TOPOLOGICAL INVARIANTS AND MILNOR FIBRE FOR $\mathscr{A}$-FINITE GERMS $\mathbb{C}^2 \to \mathbb{C}^3$

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Abstract

This note is the observation that a simple combination of known results shows that the usual analytic invariants of a finitely determined multi-germ $f\colon (\mathbb{C}^2,S) \to (\mathbb{C}^3,0)$ —namely the image Milnor number μ_I, the number of crosscaps and triple points, C and T, and the Milnor number $\mu(\Sigma)$ of the curve of double points in the target— depend only on the embedded topological type of the image of f. As a consequence one obtains the topological invariance of the sign-refined Smale invariant for immersions $j\colon S^3 \looparrowright S^5$ associated to finitely determined map germs $(\mathbb{C}^2,0) \to (\mathbb{C}^3,0)$.

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1. INTRODUCTION

Let $f\colon (\mathbb{C}^2,S)\to(\mathbb{C}^3,0)$ be an $\mathscr{A}$-finite multi-germ. For $\mathscr{A}$-finiteness and related notions, we refer the reader to Mond and Nuño-Ballesteros (2020). We write

$$X = \mathrm{im} f \subseteq (\mathbb{C}^3,0)$$

for the image of f. Associated to X, there is the *Milnor fiber*

$$\mathbb{F} = g^{-1}(\delta)\cap B,$$

where $g=0$ is a reduced equation for X, B is a small ball around the origin of $\mathbb{C}^3$, and δ is nonzero and small enough. In the previous version of this work, we related Betti numbers $b_i(\mathbb{F})$ (with coefficients in $\mathbb{Z}_2$) to analytic invariants associated to $\mathscr{A}$-finite germs, namely the numbers C,T of crosscaps and triple points and the Milnor number $\mu(D)$ of the double point curve in the source of f. The main point of finding these relations was that, together with some other relations and the fact that the Betti numbers (with any coefficients) of $\mathbb{F}$ are invariants of the embedded topological type of X (see Lê, 1973), they implied then that C,T, the image Milnor number μ_I and the Milnor number $\mu(\Sigma)$ of the double point in the target, depend only on the topological type of X. As a consequence we proved the topological invariance of the sign-refined Smale invariant for immersions $(\mathbb{C}^2,0)\to(\mathbb{C}^3,0)$ associated with finite map germs.

However, Siersma communicated to us that our expression for $b_1(\mathbb{F})$ was incorrect, and pointed us to a paper (van Straten, 2017) we were unaware of, where the correct formula is provided. After replacing our mistaken statement by the correct expression of $b_1(\mathbb{F})$, the approach we had adopted to show the topological invariances becomes hopeless.

Nevertheless, it turns out that one can combine the Euler characteristic formula (which was obtained before by other authors, as we will mention below), the Marar-Mond formulae, the topological invariance of $C-3T$ proved by Némethi and Pintér (2015), and a result of Lê (1973) to recover the topological invariances claimed above.

We shall describe now the $\mathscr{A}$-invariants of f we are dealing with: The *target double point space* of f is

$$\Sigma = \mathrm{Sing}(X).$$

Being a reduced curve with an isolated singularity, Σ has a well defined Milnor number (see Buchweitz & Greuel, 1980). The *source double point space*

$$D = f^{-1}(\Sigma)$$

is a disjoint collection of germs of plane curves with isolated singularity (Marar & Mond, 1989; Marar et al., 2012). It is immediate that the Milnor numbers of each component of D and $\mu(\Sigma)$ are holomorphic $\mathscr{A}$-invariants of f. We denote by $\mu(D)$ the sum $\sum_{z\in S}\mu(D,z)$.

Let $F = (f_s, s)\colon (\mathbb{C}^2 \times \mathbb{C}, S) \to (\mathbb{C}^3 \times \mathbb{C}, 0)$ be a stabilization of f, and let $f_s\colon U_s \to \mathbb{C}^3$ be a stable perturbation of f. The space

$$X_s = f_s(U_s)$$

is called the *disentanglement* of X. The homotopy type of X_s does not depend on the chosen stabilization, and it is that of a wedge of 2-dimensional spheres (Mond, 1991). The number of spheres,

$$\mu_I = b_2(X_s),$$

is a holomorphic $\mathscr{A}$-invariant of f known as the *image Milnor number of f*.

Being the image of a stable mapping, the disentanglement can only exhibit the following three classes of singularities: transverse double points, cross-caps, and transverse triple points. Cross-caps and triple points are isolated singularity types and give rise to holomorphic $\mathscr{A}$-invariants of f, namely,

$$C = \#\text{ cross-caps in } X_s,$$

$$T = \#\text{ triple points in } X_s.$$

2. THE BETTI NUMBERS OF $\mathbb{F}$ (CORRECTED)

In Theorem 1.1 of the previous version of this work, we mistakenly claimed that the first Betti number of $\mathbb{F}$, with $\mathbb{Z}_2$ coefficients, behaves as follows:

(1) If $C = 0$ then $b_1(\mathbb{F}, \mathbb{Z}_2) = T + 1$.

(2) If $C \neq 0$ then $b_1(\mathbb{F}, \mathbb{Z}_2) = T$.

To compute $b_1(\mathbb{F})$, we employed the homology splitting method, introduced in Siersma (1983) and Siersma (1988). A full explanation of the method, and further related references, can be found in Siersma's survey paper (see Siersma, 2001), and in the recent paper by Siersma and Tibar (2017), which presents the latest evolution of the method for a 1-dimensional critical set. However, Siersma pointed out that $b_1(\mathbb{F})$ had been computed previously by van Straten and the result differed from ours. A mistake, related to the local model around triple points, was then spotted in our computations. The correct statement, a particular case of van Straten (2017), is as follows:

Proposition 2.1. *Let $f\colon (\mathbb{C}^2, S) \to (\mathbb{C}^3, 0)$ be an $\mathscr{A}$-finite multi-germ. Then*

$$b_1(\mathbb{F}) = |S| - 1.$$

Having found $b_1(\mathbb{F})$, calculating $b_2(\mathbb{F})$ amounts to computing the Euler characteristic of $\mathbb{F}$. This has been previously achieved by Siersma and Massey, Siersma (1991), Section 4 and Massey and Siersma (1992), Section 4. It can also be deduced from T. de

Jong's formula for the Euler characteristic of the Milnor fiber (see de Jong, 1990), and formulae from de Jong and Pellikaan and from Marar and Mond for multi-germs (see Lemma 3.2). It also can be deduced from Siersma and Tibar (2017), Proposition 3.3.

In what follows, by saying that f is non regular, we mean that f is not a mono-germ of embedding.

Proposition 2.2. *[Siersma, Massey & Siersma] Let $f\colon (\mathbb{C}^2,S)\to(\mathbb{C}^3,0)$ be an $\mathscr{A}$-finite non regular multi-germ. Then*

$$\chi(\mathbb{F}) = \mu(D) + 2C - 3T.$$

The formula is not identical to the expressions in Siersma (1991) and Massey and Siersma (1992), but it is equivalent to them using the Marar-Mond formulae given below.

3. TOPOLOGICAL INVARIANTS

A key element for the proofs of topological invariance is contained in the work of Némethi and Pintér (2015), where among other things, they prove the following: If a finite *mono-germ* has $C<\infty$, then C coincides with the sign-refined Smale invariant of the associated link embedding. As a consequence of this, they obtain the C^∞ $\mathscr{A}$-invariance of the number of cross-caps, and they prove the following result.

Proposition 3.1. *If $f\colon (\mathbb{C}^2,0)\to(\mathbb{C}^3,0)$ is an $\mathscr{A}$-finite mono-germ, then $C-3T$ is invariant by topological $\mathscr{A}$-equivalence.*

Apart from the results of Némethi and Pintér (2015), we use the following adaptations of formulae for mono-germs (Marar & Mond, 1989; Mond, 1991) for multi-germs. (A previous, unpublished mono-germ version of the first formula is attributed to de Jong and Pellikaan; see Marar & Mond, 1989.) The first two formulae are used in the proof of Theorem 3.3; the third and fourth are included for completeness.

Lemma 3.2. *Let $f\colon (\mathbb{C}^2,S)\to(\mathbb{C}^3,0)$ be a non-regular $\mathscr{A}$-finite multi-germ and let $r=|S|$. Then*

$$\mu_I = \mu(D) - \mu(\Sigma) - T,$$

$$\mu(\Sigma) = \frac{1}{2}(\mu(D) - C + 2T - r + 2).$$

Let σ be the number of singular branches of the image of f. Let D^2 be the closure of the set $\{(x,x')\in\mathbb{C}^2\times\mathbb{C}^2 \mid x\neq x', f(x)=f(x')\}$ and let D^2/S_2 be the complex quotient space obtained by identifying (x,x') and (x',x). Then

$$\mu(D^2) = \mu(D) - 6T + r(r-2) + \sigma,$$

$$\mu(D^2/S_2) = \mu(\Sigma) - 4T + \frac{r(r-1)}{2} + \sigma - 1.$$

Proof.

We include only the proof of the first formula; the proofs of the others have a similar flavor and can easily be adapted from the original formulae of Marar and Mond (1989). Using the same notation as in the proof of the Euler characteristic formula $\chi(\mathbb{F}) = \mu(D)+2C-3T$, we have that $\mu_I = \chi(X_s)-1 = \chi(X_s\setminus\Sigma_s)+\chi(\Sigma_s)-1 = \chi(U_s)-\chi(D_s)+\chi(\Sigma_s)-1$. The claim then follows from the equalities $\chi(U_s) = r$, $\chi(D_s) = r-\mu(D)+3T$ and $\chi(\Sigma_s) = 1-\mu(\Sigma)+2T$. □

Theorem 3.3. *The numbers μ_I, C, T, and $\mu(\Sigma)$ and the collection $\mu(D,z), z\in S$ depend only on the embedded topological type of X. In particular, they are topological $\mathscr{A}$-invariants of f.*

Proof.

Let $f\colon(\mathbb{C}^2,S)\to(\mathbb{C}^3,0)$ and $\tilde{f}\colon(\mathbb{C}^2,\tilde{S})\to(\mathbb{C}^3,0)$ be two $\mathscr{A}$-finite mappings, and let $(X,0)$ and $(\tilde{X},0)$ be their images. Assume that there exists a homeomorphism $\psi\colon(\mathbb{C}^3,0)\to(\mathbb{C}^3,0)$ taking $(X,0)$ to $(\tilde{X},0)$. Denote the double point loci at the target and source of f and $\tilde{f}$ Σ, D, $\tilde{\Sigma}$, and $\tilde{D}$, respectively. The local homeomorphism type of $(X,0)$ at a general point of Σ (the union of two planes meeting in a line) is different from the local homeomorphism type of a point of $\tilde{X}\setminus\tilde{\Sigma}$; hence we have $\psi(\Sigma)=\tilde{\Sigma}$. Note that f and $\tilde{f}$ are the normalization mappings of $(X,0)$ and $(\tilde{X},0)$. By uniqueness of the topological normalization, there exists a homeomorphism $\phi\colon(\mathbb{C}^2,S)\to(\mathbb{C}^2,\tilde{S})$ lifting ψ. In other words, the germs f and $\tilde{f}$ are topologically $\mathscr{A}$-equivalent.

The topological $\mathscr{A}$-equivalence implies that the cardinality of S coincides with the cardinality of S'. Since $\psi(\Sigma)=\Sigma'$, we deduce that $\phi(D)=D'$. As D and D' are plane curves, this implies by Lê (1973) that the collection of Milnor numbers of the multi-germ of plane curves (D,S) coincides with that of (D',S').

Let C and $\tilde{C}$ be the number of cross-caps of f and $\tilde{f}$. Let $r=|S|$, and for $j=1,\dots r$, let $f^{(j)},\tilde{f}^{(j)}$, and $\phi^{(j)}$ be the branches of $f,\tilde{f}$, and ϕ, ordered in such a way that $f^{(j)}$ and $\tilde{f}^{(j)}$ are topologically $\mathscr{A}$-equivalent via $\phi^{(j)}$ and ψ. Observe that all $f^{(j)}$ and $\tilde{f}^{(j)}$ are $\mathscr{A}$-finite; hence, they have well-defined numbers of cross-caps and triple points: $C^{(j)},T^{(j)},\tilde{C}^{(j)}$, and $\tilde{T}^{(j)}$. Similarly, let $\mathbb{F}^{(j)},\tilde{\mathbb{F}}^{(j)},D^{(j)}$, and $\tilde{D}^{(j)}$ be the corresponding Milnor fibers and double point curves. Since $f^{(j)}$ and $\tilde{f}^{(j)}$ are topologically $\mathscr{A}$-equivalent, their images have the same embedded topological type. This implies the equality $\chi(\mathbb{F}^{(j)})=\chi(\tilde{\mathbb{F}}^{(j)})$ by Lê (1973). By the $\mathscr{A}$-equivalence of $f^{(j)}$ and $\tilde{f}^{(j)}$ we have that $D^{(j)}$ and $\tilde{D}^{(j)}$ have the same embedded topological type, and hence $\mu(D^{(j)})=\mu(\tilde{D}^{(j)})$. Comparing both Euler characteristics, we conclude that $2C^{(j)}-3T^{(j)}=2\tilde{C}^{(j)}-3\tilde{T}^{(j)}$, by virtue of Proposition 2.2. On the other hand, Proposition 3.1 states that $C^{(j)}-3T^{(j)}=\tilde{C}^{(j)}-3\tilde{T}^{(j)}$. Together they imply $C^{(j)}=\tilde{C}^{(j)}$ and $T^{(j)}=\tilde{T}^{(j)}$ and, since $C=\sum_{j=1}^{r}C^{(j)}$ and $\tilde{C}=\sum_{j=1}^{r}\tilde{C}^{(j)}$, we conclude that $C=\tilde{C}$.

So far we know that $|S|$, $\chi(\mathbb{F})$, C and the collection of Milnor numbers of the multi-germ (D,S) depend only on the embedded topological type of X. That the same applies to T,μ_I and $\mu(\Sigma)$ follows immediately from the formulae $\chi(\mathbb{F})=\mu(D)+2C-3T$,

$\mu(\Sigma) = \frac{1}{2}(\mu(D) - C + 2T - r + 2)$ and $\mu_I = \mu(D) - \mu(\Sigma) - T$. □

As a direct consequence, we obtain the following (see Némethi & Pintér, 2015, for definitions).

Corollary 3.4. *The Smale invariant is a topological $\mathscr{A}$-invariant for those immersions $j\colon S^3 \looparrowright S^5$ associated to $\mathscr{A}$-finite map germs $(\mathbb{C}^2, S) \to (\mathbb{C}^3, 0)$. More strongly, it only depends on the embedded topological type of the image of the immersion.*

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