and E be a vector bundle of rank 2 on X. Then we have

$$\operatorname{ch}(E) = 2 + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \frac{1}{6}(c_1^3 - 3c_1c_2),$$

and

$$\operatorname{td}(E) = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2,$$

where $c_i = c_i(E)$ be i-th Chern class of E.

2.4. Hirzebruch-Riemann-Roch theorem

We now state the Hirzebruch-Riemann-Roch theorem.

Theorem 2 (Hirzebruch-Riemann-Roch). *Let E be a vector bundle on a smooth projective variety X. Then we have the following formula:*

$$\chi(X,E) = \int_X \operatorname{ch}(E) \cdot \operatorname{td}(X),$$

where $\chi(X,E)$ is the Euler characteristic of E on X, $\operatorname{td}(X)$ is the Todd class of the tangent bundle on X, and $\int_X \alpha$ is the degree of the cycle class α in $H^*(X,\mathbb{Q})$.

Proof.

See [**?**, Chapter 4] or [**?**, Chapter 15]. $\qquad\square$

The Hirzebruch-Riemann-Roch formula theorem gives us an effective method for computing the Euler characteristic of a vector bundle E on a smooth projective variety X in terms of the Chern character of E and the Todd class of the tangent bundle T_X on X. In the case of projective spaces, the Chern character and the Todd class of a vector bundle on $\mathbb{P}^n$ can be considered as polynomials in one variable h. Thus the computation can be reduced to polynomials in one variable. For example, let us compute the Chern character of the Tango bundle T_n and the Todd class of the tangent bundle on $\mathbb{P}^n$.

Lemma 1. *Let $n \geq 3$ be an integer. The Chern character of the Tango bundle T_n on $\mathbb{P}^n$ is computed by the following formula:*

$$\operatorname{ch}(T_n) = (n-1) + \sum_{k=1}^{n} \frac{(-2)^k - (-1)^k(n+1)}{k!}h^k,$$

where h is the class of a hyperplane on $\mathbb{P}^n$.

Proof.

We first determine the total Chern classes of the line bundles $\mathscr{O}_{\mathbb{P}^n}(-1)$ and $\mathscr{O}_{\mathbb{P}^n}(-2)$,

$$c(\mathscr{O}_{\mathbb{P}^n}(-1)) = 1 - h, \ c(\mathscr{O}_{\mathbb{P}^n}(-2)) = 1 - 2h.$$

Then we have

$$\mathrm{ch}(\mathscr{O}_{\mathbb{P}^n}(-1)) = \sum_{k=0}^{n} \frac{1}{k!}(-h)^k, \ \mathrm{ch}(\mathscr{O}_{\mathbb{P}^n}(-2)) = \sum_{k=0}^{n} \frac{1}{k!}(-2h)^k.$$

Consider the exact sequence

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^n}(-2) \longrightarrow \mathscr{O}_{\mathbb{P}^n}^{\oplus(n+1)}(-1) \longrightarrow T_{\mathbb{P}^n}(-2) \longrightarrow 0, \tag{3}$$

we have

$$\mathrm{ch}(T_{\mathbb{P}^n}(-2)) = (n+1)\,\mathrm{ch}(\mathscr{O}_{\mathbb{P}^n}(-1)) - \mathrm{ch}(\mathscr{O}_{\mathbb{P}^n}(-2))$$

$$= (n+1) \sum_{k=0}^{n} \frac{1}{k!}(-h)^k - \sum_{k=0}^{n} \frac{1}{k!}(-2h)^k.$$

The exact sequence (1) shows us that

$$\mathrm{ch}(E_n) = \frac{1}{2}n(n+1) - \mathrm{ch}(T_{\mathbb{P}^n}(-2)).$$

Since F_2 is a trivial bundle, $\mathrm{ch}(F_2) = \mathrm{rank}(F_2) = \frac{1}{2}n(n-1) - n + 1$. By the exact sequence (2) we imply that

$$\mathrm{ch}(T_n) = \mathrm{ch}(E_n) - \mathrm{ch}(F_2)$$

$$= (2n-1) - \mathrm{ch}(T_{\mathbb{P}^n}(2))$$

$$= (n-1) + \sum_{k=1}^{n} \frac{(-2)^k - (-1)^k(n+1)}{k!} h^k.$$

$\square$

We have shown that the Chern character of the Tango bundle T_n can be considered as a polynomial in one variable h. Similarly, we can do the same thing with the Todd classes of tangent bundles on projective spaces. To do so, we need some notations regarding Stirling numbers. We refer to Roman (1984) for more details of these numbers.

The Stiring numbers, denoted by $\begin{bmatrix} n \\ k \end{bmatrix}$, where k and n are positive integers such that $0 \le k \le n$, are defined to be the coefficients of the polynomial

$$R(x) = \prod_{i=0}^{n-1}(x+i) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k.$$

For each non-negative integer k, the Stirling polynomial $S_k(x)$ (see Roman, 1984, Sect. 4.8.8) is defined by the exponential generating function

$$\left(\frac{t}{1-e^{-t}}\right)^{x+1} = \sum_{k=0}^{\infty} \frac{S_k(x)}{k!}t^k.$$

The value of the Stirling polynomial $S_k(x)$ at a positive integer n can be calculated by the Stirling number and the binomial coefficient in the following formula

$$S_k(n) = \frac{1}{\binom{n}{k}}\begin{bmatrix} n+1 \\ n-k+1 \end{bmatrix}.$$

With these notations, the Todd class of the tangent bundle on $\mathbb{P}^n$ can be expressed as

$$\mathrm{td}(T_{\mathbb{P}^n}) = \left(\frac{h}{1-e^{-h}}\right)^{n+1} = \sum_{k=0}^{n} \frac{S_k(n)}{k!}h^k. \tag{4}$$

3. PROOF OF THE MAIN RESULT

By the Hirzebruch-Riemann-Roch theorem, the Euler characteristic of the Tango bundle T_n on $\mathbb{P}^n$ is computed as follows

$$\chi(\mathbb{P}^n, T_n) = \int_{\mathbb{P}^n} \mathrm{ch}(T_n).\,\mathrm{td}(T_{\mathbb{P}^n}) = \int_{\mathbb{P}^n} \sum_{k=0}^{n} \frac{p_k}{k!}h^k.\sum_{k=0}^{n} \frac{S_k(n)}{k!}h^k,$$

where $p_0 = n-1$, $p_k = (-2)^k - (-1)^k(n+1)$, $k = 1, 2, \ldots$. It implies that

$$\chi(\mathbb{P}^n, T_n) = \sum_{k=0}^{n} \frac{p_k}{k!}\cdot\frac{S_{n-k}(n)}{(n-k)!}$$

$$= (n-1) + \sum_{k=1}^{n} \frac{p_k}{k!}\cdot\frac{S_{n-k}(n)}{(n-k)!}$$

$$= (n-1) + \frac{1}{n!}\sum_{k=1}^{n} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} p_k.$$

It remains to show that $\sum_{k=1}^{n} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} p_k = n!n$. By the definition of Stirling numbers, we have

$$xR(x+1) = x(x+1)(x+2)\ldots(x+n)$$

$$= \sum_{k=0}^{n+1} \begin{bmatrix} n+1 \\ k \end{bmatrix} x^k$$

$$. = \sum_{k=0}^{n} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} x^{k+1}.$$

Thus

$$R(x+1) = \sum_{k=0}^{n} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} x^k.$$

We can calculate directly $\begin{bmatrix} n+1 \\ 1 \end{bmatrix} = n!$ and then we have

$$\sum_{k=1}^{n} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} x^k = R(x+1) - n!.$$

Since $R(-1) = R(0) = 0$, so we have

$$\sum_{k=1}^{n} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} p_k = \sum_{k=1}^{n} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} (-2)^k - (n+1) \sum_{k=1}^{n} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} (-1)^k$$

$$= R(-1) - n! - (n+1)(R(0) - n!)$$

$$= n!n.$$

Theorem 1 is proved as desired.

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