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THE WAY TO CREATE A PLANE CREMONA MAP FROM ITS CHARACTERISTIC

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Abstract

We describe how to construct a plane Cremona map from its characteristic, which refers to the degree and the list of base-points with their respective multiplicities, and provide some examples of maps of degrees 2 and 3.

Keywords: Characteristic; Cubic maps; Plane Cremona map; Quadratic maps.

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1. INTRODUCTION

Throughout this paper, we work with $\mathbb{C}$, the field of complex numbers. Let $\mathbb{P}^2$ be the projective plane over $\mathbb{C}$. We fix homogeneous coordinates x, y, z on $\mathbb{P}^2$.
A plane Cremona map $\varphi \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ can be written explicitly as

$$\varphi([x:y:z]) = [\varphi_0(x,y,z) : \varphi_1(x,y,z) : \varphi_2(x,y,z)], \tag{1}$$

where $\varphi_i \in \mathbb{C}[x,y,z]_d$ for any $i = 0,1,2$ are homogeneous polynomials of the same degree d without a common factor.

The base-component-free linear system of plane curves of degee d with the equation

$$\lambda_0\varphi_0(x,y,z) + \lambda_1\varphi_1(x,y,z) + \lambda_2\varphi_2(x,y,z) = 0,$$

where $[\lambda_0 : \lambda_1 : \lambda_2] \in \mathbb{P}^2$, is called the *homaloidal net* $\mathscr{L}_\varphi$ defining φ.

By a slight abuse of notation, we define the base-points of the plane Cremona map φ as the base-points of the homaloidal net $\mathscr{L}_\varphi$, which are the common points (whether proper or infinitely near) that all plane curves of the homaloidal net pass through. We can, moreover, associate to each base point a number, that is the multiplicity of a general plane curve of the homaloidal net at that point. The *characteristic* of φ is a tuple that contains the degree and the list of the base-points with their respective multiplicities.

There are two known methods to describe a plane Cremona map:

- Give the rational map explicity as in (1), and check that it is invertible, for instance, by giving its inverse.
- Give the degree d of the map, the base-points $p_1, \dots, p_r$, and the multiplicities $m_1, \dots, m_r$, respectively. These uniquely determine the map (if it exists) up to automorphisms of $\mathbb{P}^2$.

This paper aims to describe through examples how these methods work. In particular, we use the latter method to create some maps of degrees 2 and 3. Accordingly, Section 2 presents theoretical arguments supporting the use of this method. The former method is used to test our results. Some remarks and computational examples are given in the final section.

2. PRELIMINARIES

We recall some definitions and basic results on algebraic geometry that will be used below. Most results in this section can be found in almost any introduction to algebraic geometry, and for a more in-depth treatment, we suggest sources such as Alberich-Carramiñana (2002), Beauville (1996), Blanc and Calabri (2016), Ciliberto (2019), Déserti (2012), and Dolgachev (2012).

2.1. Birational maps

A *birational map* $\varphi : X \dashrightarrow Y$ between two surfaces is determined by an isomorphism between two Zariski open subsets of X and Y. By a *surface*, we mean a smooth, projective, irreducible algebraic surface over $\mathbb{C}$. The maximal open subset of X where φ is well defined is called the *set of definition* of φ. The complement of the set of definition consists of finitely many points of X, called the *indeterminacy points* of φ. In particular, if the indeterminacy set of φ is empty, then φ is said to be *a birational morphism*.

Some properties of birational maps include:

- The indeterminacy locus of a birational map consists of isolated points (see Beauville, 1996, Section II.4);
- If $\varphi : X \to Y$ is a birational morphism between surfaces, then there is a sequence of blow-ups $\pi_k : X_k \to X_{k-1}$ of points $p_k \in X_k$ (k=1, ... ,r) such that

$$\varphi : X = X_r \xrightarrow{\pi_r} X_{r-1} \to \ldots \to X_1 \xrightarrow{\pi_1} X_0 = Y, \tag{2}$$

  (see Beauville, 1996, Theorem II.11);
- If $\varphi : X \dashrightarrow Y$ is a birational map between surfaces, then there is a surface Z and birational morphisms $\pi_X : Z \to X$ and $\pi_Y : Z \to Y$, which are sequences of blow-ups of points, such that the following diagram

$$\begin{array}{ccccc} & & Z & & \\ & \swarrow^{\pi_X} & \circlearrowleft & \searrow^{\pi_Y} & \\ X & & \overset{\varphi}{\dashrightarrow} & & Y. \end{array}$$

  commutes (see Beauville, 1996, Corollary II.12).

2.2. Bubble space of $\mathbb{P}^2$

We denote by $\mathscr{B}(\mathbb{P}^2)$ the *bubble space* of $\mathbb{P}^2$, which is defined as follows. Consider all surfaces X *above* $\mathbb{P}^2$, i.e., all surfaces X such that there exists a birational morphism $X \to \mathbb{P}^2$. If X_1, X_2 are two surfaces above $\mathbb{P}^2$, say $\pi_1 \colon X_1 \to \mathbb{P}^2$ and $\pi_2 \colon X_2 \to \mathbb{P}^2$ are birational morphisms, one identifies $p_1 \in X_1$ with $p_2 \in X_2$ if the birational map $(\pi_2)^{-1}\pi_1 \colon X_1 \dashrightarrow X_2$ is a local isomorphism at p_1 that sends p_1 to p_2. The bubble space $\mathscr{B}(\mathbb{P}^2)$ is the union of all points of all surfaces above $\mathbb{P}^2$ modulo, the equivalence relation generated by these identifications.

For any birational morphism $X \to \mathbb{P}^2$, there is an injective map $X \to \mathscr{B}(\mathbb{P}^2)$. Therefore we will identify points of X with their images in $\mathscr{B}(\mathbb{P}^2)$. One says that $p_1 \in \mathscr{B}(\mathbb{P}^2)$ is *infinitely near* $p_2 \in \mathscr{B}(\mathbb{P}^2)$, say $p_1 \in X_1$ and $p_2 \in X_2$, with birational morphisms $\pi_1 \colon X_1 \to \mathbb{P}^2$ and $\pi_2 \colon X_2 \to \mathbb{P}^2$, if the birational map $(\pi_2)^{-1}\pi_1 \colon X_1 \dashrightarrow X_2$ that is defined at p_1, sends p_1 to p_2, but is not a local isomorphism at p_1. In such a case we write that $p_1 \succ p_2$. Moreover, one says that p_1 is *in the first neighborhood* of p_2, or that p_1 is *infinitely near* p_2 *of* *the first order*, if $(\pi_2)^{-1}\pi_1$ corresponds locally to the blow-up of p_2. In such a case, we write that $p_1 \succ_1 p_2$. If $p_1 \succ p_2$, then one can define the *infinitesimal order* of p_1 with respect to p_2 by induction; namely, if $p_1 \succ_1 p_3$ and $p_3 \succ_k p_2$ for some k, then p_1 is *infinitely near* p_2 *of order* $k+1$. If $p_1 \succ p_2$ and $p_1 \in X_1$, then there is a unique irreducible curve $E_2 \subset X_1$ that corresponds to the exceptional curve of the blow-up of $p_2 \in X_2$. One says that p_1 is *proximate* to p_2 if $p_1 \in E_2$. In such a case, we write that $p_1 \dashrightarrow p_2$. Clearly, if $p_1 \succ_1 p_2$, then $p_1 \dashrightarrow p_2$, but the converse is not always true. If $p_1 \dashrightarrow p_2$ and $p_1 \succ_k p_2$ with $k > 1$, then we say that p_1 is *satellite* to p_2 and write $p_1 \odot p_2$. Otherwise, if p_1 is not satellite to p_2, then we denote by $p_1 \not\odot p_2$. One says that a point $p \in \mathbb{P}^2 \subset \mathscr{B}(\mathbb{P}^2)$ is a *proper point of* $\mathbb{P}^2$. Note that each point of $\mathscr{B}(\mathbb{P}^2) \setminus \mathbb{P}^2$ is infinitely near a unique point of $\mathbb{P}^2$. Note also that if $p_1 \succ_k p_k$, say

$$p_1 \succ_1 p_2 \succ_1 p_3 \succ_1 \cdots \succ_1 p_{k-1} \succ_1 p_k,$$

and $p_1 \dashrightarrow p_k$, then $p_i \dashrightarrow p_k$ for each $i = 2, \ldots, k-1$. Notice that if $p_1 \succ p_2 \in \mathbb{P}^2$ where $p_1 \in X_1$ and $\pi_1 : X_1 \to \mathbb{P}^2$ is a birational morphism, we say that a plane curve C passes through p_1 if C passes through p_2 and the strict transform of C on X_1 via π_1 passes through p_1.

Let $\varphi : X \to \mathbb{P}^2$ be a birational morphism, that is the composition of the blow-ups $\pi_1, \ldots, \pi_r$ such as in (2). Let C be a plane curve, and let C_i be the strict transform of C in S_i for $i = 1, \ldots, r$. Setting $C_0 = C$ and $m_i = \mathrm{mult}_{p_i}(C_{i-1})$ for $i = 1, \ldots, r$, one has, for each $j = 1, \ldots, r$,

$$m_j \geqslant \sum_{p_k \dashrightarrow p_j} m_k.$$

For details, see Alberich-Carramiñana (2002, Section 2.2), Casas-Alvero (2000, Theorem 3.5.3, Corollary 3.5.4), or Dolgachev (2012, Section 7.3.2).

2.3. Linear system of plane curves

Let $p_1, \ldots, p_r$ be points in $\mathscr{B}(\mathbb{P}^2)$, which can be proper or infinitely near. Fix positive integers $m_1, \ldots, m_r$, and consider the linear system

$$\mathscr{L} := \mathscr{L}(d; p_1^{m_1}, \ldots, p_r^{m_r})$$

of plane curves of degree d having multiplicity m_i at p_i, for $1 \leq i \leq r$. If the points are understood, we may simply write

$$\mathscr{L} := \mathscr{L}(d; m_1, \ldots, m_r)$$

and if some multiplicities are repeated, we may sometimes use the exponentation notation, i.e., m_i^j stays for j times m_i. We call $p_1, \ldots, p_r$ the base-points of $\mathscr{L}$ with multiplicities $m_1, \ldots, m_r$, respectively. Since the base-points can be infinitely near, the linear system $\mathscr{L}$ exists on a suitable birational model $\varphi : X \to \mathbb{P}^2$.

2.4. Plane Cremona maps

We recall that a *plane Cremona map* is a birational map between projective planes $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, and that φ is given by three homogeneous polynomials $\varphi_0, \varphi_1, \varphi_2 \in \mathbb{C}[x,y,z]$ of the same degree d, with no common factor, i.e.,

$$\varphi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2, \qquad [x:y:z] \mapsto [\varphi_0(x,y,z):\varphi_1(x,y,z):\varphi_2(x,y,z)].$$

The set of all plane Cremona maps is called the *plane Cremona group* and denoted by $\mathrm{Cr}(\mathbb{P}^2)$.

The associated linear system of $\varphi \in \mathrm{Cr}(\mathbb{P}^2)$ is the preimage of the linear system of lines of $\mathbb{P}^2$ and is the system $\mathscr{L}_\varphi$ of plane curves given by

$$\lambda_0\varphi_0(x,y,z)+\lambda_1\varphi_1(x,y,z)+\lambda_2\varphi_2(x,y,z)=0,$$

where $[\lambda_0:\lambda_1:\lambda_2] \in \mathbb{P}^2$. The integer d is the degree of the polynomial φ_i, the degree of the curves of the linear system, and is by definition the *degree* of φ.

The linear system $\mathscr{L}_\varphi$ is a net, i.e., $\dim(\mathscr{L}_\varphi)=2$, which has the property of being homaloidal, i.e., the pencil of curves of $\mathscr{L}_\varphi$ going through a generic point p in the plane has no other base-point further than p and the base-points of $\mathscr{L}_\varphi$. For that reason, we usually call $\mathscr{L}_\varphi$ the *homaloidal net* defining φ.

Notice that once three generators are chosen, a homaloidal net Λ defines a plane Cremona map ψ. If we choose any other three generators of the homaloidal net Λ, then we have another plane Cremona map ψ', which is the product of ψ by a plane projectivity. Two plane Cremona maps will be considered essentially the same if one of them is the product of the other by a projectivity. In other words, to any plane Cremona map there is associated a homaloidal net of plane curves. Conversely, any homaloidal net of plane curves generates an infinity of plane Cremona maps, any of which is the product of any other by an automorphism of $\mathbb{P}^2$.

Recall that two plane Cremona maps $\varphi, \varphi' \in \mathrm{Cr}(\mathbb{P}^2)$ are *equivalent* if there exist two automorphisms $\alpha, \alpha' \in \mathrm{Aut}(\mathbb{P}^2)$ such that

$$\varphi' = \alpha' \circ \varphi \circ \alpha.$$

Therefore, two plane Cremona maps defined by the same homaloidal net are equivalent.

Recall that the *base-points* of a plane Cremona map φ are the base-points of the homaloidal net $\mathscr{L}_\varphi$ defining φ.

In particular, let $\varphi \in \mathrm{Cr}(\mathbb{P}^2)$ and $\deg(\varphi)=d$. Let $p_1,\dots,p_r$ be all the base-points of φ with respective multiplicities $m_1,\dots,m_r$. Then, it is classically known that:

$$d^2-1=\sum_{i=1}^{r} m_i^2, \qquad 3(d-1)=\sum_{i=1}^{r} m_i, \tag{3}$$

and $(d;m_1,\dots,m_r)$ is called the *characteristic* of φ (see for instance Alberich-Carramiñana, 2002, Section 5.2).

Example 2.1. If $d=1$, there is no base-point. In fact, all automorphisms of $\mathbb{P}^2$ are linear. If $d=2$, we see that $r=3$ and that the multiplicities are $(1,1,1)=(1^3)$. If $d=3$, we see that $r=5$ and that the multiplicities are $(2,1^4)$. If $d=4$, then either $r=7$ and the multiplicities are $(3,1^6)$ or $r=6$ and the multiplicities are $(2^3,1^3)$. For higher degrees, see, for example, Blanc and Calabri (2016).

Example 2.2. If $d\geq 2$ and $m_1=d-1$, then we have $(d;d-1,1^{2d-2})$. The maps satisfying this condition are general cases of *de Jonquières maps* (see Alberich-Carramiñana, 2002, Definition 2.6.10). We note that the Enriques criterion (Alberich-Carramiñana, 2002, Theorem 5.1.1) may be used to prove that a set of $2d-1$ points $p_0,p_1,\dots,p_{2d-2}$ with assigned multiplicities $d-1,1,\dots,1$, satisfying this condition defines a de Jonquières map.

Example 2.3. If $d\geq 4$ and $m_1=d-2$, then $(d;d-2,2^{d-2},1^3)$.

Before giving examples on how to construct some quadratic and cubic plane Cremona maps in the next section, we note that not all non-negative integer solutions of equalities (3) are the characteristic of plane Cremona maps.

Example 2.4. If $d=5$, we have that all possible non-negative integer solutions of equalities (3) are $(5;4,1^8),(5;3,2^3,1^3),(5;2^6)$ and $(5;3^2,1^6)$. However, it turns out that $(5;3^2,1^6)$ is not the characteristic of any quintic plane Cremona map because the linear system associated to such a case would be reducible. (The line through the two points of multiplicity 3 would be a fixed component by Bézout's theorem.)

To decide whether or not a non-negative integer solution $(d;m_1,\dots,m_r)$ of equalities (3) is essentially the characteristic of some plane Cremona map, one can use *Hudson's test* (Alberich-Carramiñana, 2002, Definition 5.2.15, p. 134; Blanc & Calabri, 2016, Section 2.1). Basically, the Hudson's test is as follows:

Let $v=(d;m_1,m_2,\dots,m_r)$ where $m_1\geq m_2\geq\dots\geq m_r\geq 1$ is a non-negative integer solution of equalities (3). Then,

(a) if $m_1+m_2>d$, there is no plane Cremona map such that v is its characteristic;

(b) otherwise, one can easy show that $m_1+m_2+m_3>d$. Define a new tuple,

$$v_1=(d';m'_1,m'_2,m'_3,\dots,m'_r),$$

where

$$\begin{cases} d'=2d-m_1-m_2-m_3<d,\\ m'_1=d-m_2-m_3\geq 0,\\ m'_2=d-m_1-m_3\geq 0,\\ m'_3=d-m_1-m_2\geq 0,\\ m'_i=m_i \text{ for } i\in\{4,\dots,r\}.\end{cases}$$

By dropping the zero entries, reordering, and renaming, one may assume that

$$v_1 = (d'; m'_1, m'_2, \ldots, m'_s),$$

where $s \leq r$ and $m'_1 \geq m'_2 \geq \ldots \geq m'_s \geq 1$. We repeat the test for v_1 and continue. In general, we get a sequence of vectors $\{v_i|_{i=0,1,\ldots}\}$ where $v_0 = v$.

We stop if condition (i) holds at some step, namely, there is no plane Cremona map such that v is its characteristic, or in other words v does not fulfill Hudson's test. Otherwise, we continue until we reach the vector $(2;1^3)$. The process ends here, and v is the characteristic of some plane Cremona map.

3. EXAMPLES

3.1. Some examples of quadratic plane Cremona maps

In this section, we provide examples of three particularly important cases of quadratic plane Cremona maps.

Example 3.1. Let us start by constructing a quadratic plane Cremona map where the base-points are $e_1 = [1:0:0], e_2 = [0:1:0]$ and $e_3 = [0:0:1]$.

Suppose that $\sigma \in \mathrm{Cr}(\mathbb{P}^2)$ is such a map, and it is

$$\sigma\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2, \qquad [x:y:z] \mapsto [\sigma_0(x,y,z) : \sigma_1(x,y,z) : \sigma_2(x,y,z)].$$

where $\sigma_0, \sigma_1, \sigma_2 \in \mathbb{C}[x,y,z]$ are three homogeneous polynomials of degree 2 with no common factor.

A general element $(\mathscr{C})$ of the homoloidal net $\mathscr{L}_\sigma$ defining σ is an irreducible plane curve of the form

$$(\mathscr{C}) : \lambda_0\sigma_0(x,y,z) + \lambda_1\sigma_1(x,y,z) + \lambda_2\sigma_2(x,y,z) = 0, \tag{4}$$

where $[\lambda_0 : \lambda_1 : \lambda_2] \in \mathbb{P}^2$. However, $(\mathscr{C})$ can also be written as

$$(\mathscr{C}) : Ax^2 + Bxy + Cxz + Dy^2 + Eyz + Fz^2 = 0,$$

for some $A, B, C, D, E, F \in \mathbb{C}$ not all zeros.

Since e_1, e_2, e_3 are base-points of $(\mathscr{C})$, it follows that $A = D = F = 0$. In other words, $(\mathscr{C})$ is given by

$$(\mathscr{C}) : Eyz + Cxz + Bxy = 0, \tag{5}$$

for some $[E:C:B] \in \mathbb{P}^2$ and $E, C, B \neq 0$.

By (4) and (5), we consider the following map defined by

$$[x:y:z] \mapsto [yz : xz : xy].$$

The stated rational mapping is actually the ones that we are looking for, because it is invertible. Indeed, one can check that

$$\sigma \circ \sigma = \mathrm{id}_{\mathbb{P}^2}.$$

In other words, σ is an involution map, i.e., its inverse is itself.

Example 3.2. Next, we create a quadratic plane Cremona map such that not all the base-points are proper. In particular, we construct a quadratic plane Cremona map where its base-points are $p_1 = e_1 = [1:0:0], p_2 = e_2 = [0:1:0]$ and the third base-point $p_3 \succ_1 e_1$ in the direction of the line $(l): y = 0$.

Suppose that $\rho \in \mathrm{Cr}(\mathbb{P}^2)$ is such a map, and it is

$$\rho : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2, \qquad [x:y:z] \mapsto [\rho_0(x,y,z) : \rho_1(x,y,z) : \rho_2(x,y,z)].$$

where $\rho_0, \rho_1, \rho_2 \in \mathbb{C}[x,y,z]$ are three homogeneous polynomials of degree 2 with no common factor.

A general element $(\mathscr{C})$ of the homoloidal net $\mathscr{L}_\rho$ defining ρ is an irreducible plane curve of the form

$$(\mathscr{C}) : \lambda_0\rho_0(x,y,z) + \lambda_1\rho_1(x,y,z) + \lambda_2\rho_2(x,y,z) = 0, \tag{6}$$

where $[\lambda_0 : \lambda_1 : \lambda_2] \in \mathbb{P}^2$.

However, $(\mathscr{C})$ can also be written as

$$(\mathscr{C}) : Ax^2 + Bxy + Cxz + Dy^2 + Eyz + Fz^2 = 0,$$

for some $A,B,C,D,E,F \in \mathbb{C}$ not all zero.

Since p_1, p_2 are base-points of $(\mathscr{C})$, it follows that $A = D = 0$. In other words, $(\mathscr{C})$ is given by

$$(\mathscr{C}) : Bxy + Cxz + Eyz + Fz^2 = 0,$$

for some $B,C,E,F \neq 0$.

Since $p_3 \succ_1 p_1$, we work on the affine chart $U_0 = \{[x:y:z] \in \mathbb{P}^2 | x \neq 0\} \simeq \mathbb{C}^2_{\bar{y},\bar{z}}$. The point p_1 corresponds to the point $\bar{p}_1 = (0,0)$, the line (l) is locally defined by

$$\bar{y} = 0$$

and the curve $(\mathscr{C})$ is locally defined by

$$(B\bar{y} + C\bar{z}) + E\bar{y}\bar{z} + F\bar{z}^2 = 0.$$

Since $p_3 \succ_1 p_1$ in the direction of the line $(l): y=0$, we must have $C=0$. It follows that the curve $(\mathscr{C})$ is defined by

$$(\mathscr{C}): Bxy + Fz^2 + Eyz = 0, \tag{7}$$

for some $[B : F : E] \in \mathbb{P}^2$.

By (6) and (7), we can check that the following involution map is the ones that we are looking for:

$$[x : y : z] \mapsto [xy : z^2 : yz].$$

Example 3.3. Let us construct a quadratic plane Cremona map which contains only a proper base-point. Typically, the only proper base-points are $q_1 = e_3 = [0 : 0 : 1]$, the second base-point $q_2 \succ_1 q_1$ in the direction of the line $(l): x=0$ and the third base-point $q_3 \succ_1 q_2 \succ_1 q_1$ in the direction of the conic $(c): y^2 - xz = 0$.

Suppose that $\tau \in \mathrm{Cr}(\mathbb{P}^2)$ is such a map, and

$$\tau\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2, \qquad [x : y : z] \mapsto [\tau_0(x,y,z) : \tau_1(x,y,z) : \tau_2(x,y,z)].$$

where $\tau_0, \tau_1, \tau_2 \in \mathbb{C}[x,y,z]$ are three homogeneous polynomials of degree 2 with no common factor.

A general element $(\mathscr{C})$ of the homoloidal net $\mathscr{L}_\tau$ defining τ is an irreducible plane curve and of the form

$$(\mathscr{C}): \lambda_0\tau_0(x,y,z) + \lambda_1\tau_1(x,y,z) + \lambda_2\tau_2(x,y,z) = 0, \tag{8}$$

where $[\lambda_0 : \lambda_1 : \lambda_2] \in \mathbb{P}^2$.

However, $(\mathscr{C})$ can also be written as

$$(\mathscr{C}): Ax^2 + Bxy + Cxz + Dy^2 + Eyz + Fz^2 = 0,$$

for some $A,B,C,D,E,F \in \mathbb{C}$ not all zero.

Since q_1 is a base-point of $(\mathscr{C})$, it follows that $F=0$. In other words, $(\mathscr{C})$ is given by

$$(\mathscr{C}): Ax^2 + Bxy + Cxz + Dy^2 + Eyz = 0,$$

for some $A,B,C,D,E \neq 0$.

Since $q_2 \succ_1 q_1$, we work on the affine chart $U_2 = \{[x : y : z] \in \mathbb{P}^2 | z \neq 0\} \simeq \mathbb{C}^2_{\bar{x},\bar{y}}$. The point q_1 corresponds to the point $\bar{q}_1 = (0,0)$, the line (l) is locally defined by

$$(\bar{l}): \bar{x} = 0,$$

the conic (c) is locally defined by

$$(\bar{c}): \bar{y}^2 - \bar{x} = 0,$$

and the curve $(\mathscr{C})$ is locally defined by

$$(\bar{\mathscr{C}}) : (C\bar{x}+E\bar{y})+A\bar{x}^2+B\bar{x}\bar{y}+D\bar{y}^2=0.$$

In the second chart of the blow-up of $\mathbb{C}^2_{\bar{x},\bar{y}}$ at $\bar{q}_1=(0,0)$, given in coordinates by $\bar{x}=x_1y_1, \bar{y}=y_1$, the exceptional curve has the local equation

$$(E_1) : y_1=0,$$

the strict transform of the line $(\bar{l})$ is locally defined by

$$(l_1) : x_1=0,$$

the strict transform of the conic $(\bar{c})$ is locally defined by

$$(c_1) : y_1-x_1=0,$$

and the strict transform of $(\bar{\mathscr{C}})$ has the local equation

$$(\mathscr{C}_1) : E+Cx_1+Dy_1+Bx_1y_1+Ax_1^2y_1=0.$$

Since $q_2 \succ_1 q_1$ in the direction of the line $(l) : x=0$, we have $E_1 \cap l_1 \cap c_1 \cap \mathscr{C}_1 = q_2=(0,0)$. It follows that $E=0$.

In the first chart of the blow-up of $\mathbb{C}^2_{x_1,y_1}$ at $q_2=(0,0)$, given in coordinates by $x_1=x_2, y_1=x_2y_2$, the exceptional curve has the local equation

$$(E_2) : x_2=0,$$

the strict transform of the line (c_1) is locally defined by

$$(c_2) : y_2-1=0,$$

and the strict transform of $(\mathscr{C}_1)$ has the local equation

$$(\mathscr{C}_2) : C+Dy_2+Bx_2y_2+Ax_2^2y_2=0.$$

Since $q_3 \succ_1 q_2 \succ_1 q_1$ in the direction of the conic , we have $E_2 \cap c_2 \cap \mathscr{C}_2 = q_3 = (0,1)$. It follows that $C=-D$.

Therefore, the irreducible plane curve $(\mathscr{C})$ is given by

$$(\mathscr{C}) : Ax^2+Bxy+D(y^2-xz)=0, \tag{9}$$

where $[A:B:D] \in \mathbb{P}^2$.

By Equation (8) and (9), we can check that the following involution map is the one we are looking for:

$$[x:y:z] \mapsto [x^2:xy:y^2-xz].$$

Remark 3.1. *It is classifically well-known that any quadratic plane Cremona map is equivalent to one and only one among σ, ρ and τ, which are called the three* fundamental quadratic plane Cremona maps.

3.2. Some examples of cubic plane Cremona maps

Three examples of cubic plane Cremona maps are given in this section.

Example 3.4. We are now dealing with constructing a cubic plane Cremona map where the double base-point is $p_0 = [0:0:1]$, the simple base-points are $p_1 = [0:1:0], p_2 = [1:0:-1]$, $p_3 \succ_1 p_1$ in the direction of the line $(l): z = 0$, and $p_4 \succ_1 p_3 \succ_1 p_1$ in the direction of the conic $(c): x^2 + xz + yz = 0$.

Suppose that $\varphi \in \mathrm{Cr}(\mathbb{P}^2)$ is such a map, and that

$$\varphi \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2, \qquad [x:y:z] \mapsto [\varphi_0(x,y,z) : \varphi_1(x,y,z) : \varphi_2(x,y,z)].$$

where $\varphi_0, \varphi_1, \varphi_2 \in \mathbb{C}[x,y,z]$ are three homogeneous polynomials of degree 3 with no common factor.

A general element $(\mathscr{C})$ of the homoloidal net $\mathscr{L}_\varphi$ defining φ is an irreducible plane curve of the form

$$(\mathscr{C}): \lambda_0\varphi_0(x,y,z) + \lambda_1\varphi_1(x,y,z) + \lambda_2\varphi_2(x,y,z) = 0, \tag{10}$$

where $[\lambda_0 : \lambda_1 : \lambda_2] \in \mathbb{P}^2$.

However, $(\mathscr{C})$ can also be written as follows

$$(\mathscr{C}): a_1x^3 + a_2x^2y + a_3x^2z + a_4xy^2 + a_5xyz + a_6xz^2 + a_7y^3 + a_8y^2z + a_9yz^2 + a_{10}z^3 = 0,$$

for some $a_1, a_2, \ldots, a_{10} \in \mathbb{C}$ not all zero.

Since p_0 is a double base-point of $(\mathscr{C})$, it follows that $a_6, a_9, a_{10} = 0$. Since p_1 and p_2 are two simple proper base-points of $(\mathscr{C})$, we have $a_7 = 0$ and $a_3 = a_1$. In other words, $(\mathscr{C})$ is given by

$$(\mathscr{C}): a_1x^3 + a_2x^2y + a_1x^2z + a_4xy^2 + a_5xyz + a_8y^2z = 0,$$

for some $a_1, a_2, a_4, a_5, a_8 \in \mathbb{C}$

Since $p_3 \succ_1 p_1$, we work on the affine chart $U_1 = \{[x:y:z] \in \mathbb{P}^2 | y \neq 0\} \simeq \mathbb{C}^2_{\bar{x},\bar{z}}$. The point p_1 corresponds to the point $\bar{p}_1 = (0,0)$, the line (l) is locally defined by

$$(\bar{l}): \bar{z} = 0,$$

the conic (c) is locally defined by

$$(\bar{c}): \bar{x}^2 + \bar{x}\bar{z} + \bar{z} = 0,$$

and the curve $(\mathscr{C})$ is locally defined by

$$(\bar{\mathscr{C}}): a_4\bar{x} + a_8\bar{z} + a_5\bar{x}\bar{z} + a_2\bar{x}^2 + a_1\bar{x}^3 + a_1\bar{x}^2\bar{z} = 0.$$

In the first chart of the blow-up of $\mathbb{C}^2_{\bar{x},\bar{z}}$ at $\bar{p}_1 = (0,0)$, given in coordinates by $\bar{x} = x_1, \bar{z} = x_1z_1$, the exceptional curve has the local equation

$$(E_1) : x_1 = 0,$$

the strict transform of the line $(\bar{l})$ is locally defined by

$$(l_1) : z_1 = 0,$$

the strict transform of the conic $(\bar{c})$ is locally defined by

$$(c_1) : x_1z_1 + x_1 + z_1 = 0$$

and the strict transform of $(\bar{\mathscr{C}})$ has the local equation

$$(\mathscr{C}_1) : a_4 + a_8z_1 + a_2x_1 + a_5x_1z_1 + a_1x_1^2 + a_1x_1^2z_1 = 0.$$

Since $p_3 \succ_1 p_1$ in the direction of the line $(l) : z = 0$, we have $E_1 \cap l_1 \cap c_1 \cap \mathscr{C}_1 = p_3 = (0,0)$. It follows that $a_4 = 0$.

In the first chart of the blow-up of $\mathbb{C}^2_{x_1,z_1}$ at $p_3 = (0,0)$, given in coordinates by $x_1 = x_2, z_1 = x_2z_2$, the exceptional curve has the local equation

$$(E_2) : x_2 = 0,$$

the strict transform of the conic (c_1) is locally defined by

$$(c_2) : x_2z_2 + z_2 + 1 = 0,$$

and the strict transform of $(\mathscr{C}_1)$ has the local equation

$$(\mathscr{C}_2) : a_2 + a_1x_2 + a_8z_2 + a_5x_2z_2 + a_1x_2^2z_2 = 0.$$

Since $p_4 \succ_1 p_3 \succ_1 p_1$ in the direction of the conic $(c) : x^2 + xz + yz = 0$, we have $E_2 \cap c_2 \cap \mathscr{C}_2 = p_4 = (0,-1)$. It follows that $a_8 = a_2$.

Therefore, the irreducible plane curve $(\mathscr{C})$ is given by

$$(\mathscr{C}) : a_1x^2(x+z) + a_2y(x^2 + yz) + a_5xyz = 0, \tag{11}$$

where $[a_1 : a_2 : a_5] \in \mathbb{P}^2$.

By (10) and (11), we can check that the following map is the one that we are looking for:

$$[x : y : z] \mapsto [x^2(x+z) : y(x^2 + yz) : xyz].$$

Its inverse is of the following form:

$$[x : y : z] \mapsto [(x+z)(xy - z^2) : (y+z)(xy - z^2) : z(x+z)^2].$$

Example 3.5. A cubic plane Cremona map where the double base-point is $p_0 = [0:0:1]$, the simple base-points are $p_1 = [1:0:0], p_2 = [0:1:0]$ and $p_4 \succ_1 p_3 \succ_1 p_1$ in the direction of the line $(l): y-z=0$. In other words, p_1, p_3, p_4 are collinear and constructed as follows:

Suppose that $\varphi \in \mathrm{Cr}(\mathbb{P}^2)$ is such a map, and that

$$\varphi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2, \qquad [x:y:z] \mapsto [\varphi_0(x,y,z) : \varphi_1(x,y,z) : \varphi_2(x,y,z)].$$

where $\varphi_0, \varphi_1, \varphi_2 \in \mathbb{C}[x,y,z]$ are three homogeneous polynomials of degree 3 with no common factor.

A general element $(\mathscr{C})$ of the homoloidal net $\mathscr{L}_\varphi$ defining φ is an irreducible plane curve of the form

$$(\mathscr{C}): \lambda_0\varphi_0(x,y,z) + \lambda_1\varphi_1(x,y,z) + \lambda_2\varphi_2(x,y,z) = 0, \tag{12}$$

where $[\lambda_0 : \lambda_1 : \lambda_2] \in \mathbb{P}^2$.

However, $(\mathscr{C})$ can also be written as follows

$$(\mathscr{C}): a_1x^3 + a_2x^2y + a_3x^2z + a_4xy^2 + a_5xyz + a_6xz^2 + a_7y^3 + a_8y^2z + a_9yz^2 + a_{10}z^3 = 0,$$

for some $a_1, a_2, \ldots, a_{10} \in \mathbb{C}$ not all zero.

Since p_0 is a double base-point of $(\mathscr{C})$, it follows that $a_6, a_9, a_{10} = 0$. Since p_1 and p_2 are two simple proper base-points of $(\mathscr{C})$, we have $a_1 = 0$ and $a_7 = 0$. In other words, $(\mathscr{C})$ is given by

$$(\mathscr{C}): a_2x^2y + a_3x^2z + a_4xy^2 + a_5xyz + a_8y^2z = 0,$$

for some $a_2, a_3, a_4, a_5, a_8 \in \mathbb{C}$.

Since $p_3 \succ_1 p_1$, we work on the affine chart $U_0 = \{[x:y:z] \in \mathbb{P}^2 | x \neq 0\} \simeq \mathbb{C}^2_{\bar{y},\bar{z}}$. The point p_1 corresponds to the point $\bar{p}_1 = (0,0)$, the line (l) is locally defined by

$$(\bar{l}): \bar{y} - \bar{z} = 0,$$

and the curve $(\mathscr{C})$ is locally defined by

$$(\bar{\mathscr{C}}): a_2\bar{y} + a_3\bar{z} + a_4\bar{y}^2 + a_5\bar{y}\bar{z} + a_8\bar{y}^2\bar{z} = 0.$$

In the first chart of the blow-up of $\mathbb{C}^2_{\bar{y},\bar{z}}$ at $\bar{p}_1 = (0,0)$, given in coordinates by $\bar{y} = y_1, \bar{z} = y_1z_1$, the exceptional curve has the local equation

$$(E_1): y_1 = 0,$$

the strict transform of the line $(\bar{l})$ is locally defined by

$$(l_1) : 1 - z_1 = 0,$$

and the strict transform of $(\bar{\mathscr{C}})$ has the local equation

$$(\mathscr{C}_1) : a_2 + a_3 z_1 + a_4 y_1 + a_5 y_1 z_1 + a_8 y_1^2 z_1 = 0.$$

Since $p_3 \succ_1 p_1$ in the direction of the line $(l) : y - z = 0$, we have $E_1 \cap l_1 \cap \mathscr{C}_1 = p_3 = (0,1)$. It follows that $a_3 = -a_2$.

Change coordinates: $y_1 = Y, z_1 = Z + 1$. In the new coordinates, the point p_3 correspond to the point $\bar{p}_3 = (0,0)$, the line (l_1) is given by

$$(\bar{l}_1) : Z = 0,$$

and the plane curve $(\mathscr{C}_1)$ has the local equation

$$(\bar{\mathscr{C}}_1) : -a_2 Z + a_4 Y + a_5 Y(Z+1) + a_8 Y^2 (1+Z) = 0.$$

In the first chart of the blow-up of $\mathbb{C}^2_{Y,Z}$ at $\bar{p}_3 = (0,0)$, given in coordinates by $Y = y_2, Z = y_2 z_2$, the exceptional curve has the local equation

$$(E_2) : y_2 = 0,$$

the strict transform of the line $(\bar{l}_1)$ is locally defined by

$$(l_2) : z_2 = 0,$$

and the strict transform of $(\bar{\mathscr{C}}_1)$ has the local equation

$$(\mathscr{C}_2) : (a_4 + a_5) + a_8 y_2 - a_2 z_2 + (a_5 + a_8) y_2 z_2 = 0.$$

Since $p_4 \succ_1 p_3 \succ_1 p_1$ in the direction of the line $(l) : y - z = 0$, we have $E_2 \cap l_2 \cap \mathscr{C}_2 = p_4 = (0,0)$. It follows that $a_5 = -a_4$.

Therefore, the irreducible plane curve $(\mathscr{C})$ is given by

$$(\mathscr{C}) : a_2 x^2 (y - z) + a_4 xy(y - z) + a_8 y^2 z = 0, \tag{13}$$

where $[a_2 : a_4 : a_8] \in \mathbb{P}^2$.

By (12) and (13), we can check that the following map is the one that we are looking for:

$$[x : y : z] \mapsto [x^2(y - z) : xy(y - z) : y^2 z].$$

Its inverse is of the following form:

$$[x : y : z] \mapsto [(xz + y^2)x : (xz + y^2)y : xyz].$$

Example 3.6. Finally, we create a cubic plane Cremona map where the double base-point is $p_0 = [0:0:1]$, the simple base-points are $p_1 = [1:0:0], p_2 = [0:1:0]$, and $p_3 \succ_1 p_1$ in the direction of the line $(l): z = 0$, which is the line passing through p_1, p_2, (in other words p_1, p_2, p_3 are collinear) and $p_4 \succ_1 p_3 \succ_1 p_1$ in the direction of the conic $(c): y^2 - xz = 0$.

Suppose that $\varphi \in \mathrm{Cr}(\mathbb{P}^2)$ is such a map, and that

$$\varphi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2, \qquad [x:y:z] \mapsto [\varphi_0(x,y,z) : \varphi_1(x,y,z) : \varphi_2(x,y,z)].$$

where $\varphi_0, \varphi_1, \varphi_2 \in \mathbb{C}[x,y,z]$ are three homogeneous polynomials of degree 3 with no common factor.

A general element $(\mathscr{C})$ of the homoloidal net $\mathscr{L}_\varphi$ defining φ is an irreducible plane curve of the form

$$(\mathscr{C}): \lambda_0\varphi_0(x,y,z) + \lambda_1\varphi_1(x,y,z) + \lambda_2\varphi_2(x,y,z) = 0, \tag{14}$$

where $[\lambda_0 : \lambda_1 : \lambda_2] \in \mathbb{P}^2$.

However, $(\mathscr{C})$ can also be written as follows

$$(\mathscr{C}): a_1x^3 + a_2x^2y + a_3x^2z + a_4xy^2 + a_5xyz + a_6xz^2 + a_7y^3 + a_8y^2z + a_9yz^2 + a_{10}z^3 = 0,$$

for some $a_1, a_2, \ldots, a_{10} \in \mathbb{C}$ not all zero.

Since p_0 is a double base-point of $(\mathscr{C})$, it follows that $a_6, a_9, a_{10} = 0$. Since p_1 and p_2 are two simple proper base-points of $(\mathscr{C})$, we have $a_1 = 0$ and $a_7 = 0$. In other words, $(\mathscr{C})$ is given by

$$(\mathscr{C}): a_2x^2y + a_3x^2z + a_4xy^2 + a_5xyz + a_8y^2z = 0,$$

for some $a_2, a_3, a_4, a_5, a_8 \in \mathbb{C}$.

Since $p_3 \succ_1 p_1$, we work on the affine chart $U_0 = \{[x:y:z] \in \mathbb{P}^2 | x \neq 0\} \simeq \mathbb{C}^2_{\bar{y},\bar{z}}$. The point p_1 corresponds to the point $\bar{p}_1 = (0,0)$, the line (l) is locally defined by

$$(\bar{l}): \bar{z} = 0,$$

the conic (c) is locally defined by

$$(\bar{c}): \bar{y}^2 - \bar{z} = 0,$$

and the curve $(\mathscr{C})$ is locally defined by

$$(\bar{\mathscr{C}}): a_2\bar{y} + a_3\bar{z} + a_4\bar{y}^2 + a_5\bar{y}\bar{z} + a_8\bar{y}^2\bar{z} = 0.$$

In the first chart of the blow-up of $\mathbb{C}^2_{\bar{y},\bar{z}}$ at $\bar{p}_1 = (0,0)$, given in coordinates by $\bar{y} = y_1, \bar{z} = y_1z_1$, the exceptional curve has the local equation

$$(E_1): y_1 = 0,$$

the strict transform of the line $(\bar{l})$ is locally defined by

$$(l_1) : z_1 = 0,$$

the strict transform of the conic $(\bar{c})$ is locally defined by

$$(c_1) : y_1 - z_1 = 0,$$

and the strict transform of $(\bar{\mathscr{C}})$ has the local equation

$$(\mathscr{C}_1) : a_2 + a_3 z_1 + a_4 y_1 + a_5 y_1 z_1 + a_8 y_1^2 z_1 = 0.$$

Since $p_3 \succ_1 p_1$ in the direction of the line $(l) : z = 0$, we have $E_1 \cap l_1 \cap c_1 \cap \mathscr{C}_1 = p_3 = (0,0)$. It follows that $a_2 = 0$.

In the first chart of the blow-up of $\mathbb{C}^2_{y_1,z_1}$ at $p_3 = (0,0)$, given in coordinates by $y_1 = y_2, z_1 = y_2 z_2$, the exceptional curve has the local equation

$$(E_2) : y_2 = 0,$$

the strict transform of the line (c_1) is locally defined by

$$(c_2) : 1 - z_2 = 0,$$

and the strict transform of $(\mathscr{C}_1)$ has the local equation

$$(\mathscr{C}_2) : a_4 + a_3 z_2 + a_5 y_2 z_2 + a_8 y_2^2 z_2 = 0.$$

Since $p_4 \succ_1 p_3 \succ_1 p_1$ in the direction of the conic $(c) : y^2 - xz = 0$, we have $E_2 \cap c_2 \cap \mathscr{C}_2 = p_4 = (0,1)$. It follows that $a_3 = -a_4$.

Therefore, the irreducible plane curve $(\mathscr{C})$ is given by

$$(\mathscr{C}) : a_5 xyz + a_8 y^2 z + a_4 x(y^2 - xz) = 0, \tag{15}$$

where $[a_5 : a_8 : a_4] \in \mathbb{P}^2$.

By (14) and (15), we can check that the following map is the one we are looking for:

$$[x : y : z] \mapsto [xyz : y^2 z : x(y^2 - xz)].$$

Its inverse is of the following form:

$$[x : y : z] \mapsto [(x^2 + yz)x : y(x^2 + yz) : xy^2].$$

Remark 3.2. *Let ψ_{19} be Type 19 in Calabri and Nguyen (2020), that is*

$$\psi_{19} = [x(x^2 + xz + yz) : y(x^2 + xz + yz) : xyz].$$

Then, ψ_{19} is equivalent to our equation in Example 3.4. In deed, one has

$$[x + z : z + y : z] \circ [x^2(x + z) : y(x^2 + yz) : xyz] = [x(x^2 + xz + yz) : y(x^2 + xz + yz) : xyz].$$

Remark 3.3. *Examples 3.5, 3.6 are Types 17, 18, respectively, in Calabri and Nguyen (2020).*

Remark 3.4. *In exactly the same way, we created 31 explicit formulas of cubic plane Cremona maps and 449 explicit formulas of quartic plane de Jonquières maps. Lists of these maps are given in Calabri and Nguyen (2020) and Nguyen (2020).*

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